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ON AUSLANDER-REITEN SEQUENCES FOR BOREL-SCHUR ALGEBRAS

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Abstract: We determine Auslander-Reiten sequences for a large class of simple modules over Borel-Schur algebras. We also investigate functors between Borel-Schur algebras of different ranks. A partial information on the structure of the socles of Borel-Schur algebras is given.

Keywords: Auslander-Reiten sequences, Schur algebra, Borel-Schur algebra.
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1. Introduction

Consider the general linear group $GL_n(\mathbb{K})$ where $\mathbb{K}$ is an infinite field, and let B^+ be the Borel subgroup of $GL_n(\mathbb{K})$ consisting of all upper triangular matrices in $GL_n(\mathbb{K})$. The Schur algebras $S(n,r)$ and $S(B^+) := S(B^+, n, r)$ corresponding to $GL_n(\mathbb{K})$ and B^+, respectively, are powerful tools in the study of polynomial representations of $GL_n(\mathbb{K})$ and B^+. In particular, the simple modules of $S(B^+)$ labelled by partitions induce to Weyl modules for $S(n,r)$, and Weyl modules are central objects of study. In the recent paper [8], Borel-Schur algebras were crucial to construct resolutions of Weyl modules. Therefore one would like to understand homological properties of $S(B^+)$.

Auslander-Reiten sequences, also known as almost split sequences, are an important invariant of the module category of a finite-dimensional algebra. They provide part of a presentation of the module category: one takes isomorphism classes of indecomposable modules; then one takes homomorphisms f between indecomposables that do not have any non-trivial factorisations, that is if $f = g \circ h$ then one of g or h must be split. Such homomorphisms

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are called irreducible, and they are precisely the components of maps occuring in Auslander-Reiten sequences. Hence if one knows the Auslander-Reiten sequences, then one knows all maps which can be obtained as compositions of irreducible maps, and further more, one obtains some relations between irreducible maps which come from the short exact sequences. These are known as 'mesh relations', and for many algebras of finite type, this gives a complete presentation of the module category in this sense.

Auslander-Reiten sequences have many applications, such as understanding distinguished modules. To do so, one would to like to know their position in the Auslander-Reiten quiver.

In this paper, we determine Auslander-Reiten sequences for a large class of simple $S(B^+)$-modules. For this we use the first two steps of a minimal projective resolution for these modules, which were found in [7]. Then we construct the Auslander-Reiten sequence as a pushout, by following the methods explained in [2, 4]. We are able to do this, for an arbitrary n, under some combinatorial conditions. We note that when these are satisfied, the relevant simple module does not occur in the socle of $S(B^+)$.

On the way, we investigate functors between Borel-Schur algebras of different ranks. We obtain that Auslander-Reiten sequences are not usually preserved by induction.

The paper is organized as follows. Section 2 recalls the definitions of the algebras, and some basic background. In Section 3, we construct Auslander-Reiten sequences ending in a simple module $\mathbb{K}_\lambda$ where λ satisfies a condition given in (3.4). As a by-product we see that this condition imples that $\mathbb{K}_\lambda$ does not occur in the (left) socle of the algebra. The main result is Theorem 3.6.

In Section 4 we consider $n = 2$ and find Auslander-Reiten sequences ending in an arbitrary simple module, that is we deal with the cases missing in Section 3. As an easy consequence of the results in this section we can obtain a necessary and sufficient condition for a simple module to occur in the socle of $S(B^+, 2, r)$. Section 5 considers some cases not covered in Section 3 when $n = 3$.

In Section 6 we discuss reduction of rank. This may be of more general interest. In fact, we prove that if $m < n$, then the induction functor from $S(B^+, m, r)$ to $S(B^+, n, r)$ is exact and preserves irreducible and indecomposable modules. Section 7 summarizes what we have obtained about the socle of the Borel-Schur algebra.

2. Notation and basic results

In this section we establish the notation we will use and give some basic results. We will follow [7] and any undefined term may be found in this reference.

$\mathbb{K}$ is an infinite field of arbitrary characteristic, n and r are arbitrary fixed positive integers and p is any prime number.

For any natural number s, we denote by $\mathbf{s}$ the set $\{1, \ldots, s\}$ and by Σ_s the symmetric group on $\mathbf{s}$. Define the sets of multi-indices $I(n,r)$ and of compositions $\Lambda(n,r)$ by

$$I(n,r) = \{i = (i_1, \ldots, i_r) \mid i_\rho \in \mathbf{n} \text{ all } \rho \in \mathbf{r}\}$$
$$\Lambda(n,r) = \{\, \lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_\nu \in \mathbb{Z},\ \lambda_\nu \geq 0\ (\nu \in \mathbf{n}),\ \sum_{\nu \in \mathbf{n}} \lambda_\nu = r\}.$$

We will often write I instead of $I(n,r)$ and Λ instead of $\Lambda(n,r)$.

Given $i \in I$ and $\lambda \in \Lambda$, we say that i has *weight* λ and write $i \in \lambda$ if $\lambda_\nu = \#\{\,\rho \in \mathbf{r} \mid i_\rho = \nu\}$, for $\nu \in \mathbf{n}$.

The group Σ_r acts on the right of I and of $I \times I$, respectively, by $i\pi = (i_{\pi 1}, \ldots, i_{\pi r})$ and $(i,j)\pi = (i\pi, j\pi)$, all $\pi \in \Sigma_r$ and i, $j \in I$. If i and j are in the same Σ_r-orbit of I we write $i \sim j$. Also $(i,j) \sim (i',j')$ means these two pairs are in the same Σ_r-orbit of $I \times I$. We denote the stabilizer of i in Σ_r by Σ_i, that is $\Sigma_i = \{\,\pi \in \Sigma_r \mid i\pi = i\}$. We write $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j$. Given i, $j \in I$, then $i \leq j$ means $i_\rho \leq j_\rho$ for all $\rho \in \mathbf{r}$, and $i < j$ means $i \leq j$ and $i \neq j$.

We use $\trianglelefteq$ for the "dominance order" on Λ, that is $\alpha \trianglelefteq \beta$ if $\sum_{\nu=1}^{\mu} \alpha_\nu \leq \sum_{\nu=1}^{\mu} \beta_\nu$ for all $\mu \in \mathbf{n}$. Obviously if $i \in \alpha$ and $j \in \beta$ (where α, $\beta \in \Lambda$), then $i \leq j$ implies $\beta \trianglelefteq \alpha$.

Given $\lambda \in \Lambda$, we consider in I the special element

$$l = l(\lambda) = (\underbrace{1, \ldots, 1}_{\lambda_1}, \underbrace{2, \ldots, 2}_{\lambda_2}, \ldots, \underbrace{n, \ldots, n}_{\lambda_n}).$$

Clearly $\Sigma_{l(\lambda)}$ is the parabolic subgroup associated with λ

$$\Sigma_\lambda = \Sigma_{\{1,\ldots,\lambda_1\}} \times \Sigma_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \Sigma_{\{\lambda_1+\cdots+\lambda_{n-1}+1,\ldots,r\}}.$$

For each $\nu \in \mathbf{n-1}$, and each non-negative integer $m \leq \lambda_{\nu+1}$, we define

$$\lambda(\nu,m) = (\lambda_1, \ldots, \lambda_\nu + m, \lambda_{\nu+1} - m, \ldots, \lambda_n) \in \Lambda,$$

and write $l(\nu,m)$ for $l\,(\lambda\,(\nu,m))$. We have $l(\nu,m) \leq l$.

For the notation of λ-tableaux the reader is referred to [7]. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$, we choose the basic λ-tableau

$$T^\lambda = \begin{array}{cccccc} 1 & 2 & \ldots & \lambda_1 & & \\ \lambda_1+1 & \lambda_1+2 & \ldots & \ldots & \ldots & \lambda_1+\lambda_2 \\ \ldots & & & & & \\ \lambda_1+\cdots+\lambda_{n-1}+1 & \ldots & \ldots & \ldots & r & \end{array}$$

The row-stabilizer of T^λ, i.e. the subgroup of Σ_r consisting of all those $\pi \in \Sigma_r$ which preserve the rows of T^λ is the parabolic subgroup Σ_λ.

Given $i \in I$, we define the λ-tableau T_i^λ as

$$T_i^\lambda = \begin{array}{cccccc} i_1 & i_2 & \ldots & i_{\lambda_1} & & \\ i_{\lambda_1+1} & i_{\lambda_1+2} & \ldots & \ldots & \ldots & i_{\lambda_1+\lambda_2} \\ \ldots & & & & & \\ i_{\lambda_1+\ldots\lambda_{n-1}+1} & \ldots & \ldots & \ldots & i_r. & \end{array}$$

Then T_l^λ has only 1's in the first row, 2's in the second row, $\ldots$, n's in row n. Notice also that $T_{l(\nu,m)}^\lambda$ differs from T_l^λ only by the first m entries of row $\nu+1$: these entries are all equal to ν.

We say that a λ-tableau T_i^λ is row-semistandard if the entries in each row of T_i^λ are weakly increasing from left to right. We define

$$I(\lambda) := \left\{ i \in I \,\middle|\, i \le l(\lambda) \text{ and } T_i^\lambda \text{ is row-semistandard } \right\}$$

and

$$J(\lambda) := \left\{ j \in I \,\middle|\, j \ge l(\lambda) \text{ and } T_j^\lambda \text{ is row-semistandard} \right\}.$$

The following obvious fact will be used later in this paper:

$$\text{If } \lambda_n \neq 0 \text{ and } m \le \lambda_n, \text{ then } J(\lambda) = \{ j \in J(\lambda(n-1,m)) \,|\, j \ge l(\lambda) \}. \tag{2.1}$$

Next we recall the definition of Schur algebra and of Borel-Schur algebra as they were introduced in [5].

The general linear group $\mathrm{GL}_n(\mathbb{K})$ acts on $\mathbb{K}^n$ by multiplication. So $\mathrm{GL}_n(\mathbb{K})$ acts on the r-fold tensor product $(\mathbb{K}^n)^{\otimes r}$ by the rule

$$g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r, \text{ all } g \in \mathrm{GL}_n(\mathbb{K}),\ v_1, \ldots, v_r \in \mathbb{K}^n.$$

Extending by linearity this action to the group algebra $\mathbb{K}\mathrm{GL}_n(\mathbb{K})$, we obtain a homomorphism of algebras

$$T\colon \mathbb{K}\mathrm{GL}_n(\mathbb{K}) \to \mathrm{End}_{\mathbb{K}}\left((\mathbb{K}^n)^{\otimes r}\right).$$

The image of T, i.e. $T(\mathbb{K}\mathrm{GL}_n(\mathbb{K}))$ is called the *Schur algebra* for $\mathbb{K}$, n, r and is denoted by $S(n,r)$. Let $B^+ = B^+_{\mathbb{K}}(n,r)$ denote the Borel subgroup of $\mathrm{GL}_n(\mathbb{K})$ consisting of all upper triangular matrices in $\mathrm{GL}_n(\mathbb{K})$. The *Borel-Schur algebra* $S(B^+) = S(B^+, n, r)$ is the subalgebra $T(\mathbb{K}B^+)$ of $S(n,r)$.

