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CHARACTERIZATION THEOREM FOR LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON NON-UNIFORM LATTICES

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Abstract: It is stated and proved a characterization theorem for Laguerre-Hahn orthogonal polynomials on non-uniform lattices. This theorem proves the equivalence between the Riccati equation for the formal Stieltjes function, linear first-order difference relations for the orthogonal polynomials as well as for the associated polynomials of the first kind, and linear first-order difference relations for the functions of the second kind.

Keywords: Laguerre-Hahn orthogonal polynomials; Divided difference operator; non-uniform lattices; Riccati difference equation; Structure relations.
AMS Subject Classification (2000): 33C45; 33C47; 33D45.

1. Introduction

The present paper concerns orthogonal polynomials of a discrete variable on non-uniform lattices (commonly denoted by snul). These lattices are associated with divided differences operators such as the Wilson or Askey-Wilson operator ([2, Section 5], and [3, 12, 17, 18]). Specifically, we focus our attention on the so-called Laguerre-Hahn orthogonal polynomials. The Laguerre-Hahn orthogonal polynomials on non-uniform lattices were introduced by A. Magnus in [14], as the ones for which the formal Stieltjes function satisfies a Riccati difference equation with polynomial coefficients, with the difference operator taken as a general divided difference operator given by [14, Eq. (1.1)] (see Section 2 of the present paper for the precise definitions and main properties). In this pioneering work, Magnus establishes difference relations as well as representations for the Laguerre-Hahn orthogonal polynomials and he proves that, under certain restrictions on the degrees of the coefficient of the Riccati difference equation, the Laguerre-Hahn orthogonal polynomials are the associated Askey-Wilson polynomials [1, 2].

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As it is well known from the setting of continuous orthogonality, Laguerre-Hahn orthogonal polynomials inherit many properties from the classical and semi-classical families [5, 7, 13, 16]. Indeed, one of the research topics within the Laguerre-Hahn theory of a discrete variable is the so-called structure relations, that is, linear difference relations involving the orthogonal polynomials (see [4, 8, 10, 11] and their lists of references). In the semi-classical case, it was proven in [15] the characterization of semi-classical orthogonal polynomials on non-uniform lattices in terms of structure relations. A more recent contribution, [9], proves the characterization of classical polynomials on non-uniform lattices in terms of two types of structure relations, using the so-called functional approach.

In the present paper we show a characterization theorem for Laguerre-Hahn orthogonal polynomial on arbitrary non-uniform lattices. Our main result is given in Theorem 2, where it is shown the equivalence between:

(a) the Riccati difference equation for the formal Stieltjes function, S;
(b) linear first-order difference relations for orthogonal polynomials related to S, as well as for the associated polynomials of the first kind;
(c) linear first-order difference relations for the functions of the second kind related to S.

The difference relations contained in Theorem 2 for Laguerre-Hahn families extend some of the difference relations for the classical families given in [9, 15, 22].

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show the main results of the paper, namely, the equivalence between the above referred conditions (a), (b) and (c), stated in Theorem 2 Section 4 is devoted to the proof of Theorem 2.

2. Preliminary results

2.1. The operators $\mathbb{D}, \mathbb{E}_j, \mathbb{M}$ and the related non-uniform lattices. We consider the divided difference operator $\mathbb{D}$ given in [14], involving the values of a function at two points, with the fundamental property that $\mathbb{D}$ leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n. The operator $\mathbb{D}$, defined on the space of arbitrary functions, is given by

$$\mathbb{D}f(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}, \tag{1}$$

where, at this stage, y_1 and y_2 are still unknown functions. To define them, one starts by using the property that $\mathbb{D}f$ is a polynomial of degree $n-1$ whenever f is a polynomial of degree n. Then, applying $\mathbb{D}$ to $f(x)=x^2$ and $f(x)=x^3$, one obtains, respectively,

$$y_1(x)+y_2(x) = \text{polynomial of degree } 1\,, \tag{2}$$

$$(y_1(x))^2+y_1(x)y_2(x)+(y_2(x))^2 = \text{polynomial of degree } 2\,, \tag{3}$$

the later condition being equivalent to $y_1(x)y_2(x)$ = polynomial of degree less or equal than 2. The conditions (2) and (3) define y_1 and y_2 as the two roots of a quadratic equation

$$\hat{a}y^2+2\hat{b}xy+\hat{c}x^2+2\hat{d}y+2\hat{e}x+\hat{f}=0\,,\quad \hat{a}\neq 0\,. \tag{4}$$

Some identities involving y_1 and y_2, following from the fact that y_1, y_2 are the roots of (4):

$$y_1(x)+y_2(x) = -2(\hat{b}x+\hat{d})/\hat{a}\,,$$
$$y_1(x)y_2(x) = (\hat{c}x^2+2\hat{e}x+\hat{f})/\hat{a}\,.$$

There are four primary classes of lattices and related divided difference operators (1):

(i) the linear lattice, related to the forward difference operator [19, Chapter 2, Section 12] ;
(ii) the q-linear lattice, related to the q-difference operator [12] ;
(iii) the quadratic lattice, related to the Wilson operator [2] ;
(iv) the q-quadratic lattice, related to the Askey-Wilson operator [2].

This classification of lattices is done according to the two parameters $\lambda=\hat{b}^2-\hat{a}\hat{c}$ and $\tau=\left((\hat{b}^2-\hat{a}\hat{c})(\hat{d}^2-\hat{a}\hat{f})-(\hat{b}\hat{d}-\hat{a}\hat{e})^2\right)/\hat{a}$, assuming $\hat{a}\hat{c}\neq 0$: $\lambda=\tau=0$ in case (i); $\lambda>0, \tau=0$ in case (ii); $\lambda=0, \tau<0$ in case (iii); $\lambda\tau<0$ in case (iv).

