CHARACTERIZATION THEOREM FOR LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON NON-UNIFORM LATTICES

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ABSTRACT: It is stated and proved a characterization theorem for Laguerre-Hahn orthogonal polynomials on non-uniform lattices. This theorem proves the equivalence between the Riccati equation for the formal Stieltjes function, linear first-order difference relations for the orthogonal polynomials as well as for the associated polynomials of the first kind, and linear first-order difference relations for the functions of the second kind.

KEYWORDS: Laguerre-Hahn orthogonal polynomials; Divided difference operator; non-uniform lattices; Riccati difference equation; Structure relations.

AMS SUBJECT CLASSIFICATION (2000): 33C45; 33C47; 33D45.

1. Introduction

The present paper concerns orthogonal polynomials of a discrete variable on non-uniform lattices (commonly denoted by snul). These lattices are associated with divided differences operators such as the Wilson or Askey-Wilson operator ([2, Section 5], and [3, 12, 17, 18]). Specifically, we focus our attention on the so-called Laguerre-Hahn orthogonal polynomials. The Laguerre-Hahn orthogonal polynomials on non-uniform lattices were introduced by A. Magnus in [14], as the ones for which the formal Stieltjes function satisfies a Riccati difference equation with polynomial coefficients, with the difference operator taken as a general divided difference operator given by [14, Eq. (1.1)] (see Section 2 of the present paper for the precise definitions and main properties). In this pioneering work, Magnus establishes difference relations as well as representations for the Laguerre-Hahn orthogonal polynomials and he proves that, under certain restrictions on the degrees of the coefficient of the Riccati difference equation, the Laguerre-Hahn orthogonal polynomials are the associated Askey-Wilson polynomials [1, 2].

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As it is well known from the setting of continuous orthogonality, Laguerre-Hahn orthogonal polynomials inherit many properties from the classical and semi-classical families \[5, 7, 13, 16\]. Indeed, one of the research topics within the Laguerre-Hahn theory of a discrete variable is the so-called structure relations, that is, linear difference relations involving the orthogonal polynomials (see \[4, 8, 10, 11\] and their lists of references). In the semi-classical case, it was proven in \[15\] the characterization of semi-classical orthogonal polynomials on non-uniform lattices in terms of structure relations. A more recent contribution, \[9\], proves the characterization of classical polynomials on non-uniform lattices in terms of two types of structure relations, using the so-called functional approach.

In the present paper we show a characterization theorem for Laguerre-Hahn orthogonal polynomial on arbitrary non-uniform lattices. Our main result is given in Theorem 2, where it is shown the equivalence between:

(a) the Riccati difference equation for the formal Stieltjes function, \(S\);
(b) linear first-order difference relations for orthogonal polynomials related to \(S\), as well as for the associated polynomials of the first kind;
(c) linear first-order difference relations for the functions of the second kind related to \(S\).

The difference relations contained in Theorem 2 for Laguerre-Hahn families extend some of the difference relations for the classical families given in \[9, 15, 22\].

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show the main results of the paper, namely, the equivalence between the above referred conditions (a), (b) and (c), stated in Theorem 2 Section 4 is devoted to the proof of Theorem 2.

2. Preliminary results

2.1. The operators \(\mathbb{D}, \mathbb{E}_j, \mathbb{M}\) and the related non-uniform lattices. We consider the divided difference operator \(\mathbb{D}\) given in \[14\], involving the values of a function at two points, with the fundamental property that \(\mathbb{D}\) leaves a polynomial of degree \(n - 1\) when applied to a polynomial of degree \(n\). The operator \(\mathbb{D}\), defined on the space of arbitrary functions, is given by

\[
\mathbb{D}f(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)},
\]
where, at this stage, \(y_1\) and \(y_2\) are still unknown functions. To define them, one starts by using the property that \(Df\) is a polynomial of degree \(n - 1\) whenever \(f\) is a polynomial of degree \(n\). Then, applying \(D\) to \(f(x) = x^2\) and \(f(x) = x^3\), one obtains, respectively,

\[
y_1(x) + y_2(x) = \text{polynomial of degree 1},
\]

\[
(y_1(x))^2 + y_1(x)y_2(x) + (y_2(x))^2 = \text{polynomial of degree 2},
\]

the later condition being equivalent to \(y_1(x)y_2(x) = \text{polynomial of degree less or equal than 2}\). The conditions (2) and (3) define \(y_1\) and \(y_2\) as the two roots of a quadratic equation

\[
\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0, \quad \hat{a} \neq 0.
\]