Associated with each pair $(i,j) \in I \times I$, there is a well defined element $\xi_{i,j}$ of $S(n,r)$ (see [5]). These elements have the property that $\xi_{i,j} = \xi_{k,h}$ if and only if $(i,j) \sim (k,h)$. If we eliminate repetitions in the set $\{\, \xi_{i,j} \mid (i,j) \in I \times I\}$ then we obtain a basis of $S(n,r)$. Also $S(B^+) = \mathbb{K}\{\, \xi_{i,j} \mid i \leq j,\ (i,j) \in I \times I\}$.

If i has weight $\alpha \in \Lambda$, we write $\xi_{i,i} = \xi_\alpha$. The set $\{\, \xi_\alpha \mid \alpha \in \Lambda\}$ is a set of orthogonal idempotents and $1_{S(n,r)} = \sum_{\alpha\in\Lambda} \xi_\alpha$.

A formula for the product of two basis elements is the following (see [5]): $\xi_{i,j}\xi_{k,h} = 0$, unless $j \sim k$; and

$$\xi_{i,j}\xi_{j,h} = \sum_{\sigma} \left[\Sigma_{i\sigma,h} : \Sigma_{i\sigma,j,h}\right] \xi_{i\sigma,h} \tag{2.2}$$

where the sum is over a transversal $\{\sigma\}$ of the set of all double cosets $\Sigma_{i,j}\sigma\Sigma_{j,h}$ in Σ_j.

Observation 2.1. *(1) $\xi_\alpha\xi_{i,j} = \xi_{i,j}$ or zero, according to $i \in \alpha$ or $i \notin \alpha$. Similarly, $\xi_{i,j}\xi_\beta = \xi_{i,j}$ or zero, according to $j \in \beta$ or $j \notin \beta$.*
(2) If $\Sigma_{i,j}\Sigma_{j,h} = \Sigma_j$, then the product is a scalar multiple of $\xi_{i,h}$.

Here we are particularly interested in products of the type $\xi_{l(\nu,m),l}\xi_{l,j}$, for $l = l(\lambda)$, and $j \in J(\lambda)$, for some $\lambda \in \Lambda$.

Lemma 2.2. *Let $\lambda \in \Lambda$, $\nu \in \mathbf{n-1}$, $0 \leq m \leq \lambda_{\nu+1}$, and $j \in J(\lambda)$. If the $\nu+1$-st row of T^λ_j is constant then $\Sigma_{l(\nu,m),l}\Sigma_{l,j} = \Sigma_l$.*

Proof: We have $\Sigma_l = \Sigma_\lambda$. Now we know that $\Sigma_{l(\nu,m)}$ differs from Σ_λ only in factors ν and $\nu+1$ and in these it is

$$\Sigma_{\{t+1,\ldots,t+\lambda_\nu+m\}} \times \Sigma_{\{t+\lambda_\nu+m+1,\ldots,t+\lambda_\nu+\lambda_{\nu+1}\}},$$

where $t = \lambda_1 + \cdots + \lambda_{\nu-1}$. It follows that the intersection $\Sigma_{l(\nu,m),l}$ differs from Σ_λ only in factor $\nu+1$ and this is

$$\Sigma_{\{t+\lambda_\nu+1,\ldots,t+\lambda_\nu+m\}} \times \Sigma_{\{t+\lambda_\nu+m+1,\ldots,t+\lambda_\nu+\lambda_{\nu+1}\}}.$$

We can write $\Sigma_{l,j} = U_1 \times \cdots \times U_n$, where U_s is a subgroup of Σ_{λ_s}. Therefore the product $\Sigma_{l(\nu,m),l}\Sigma_{l,j} = \Sigma_l$ when the product of the two $(\nu+1)$-st factors is $\Sigma_{\lambda_{\nu+1}}$. This holds if $U_{\nu+1} = \Sigma_{\lambda_{\nu+1}}$, i.e., if the $(\nu+1)$-st row of T^λ_j is constant. ∎

Lemma 2.3. *Let $\lambda \in \Lambda$, $\nu \in \mathbf{n-1}$, $0 \le m \le \lambda_{\nu+1}$. Given $j \in J(\lambda)$, suppose that the $(\nu+1)$-st row of T_j^λ is constant with all entries equal to c, and that c occurs exactly a times in row ν. Then*

$$\xi_{l(\nu,m),l}\xi_{l,j} = \binom{a+m}{m}\xi_{l(\nu,m),j}.$$

If $\nu = n-1$ then the hypothesis holds for all $j \in J(\lambda)$.

Proof: From Lemma 2.2 and Observation 2.1, we know that

$$\xi_{l(\nu,m),l}\xi_{l,j} = \left[\Sigma_{l(\nu,m),j} : \Sigma_{l(\nu,m),l,j}\right]\xi_{l(\nu,m),j}.$$

Now $\Sigma_{l(\nu,m),j}$ and $\Sigma_{l(\nu,m),j,l}$ differ only in factors ν and $\nu+1$. If the entries of row ν of T^λ where c occurs in T_j^λ are $t_1, \ldots, t_a$, then factors ν and $\nu+1$ of $\Sigma_{l(\nu,m),j}$ and $\Sigma_{l(\nu,m),l,j}$ are, respectively,

$$\cdots \times \Sigma_{\{t_1,\ldots,t_a,\lambda_1+\cdots+\lambda_\nu+1,\ldots,\lambda_1+\cdots+\lambda_\nu+m\}} \times \cdots$$

and

$$\cdots \times \Sigma_{\{t_1,\ldots,t_a\}} \times \Sigma_{\{\lambda_1,\ldots,\lambda_\nu+1,\ldots,\lambda_1+\cdots+\lambda_\nu+m\}} \times \cdots.$$

Therefore $\left[\Sigma_{l(\nu,m),j} : \Sigma_{l(\nu,m),l,j}\right] = \binom{a+m}{m}$. ∎

Given $\lambda \in \Lambda$, let $\mathbb{K}_\lambda$ denote the one-dimensional $S(B^+)$-module $\mathbb{K}$, where ξ_λ acts as identity and all the other basis elements, $\xi_{i,j}$, where $i \le j$ and $(i,j) \not\sim (l,l)$, act as zero. It is well known (see [7]), that:

(1) $\{\, \mathbb{K}_\lambda \,|\, \lambda \in \Lambda\}$ is a full set of irreducible $S(B^+)$-modules.
(2) $S(B^+)\,\xi_\lambda$ is a projective cover of $\mathbb{K}_\lambda$.
(3) $S(B^+)\,\xi_\lambda$ has a $\mathbb{K}$-basis $\{\, \xi_{i,l} \,|\, i \in I(\lambda)\}$.
(4) $\mathbb{K}_\lambda$ is a projective $S(B^+)$-module if and only if $\lambda = (r,0,\ldots,0)$. This assertion follows from the fact that $\#I(\lambda) = 1$ if and only if $\lambda = (r,0,\ldots,0)$.

To calculate an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$, we need to know the first two steps of a minimal projective resolution of $\mathbb{K}_\lambda$. For this, define

$$P_0 := S\left(B^+\right)\xi_\lambda; \quad P_1 := \begin{cases} \bigoplus\limits_{\nu\in\mathbf{n-1}} S(B^+)\,\xi_{\lambda(\nu,1)}, & \text{if } \operatorname{char}\mathbb{K} = 0; \\ \bigoplus\limits_{\nu\in\mathbf{n-1}} \bigoplus\limits_{1\le p^{d_\nu}\le\lambda_{\nu+1}} S(B^+)\,\xi_{\lambda(\nu,p^{d_\nu})}, & \text{if } \operatorname{char}\mathbb{K} = p. \end{cases}$$

Then (see [7, (5.4)]) the first two steps of a minimal projective resolution of $\mathbb{K}_\lambda$ are

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} \mathbb{K}_\lambda \to 0. \tag{2.3}$$

Here p_0 is the $S(B^+)$-homomorphism defined on the generator by $p_0(\xi_\lambda) = 1$. Furthermore, p_1 is the $S(B^+)$-homomorphism defined on generators, by

$$p_1(\xi_{\lambda(\nu,1)}) = \xi_{l(\nu,1),l}$$

when $\mathrm{char}(\mathbb{K}) = 0$, and

$$p_1(\xi_{\lambda(\nu,p^{d_\nu})}) = \xi_{l(\nu,p^{d_\nu}),l}$$

when $\mathrm{char}(\mathbb{K}) = p$.

3. Auslander-Reiten sequences

In this section we give an overview of some results and definitions connected to the notion of Auslander-Reiten sequences. Let A be a finite dimensional algebra over $\mathbb{K}$.

A short exact sequence

$$(E) \quad 0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$$

is said to be *Auslander-Reiten* if

(i) (E) is not split;
(ii) S and N are indecomposable;
(iii) If X is an indecomposable A-module and $h\colon X \to S$ is a non-invertible homomorphism of A-modules, then h factors through g.

Alternatively, (E) satisfies (i) and (ii), and if X is an indecomposable A-module then any non-invertible A-module homomorphism $h\colon N \to X$ factors through f.

Theorem 3.1 ([1]). *Given any non-projective indecomposable A-module S, there is an Auslander-Reiten sequence (E) ending with S. Moreover, (E) is determined by S, uniquely up to isomorphism of short exact sequences.*

Several recipes where given in the end of the 80's for the construction of Auslander-Reiten sequences. In this paper we will follow such a recipe due to M. Auslander described in the article [2] of M.C.R. Butler, and we will construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$, for many $\lambda \in \Lambda(n, r)$. The same results can be obtained using J.A. Green's recipe [4]. We will use two contravariant functors

$$D,\ (\cdot)^t\colon \mathrm{mod}S(B^+) \to \mathrm{mod}S(B^+)^{op}$$

where for every $X \in \mathrm{mod}S(B^+)$

$$X^t := \mathrm{Hom}_{S(B^+)}(X, S(B^+)), \qquad D\,(X) := \mathrm{Hom}_{\mathbb{K}}(X, \mathbb{K}).$$

Recall that $S(B^+)$ acts on the right of X^t and $D\,(X)$, respectively, by

$$(\phi\xi)\,(x) = \phi\,(x)\,\xi, \qquad (\psi\xi)\,(x) = \psi\,(\xi x)\,,$$

where $\phi \in X^t$, $\psi \in D\,(X)$, $\xi \in S\,(B^+)$, and $x \in X$.

Consider the Nakayama functor [3, p.10]

$$D(\cdot)^t \colon \mathrm{mod}S(B^+) \to \mathrm{mod}S(B^+).$$

This is a covariant right exact functor which turns projectives into injectives.