We would like to remark [15, Section 2], where it is given a geometric interpretation of the lattices. For the quadratic class of lattices (the so-called snul), it is possible to have a parametric representation of the conic (4), say $\{x(s), y(s)\}$, such that $y_1(x(s))=y(s)=x(s-1/2)$ and $y_2(x(s))=y(s+1)=x(s+1/2)$, ading to [3, 17, 18]

$$\begin{cases} x(s)=c_4s^2+c_5s+c_6\,, & \text{if } \lambda=0, \tau<0\,,\\ x(s)=c_1q^s+c_2q^{-s}+c_3\,, & \text{if } \lambda\tau<0\,,\quad q+q^{-1}=4\hat{b}^2/(\hat{a}\hat{c})-2\,. \end{cases}$$

Note that each of the operators in (i)-(iv) is an extension of the preceding one, which is recovered as a particular case or as a limit case, up to a linear transformation of the variable.

In the present paper we shall operate with the divided difference operator given in its general form (1). By defining the operators $\mathbb{E}_1$ and $\mathbb{E}_2$, acting on arbitrary functions f, as

$$\mathbb{E}_1 f(x) = f(y_1(x)) \, , \;\; \mathbb{E}_2 f(x) = f(y_2(x)) \, ,$$

(1) is given by

$$\mathbb{D} f(x) = \frac{\mathbb{E}_2 f(x) - \mathbb{E}_1 f(x)}{y_2(x) - y_1(x)} \, .$$

We define the companion operator of $\mathbb{D}$ as

$$\mathbb{M} f(x) = \frac{\mathbb{E}_1 f(x) + \mathbb{E}_1 f(x)}{2} \, . \tag{5}$$

Some useful identities involving $\mathbb{D}, \mathbb{M}$ and $\mathbb{E}_1, \mathbb{E}_2$ are listed below:

$$\mathbb{D}(gf) = \mathbb{D}g \, \mathbb{M}f + \mathbb{M}g \, \mathbb{D}f \, , \tag{6}$$

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g \, \mathbb{M}f - \mathbb{D}f \, \mathbb{M}g}{\mathbb{E}_1 f \, \mathbb{E}_2 f} \, , \tag{7}$$

$$\mathbb{D}(1/f) = \frac{-\mathbb{D}f}{\mathbb{E}_1 f \, \mathbb{E}_2 f} \, , \tag{8}$$

$$\mathbb{M}(gf) = \mathbb{M}g \, \mathbb{M}f + \mathbb{D}g \, \mathbb{D}f \, \frac{(y_1 - y_2)^2}{4} \, ,$$

$$\mathbb{M}(g/f) = \frac{\mathbb{E}_1 g \, \mathbb{E}_2 f + \mathbb{E}_2 g \, \mathbb{E}_1 f}{2\mathbb{E}_1 f \, \mathbb{E}_2 f} \, , \tag{9}$$

$$\mathbb{M}(1/f) = \frac{\mathbb{M}f}{\mathbb{E}_1 f \, \mathbb{E}_2 f} \, . \tag{10}$$

Eq. (6) has the equivalent forms:

$$\mathbb{D}(gf) = \mathbb{D}g \, \mathbb{E}_1 f + \mathbb{D}f \, \mathbb{E}_2 g \, , \tag{11}$$

$$\mathbb{D}(gf) = \mathbb{D}g \, \mathbb{E}_2 f + \mathbb{D}f \, \mathbb{E}_1 g \, .$$

Also, one has two equivalent forms for (7):

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g\,\mathbb{E}_1 f - \mathbb{D}f\,\mathbb{E}_1 g}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,, \tag{12}$$

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g\,\mathbb{E}_2 f - \mathbb{D}f\,\mathbb{E}_2 g}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,. \tag{13}$$

2.2. Laguerre-Hahn orthogonal polynomials and auxiliary results. We shall consider formal orthogonal polynomials related to a (formal) Stieltjes function defined by

$$S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1} \tag{14}$$

where (u_n), the sequence of moments, is such that $\det\left[u_{i+j}\right]_{i,j=0}^{n} \neq 0$, $n \geq 0$, $u_0 = 1$. The orthogonal polynomials related to S, $P_n, n \geq 0$, are the diagonal Padé denominators of (14), thus the numerator polynomial (of degree $n-1$), henceforth denoted by $P_{n-1}^{(1)}$, and the denominator P_n (of degree n) are determined through

$$S(x) - P_{n-1}^{(1)}(x)/P_n(x) = \mathcal{O}(x^{-2n-1})\,, \quad x \to \infty\,. \tag{15}$$

Throughout the paper we consider each P_n monic, and we will denote the sequence of monic polynomials $\{P_n\}_{n\geq 0}$ by SMOP.

Monic orthogonal polynomials satisfy a three term recurrence relation [20]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots\,, \tag{16}$$

with $P_{-1}(x) = 0,\ P_0(x) = 1$, and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = u_0 = 1$.

The sequence $\{P_n^{(1)}\}_{n\geq 0}$, also known as the sequence of associated polynomials of the first kind, satisfies the three term recurrence relation

$$P_n^{(1)}(x) = (x - \beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n = 1, 2, \ldots$$

with $P_{-1}^{(1)}(x) = 0,\ P_0^{(1)}(x) = 1$.

An equivalent form of (15), often encountered in the literature of orthogonal polynomials (see, for example, [21] and its list of references), is given by

$$q_n = P_n S - P_{n-1}^{(1)}\,, \quad n \geq 1\,,\ q_0 = S\,, \tag{17}$$

where $q_n, n \geq 0$, are the so-called functions of the second kind corresponding to $\{P_n\}_{n\geq 0}$. The sequence $\{q_n\}_{n\geq 0}$ also satisfies a three term recurrence relation,

$$q_{n+1}(x) = (x - \beta_n)q_n(x) - \gamma_n q_{n-1}(x)\,, \quad n = 0, 1, 2, ... \tag{18}$$

with initial conditions $q_{-1} = 1,\ q_0(x) = S(x)$.

We will make use of the following result (see [6]).