Fix $\lambda \in \Lambda(n, r)$, $\lambda \neq (r, 0, \ldots, 0)$. Then $\mathbb{K}_\lambda$ is indecomposable and non-projective. Consider the first two steps of the minimal projective resolution (2.3) of $\mathbb{K}_\lambda$

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} \mathbb{K}_\lambda \to 0.$$

Applying the Nakayama functor we get from this the exact sequence

$$0 \to \tau\mathbb{K}_\lambda \to DP_1^t \xrightarrow{Dp_1^t} DP_0^t \xrightarrow{Dp_0^t} D\mathbb{K}_\lambda^t \to 0, \tag{3.1}$$

where $\tau\mathbb{K}_\lambda = \mathrm{Ker}Dp_1^t \cong D\,(\mathrm{Coker}p_1^t)$, that is τ is the Auslander-Reiten translation.

Given an $S\,(B^+)$-homomorphism $\theta \colon \mathbb{K}_\lambda \to DP_0^t$, consider the short exact sequence obtained from (3.1) by pullback along θ:

$$0 \to \tau\mathbb{K}_\lambda \xrightarrow{f} E\,(\theta) \xrightarrow{g} \mathbb{K}_\lambda \to 0. \tag{3.2}$$

Here $E\,(\theta) = \{\,(z, c) \in DP_1^t \oplus \mathbb{K}_\lambda \mid Dp_1^t\,(z) = \theta\,(c)\}$ is an $S(B^+)$-submodule of $DP_1^t \oplus \mathbb{K}_\lambda$, and f, g are the homomorphisms of $S(B^+)$-modules defined by $g\,(z, c) = c$, $f\,(v) = (v, 0)$, for all $z \in Dp_1^t$, $v \in \tau\mathbb{K}_\lambda$, and $c \in \mathbb{K}_\lambda$.

If we choose θ according to Green's or Auslander's recipe, then (3.2) is an Auslander-Reiten sequence. Note that DP_0^t has simple socle isomorphic to the 1-dimensional module $\mathbb{K}_\lambda$, so one can take for θ any non-zero $S(B^+)$-homomorphism. Before constructing such sequences, we will determine $\mathbb{K}$-bases of P_0 and P_1 adapted to our calculations.

Notice first that $(S\,(B^+)\,\xi_\alpha)^t$ and $\xi_\alpha S\,(B^+)$ are isomorphic right $S\,(B^+)$-modules for every $\alpha \in \Lambda$. So we will identify these two $S\,(B^+)$-modules. We will also identify $\left(\bigoplus_{\alpha\in\Lambda'} S\,(B^+)\,\xi_\alpha\right)^t$ with $\bigoplus_{\alpha\in\Lambda'} \xi_\alpha S\,(B^+)$, for every family Λ' of elements in Λ.

Lemma 3.2. *Let $\alpha \in \Lambda$. Then $\left\{\,\xi_{l(\alpha),j} \,\middle|\, j \in J\,(\alpha)\right\}$ is a $\mathbb{K}$-basis of $\xi_\alpha S\,(B^+)$.*

Proof: We know that $\xi_\alpha S(B^+)$ is spanned by $\left\{ \xi_{l(\alpha),j} \,\middle|\, j \in I(n,r),\ j \geq l(\alpha) \right\}$. As $\xi_{l(\alpha),j} = \xi_{l(\alpha),i}$ if and only if $i\pi = j$, for some π in the stabilizer of $l(\alpha)$ in Σ_r and this stabilizer coincides with the row stabilizer of T^α, the result follows. ■

Fix $\lambda \in \Lambda(n,r)$ and consider the result of the application of $(\cdot)^t$ to (2.3). Then

$$P_1^t \cong \begin{cases} \bigoplus\limits_{\nu=1}^{n-1} \xi_{\lambda(\nu,1)} S(B^+), & \text{if } \operatorname{char} \mathbb{K} = 0 \\ \bigoplus\limits_{\nu=1}^{n-1} \bigoplus\limits_{1 \leq p^{d_\nu} \leq \lambda_{\nu+1}} \xi_{\lambda(\nu,p^{d_\nu})} S(B^+), & \text{if } \operatorname{char} \mathbb{K} = p. \end{cases}$$

Thus a $\mathbb{K}$-basis of P_1^t is given by

$$B_1 := \left\{ \xi_{l(\nu,1),j} \,\middle|\, j \in J(\lambda(\nu,1)),\ \nu \in \mathbf{n-1} \right\}, \qquad \text{if } \operatorname{char} \mathbb{K} = 0,$$

$$B_2 := \left\{ \xi_{l(\nu,p^{d'}),j} \,\middle|\, \begin{array}{l} j \in J(\lambda(\nu,p^{d'})), \\ 1 \leq p^{d'} \leq \lambda_{\nu+1},\ \nu \in \mathbf{n-1} \end{array} \right\}, \quad \text{if } \operatorname{char} \mathbb{K} = p. \tag{3.3}$$

With the above identifications of the projective modules, the map $p_1^t \colon P_0^t \to P_1^t$ becomes

$$p_1^t(\eta) = \begin{cases} \sum\limits_{\nu=1}^{n-1} \xi_{l(\nu,1),l}\eta, & \text{if } \operatorname{char} \mathbb{K} = 0 \\ \sum\limits_{\nu=1}^{n-1} \sum\limits_{1 \leq p^{d'} \leq \lambda_{\nu+1}} \xi_{l(\nu,p^{d'}),l}\eta, & \text{if } \operatorname{char} \mathbb{K} = p, \end{cases}$$

where $\eta \in P_0^t$.

To construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$, it is convenient to obtain, from B_1 and B_2, new bases for P_1^t containing $p_1^t\left(\xi_{l(\lambda),j}\right)$, $j \in J(\lambda)$. Suppose λ satisfies conditions

$$\begin{cases} \lambda_n \neq 0, & \text{if } \operatorname{char} \mathbb{K} = 0, \\ \lambda_n \neq 0,\ \lambda_{n-1} < p^{d+1} - 1, & \text{if } \operatorname{char} \mathbb{K} = p \text{ and } p^d \leq \lambda_n < p^{d+1}. \end{cases} \tag{3.4}$$

Given $j \in J(\lambda)$, as $j \geq l = l(\lambda)$, the nth row of T_j^λ is constant with all entries equal to n, and its $(n-1)$st row has a entries equal to n and $\lambda_{n-1} - a$ entries equal to $n-1$, for some $0 \leq a \leq \lambda_{n-1}$.

We shall look first at the case $\operatorname{char}\mathbb{K} = 0$. Then by Lemma 2.3

$$\xi_{l(n-1,1),l}\xi_{l,j} = (a+1)\,\xi_{l(n-1,1),j},$$

and $a+1 \neq 0$. Using this, we shall show that we can replace $\xi_{l(n-1,1),j}$ by $p_1^t(\xi_{l,j})$ in B_1 and obtain a new basis for P_1^t. Notice that $J(\lambda) \subset J(\lambda(n-1,1))$, and so $\xi_{l(n-1,1),j} \in B_1$. On the other hand, $\xi_{l,j} \in P_0^t = \xi_\lambda S(B^+)$, and

$$p_1^t(\xi_{l,j}) = \xi_{l(n-1,1),l}\xi_{l,j} + \sum_{\nu=1}^{n-2} \xi_{l(\nu,1),l}\xi_{l,j} = (a+1)\,\xi_{l(n-1,1),j} + \sum_{\nu=1}^{n-2} \xi_{l(\nu,1),l}\xi_{l,j}.$$

Now $\xi_{l(\nu,1),l}\xi_{l,j}$ is a linear combination of basis elements of the type $\xi_{l(\nu,1),j\pi}$, for some $\pi \in \Sigma_r$. As, for $\nu = 1,\dots,n-2$, we have that $l(\nu,1) \not\sim l(n-1,1)$, we get that $\xi_{l(n-1,1),j}$ is always different from $\xi_{l(\nu,1),i}$, for any $i \in I(n,r)$. Therefore, we can replace $\xi_{l(n-1,1),j}$ in B_1 by $p_1^t(\xi_{l,j})$ and still get a basis for P_1^t. We have proved the following:

Proposition 3.3. *If* $\operatorname{char}\mathbb{K} = 0$ *and* $\lambda_n \neq 0$, *then*

$$\begin{aligned}\overline{B}_1 = &\left\{\, \xi_{l(\nu,1),j} \,\middle|\, j \in J(\lambda(\nu,1)),\ \nu = 1,\dots,n-2 \right\}\\ &\cup \left\{\, \xi_{l(n-1,1),j} \,\middle|\, j \in J(\lambda(n-1,1)) \setminus J(\lambda) \right\} \cup \left\{\, p_1^t(\xi_{l,j}) \,\middle|\, j \in J(\lambda) \right\}\end{aligned}$$

is a basis of P_1^t. *In particular,* p_1^t *is a monomorphism.*

Suppose now that $\operatorname{char}\mathbb{K} = p$ and that λ satisfies condition (3.4). We will apply Lemma 2.3, together with the following well known consequence of Lucas' Theorem:

Proposition 3.4. *Assume* m *and* q *are positive integers and that*

$$\begin{aligned} m &= m_0 + m_1 p + \dots + m_t p^t \\ q &= q_0 + q_1 p + \dots + q_s p^s \end{aligned}$$

are the p*-adic expansions of* m *and* q. *Then* p *divides* $\binom{m}{q}$ *if and only if* $m_\nu < q_\nu$ *for some* ν.

Given $j \in J(\lambda)$, we denote by $a = a(j)$ the number of entries equal to n in the $(n-1)$st row of T_j^λ. Let $a_0, \dots, a_d$ be the coefficients in the p-adic expansion of a, with a_d possibly equal to zero. As $a \leq \lambda_{n-1} < p^{d+1} - 1$, and all the coefficients in the p-adic expansion of $p^{d+1}-1$ are equal to $p-1$, we get that there is some $a_t \neq p-1$. Let

$$m(j) := \min\{\, t \mid a_t < p-1 \,\} \tag{3.5}$$

and define

$$J(\lambda, d') = \{ j \in J(\lambda) \,|\, m(j) = d' \}.$$

Obviously $J(\lambda) = \dot{\bigcup}_{0 \le d' \le d} J(\lambda, d')$.