Lemma 1. *Let $\{P_n\}_{n\geq 0}$ be a SMOP and let $\{P_n^{(1)}\}_{n\geq 0}$ be the sequence of associated polynomials of the first kind. The following holds:*

$$\mathbb{E}_j P_n^{(1)}\, \mathbb{E}_j P_n - \mathbb{E}_j P_{n+1}\, \mathbb{E}_j P_{n-1}^{(1)} = \prod_{k=0}^{n} \gamma_k\,, \quad j = 1, 2\,, \quad n \geq 0\,. \tag{19}$$

Therefore, for each $j = 1, 2$, $\mathbb{E}_j P_n^{(1)}$ and $\mathbb{E}_j P_{n+1}$ do not share zeroes.

Proof: Eq. (19) follows from the application of the operator $\mathbb{E}_j$, $j = 1, 2$, to the identity

$$P_n^{(1)} P_n - P_{n+1} P_{n-1}^{(1)} = \prod_{k=0}^{n} \gamma_k\,, \quad n \geq 0\,.$$

From (19) there follows the statement concerning the zeros. ■

Definition 1. A SMOP $\{P_n\}_{n\geq 0}$ related to a Stieltjes function S (14) is said to be *Laguerre-Hahn* if S satisfies a Riccati equation

$$A(x)\, \mathbb{D}S(x) = B(x)\, \mathbb{E}_1 S(x)\, \mathbb{E}_2 S(x) + C(x)\, \mathbb{M}S(x) + D(x)\,, \tag{20}$$

where A, B, C, D are polynomials in x, $A \neq 0$.
If $B \equiv 0$, then $\{P_n\}_{n\geq 0}$ is said to be *semi-classical*.

We will make use of the Theorem that follows.

Theorem 1. *Let $\{f_n\}$ be a sequence of functions satisfying a three term recurrence relation*

$$f_{n+1}(x) = (x - \beta_n) f_n(x) - \gamma_n f_{n-1}(x)\,, \ \ \gamma_n \neq 0,\ n \geq 0\,. \tag{21}$$

Let $g_n = f_{n+1}/f_n$ satisfy for all $n \geq 0$

$$A_n(x)\, \mathbb{D}g_n(x) = B_n(x)\, \mathbb{E}_1 g_n(x)\, \mathbb{E}_2 g_n(x) + C_n\, \mathbb{M}g_n(x) + D_n(x)\,, \tag{22}$$

with $\mathbb{D}, \mathbb{M}$ *the operators defined in (1) and (5), and* A_n, B_n, C_n, D_n *bounded degree polynomials. Then, for all* $n \geq 0$*, the following relations hold:*

$$A_{n+1} = A_n - \frac{(y_1 - y_2)^2}{2} \frac{D_n}{\gamma_{n+1}} \,, \tag{23}$$

$$B_{n+1} = \frac{D_n}{\gamma_{n+1}} \,, \tag{24}$$

$$C_{n+1} = -C_n - 2\mathbb{M}(x - \beta_{n+1}) \frac{D_n}{\gamma_{n+1}} \,, \tag{25}$$

$$D_{n+1} = A_n + \gamma_{n+1} B_n + \mathbb{M}(x - \beta_{n+1}) C_n + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) \frac{D_n}{\gamma_{n+1}}. \tag{26}$$

Proof: From (21) we get $g_n = (x - \beta_n) - \gamma_n / g_{n-1}$, thus, writing the above equation to $n+1$,

$$g_{n+1} = (x - \beta_{n+1}) - \gamma_{n+1}/g_n \,. \tag{27}$$

Applying $\mathbb{D}$ to (27) and using $\mathbb{D}(1/g_n) = -\mathbb{D}g_n / (\mathbb{E}_1 g_n \mathbb{E}_2 g_n)$ (cf. (8)) we get

$$\mathbb{D}g_{n+1} = 1 + \gamma_{n+1} \frac{\mathbb{D}g_n}{\mathbb{E}_1 g_n \, \mathbb{E}_2 g_n} \,.$$

Now we multiply the above equation by A_n and use (22), as well as $\mathbb{M}(1/g_n) = \mathbb{M}g_n / (\mathbb{E}_1 g_n \, \mathbb{E}_2 g_n)$ (cf. (10)), thus obtaining

$$A_n \, \mathbb{D}g_{n+1} = A_n + \gamma_{n+1} B_n + \gamma_{n+1} C_n \, \mathbb{M}(1/g_n) + \frac{\gamma_{n+1} D_n}{\mathbb{E}_1 g_n \, \mathbb{E}_2 g_n} \,. \tag{28}$$

Note that from (27) we have

$$\mathbb{M}(1/g_n) = \frac{\mathbb{M}(x - \beta_{n+1})}{\gamma_{n+1}} - \frac{\mathbb{M}g_{n+1}}{\gamma_{n+1}} \,. \tag{29}$$

Also,

$$\frac{\gamma_{n+1} D_n}{\mathbb{E}_1 g_n \mathbb{E}_2 g_n} = \frac{D_n}{\gamma_{n+1}} \left(y_1 - \beta_{n+1} - \mathbb{E}_1 g_{n+1}\right) \left(y_2 - \beta_{n+1} - \mathbb{E}_2 g_{n+1}\right) \,,$$

and some computations yield

$$\begin{aligned} \frac{\gamma_{n+1} D_n}{\mathbb{E}_1 g_n \, \mathbb{E}_2 g_n} = \frac{D_n}{\gamma_{n+1}} \big((y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) + (y_1 - y_2)^2 / 2 \, \mathbb{D}g_{n+1} \\ -2\mathbb{M}(x - \beta_{n+1}) \mathbb{M}g_{n+1} + \mathbb{E}_1 g_{n+1} \mathbb{E}_2 g_{n+1} \big) \,. \end{aligned} \tag{30}$$

The substitution of (29) and (30) into (28) yields

$$\left(A_n - 2\frac{(y_1-y_2)^2}{4}\frac{D_n}{\gamma_{n+1}}\right)\mathbb{D}g_{n+1} = \frac{D_n}{\gamma_{n+1}}\mathbb{E}_1 g_{n+1}\mathbb{E}_2 g_{n+1}$$
$$+\left(-C_n - 2\mathbb{M}(x-\beta_{n+1})\frac{D_n}{\gamma_{n+1}}\right)\mathbb{M}g_{n+1}$$
$$+\left(A_n + \gamma_{n+1}B_n + \mathbb{M}(x-\beta_{n+1})C_n + (y_1-\beta_{n+1})(y_2-\beta_{n+1})\frac{D_n}{\gamma_{n+1}}\right).$$

The comparison between the above equation and (22) written to $n+1$ gives us (23)-(26). ∎

3. Characterization theorem

Theorem 2. *Let S be a Stieltjes function, let $\{P_n\}_{n\geq 0}$ be the corresponding SMOP, and let $\{P_n^{(1)}\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ be the sequence of associated polynomials of the first kind and the sequence of functions of the second kind, respectively. The following statements are equivalent:*

(a) *S satisfies the Riccati equation (20),*

$$A\,\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D\,,$$

where A, B, C, D are polynomials;

(b) *P_n and $P_n^{(1)}$ satisfy the difference relations for all $n\geq 1$,*

$$\begin{cases} A\,\mathbb{D}P_n = l_{n-1}\mathbb{E}_1 P_n - C/2\,\mathbb{E}_2 P_n - B\,\mathbb{E}_2 P_{n-1}^{(1)} + \Theta_{n-1}\mathbb{E}_1 P_{n-1}\,, \\ A\,\mathbb{D}P_{n-1}^{(1)} = l_{n-1}\mathbb{E}_1 P_{n-1}^{(1)} + C/2\,\mathbb{E}_2 P_{n-1}^{(1)} + D\,\mathbb{E}_2 P_n + \Theta_{n-1}\mathbb{E}_1 P_{n-2}^{(1)}\,; \end{cases} \tag{31}$$

(c) *q_n satisfies for all $n\geq 0$,*

$$A\,\mathbb{D}q_n = l_{n-1}\,\mathbb{E}_1 q_n + (B\,\mathbb{E}_1 S + C/2)\,\mathbb{E}_2 q_n + \Theta_{n-1}\,\mathbb{E}_1 q_{n-1}\,, \tag{32}$$

where l_n, Θ_n are polynomials of uniformly bounded degrees satisfying the initial conditions $l_{-1} = C/2$, $\Theta_{-1} = D$.

The proof of Theorem 2 will be given at the next section.

Remark . The characterizations stated in Theorem 1 are not uniquely represented. One can also deduce that the following statements (a), (b), (c) are equivalent:

(a) S satisfies the Riccati equation (20),

$$A\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D\,;$$

(b) P_n and $P_n^{(1)}$ satisfy the difference relations for all $n \geq 1$,

$$\begin{cases} A\,\mathbb{D}P_n = l_{n-1}\mathbb{E}_2 P_n - C/2\,\mathbb{E}_1 P_n - B\,\mathbb{E}_1 P_{n-1}^{(1)} + \Theta_{n-1}\mathbb{E}_2 P_{n-1}\,, \\ A\,\mathbb{D}P_{n-1}^{(1)} = l_{n-1}\mathbb{E}_2 P_{n-1}^{(1)} + C/2\,\mathbb{E}_1 P_{n-1}^{(1)} + D\,\mathbb{E}_1 P_n + \Theta_{n-1}\mathbb{E}_2 P_{n-2}^{(1)}\,; \end{cases} \tag{33}$$

(c) q_n satisfies for all $n \geq 0$,

$$A\,\mathbb{D}q_n = l_{n-1}\,\mathbb{E}_2 q_n + (B\,\mathbb{E}_2 S + C/2)\,\mathbb{E}_1 q_n + \Theta_{n-1}\,\mathbb{E}_2 q_{n-1}\,. \tag{34}$$

Therefore, we deduce the result that follows.

Theorem 3. *Let S be a Stieltjes function satisfying the Riccati equation*

$$A\,\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D\,,$$

where A, B, C, D are polynomials. Let $\{P_n\}_{n\geq 0}$ be the SMOP related to S, and let $\{P_n^{(1)}\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ be the sequence of associated polynomials of the first kind and the sequence of functions of the second kind, respectively. The following relations hold, for all $n \geq 0$:

$$A\,\mathbb{D}P_{n+1} = (l_n - C/2)\mathbb{M}P_{n+1} - B\,\mathbb{M}P_n^{(1)} + \Theta_n\mathbb{M}P_n\,, \tag{35}$$

$$A\,\mathbb{D}P_n^{(1)} = (l_n + C/2)\mathbb{M}P_n^{(1)} + D\,\mathbb{M}P_{n+1} + \Theta_n\mathbb{M}P_{n-1}^{(1)}\,, \tag{36}$$

$$A\,\mathbb{D}q_n = (l_{n-1} + C/2)\mathbb{M}q_n + B\,(\mathbb{M}S\,\mathbb{M}q_n - \mathbb{M}(Sq_n)) + \Theta_{n-1}\mathbb{M}q_{n-1}\,. \tag{37}$$

Proof: Sum (31) and (33) to get (35) and (36). Following the same idea, sum (32) and (34) to get (37). ■

Remark . The equations (35)-(37) extend the ones given in [22] for the semi-classical case.

Corollary 1. *The polynomials l_n, Θ_n of Theorems 2, 3 satisfy, for all $n \geq 0$,*

$$l_{n+1} + l_n + \mathbb{M}(x - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0\,, \tag{38}$$

$$\begin{aligned} \Theta_{n+1} = A + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} \\ + \left(\gamma_{n+1} - \frac{(y_1 - y_2)^2}{4}\right)\frac{\Theta_{n-1}}{\gamma_n} + 2\mathbb{M}(x - \beta_{n+1})l_n\,, \end{aligned} \tag{39}$$

with initial conditions $l_{-1} = C/2, \Theta_{-1} = D$.