Proposition 3.5. *Suppose that* $\operatorname{char} \mathbb{K} = p$ *and that* $\lambda \in \Lambda(n, r)$ *satisfies condition* (3.4)*. Then*

$$\begin{aligned} \overline{B}_2 = & \left\{ \xi_{l\left(\nu, p^{d'}\right), j} \,\middle|\, j \in J(\lambda(\nu, p^{d'})),\ 1 \le p^{d'} \le \lambda_{\nu+1},\ \nu = 1, \dots, n-2 \right\} \\ & \cup \left\{ \xi_{l\left(n-1, p^{d'}\right), j} \,\middle|\, j \in J(\lambda(n-1, p^{d'})) \setminus J(\lambda, d'),\ 0 \le d' \le d \right\} \\ & \cup \left\{ p_1^t(\xi_{l,j}) \,|\, j \in J(\lambda) \right\} \end{aligned}$$

is a $\mathbb{K}$*-basis for* P_1^t*. In particular,* p_1^t *is a monomorphism.*

Proof: Let $j \in J(\lambda)$. Just like in the characteristic zero case, we consider the basis element $\xi_{l,j}$ of P_0^t and look at

$$p_1^t(\xi_{l,j}) = \sum_{\nu=1}^{n-1} \sum_{1 \le p^{d'} \le \lambda_{\nu+1}} \xi_{l\left(\nu, p^{d'}\right), l} \xi_{l,j}.$$

By Lemma 2.3 for any d' such that $0 \le p^{d'} \le \lambda_n$ we have

$$\xi_{l\left(n-1, p^{d'}\right), l} \xi_{l,j} = \binom{a + p^{d'}}{p^{d'}} \xi_{l\left(n-1, p^{d'}\right), j}.$$

Here a is the number of times n occurs in row $n-1$ of T_j^λ. It follows from Proposition 3.4 and the definition of $m(j)$ (see (3.5)), that p does not divide $\binom{a+p^{m(j)}}{p^{m(j)}}$ and divides all $\binom{a+p^{d'}}{p^{d'}}$ for $d' < m(j)$. Therefore

$$p_1^t(\xi_{l,j}) = \sum_{\nu=1}^{n-2} \sum_{1 \le p^{d'} \le \lambda_{\nu+1}} \xi_{l\left(\nu, p^{d'}\right), l} \xi_{l,j} + \sum_{p^{m(j)} \le p^{d'} \le \lambda_n} \binom{a + p^{d'}}{p^{d'}} \xi_{l\left(n-1, p^{d'}\right), j} \tag{3.6}$$

and the coefficient of $\xi_{l\left(n-1, p^{m(j)}\right), j}$ in this sum is non-zero. As, for $\nu \neq n-1$, we have $l\left(n-1, p^{m(j)}\right) \not\sim l\left(\nu, p^{d'}\right)$ it follows that $\xi_{l(n-1, p^{m(j)}), j}$ does not appear in the basis expansion of $\xi_{l\left(\nu, p^{d'}\right), l} \xi_{l,h}$, for any $h \in J(\lambda)$. Also, if $d' \neq m(j)$, then $l\left(n-1, p^{d'}\right) \not\sim l\left(n-1, p^{m(j)}\right)$ and so $\xi_{l\left(n-1, p^{d'}\right), h} \neq \xi_{l\left(n-1, p^{m(j)}\right), j}$ for all $h \in J(\lambda)$.

Finally, suppose $h \in J(\lambda)$ satisfies

$$(\, l(n-1, p^{m(j)}), j \,) \sim (\, l(n-1, p^{m(h)}), h \,).$$

Then $h = j\pi$ for some $\pi \in \Sigma_{\lambda\left(n-1,p^{m(j)}\right)}$. But, since both h, $j \geq l$, we can not move any entry $n-1$ in row $(n-1)$ of T_j^λ to row n to obtain T_h^λ. This implies that π belongs to the row stabilizer of T^λ. As both T_j^λ and T_h^λ are row semistandard we get $h = j$. Therefore $\xi_{l\left(n-1,p^{m(j)}\right),j}$ appears only once in $\overline{B}_2$: in the expression (3.6) of $p_1^t(\xi_{l,j})$ with the coefficient $\binom{a+p^{m(j)}}{p^{m(j)}}$. Hence, we can replace $\xi_{l\left(n-1,p^{m(j)}\right),j}$ by $p_1^t(\xi_{l,j})$ in B_2 for all $j \in J(\lambda)$ and still have a basis for P_1^t. ■

It is now easy to obtain an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ for λ satisfying (3.4). Denote, respectively, by B_1^* and B_2^* the $\mathbb{K}$-basis of $D(P_1^t)$ dual to $\overline{B}_1$ and $\overline{B}_2$. For $j \in J(\lambda)$, we denote by $z_{l,j}$ the element in B_1^* (respectively, in B_2^*) that is dual to $p_1^t(\xi_{l,j})$. Let U_λ be the subspace of $D(P_1^t)$ with $\mathbb{K}$-basis $B_1^* \setminus \{ \, z_{l,j} \,|\, j \in J(\lambda)\}$ if $\operatorname{char}\mathbb{K} = 0$ or $B_2^* \setminus \{ \, z_{l,j} \,|\, j \in J(\lambda)\}$ if $\operatorname{char}\mathbb{K} = p$. Then U_λ is in fact a $S(B^+)$-submodule. Define

$$E(\lambda) = \left\{ \, (z, c) \in D\left(P_1^t\right) \oplus \mathbb{K}_\lambda \,\middle|\, z \in (U_\lambda + c z_{l,l}) \right\}.$$

Then we have the following result.

Theorem 3.6. *Suppose that $\lambda \in \Lambda(n, r)$ satisfies* (3.4)*. Then the sequence*

$$0 \to U_\lambda \xrightarrow{f} E(\lambda) \xrightarrow{g} \mathbb{K}_\lambda \to 0, \tag{3.7}$$

where f and g are defined by

$$f(z) = (z, 0), \ \forall z \in U_\lambda; \qquad g(z', c) = c, \ \forall (z', c) \in E(\lambda),$$

is an Auslander-Reiten sequence.

Proof: Notice first that since $\lambda \neq (r, 0, \ldots, 0)$, we have that $\mathbb{K}_\lambda$ is not projective. Hence an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ exists.

Now we will prove the theorem in the case $\operatorname{char}\mathbb{K} = 0$. The case of $\operatorname{char}\mathbb{K} = p$ is similar. To do this we will follow Auslander's recipe given in [2].

Transferring Butler's notation to our setting, we have $C = \mathbb{K}_\lambda$, $\Lambda = S(B^+)$, $\operatorname{rad}\left(\operatorname{End}_{S(B^+)}(\mathbb{K}_\lambda)\right) = 0$, and $C_0 = 0$. Choose $x_0 = \xi_\lambda \in P_0$ and $s = \operatorname{id}_{\mathbb{K}_\lambda} \in D(\mathbb{K}_\lambda)$. Notice, that $\operatorname{id}_{\mathbb{K}_\lambda}$ satisfies $\operatorname{id}_{\mathbb{K}_\lambda}(C_0) = 0$ and $\operatorname{id}_{\mathbb{K}_\lambda}(p_0(\xi_\lambda)) =$

$\mathrm{id}_{\mathbb{K}_\lambda}(1_{\mathbb{K}}) \neq 0$. Now it is sufficient to take $\theta \in \mathrm{Hom}_{S(B^+)}(\mathbb{K}_\lambda, D(P_0^t))$ such that

$$\theta(c)(\eta) = \mathrm{id}_{\mathbb{K}_\lambda}((\xi_\lambda \eta)c) = \eta c \text{ for all } \eta \in P_0^t = \xi_\lambda S(B^+) \text{ and all } c \in \mathbb{K}_\lambda.$$

Note that as P_0^t has $\mathbb{K}$-basis $\{\, \xi_{l,j} \mid j \in J(\lambda)\}$, and $\xi_{l,j}c = c$ or 0, according to $j = l$ or $j \neq l$, we have that θ is completely determined by saying that

$$\theta(c)(\xi_{l,j}) = \begin{cases} c, & \text{if } j = l \\ 0, & \text{if } j \neq l, \end{cases} \tag{3.8}$$

where $j \in J(\lambda)$ and $c \in \mathbb{K}_\lambda$. Given $z \in D(P_1^t)$ we can write z as a linear combination of the elements of B_1^*. Then for any $c \in \mathbb{K}_\lambda$, we have $D(p_1^t)(z) = \theta(c)$ if and only if $zp_1^t = \theta(c)$, which in turn holds if and only if for all $j \in J(\lambda)$ there holds

$$zp_1^t(\xi_{l,j}) = \begin{cases} c, & \text{if } j = l \\ 0, & \text{if } j \neq l. \end{cases}$$

Thus $z = cz_{l,l} + u$ for some $u \in U_\lambda$. Hence

$$E(\theta) = \left\{ (z, c) \in D\left(P_1^t\right) \oplus \mathbb{K}_\lambda \,\middle|\, Dp_1^t(z) = \theta(c) \right\} = E(\lambda).$$

In a similar way, we see that $z \in \tau\mathbb{K}_\lambda = \ker(Dp_1^t)$ if and only if $zp_1^t = 0$, that is if and only if $z \in U_\lambda$. Therefore $\tau\mathbb{K}_\lambda = U_\lambda$. ■

4. The case $n = 2$

In this section we study the construction of an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ in the particular case of $n = 2$. We will show that it is very easy to obtain such sequences with no restriction on λ or the characteristic of $\mathbb{K}$.

Let $\lambda = (\lambda_1, \lambda_2)$. Since $\mathbb{K}_\lambda$ is non-projective if and only if $\lambda_2 \neq 0$, all compositions we are interested in satisfy this condition. So in this section we assume that $\lambda_2 \neq 0$. In particular, the construction of Auslander-Reiten sequences in the characteristic zero and $n = 2$ case is completely answered in Theorem 3.6.

Suppose now that $\mathrm{char}\,\mathbb{K} = p$ and d is such that $p^d \leq \lambda_2 < p^{d+1}$. Given $j \in J(\lambda)$, recall that $a(j)$ is the number of 2's in the first row of T_j^λ. If

$$a = a(j) = (p-1) + (p-1)p + \cdots + (p-1)p^d + \dots \tag{4.1}$$

is the p-adic expansion of a then, by Proposition 3.4, for all $0 \leq d' \leq d$ the binomial coefficient $\binom{a+p^{d'}}{p^{d'}}$ is divisible by p. Hence

$$p_1^t(\xi_{l,j}) = \sum_{d'=0}^{d} \xi_{l\left(1,p^{d'}\right),l}\xi_{l,j} = \sum_{d'=0}^{d} \binom{a+p^{d'}}{p^{d'}} \xi_{l\left(1,p^{d'}\right),j} = 0.$$

Next we suppose that $a = a(j)$ has p-adic expansion

$$a = a_0 + a_1 p + \cdots + a_s p^s \tag{4.2}$$

with $a_t \neq p-1$ for some $0 \leq t \leq d$. Define

$$m(j) = \min\{\, t \,|\, t \leq d \text{ and } a_t < p-1\}$$

and

$$\hat{J}(\lambda) = \left\{\, j \in J(\lambda) \,\middle|\, a(j) \neq (p-1) + (p-1)p + \cdots + (p-1)p^d + \dots \right\}.$$

For $0 \leq d' \leq d$ we denote by $\hat{J}(\lambda, d')$ the subset of those $j \in \hat{J}(\lambda)$ such that $m(j) = d'$. Then $\hat{J}(\lambda) = \dot{\bigcup}_{0\leq d'\leq d}\hat{J}(\lambda, d')$ and $\hat{J}(\lambda, d') \subset J(\lambda(1, p^{d'}))$. Now with a proof completely analogous to the proof of Proposition 3.5, we see that, for $j \in \hat{J}(\lambda, d')$, the element $\xi_{l(1,p^{m(j)}),j}$ in B_2 can be replaced by $p_1^t(\xi_{l,j})$ and the resulting set $\overline{B}_2$ is a new basis for P_1^t. This proves the following result.