Proof: Multiply (32), written to $n+1$, by $\mathbb{E}_2 q_n$ and subtract to (32) multiplied by $\mathbb{E}_2 q_{n+1}$. Then, multiply the resulting equation by $1/(\mathbb{E}_1 q_n \mathbb{E}_2 q_n)$, thus obtaining

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = l_n \mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) - l_{n-1}\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n - \Theta_{n-1}\mathbb{E}_1\left(\frac{q_{n-1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right), \quad (40)$$

where we used the property (13). From the recurrence relation for q_n there holds

$$\frac{q_{n-1}}{q_n} = \frac{(x-\beta_n)}{\gamma_n} - \frac{1}{\gamma_n}\frac{q_{n+1}}{q_n},$$

thus

$$\mathbb{E}_1\left(\frac{q_{n-1}}{q_n}\right) = \frac{1}{\gamma_n}\mathbb{E}_1(x-\beta_n) - \frac{1}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right). \quad (41)$$

The substitution of (41) in (40) yields

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = l_n \mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) - l_{n-1}\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n - \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1(x-\beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right). \quad (42)$$

On the other hand, if we proceed as above, but starting with the eq. (32) and using the property (12), we obtain

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = l_n \mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) - l_{n-1}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n - \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_2(x-\beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right). \quad (43)$$

From the sum of (42) with (43) there follows

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = (l_n - l_{n-1})\mathbb{M}\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) - \frac{\Theta_{n-1}}{2\gamma_n}\left(\mathbb{E}_1(x-\beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \mathbb{E}_2(x-\beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\right). \quad (44)$$

The use of

$$\mathbb{E}_1(x-\beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right)+\mathbb{E}_2(x-\beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) = 2\mathbb{M}(x-\beta_n)\mathbb{M}\left(\frac{q_{n+1}}{q_n}\right)-2\frac{(y_1-y_2)^2}{4}\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right)$$

in (44) gives us the Riccati equation for $g_n = q_{n+1}/q_n$,

$$A_n\,\mathbb{D}g_n = B_n\,\mathbb{E}_1 g_n\,\mathbb{E}_2 g_n + C_n\,\mathbb{M}g_n + D_n$$

with

$$\begin{aligned} A_n &= A - \frac{(y_1-y_2)^2}{4}\frac{\Theta_{n-1}}{\gamma_n}\,,\\ B_n &= \frac{\Theta_{n-1}}{\gamma_n}\,,\\ C_n &= l_n - l_{n-1} - \mathbb{M}(x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n}\,,\\ D_n &= \Theta_n\,. \end{aligned}$$

Now we use Theorem 1. Taking into account the relations (23)-(26) for A_n, B_n, C_n, D_n, there follows, for all $n \geq 0$,

$$l_{n+1} - l_{n-1} - \mathbb{M}(x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n} + \mathbb{M}(x-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0\,, \tag{45}$$

$$\begin{aligned} \Theta_{n+1} = {} & A + (y_1-\beta_{n+1})(y_2-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}}\\ & + \left(\gamma_{n+1} - \frac{(y_1-y_2)^2}{4} - \mathbb{M}(x-\beta_n)\mathbb{M}(x-\beta_{n+1})\right)\frac{\Theta_{n-1}}{\gamma_n}\\ & + \mathbb{M}(x-\beta_{n+1})(l_n - l_{n-1})\,. \end{aligned} \tag{46}$$

To deduce (38) we write (45) in the equivalent form

$$M_{n+1} = M_n\,, \quad n \geq 0 \quad \text{and} \quad M_{n+1} = l_{n+1} + l_n + \mathbb{M}(x-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}}\,,$$

from which there follows $M_{n+1} = M_0$, $n \geq 0$. The use of the initial conditions $l_0 + l_{-1} + \mathbb{M}(x-\beta_0)\frac{\Theta_{-1}}{\gamma_0} = 0$ yield (38).

Using (38) written to $n-1$ in (46) we obtain (39). ■

4. Proof of Theorem 2

Proof of $(a) \Rightarrow (b)$**.**

If we use $S = \dfrac{q_n}{P_n} + \dfrac{P^{(1)}_{n-1}}{P_n}$, $n \geq 1$ (cf. (17)), then (20) yields

$$M_n = -A\mathbb{D}\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + B\mathbb{E}_1\left(\frac{P^{(1)}_{n-1}}{P_n}\right)\mathbb{E}_2\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + C\mathbb{M}\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + D\,, \quad (47)$$

where

$$M_n = A\mathbb{D}\left(\frac{q_n}{P_n}\right) - B\left[\mathbb{E}_1\left(\frac{q_n}{P_n}\right)\mathbb{E}_2\left(\frac{q_n}{P_n}\right) + \mathbb{E}_1\left(\frac{q_n}{P_n}\right)\mathbb{E}_2\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + \right.$$
$$\left. \mathbb{E}_1\left(\frac{P^{(1)}_{n-1}}{P_n}\right)\mathbb{E}_2\left(\frac{q_n}{P_n}\right)\right] - C\mathbb{M}\left(\frac{q_n}{P_n}\right)\,.$$

By multiplying both hand sides of (47) by $\mathbb{E}_1 P_n \mathbb{E}_2 P_n$ and using the properties (12) and (9), we obtain

$$\begin{aligned} M_n\,\mathbb{E}_1 P_n\,\mathbb{E}_2 P_n = &-A\,\mathbb{D}P^{(1)}_{n-1}\,\mathbb{E}_1 P_n + A\,\mathbb{D}P_n\,\mathbb{E}_1 P^{(1)}_{n-1} + B\,\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_2 P^{(1)}_{n-1} \\ &+ \frac{C}{2}\left(\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_2 P_n + \mathbb{E}_1 P_n\,\mathbb{E}_2 P^{(1)}_{n-1}\right) + D\mathbb{E}_1 P_n\,\mathbb{E}_2 P_n\,. \end{aligned}$$

Now let us write

$$\begin{aligned} &-A\,\mathbb{D}P^{(1)}_{n-1}\,\mathbb{E}_1 P_n + A\,\mathbb{D}P_n\,\mathbb{E}_1 P^{(1)}_{n-1} + B\,\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_2 P^{(1)}_{n-1} \\ &\quad + \frac{C}{2}\left(\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_2 P_n + \mathbb{E}_1 P_n\,\mathbb{E}_2 P^{(1)}_{n-1}\right) + D\,\mathbb{E}_1 P_n\,\mathbb{E}_2 P_n = \hat{\Theta}_{n-1}\,, \end{aligned} \quad (48)$$

where $\hat{\Theta}_{n-1}$ is a bounded degree polynomial, as $q_n(x) = \mathcal{O}(x^{-2n-1})$. One has $\deg(\hat{\Theta}_{n-1}) = \max\{\deg(A) - 2, \deg(B) - 2, \deg(C) - 1\}$.

Taking into account $\mathbb{E}_1(P^{(1)}_{n-1})\mathbb{E}_1(P_{n-1}) - \mathbb{E}_1(P_n)\mathbb{E}_1(P^{(1)}_{n-2}) = \prod_{k=0}^{n-1}\gamma_k$, $n \geq 1$, (cf. (19)), then (48) can be written as

$$\begin{aligned} &-A\,\mathbb{D}P^{(1)}_{n-1}\,\mathbb{E}_1 P_n + A\,\mathbb{D}P_n\,\mathbb{E}_1 P^{(1)}_{n-1} + B\,\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_2 P^{(1)}_{n-1} \\ &\quad + \frac{C}{2}\left(\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_2 P_n + \mathbb{E}_1 P_n\,\mathbb{E}_2 P^{(1)}_{n-1}\right) + D\mathbb{E}_1 P_n\,\mathbb{E}_2 P_n \\ &\quad = \Theta_{n-1}\left(\mathbb{E}_1 P^{(1)}_{n-1}\,\mathbb{E}_1 P_{n-1} - \mathbb{E}_1 P_n\,\mathbb{E}_1 P^{(1)}_{n-2}\right)\,, \end{aligned} \quad (49)$$

where $\Theta_{n-1} = \hat{\Theta}_{n-1}/\prod_{k=0}^{n-1}\gamma_k$.

Now, let us write (49) as

$$\left\{A\,\mathbb{D}P_n + C/2\,\mathbb{E}_2P_n + B\,\mathbb{E}_2P_{n-1}^{(1)} - \Theta_{n-1}\mathbb{E}_1P_{n-1}\right\}\mathbb{E}_1P_{n-1}^{(1)}$$
$$= \left\{A\,\mathbb{D}P_{n-1}^{(1)} - C/2\,\mathbb{E}_2P_{n-1}^{(1)} - D\,\mathbb{E}_2P_n - \Theta_{n-1}\mathbb{E}_1P_{n-2}^{(1)}\right\}\mathbb{E}_1P_n\,. \quad (50)$$

Since $\mathbb{E}_1P_{n-1}^{(1)}$ and $\mathbb{E}_1P_n$ do not have common zeroes, for all $n \geq 1$, then there exists a polynomial, l_{n-1}, such that

$$\begin{cases} A\,\mathbb{D}P_n + C/2\,\mathbb{E}_2P_n + B\,\mathbb{E}_2P_{n-1}^{(1)} - \Theta_{n-1}\mathbb{E}_1P_{n-1} = l_{n-1}\mathbb{E}_1P_n\,, \\ A\,\mathbb{D}P_{n-1}^{(1)} - C/2\,\mathbb{E}_2P_{n-1}^{(1)} - D\,\mathbb{E}_2P_n - \Theta_{n-1}\mathbb{E}_1P_{n-2}^{(1)} = l_{n-1}\mathbb{E}_1P_{n-1}^{(1)}\,, \end{cases}$$

that is, we get (31).

Proof of $(b) \Rightarrow (a)$**.**

Let us define $\psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}$. From the three term recurrence relation for $\{P_n\}$ and $\{P_n^{(1)}\}$, there follows that ψ_n satisfies

$$\psi_n = (x-\beta_n)\psi_{n-1} - \gamma_n\psi_{n-2}\,, \quad n \geq 1\,,\ \ \psi_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \ \psi_0 = \begin{bmatrix} x-\beta_0 \\ 1 \end{bmatrix}. \quad (51)$$

With the notation $\mathbb{D}\psi_n = \begin{bmatrix} \mathbb{D}P_{n+1} \\ \mathbb{D}P_n^{(1)} \end{bmatrix}$, $\mathbb{E}_j\psi_n = \begin{bmatrix} \mathbb{E}_jP_{n+1} \\ \mathbb{E}_jP_n^{(1)} \end{bmatrix}$, $j = 1, 2$, (31) reads as

$$A\,\mathbb{D}\psi_{n-1} = l_{n-1}\mathbb{E}_1\psi_{n-1} + \mathcal{C}\,\mathbb{E}_2\psi_{n-1} + \Theta_{n-1}\mathbb{E}_1\psi_{n-2}\,, \quad (52)$$

where $\mathcal{C} = \begin{bmatrix} -C/2 & -B \\ D & C/2 \end{bmatrix}$.

In turn, (52) reads as

$$A\left(\frac{\psi_{n-1}(y_2) - \psi_{n-1}(y_1)}{y_2 - y_1}\right) = l_{n-1}\psi_{n-1}(y_1) + \mathcal{C}\psi_{n-1}(y_2) + \Theta_{n-1}\psi_{n-2}(y_1)\,,$$

that is,

$$\mathcal{A}_n\psi_{n-1}(y_1) + \mathcal{B}\psi_{n-1}(y_2) = \mathcal{C}_n\psi_{n-2}(y_1) \quad (53)$$

with

$$\mathcal{A}_n = (-\frac{A}{y_2-y_1} - l_{n-1})I,\ \mathcal{B} = \frac{A}{y_2-y_1}I - \mathcal{C},\ \mathcal{C}_n = \Theta_{n-1}I,$$

and I denoting the identity matrix of order 2.