Proposition 4.1. *Suppose that* $\operatorname{char}\mathbb{K} = p$ *and* $\lambda = (\lambda_1, \lambda_2)$, *with* $\lambda_2 \neq 0$. *Then*

$$\overline{B}_2 = \left\{\, \xi_{l\left(1,p^{d'}\right),j} \,\middle|\, j \in J(\lambda(1,p^{d'})) \setminus \hat{J}(\lambda, d'),\ 0 \leq d' \leq d \right\} \cup \left\{\, p_1^t(\xi_{l,j}) \,\middle|\, j \in \hat{J}(\lambda) \right\}$$

is a $\mathbb{K}$*-basis for* P_1^t.

We also have that $\left\{\, \xi_{l,j} \,\middle|\, j \in J(\lambda) \setminus \hat{J}(\lambda) \right\}$ *is a* $\mathbb{K}$*-basis for* $\ker(p_1^t)$. *In particular,* p_1^t *is injective if and only if* $p^d \leq \lambda_2 < p^{d+1}$ *and* $\lambda_1 < p^{d+1} - 1$, *i.e., if and only if* λ *satisfies condition 3.4.*

Denote by B_2^* the basis of $D(P_1^t)$ dual to $\overline{B}_2$. We write $z_{l,j}$ for the element dual to $p_1^t(\xi_{l,j})$, where $j \in \hat{J}(\lambda)$. Let U_λ be the $S(B^+)$-submodule of $D(P_1^t)$ with $\mathbb{K}$-basis $B_2^* \setminus \left\{\, z_{l,j} \,\middle|\, j \in \hat{J}(\lambda) \right\}$ and

$$E(\lambda) = \left\{\, (z,c) \in D\left(P_1^t\right) \oplus \mathbb{K}_\lambda \,\middle|\, z \in (U_\lambda + cz_{l,l}) \right\}.$$

Then, adapting the proof of Theorem 3.6, we can conclude the following result

Theorem 4.2. *Suppose that* $\operatorname{char}\mathbb{K} = p$ *and* $\lambda = (\lambda_1, \lambda_2)$, *with* $\lambda_2 \neq 0$. *Then the sequence*

$$0 \to U_\lambda \xrightarrow{f} E(\lambda) \xrightarrow{g} \mathbb{K}_\lambda \to 0,$$

where

$$f(z) = (z, 0), \qquad g(z', c) = c, \ \forall z \in U_\lambda, (z', c) \in E(\lambda)$$

is an Auslander-Reiten sequence.

5. Some results in the case $n = 3$

We will consider fields of characteristic p, $n = 3$, and $\lambda \in \Lambda(3, r)$ with $\lambda_3 \neq 0$. Define d by $p^d \leq \lambda_3 < p^{d+1}$. If $\lambda_2 < p^{d+1} - 1$, we know from Theorem 3.6 an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$. In this section we study the construction of Auslander-Reiten sequences for λ with $\lambda_2 = 2p^{d+1} - 1$.

In this case p_1^t may not be injective. Our first step will be again to determine a basis for P_1^t, which contains a basis of $\operatorname{Im}(p_1^t)$. Recall that

$$B_2 = \left\{ \xi_{l\left(\nu, p^{d'}\right), j} \,\middle|\, j \in J\left(\lambda(\nu, p^{d'})\right), \ 1 \leq p^{d'} \leq \lambda_{\nu+1}, \ \nu = 1, 2 \right\}$$

is a $\mathbb{K}$-basis of P_1^t. We will end this section by explaining how to replace some of these elements $\xi_{l(\nu, p^{d'}), j}$ by elements of the form $p_1^t(\xi_{l,h})$ and obtain a new basis of P_1^t. The construction of the Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ is then similar to the one in the previous sections.

Given $j \in J(\lambda)$, suppose that the number of entries equal to 3 in the second row of T_j^λ is $a = a(j)$. Let $a_0, \ldots, a_{d+1}$ be the coefficients of the p-adic expansion of a. If $a_t \neq p - 1$, for some $t \leq d$, let $m = m(j) = \min\{ t \mid a_t < p - 1 \}$. Then

$$p_1^t(\xi_{l,j}) = \sum_{d'=0}^{d+1} \xi_{l\left(1, p^{d'}\right), l} \xi_{l,j} + \sum_{d'=m(j)}^{d} \binom{a + p^{d'}}{p^{d'}} \xi_{l\left(2, p^{d'}\right), j}$$

and p does not divide $\binom{a+p^{m(j)}}{p^{m(j)}}$. Now, like in the proof of Proposition 3.3, it is simple to see that if we replace $\xi_{l(2, p^{m(j)}), j}$ by $p_1^t(\xi_{l,j})$ in B_2 for all j's satisfying these conditions, we obtain a new basis B_2' for P_1^t.

The problem arises when the p-adic expansion of a is

$$a = (p-1) + (p-1)p + \cdots + (p-1)p^d + cp^{d+1}, \ c = 0, 1. \tag{5.1}$$

If $a = a(j)$ satisfies (5.1), we say that j is a *critical* element of $J(\lambda)$. In this case we have

$$p_1^t(\xi_{l,j}) = \sum_{d'=0}^{d+1} \xi_{l\left(1,p^{d'}\right),l}\xi_{l,j}.$$

If $c = 0$, then the second row of T_j^λ is not constant and we can not use the multiplication formula in Lemma 2.3 to calculate $\xi_{l(1,p^{d'}),l}\xi_{l,j}$. We will use a different version of the multiplication formula (2.2) to study these products (see [5, (2.7)]). Given $i,\ j,\ k \in I(3,r)$ the double cosets $\Sigma_{i,j}\sigma\Sigma_{j,k}$ in Σ_j correspond one-to-one to the $\Sigma_{j,k}$-orbits of $i\Sigma_j$. So (2.2) becomes

$$\xi_{i,j}\xi_{j,k} = \sum_h \left[\Sigma_{h,k} : \Sigma_{h,j,k}\right]\xi_{h,k}, \tag{5.2}$$

where the sum is over a transversal $\{h\}$ of the $\Sigma_{j,k}$-orbits in the set $i\Sigma_j$. Now we fix a critical $j \in J(\lambda)$ such that $a = a(j) = p^{d+1} - 1$. Suppose that the number of entries equal to 2 and the number of entries equal to 3 in the first row of T_j^λ are $t_2 = t_2(j)$ and $t_3 = t_3(j)$, respectively. Thus we have

$$T_j^\lambda = \begin{array}{l} 1\ldots 1\overbrace{2\ldots\ldots 2}^{t_2}\overbrace{3\ldots\ldots\ldots 3}^{t_3} \\ \\ 2\ldots\ldots 2\underbrace{3\ldots\ldots\ldots\ldots\ldots\ldots 3}_{a} \\ \\ 3\ldots\ldots\ldots\ldots\ldots 3 \end{array} \tag{5.3}$$

Applying (5.2) in the case of our composition, we obtain

$$\xi_{l\left(1,p^{d'}\right),l}\xi_{l,j} = \sum_h \binom{t_2+s}{s}\binom{t_3+t}{t}\xi_{h,j} \tag{5.4}$$

where

$$h = (\underbrace{1,\ldots,1}_{\lambda_1+s},\underbrace{2,\ldots,2}_{\lambda_2-a-s},\underbrace{1,\ldots,1}_{t},\underbrace{2,\ldots,2}_{a-t},\underbrace{3,\ldots,3}_{\lambda_3})$$

that is

$$T_h^\lambda = \begin{array}{l} 1\ldots\ldots\ldots\ldots\ldots 1 \\ \\ \underbrace{1\ldots\ldots 1}_{s}2\ldots 2\overbrace{\underbrace{1\ldots\ldots 1}_{t}2\ldots 2}^{a}, \\ \\ 3\ldots\ldots\ldots\ldots 3 \end{array}$$

and $s+t=p^{d'}$, $t<p^{d+1}$. Note that all the $\xi_{h,j}$ in (5.4) are distinct.

Remark 5.1. *Suppose that j_1 and j_2 are critical elements of $J(\lambda)$. Then $(h,j_1)\sim(h',j_2)$ implies $j_1\sim j_2$. Hence if $j_1\nsim j_2$ all the basis elements ξ_{h,j_1} appearing in the B_2-expansion of $p_1^t(\xi_{l,j_1})$ are distinct from those appearing in the B_2-expansion of $p_1^t(\xi_{l,j_2})$.*

Thus given j defined by (5.3), we only have to study the linear independence of $\{p_1^t(\xi_{l,j}),\ p_1^t(\xi_{l,j'})\}$, where j' is a critical element of $J(\lambda)$ and $j'\sim j$. Hence

$$T_{j'}^{\lambda}=\begin{array}{l} 1\ldots 1\overbrace{2\ldots\ldots\ldots 2}^{t_2+p^{d+1}}\overbrace{3\ldots\ldots 3}^{t_3-p^{d+1}}\\ 3\ldots\ldots\ldots\ldots\ldots 3\\ 3\ldots\ldots\ldots 3\end{array}\quad. \tag{5.5}$$

Note that if $t_3<p^{d+1}$, then j' and $p_1^t(\xi_{l,j'})$ are not defined. Thus we will assume that $t_3\geq p^{d+1}$.

Recall that, from Lemma (2.3), we have

$$p_1^t(\xi_{l,j'})=\sum_{0\leq d'\leq d+1}\binom{t_3-p^{d+1}+p^{d'}}{p^{d'}}\xi_{l\left(1,p^{d'}\right),j'}. \tag{5.6}$$

Before we proceed we need a technical result. Its proof is an easy consequence of Proposition 3.4.

Lemma 5.2. *Let $0\leq m\leq d+1$. Then p divides all the products $\binom{t_2+s}{s}\binom{t_3+t}{t}$ with $s+t=p^{d'}$ and $0\leq d'\leq m$ if and only if the p-adic expansions of t_2 and t_3 have the form*

$$\begin{aligned} t_2&=(p-1)+(p-1)p+\cdots+(p-1)p^m+c'_{m+1}p^{m+1}+\ldots\\ t_3&=(p-1)+(p-1)p+\cdots+(p-1)p^m+c''_{m+1}p^{m+1}+\ldots. \end{aligned} \tag{5.7}$$

Proof: Suppose p divides all the products $\binom{t_2+s}{s}\binom{t_3+t}{t}$ with $s+t=p^{d'}$ and $0\leq d'\leq m$. Taking $s=p^{d'}$ and $t=0$ we get $\binom{t_3+t}{t}=1$. Thus, $\binom{t_2+p^{d'}}{p^{d'}}$ is divisible by p for any $0\leq d'\leq m$. It follows from Proposition 3.4 that the coefficient of $p^{d'}$ in the p-adic expansion of t_2 is $(p-1)$ for any $0\leq d'\leq m$. The case of t_3 is proved similarly.

Now, suppose that t_2 and t_3 satisfy (5.7) and $0\leq d'\leq p^m$, $s+t=p^{d'}$. If $s=0$, then $t=p^{d'}$ and $\binom{t_3+p^{d'}}{p^{d'}}$ is divisible by p by Proposition 3.4. Suppose $s\neq 0$ and s_i is the first non-zero coefficient in the p-adic expansion of s. Then the ith coefficient of t_2+s in its p-adic expansion is s_i-1. Since $s_i-1<s_i$, we get, from Proposition 3.4, that $\binom{t_2+s}{s}$ is divisible by p. ■

As an immediate consequence of Lemma (5.2) we get the following result.

Lemma 5.3. *Given j and j' as above we have:*

(i) $p_1^t(\xi_{l,j})=0$ if and only if the p-adic expansions of t_2 and t_3 are

$$\begin{aligned} t_2 &= (p-1)+(p-1)p+\cdots+(p-1)p^{d+1}+c'_{d+2}p^{d+2}+\dots \\ t_3 &= (p-1)+(p-1)p+\cdots+(p-1)p^{d}+c''_{d+1}p^{d+1}+\dots; \end{aligned}$$

(ii) $p_1^t(\xi_{l,j'})=0$ if and only if the p-adic expansion of t_3 is of the form

$$(p-1)+(p-1)p+\cdots+(p-1)p^d+0\cdot p^{d+1}+\dots.$$

Suppose now that $p_1^t(\xi_{l,j})\neq 0$. Let

$$b:=b(j):=\min\left\{0\leq d'\leq d+1\,\middle|\,\xi_{l(1,p^{d'}),l}\xi_{l,j}\neq 0\right\}.$$

Then p divides all the products $\binom{t_2+s}{s}\binom{t_3+t}{t}$ with $s+t=p^{d'}$ and $0\leq d'<b$, and there are s and t such that $s+t=p^b$ and $\binom{t_2+s}{s}\binom{t_3+t}{t}$ is non-zero in $\mathbb{K}$. From Lemma 5.2, we obtain that the p-adic expansions of t_2 and t_3 should be of the form

$$\begin{aligned} t_2 &= (p-1)+\cdots+(p-1)p^{b-1}+c'_bp^b+\dots \\ t_3 &= (p-1)+\cdots+(p-1)p^{b-1}+c''_bp^b+\dots, \end{aligned} \tag{5.8}$$

where either c'_b or c''_b is different from $p-1$.

Lemma 5.4. *Given j, j' and $b=b(j)$ as above, we have $\xi_{l\left(1,p^{d'}\right),l}\xi_{l,j}=0$, if $d'<b$, and*

$$\xi_{l(1,p^b),l}\xi_{l,j}=\begin{cases}\binom{t_2+p^b}{p^b}\xi_{l(1,p^b),j}+\binom{t_3+p^b}{p^b}\xi_{i,j}, & \text{if } b<d+1\\ \binom{t_2+p^{d+1}}{p^{d+1}}\xi_{l(1,p^{d+1}),j}, & \text{if } b=d+1,\end{cases}$$

where

$$T_i^\lambda=\begin{matrix}1\,.........\,1 \\ \underbrace{2\,......\,2}_{p^{d+1}}\underbrace{1\,......\,1}_{p^b}2\,.........\,2 \\ 3\,.........\,3\end{matrix}\ .$$

Proof: In the conditions of the lemma, we see that t_2 and t_3 are as in (5.8). Suppose $b < d+1$. Given $0 < s < p^b$, if s_i is the first non-zero coefficient in the p-adic expansion of s, then the ith coefficient in the p-adic expansion of $t_2 + s$ is $s_i - 1$. Therefore p divides $\binom{t_2+s}{s}$. Hence, when we apply (5.4) to $\xi_{l(1,p^b),l}\xi_{l,j}$, only the summands corresponding to $s = p^b$, $t = 0$ and $s = 0$, $t = p^b$ remain.

If $b = d+1$ a similar argument applies. Only this time, the condition $t < p^{d+1}$ in (5.4) implies that we are left only with the summand that corresponds to $s = p^b$, $t = 0$. ∎

Lemma 5.5. *Given j, j' and $b = b(j)$ as above we have:*

(i) if $b < d+1$ and $p_1^t(\xi_{l,j'}) \neq 0$, then $p_1^t(\xi_{l,j})$ and $p_1^t(\xi_{l,j'})$ are linearly independent;

(ii) if $b = d+1$, then $p_1^t(\xi_{l,j})$ and $p_1^t(\xi_{l,j'})$ are linearly dependent.

Proof: Suppose $b < d+1$. Using Lemma 5.4, we only have to make sure that $\xi_{l(1,p^b),j'}$ is different from $\xi_{l(1,p^b),j}$ and $\xi_{i,j}$. Note first that $j' \neq j\pi$ for any $\pi \in \Sigma_{\lambda(1,p^b)}$. In fact, π of $\Sigma_{\lambda(1,p^b)}$ can move at most p^b 2's from the second row of T_j^λ to its first row. But $T_{j'}^\lambda$ is obtained from T_j^λ by moving exactly p^{d+1} 2's from the second to the first row. As $p^{d+1} > p^b$ this can not be achieved by application of $\pi \in \Sigma_{\lambda(1,p^b)}$. Thus $\left(l\left(1,p^b\right),j\right) \not\sim \left(l\left(1,p^b\right),j'\right)$. In a similar way, we see that $\left(l(1,p^b),j'\right) \not\sim (i,j)$, since no σ satisfying $j\sigma = j'$ can move the p^b 1's from the second row of T_i^λ to the first p^b positions of this row.

Now consider the case $b = d+1$. In this case, the permutations that permute the p^{d+1} 2's in the second row of T_j^λ with the first p^{d+1} 3's in the first row belong to $\Sigma_{\lambda(1,p^{d+1})}$. So $\xi_{l(1,p^{d+1}),j'} = \xi_{l(1,p^{d+1}),j}$.

Notice also that $b = d+1$ implies that

$$t_3 = (p-1) + \cdots + (p-1)p^d + c''_{d+1}p^{d+1} + \dots.$$

Thus

$$t_3 - p^{d+1} + p^{d'} = (p-1) + \cdots + (p-1)p^{d'-1} + c''_{d+1}p^{d+1} + \dots.$$

Hence

$$p_1^t(\xi_{l,j'}) = \binom{t_3}{p^{d+1}}\xi_{l(1,p^{d+1}),j'} = \binom{t_3}{p^{d+1}}\xi_{l(1,p^{d+1}),j}.$$

∎

At this point, we should remark that if $t_3 \neq 0$, then $j \notin J\left(\lambda\left(1, p^b\right)\right)$ for any b, since $T_j^{\lambda(1,p^b)}$ is not row semistandard. But $\xi_{l(1,p^b),j} = \xi_{l(1,p^b),j\pi}$, for any π in the row stabilizer of $T^{\lambda(1,p^b)}$, and we can choose π such that $j\pi \in J\left(\lambda(1,p^b)\right)$. So $\xi_{l(1,p^b),j} = \xi_{l(1,p^b),j\pi} \in B_2$. In the particular case of $b = d+1$, we have $\xi_{l(1,p^{d+1}),j} = \xi_{l(1,p^{d+1}),j'}$.

The following is the main result of this section.

Proposition 5.6. *Let* $\operatorname{char}\mathbb{K} = p$, *$d$ a natural number, and* $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda(3, r)$ *with* $p^d \leq \lambda_3 < p^{d+1}$. *Suppose* $\lambda_2 = 2p^{d+1}-1$. *Let B_2 be the basis* (3.3) *of P_1^t. Then we obtain from B_2 a new basis $\hat{B}_2$ of P_1^t in the following way: given $j \in J(\lambda)$, suppose the p-adic expansion of $a(j)$ is $\sum_{q=0}^{d+1} a_q p^q$. Then:*

(a) If $a_q \neq p-1$, for some $q \leq d$, we replace $\xi_{l(2,p^{m(j)}),j}$ by $p_1^t(\xi_{l,j})$ in B_2.

(b) If $a(j) = p^{d+1}-1$, let t_2 and t_3 be the number of 2's and 3's in the first row of T_j^λ, respectively.

(i) If the p-adic expansions of t_2 and t_3 are:

$$
\begin{aligned}
t_2 &= (p-1) + \cdots + (p-1)p^{b-1} + c'_b p^b + \ldots \\
t_3 &= (p-1) + \cdots + (p-1)p^{b-1} + c''_b p^b + \ldots \\
&\neq (p-1) + \cdots + (p-1)p^d + 0 \cdot p^{d+1} + \ldots
\end{aligned}
$$

and $c'_b \neq p-1$ or $c''_b \neq p-1$ for some $b < d+1$, then in B_2, we replace $\xi_{l(1,p^b),j}$ and $\xi_{l(1,p^{d+1}),j'}$ by $p_1^t(\xi_{l,j})$ and $p_1^t(\xi_{l,j'})$, respectively.

(ii) If the p-adic expansions of t_2 and t_3 are:

$$
\begin{aligned}
t_2 &= (p-1) + \cdots + (p-1)p^{b-1} + c'_b p^b + \ldots \\
t_3 &= (p-1) + \cdots + (p-1)p^d + 0 \cdot p^{d+1} + \ldots
\end{aligned}
$$

for some $b < d+1$ such that $c'_b \neq p-1$, then $p_1^t(\xi_{l,j'}) = 0$ and we replace $\xi_{l(1,p^b),j}$ by $p_1^t(\xi_{l,j})$ in B_2.

(iii) If the p-adic expansions of t_2 and t_3 are:

$$
\begin{aligned}
t_2 &= (p-1) + \cdots + (p-1)p^d + c'_{d+1} p^{d+1} + \ldots \\
t_3 &= (p-1) + \cdots + (p-1)p^d + c''_{d+1} p^{d+1} + \ldots
\end{aligned}
$$

with $c''_{d+1} \neq 0$, then we replace $\xi_{l(1,p^{d+1}),j}$ by $p_1^t(\xi_{l,j'})$ in B_2. In this case $\xi_{l(1,p^{d+1}),j} = \xi_{l(1,p^{d+1}),j'}$ and $p_1^t(\xi_{l,j})$ is a multiple of $p_1^t(\xi_{l,j'})$.