Taking $n+1$ in (53) and using the recurrence relation (51) we get

$$\tilde{\mathcal{A}}_n\psi_{n-1}(y_1)+\tilde{\mathcal{B}}_n\psi_{n-1}(y_2)=\tilde{\mathcal{C}}_n\psi_{n-2}(y_1)+\tilde{\mathcal{D}}_n\psi_{n-2}(y_2)\,, \tag{54}$$

with

$$\tilde{\mathcal{A}}_n=(y_1-\beta_n)\mathcal{A}_{n+1}-\mathcal{C}_{n+1},\ \tilde{\mathcal{B}}_n=(y_2-\beta_n)\mathcal{B},\ \tilde{\mathcal{C}}_n=\gamma_n\mathcal{A}_{n+1},\ \tilde{\mathcal{D}}_n=\gamma_n\mathcal{B}\,.$$

Now, we gather (53) and (54) in the system

$$\mathcal{E}_n\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}=\mathcal{F}_n\begin{bmatrix}\psi_{n-2}(y_1)\\ \psi_{n-2}(y_2)\end{bmatrix}\,, \tag{55}$$

where $\mathcal{E}_n$ and $\mathcal{F}_n$ are the block matrices

$$\mathcal{E}_n=\begin{bmatrix}\mathcal{A}_n & \mathcal{B}_n\\ \tilde{\mathcal{A}}_n & \tilde{\mathcal{B}}_n\end{bmatrix}\,,\quad \mathcal{F}_n=\begin{bmatrix}\mathcal{C}_n & 0_{2\times 2}\\ \tilde{\mathcal{C}}_n & \tilde{\mathcal{D}}_n\end{bmatrix}\,.$$

Note that $\mathcal{E}_n$ is invertible,

$$\mathcal{E}_n^{-1}=\frac{\gamma_{n-1}}{\gamma_n\Theta_{n-2}}\begin{bmatrix}(y_2-\beta_n)I & -I\\ -\mathcal{B}^{-1}\tilde{\mathcal{A}}_n & \mathcal{B}^{-1}\mathcal{A}_n\end{bmatrix}\,. \tag{56}$$

From (55) there follows

$$\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}=\mathcal{G}_n\begin{bmatrix}\psi_{n-2}(y_1)\\ \psi_{n-2}(y_2)\end{bmatrix}\,,\quad \mathcal{G}_n=\mathcal{E}_n^{-1}\mathcal{F}_n\,, \tag{57}$$

being $\mathcal{G}_n$ an invertible matrix as is a product of invertible matrices.

Take $n+1$ in (57). On the one hand we have

$$\begin{bmatrix}\psi_{n}(y_1)\\ \psi_{n}(y_2)\end{bmatrix}=\mathcal{G}_{n+1}\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix} \tag{58}$$

and, on the other hand, using the three term recurrence relation (51), we have

$$\begin{bmatrix}\psi_{n}(y_1)\\ \psi_{n}(y_2)\end{bmatrix}=\begin{bmatrix}(y_1-\beta_n)I & 0\\ 0 & (y_2-\beta_n)I\end{bmatrix}\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}-\gamma_n\begin{bmatrix}\psi_{n-2}(y_1)\\ \psi_{n-2}(y_2)\end{bmatrix}\,,$$

thus,

$$\begin{bmatrix}\psi_{n}(y_1)\\ \psi_{n}(y_2)\end{bmatrix}=\left(\begin{bmatrix}(y_1-\beta_n)I & 0\\ 0 & (y_2-\beta_n)I\end{bmatrix}-\gamma_n\mathcal{G}_n^{-1}\right)\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}\,. \tag{59}$$

Consequently, (58) and (59) yield

$$\mathcal{G}_{n+1}=\begin{bmatrix}(y_1-\beta_n)I & 0\\ 0 & (y_2-\beta_n)I\end{bmatrix}-\gamma_n\mathcal{G}_n^{-1}\,. \tag{60}$$

Let us compute $\mathcal{G}_n^{-1}$. Taking into account (56), we obtain

$$\mathcal{G}_n = \begin{bmatrix} X_n I & Y_n \mathcal{B} \\ U_n \mathcal{B}^{-1} & V_n I \end{bmatrix}, \tag{61}$$

where X_n, Y_n, U_n, V_n are the functions given by

$$X_n = \alpha_n \left((y_2 - \beta_n)\Theta_{n-1} + \gamma_n(\frac{A}{y_2 - y_1} + l_n) \right),$$

$$Y_n = -\gamma_n \alpha_n ,$$

$$U_n = \alpha_n \left((\frac{A}{y_2 - y_1} + l_n) \left[\gamma_n(\frac{A}{y_2 - y_1} + l_{n-1}) + (y_1 - \beta_n)\Theta_{n-1} \right] + \Theta_n \Theta_{n-1} \right),$$

$$V_n = -\gamma_n \alpha_n (\frac{A}{y_2 - y_1} + l_{n-1}),$$

with $\alpha_n = \dfrac{\gamma_{n-1}}{\gamma_n \Theta_{n-2}}$. Therefore, it turns out that

$$\mathcal{G}_n^{-1} = \frac{1}{\delta_n} \begin{bmatrix} V_n I & -Y_n \mathcal{B} \\ -U_n \mathcal{B}^{-1} & X_n I \end{bmatrix}, \tag{62}$$

where δ_n is the function given by $\delta_n = X_n V_n - Y_n U_n$.