(iv) If the p-adic expansions of t_2 *and* t_3 *are:*

$$t_2 = (p-1) + \cdots + (p-1)p^d + c'_{d+1}p^{d+1} + \dots$$
$$t_3 = (p-1) + \cdots + (p-1)p^d + 0 \cdot p^{d+1} + \dots$$

with $c'_{d+1} \neq p-1$*, then* $p_1^t(\xi_{l,j'}) = 0$ *and we replace* $\xi_{l(1,p^{d+1}),j'}$ *by* $p_1^t(\xi_{l,j})$ *in* B_2

(c) If $a(j) = 2p^{d+1} - 1$ *with* $t_2 < p^{d+1}$*.*

(i) If the p-adic expansion of t_3 *is:*

$$t_3 = (p-1) + \cdots + (p-1)p^{b-1} + c_b p^b + \dots$$

with $b \leq d+1$ *and* $c_b \neq p-1$*, then we replace* $\xi_{l(1,p^b),j}$ *by* $p_1^t(\xi_{l,j})$ *in* B_2*.*

(ii) If the p-adic expansion of t_2 *is*

$$t_3 = (p-1) + \cdots + (p-1)p^{d+1} + \dots,$$

then $p_1^t(\xi_{l,j}) = 0$*.*

Proof: Putting together Lemma 5.3, Lemma 5.5, and the result obtained for j not a critical element of $J(\lambda)$, we obtain (a) and (b).

If $a(j) = 2p^{d+1}-1$ and $t_2 \geq p^{d+1}$, then there is $\tilde{\jmath} \in J(\lambda)$ with $a(\tilde{\jmath}) = p^{d+1}-1$ such that $j = \tilde{\jmath}'$. Thus this case is considered in (b). Now suppose that $t_2 < p^{d+1}$. By the formula similar to (5.6), we get

$$p_1^t(\xi_{l,j}) = \sum_{0 \leq d' \leq d+1} \binom{t_3 + p^{d'}}{p^{d'}} \xi_{l(1,p^{d'}),j}. \tag{5.9}$$

If the p-adic expansion of t_3 is of the form

$$t_3 = (p-1) + \cdots + (p-1)p^{d+1} + \dots,$$

then (5.9) implies that $p_1^t(\xi_{l,j}) = 0$. Otherwise, let b be the first coefficient of the p-adic expansion of t_3 different from zero. Then the coefficient of $\xi_{l(1,p^b),j}$ in (5.9) is non-zero. Thus we can replace $\xi_{l(1,p^b),j}$ by $p_1^t(\xi_{l,j})$ in B_2. ■

To construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$, we repeat the procedure used in the previous cases. Let $\hat{B}_2^*$ be the basis of $D(P_1^t)$ dual to $\hat{B}_2$. Define U_λ as the $S(B^+)$-submodule of $D(P_1^t)$ spanned by the elements of $\hat{B}_2^*$ which do not correspond to $p_1^t(\xi_{l,j})$. If $E(\lambda)$, f, and g are defined as in Theorem 3.6, then an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$ is

$$0 \to U_\lambda \xrightarrow{f} E(\lambda) \xrightarrow{g} \mathbb{K}_\lambda \to 0.$$

6. The functors $S\left(B_n^+\right)$-mod $\rightleftarrows S\left(B_m^+\right)$-mod

Let m be a positive integer with $m < n$ and write $S(B_n^+)$ for $S(B^+, n, r)$ and $S(B_m^+)$ for $S(B^+, m, r)$. In this section we consider a pair of exact functors

$$S\left(B_n^+\right)\text{-mod} \underset{G}{\overset{F}{\rightleftarrows}} S\left(B_m^+\right)\text{-mod}\ ,$$

which are useful in the construction of short exact non-split sequences.

Given $\lambda \in \Lambda(n, r)$, suppose $\lambda_m \neq 0$ and $\lambda_{m+1} = \cdots = \lambda_n = 0$. As $\lambda_n = 0$, we do not know how to construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$. But, since $\lambda_m \neq 0$, we can use Theorem 3.6 to construct an Auslander-Reiten sequence ending with $\mathbb{K}_{\overline{\lambda}} = F\left(\mathbb{K}_\lambda\right)$ in $S(B_m^+)$-mod. Then using G we lift this sequence to $S(B_n^+)$-mod. We obtain an exact non-split sequence, which ends with $\mathbb{K}_\lambda$ and starts with an indecomposable module, but which, in general, is not an Auslander-Reiten sequence. We do not know at the moment when this is an Auslander-Reiten sequence in $S(B_n^+)$-mod.

Denote by $\Lambda^*(m, r)$ the image of $\Lambda(m, r)$ in $\Lambda(n, r)$ under the natural embedding, i.e. the elements of $\Lambda^*(m, r)$ are of the form $(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$. Define the idempotent $e \in S(B_n^+)$ by

$$e := \sum_{\alpha \in \Lambda^*(m,r)} \xi_\alpha .$$

As $n > m$, we may regard $I(m, r)$ as a subset of $I(n, r)$. As a consequence, $S(B_m^+)$ can be regarded as a subalgebra of $S(B_n^+)$ (see [6, § 6.5]). In fact, we have that

$$S(B_m^+) = eS(B_n^+)e.$$

Therefore we have an exact functor (cf. [6, § 6.2])

$$\begin{aligned} F\colon S(B_n^+)\text{-mod} &\to S(B_m^+)\text{-mod} \\ V &\mapsto eV \\ (\theta\colon V \to V') &\mapsto \theta|_{eV}. \end{aligned}$$

Suppose that $\lambda \in \Lambda^*(m, r)$ satisfies $\lambda_m \neq 0$ and, moreover, $\lambda_{m-1} < p^{d+1} - 1$ if $\operatorname{char}\mathbb{K} = p$ and $p^d \leq \lambda_m < p^{d+1}$. We write $\overline{\lambda}$ for the preimage of λ in $\Lambda(m, r)$. As $\lambda_n = 0$ we do not know how to construct an Auslander-Reiten sequence ending with $\mathbb{K}_\lambda$. But if we consider the $S(B_m^+)$-irreducible module $\mathbb{K}_{\overline{\lambda}}$, we have $\mathbb{K}_{\overline{\lambda}} = F\left(\mathbb{K}_\lambda\right)$ and, from Theorem 3.6, we know an Auslander-Reiten sequence in $S(B_m^+)$-mod:

$$0 \to U_{\overline{\lambda}} \to E(\overline{\lambda}) \to \mathbb{K}_{\overline{\lambda}} \to 0. \tag{6.1}$$

We want to lift this sequence to $S(B_n^+)$-mod. For this we consider the functor

$$G := S(B_n^+)e \otimes_{S(B_m^+)} - \colon S(B_m^+)\text{-mod} \to S(B_n^+)\text{-mod}.$$

It is well known (see [6, 6.2d]) that for $M \in S(B_m^+)$-mod there holds $eG(M) = \{\, e \otimes m \mid m \in M \}$ and the map

$$\begin{aligned} M &\to eG(M) = F(G(M)) \\ m &\mapsto e \otimes m \end{aligned}$$

gives an isomorphism of $S(B_m^+)$-modules. We will prove next that, in our particular setting, G is an exact functor, and takes indecomposable modules into indecomposable ones.

Lemma 6.1. *Given an $S(B_m^+)$-module M, we have:*

(i) $ev = v$, for all $v \in G(M)$;

(ii) $\dim_{\mathbb{K}} M = \dim_{\mathbb{K}} G(M)$.

Proof: (i) We know that $G(M)$ is the $\mathbb{K}$-span of the set

$$\{\, \xi_{ij} e \otimes m \mid i \leq j,\ i, j \in I(n,r),\ m \in M \}\,.$$

Given i, $j \in I(n,r)$, with $i \leq j$, we get $\xi_{ij}e = 0$, if $j \notin I(m,r)$. If $j \in I(m,r)$, then for every $\rho \in \mathbf{r}$, we have $i_\rho \leq j_\rho \leq m$. Therefore $i \in I(m,r)$ and $\xi_{ij} \in S(B_m^+)$. Hence

$$\xi_{ij} e \otimes m = \xi_{ij} \otimes m = 1 \otimes \xi_{ij} m.$$

This shows that $G(M)$ is the $\mathbb{K}$-span of the set

$$\{\, 1 \otimes m \mid m \in M \}\,.$$

Obviously, we have $e(1 \otimes m) = 1 \otimes em = 1 \otimes m$ for all $m \in M$. Thus $ev = v$ for all $v \in G(M)$.

(ii) We may define linear maps

$$\begin{aligned} M &\to G(M) \qquad & G(M) &\to M \\ m &\mapsto 1 \otimes m \qquad & \xi \otimes m &\mapsto e\xi m. \end{aligned}$$

Using (i) it is immediate to see that these two maps are inverse to each other. Hence M and $G(M)$ are isomorphic as $\mathbb{K}$-vector spaces. ■

Proposition 6.2. *The functor G is exact.*

Proof: Let
$$0 \longrightarrow M_2 \xrightarrow{t_1} M_1 \xrightarrow{t_0} M_0 \longrightarrow 0 \tag{6.2}$$
be an exact sequence in the category of $S(B_m^+)$-modules. As G is a right exact functor, we have that
$$0 \longrightarrow \ker G(t_1) \longrightarrow G(M_2) \xrightarrow{G(t_1)} G(M_1) \xrightarrow{G(t_0)} G(M_0) \longrightarrow 0$$
is an exact sequence in the category of $S(B_n^+)$-modules. Therefore
$$\dim_{\mathbb{K}} \ker G(t_1) = \dim_{\mathbb{K}} G(M_2) - \dim_{\mathbb{K}} G(M_1) + \dim_{\mathbb{K}} G(M_0).$$
Applying Lemma 6.1 and using the fact that (6.2) is an exact sequence, we get
$$\dim_{\mathbb{K}} \ker G(t_1) = \dim_{\mathbb{K}} M_2 - \dim_{\mathbb{K}} M_1 + \dim_{\mathbb{K}} M_0 = 0.$$
This shows that G is an exact functor. ∎

Proposition 6.3. *Let $\mu \in \Lambda(m,r)$. Then $G(\mathbb{K}_\mu)$ is isomorphic to $\mathbb{K}_{\mu'}$, where $\mu' = (\mu_1, \dots, \mu_m, 0, \dots, 0)$. In particlular, G preserves irreducible modules.*

Proof: By Lemma 6.1, we know that $G(\mathbb{K}_\mu)$ is a one-dimensional module. Thus $G(\mathbb{K}_\mu) \cong \mathbb{K}_\beta$ for some $\beta \in \Lambda(n,r)$. We know that $F(G(\mathbb{K}_\mu))$ is isomorphic to $\mathbb{K}_\mu$. Therefore $\mathbb{K}_\mu$ is isomorphic to $F(\mathbb{K}_\beta)$. But
$$F(\mathbb{K}_\beta) \cong \begin{cases} 0, & \beta \notin \Lambda^*(m,r) \\ \mathbb{K}_{\overline{\beta}}, & \beta \in \Lambda^*(m,r). \end{cases}$$
Thus $\overline{\beta} = \mu$ and this is equivalent to $\beta = \mu'$. ∎

Proposition 6.4. *Let M be an indecomposable $S(B_m^+)$-module. Then $G(M)$ is an indecomposable $S(B_n^+)$-module.*

Proof: Suppose $G(M) = U_1 \oplus U_2$, direct sum of non-zero $S(B_n^+)$-modules. By Lemma 6.1, we have that $eU_1 = U_1$ and $eU_2 = U_2$. Therefore $M \cong eG(M) = U_1 \oplus U_2$, which contradicts the fact that M is indecomposable. ∎

Applying G to the Auslander-Reiten sequence (6.1), by Propositions 6.2 and 6.3, we get the exact sequence
$$0 \to G(U_{\overline{\lambda}}) \xrightarrow{G(f)} G(E(\overline{\lambda})) \xrightarrow{G(g)} \mathbb{K}_\lambda \to 0, \tag{6.3}$$
where the module $G(U_{\overline{\lambda}})$ is indecomposable by Proposition 6.4.