Taking into account (61) and (62), (60) reads

$$X_{n+1} = (y_1 - \beta_n) - \gamma_n V_n / \delta_n , \tag{63}$$

$$Y_{n+1} = \gamma_n Y_n / \delta_n , \tag{64}$$

$$U_{n+1} = \gamma_n U_n / \delta_n \tag{65}$$

$$V_{n+1} = (y_2 - \beta_n) - \gamma_n X_n / \delta_n . \tag{66}$$

From (63)-(66) there follows that $\delta_{n+1} = X_{n+1} V_{n+1} - Y_{n+1} U_{n+1}$ is given by

$$\delta_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n) - \gamma_n \left((y_1 - \beta_n) X_n + (y_2 - \beta_n) V_n \right) \frac{1}{\delta_n} + \frac{\gamma_n^2}{\delta_n}.$$

Now we proceed analogously with Magnus [14, 15]. Write $\delta_n = \mu_n / \mu_{n-1}$. Then, we obtain

$$\mu_n X_{n+1} = (y_1 - \beta_n)\mu_n - \gamma_n \mu_{n-1} V_n ,$$

$$\mu_n V_{n+1} = (y_2 - \beta_n)\mu_n - \gamma_n \mu_{n-1} X_n ,$$

$$\mu_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n)\mu_n - \gamma_n \mu_{n-1}((y_1 - \beta_n) X_n + (y_2 - \beta_n) V_n) + \gamma_n^2 \mu_{n-1}.$$

The change of variables

$$\hat{X}_{n+1} = \mu_n X_{n+1},\ \hat{V}_{n+1} = \mu_n V_{n+1}$$

yields the relations

$$\begin{aligned}\hat{X}_{n+1} &= (y_1-\beta_n)\mu_n - \gamma_n \hat{V}_n\,,\\ \hat{V}_{n+1} &= (y_2-\beta_n)\mu_n - \gamma_n \hat{X}_n\,,\\ \mu_{n+1} &= (y_1-\beta_n)(y_2-\beta_n)\mu_n - \gamma_n\left((y_1-\beta_n)\hat{X}_n + (y_2-\beta_n)\hat{V}_n\right) + \gamma_n^2\mu_{n-1}\,.\end{aligned}$$

Remark that the above recurrence relations for $\hat{X}_n, \hat{V}_n$ and μ_n are precisely the recurrence relations satisfied by the products of solutions of the three term recurrence relation (16) at y_1 and y_2. Indeed, if

$$\xi_{n+1} = (y_1-\beta_n)\xi_n - \gamma_n\xi_{n-1}\,,\quad \eta_{n+1} = (y_2-\beta_n)\eta_n - \gamma_n\eta_{n-1}\,,$$

then the above recurrence relation for $\hat{X}_n, \hat{V}_n, \mu_n$ is precisely the relation for $\xi_n\eta_{n-1}$, $\xi_{n-1}\eta_n, \xi_n\eta_n$, respectively. Taking into account that a basis of the three term recurrence relation $\tau_{n+1} = (x-\beta_n)\tau_n - \gamma_n\tau_{n-1}$ is constituted by $\{P_n\}$ and $\{q_n\}$, (cf. (16) and (18)), the following must hold: ξ_n must be a combination of $P_n(y_1)$ and $q_n(y_1)$, and η_n must be a combination of $P_n(y_2)$ and $q_n(y_2)$. Thus, there are four choices to be considered:

$$\begin{aligned}&(i)\ \xi_n = P_n(y_1),\ \eta_n = P_n(y_2),\\ &(ii)\ \xi_n = P_n(y_1),\ \eta_n = q_n(y_2)\,,\\ &(iii)\ \xi_n = q_n(y_1),\ \eta_n = P_n(y_2)\,,\\ &(iv)\ \xi_n = q_n(y_1),\ \eta_n = q_n(y_2).\end{aligned}$$

Therefore, we obtain

$$\mu_n = \alpha P_n(y_1)P_n(y_2) + \beta P_n(y_1)q_n(y_2) + \gamma q_n(y_1)P_n(y_2) + \delta q_n(y_1)q_n(y_2)\,. \quad (67)$$

Taking $n=0$ in (67) we obtain

$$\mu_0 = \alpha + \beta q_0(y_2) + \gamma q_0(y_1) + \delta q_0(y_1)q_0(y_1)\,,$$

and such a relation is $A\mathbb{D}S = B\mathbb{E}_1 S\,\mathbb{E}_2 S + C\mathbb{M}S + D$, with

$$A = \frac{(\gamma-\beta)}{2}(y_2-y_1),\ B=\delta,\ C=\gamma+\beta,\ D=\alpha-\mu_0\,.$$

Proof of $(a)\Rightarrow(c)$**.**
Note that $A\,\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D$ is

$$A\,\mathbb{D}q_n = l_{n-1}\,\mathbb{E}_1 q_n + (B\,\mathbb{E}_1 S + C/2)\,\mathbb{E}_2 q_n + \Theta_{n-1}\,\mathbb{E}_1 q_{n-1}$$

with $n=0$, since $q_{-1}=1, q_0=S, l_{-1}=C/2, \Theta_{-1}=D$.

Let us now deduce the above difference equation for $n\geq 1$.

Applying $A\mathbb{D}$ to $q_n=P_nS-P^{(1)}_{n-1},\ n\geq 1$ (cf. (17)), and using the property (11) we obtain

$$A\,\mathbb{D}q_n = A\,\mathbb{D}P_n\,\mathbb{E}_1S + A\,\mathbb{D}S\,\mathbb{E}_2P_n - A\,\mathbb{D}P^{(1)}_{n-1}\,.$$

Using the equations (31) as well as (20) in the above equation, we obtain

$$A\,\mathbb{D}q_n = l_{n-1}\mathbb{E}_1(P_nS-P^{(1)}_{n-1}) + (B\,\mathbb{E}_1S + C/2\,)\,\mathbb{E}_2(P_nS-P^{(1)}_{n-1}) \\ + \Theta_{n-1}\mathbb{E}_1(P_{n-1}S-P^{(1)}_{n-2})\,,$$

thus (32) follows.

Proof of $(c)\Rightarrow(a)$**.**

Take $n=0$ in (32).

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