The sequence (6.3) is not split. In fact, applying F to (6.3), we obtain an exact sequence isomorphic to (6.1). If (6.3) were split, then (6.1) would also be split. Contradiction.

So the sequence (6.3) is a short exact non-split sequence, ending and starting with indecomposable modules. But it needs not be Auslander-Reiten in general.

Example 6.5. Assume $m = 2$, $n = 3$, $r = 3$, and $\mathrm{char}(\mathbb{K}) \neq 2$. Consider $\lambda = (1, 2, 0)$. Then $\overline{\lambda} = (1, 2)$. First we claim that the Auslander-Reiten translate $\tau(\mathbb{K}_{(1,2)})$ is simple, isomorphic to $\mathbb{K}_{(2,1)}$. Since $\mathrm{char}\,\mathbb{K} \neq 2$, we get $P_1^t = \xi_{(2,1)} S(B_2^+)$, so the cokernel of p_1^t has a basis labelled by the set $J((2, 1)) \setminus J(\overline{\lambda})$.

The set $J((2, 1))$ is in bijective correspondence with

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and the set $J(\overline{\lambda})$ is in bijective correspondence with

$$\begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline\end{array}$$

Therefore the cokernel of p_1^t has basis labelled by $j = (1, 1, 2)$. Taking duals, gives that $\tau(\mathbb{K}_{(1,2)}) \cong \mathbb{K}_{(2,1)}$.

Let $0 \to \mathbb{K}_{(2,1)} \to E \to \mathbb{K}_{(1,2)} \to 0$ be the Auslander-Reiten sequence in $S(B_2^+)$-mod. Applying the functor G, this gives the non-split exact sequence of $S(B_3^+)$-modules

$$0 \to \mathbb{K}_{(2,1,0)} \xrightarrow{f} G(E) \xrightarrow{g} \mathbb{K}_{(1,2,0)} \to 0$$

and $G(E)$ is indecomposable of length 2.

There is an indecomposable $S(B_3^+)$-module X with simple head $\mathbb{K}_{(2,0,1)}$ and simple socle $\mathbb{K}_{(2,1,0)}$, namely the quotient of $S(B_3^+)\xi_{(2,0,1)}$ module the square of the radical.

So there is a 1-1 homomorphism $h : \mathbb{K}_{(2,1,0)} \to X$. Assume for a contradiction that there is some homomorphism $\psi : G(E) \to X$ with $\psi \circ f = h$. Then ψ must be 1-1 (since the socle of $G(E)$ is equal to the image of f), and then ψ is an isomorphism, by dimensions. But this is a contradiction, since $G(E)$ and X have non-isomorphic heads.

7. The socle of $S^+(n,r)$

In the previous sections we studied the kernel of the map p_1^t. Since this kernel can be identified with $\mathrm{Hom}_{S(B^+)}(\mathbb{K}_\lambda, S(B^+))$, it provides information on the socle of the Borel-Schur algebra $S(B^+)$. Namely, p_1^t is non-injective if and only if $\mathbb{K}_\lambda$ is in the socle of $S(B^+)$. In this section we collect some facts on the socle of $S(B^+)$. We will use the usual notation $\Lambda^+(n,r)$ for the subset of partitions in $\Lambda(n,r)$.

We start with the following auxiliary result.

Lemma 7.1. *Suppose $\nu \in \Lambda(n,r) \setminus \Lambda^+(n,r)$ and let M be an $S(n,r)$-module. Then $\mathrm{Hom}_{S(B^+)}(\mathbb{K}_\nu, M) = 0$, where we consider M as an $S(B^+)$-module by restriction.*

Proof: Let $f\colon \mathbb{K}_\nu \to M$ be an $S(B^+)$-homomorphism and $c \in \mathbb{K}_\nu$. Then $\xi_{ij} f(c) = f(\xi_{ij} c) = 0$ for all $\xi_{ij} \in S(B^+)$ different from ξ_ν. By [5, Theorem 5.2] we have

$$S(n,r) = \sum_{\lambda \in \Lambda^+(n,r)} S(B^-)\xi_\lambda S(B^+),$$

where $S(B^-)$ denotes the lower Borel subalgebra of the Schur algebra $S(n,r)$. Since $\nu \notin \Lambda^+(n,r)$, we get that $S(n,r)f(c) = 0$. This shows that $f(c) = 0$ for all $c \in \mathbb{K}_\nu$ and thus f is the zero map. ∎

As a simple consequence we get:

Proposition 7.2. *Let $\nu \in \Lambda(n,r) \setminus \Lambda^+(n,r)$. Then $\mathrm{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(B^+)) = 0$.*

Proof: The embedding $S(B^+) \hookrightarrow S(n,r)$ induces the injective map

$$\mathrm{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(B^+)) \to \mathrm{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(n,r)).$$

Now Lemma 7.1 implies that the vector space $\mathrm{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(n,r))$ is trivial. Thus also $\mathrm{Hom}_{S(B^+)}(\mathbb{K}_\nu, S(B^+))$ vanishes. ∎

Combining the results of the previous sections and Proposition 7.2, we get the following theorem.

Theorem 7.3. *(1) The module $\mathbb{K}_{(r,0,\dots,0)}$ is a direct summand of the socle of $S(B^+)$ independently of* $\mathrm{char}\,\mathbb{K}$.

(2) Suppose $\mathrm{char}\,\mathbb{K} = 0$ *and $n = 2$. Then the socle of $S(B^+)$ is a direct sum of several copies of $\mathbb{K}_{(r,0)}$.*

(3) Suppose $\operatorname{char}\mathbb{K} = p$ *and* $n = 2$. *Then* $\mathbb{K}_\lambda$ *is a direct summand of the socle of* $S(B^+)$ *if and only if* $\lambda = (r, 0)$ *or* λ *is a partition satisfying*

$$\lambda_1 \geq p^{\lfloor \log_p \lambda_2 \rfloor + 1} - 1.$$

(4) Suppose $n \geq 3$ *and* $\operatorname{char}\mathbb{K} = 0$. *Then all the composition factors of the socle of* $S(B^+)$ *are of the form* $\mathbb{K}_\lambda$ *with* λ *a partition such that* $\lambda_n = 0$.

(5) Suppose $n \geq 3$ *and* $\operatorname{char}\mathbb{K} = p$. *Then all the composition factors of the socle of* $S(B^+)$ *are of the form* $\mathbb{K}_\lambda$ *with* λ *a partition such that either* $\lambda_n = 0$ *or*

$$\lambda_{n-1} \geq p^{\lfloor \log_p \lambda_n \rfloor + 1} - 1.$$

(6) Suppose $n = 3$, $\operatorname{char}\mathbb{K} = p \geq 3$, *and* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *satisfies*

$$\lambda_1 \geq p^{\lfloor \log_p \lambda_3 \rfloor + 2} - 1, \ \lambda_2 = 2p^{\lfloor \log_p \lambda_3 \rfloor + 1} - 1. \tag{7.1}$$

Then the module $\mathbb{K}_\lambda$ *has a non-zero multiplicity in the socle of* $S(B^+)$.

Proof: The simple module $\mathbb{K}_{(r,0,\dots,0)}$ is isomorphic to its projective cover $S(B^+)\xi_{(r,0,\dots,0)}$. Thus $\operatorname{Hom}_{S(B^+)}(\mathbb{K}_{(r,0,\dots,0)}, S(B^+)) \cong \xi_{(r,0,\dots,0)}S(B^+)$ is non trivial. This shows that $\mathbb{K}_{(r,0,\dots,0)}$ is a direct summand of the socle of $S(B^+)$.

Now the claims (2)-(5) follow from Propositions 3.3, 3.5, and 4.1.

To prove (6), we denote $\lfloor \log_p \lambda_3 \rfloor$ by d. Then $p^d \leq \lambda_3 < p^{d+1}$. Since λ satisfies (7.1) there is $j \in J(\lambda)$ such that $a(j) = \lambda_2 = 2p^{d+1} - 1$ and $t_3 = t_3(j) = p^{d+2} - 1$. Then by Proposition 5.6(c)(ii) we get $p_1^t(\xi_{l,j}) = 0$. This shows that $\ker(p_1^t)$ is non-trivial, and therefore there is an embedding of $\mathbb{K}_\lambda$ into the socle of $S(B^+)$. ■

References

[1] Maurice Auslander and Idun Reiten, *Representation theory of Artin algebras. III. Almost split sequences*, Comm. Algebra **3** (1975), 239–294. MR 0379599 (52 #504)

[2] M. C. R. Butler, *The construction of almost split sequences. I*, Proc. London Math. Soc. (3) **40** (1980), no. 1, 72–86. MR **560996 (82d:**16004a)

[3] Peter Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 831, Springer, Berlin, 1980, pp. 1–71. MR **607140 (82i:**16030)

[4] J. A. Green, *Notes on almost split sequences, I*, preprint, 1983.

[5] ———, *On certain subalgebras of the Schur algebra*, J. Algebra **131** (1990), no. 1, 265–280. MR **1055008 (91b:**20019)

[6] ———, *Polynomial representations of* GL_n, augmented ed., Lecture Notes in Mathematics, vol. 830, Springer, Berlin, 2007, With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, Green and M. Schocker. MR **2349209 (2009b:**20084)

[7] Ana Paula Santana, *The Schur algebra $S(B^+)$ and projective resolutions of Weyl modules*, J. Algebra **161** (1993), no. 2, 480–504. MR **1247368 (95a:**20046)

[8] Ana Paula Santana and Ivan Yudin, *Characteristic-free resolutions of Weyl and Specht modules*, Adv. Math. **229** (2012), no. 4, 2578–2601. MR 2880232

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