Some identities involving \(y_1\) and \(y_2\), following from the fact that \(y_1, y_2\) are the roots of (4):

\[
y_1(x) + y_2(x) = -2(\hat{b}x + \hat{d})/\hat{a},
\]

\[
y_1(x)y_2(x) = (\hat{c}x^2 + 2\hat{e}x + \hat{f})/\hat{a}.
\]

There are four primary classes of lattices and related divided difference operators (1):

(i) the linear lattice, related to the forward difference operator [19, Chapter 2, Section 12];

(ii) the \(q\)-linear lattice, related to the \(q\)-difference operator [12];

(iii) the quadratic lattice, related to the Wilson operator [2];

(iv) the \(q\)-quadratic lattice, related to the Askey-Wilson operator [2].

This classification of lattices is done according to the two parameters \(\lambda = \hat{b}^2 - \hat{a}\hat{c}\) and \(\tau = \left((\hat{b}^2 - \hat{a}\hat{c})(\hat{d}^2 - \hat{a}\hat{f}) - (\hat{b}\hat{d} - \hat{a}\hat{e})^2\right)/\hat{a}\), assuming \(\hat{a}\hat{c} \neq 0\): \(\lambda\) = \(\tau\) = 0 in case (i); \(\lambda > 0, \tau = 0\) in case (ii); \(\lambda = 0, \tau < 0\) in case (iii); \(\lambda\tau < 0\) in case (iv).

We would like to remark [15, Section 2], where it is given a geometric interpretation of the lattices. For the quadratic class of lattices (the so-called snul), it is possible to have a parametric representation of the conic (4), say \(\{x(s), y(s)\}\), such that \(y_1(x(s)) = y(s) = x(s-1/2)\) and \(y_2(x(s)) = y(s+1) = x(s+1/2)\), ading to [3, 17, 18]

\[
\begin{align*}
x(s) &= c_4s^2 + c_5s + c_6, \quad \text{if } \lambda = 0, \tau < 0, \\
x(s) &= c_1q^s + c_2q^{-s} + c_3, \quad \text{if } \lambda \tau < 0, \quad q + q^{-1} = 4\hat{b}^2/(\hat{a}\hat{c}) - 2.
\end{align*}
\]
Note that each of the operators in (i)-(iv) is an extension of the preceding one, which is recovered as a particular case or as a limit case, up to a linear transformation of the variable.

In the present paper we shall operate with the divided difference operator given in its general form (1). By defining the operators $E_1$ and $E_2$, acting on arbitrary functions $f$, as

$$E_1 f(x) = f(y_1(x)), \quad E_2 f(x) = f(y_2(x)),$$

(1) is given by

$$D f(x) = \frac{E_2 f(x) - E_1 f(x)}{y_2(x) - y_1(x)}.$$

We define the companion operator of $D$ as

$$M f(x) = \frac{E_1 f(x) + E_2 f(x)}{2}.$$  (5)

Some useful identities involving $D, M$ and $E_1, E_2$ are listed below:

$$D(gf) = Dg M f + M g D f,$$  (6)

$$D(g/f) = \frac{Dg M f - D f M g}{E_1 f E_2 f},$$  (7)

$$D(1/f) = \frac{-D f}{E_1 f E_2 f},$$  (8)

$$M(gf) = M g M f + D g D f \frac{(y_1 - y_2)^2}{4},$$

$$M(g/f) = \frac{E_1 g E_2 f + E_2 g E_1 f}{2E_1 f E_2 f},$$  (9)

$$M(1/f) = \frac{M f}{E_1 f E_2 f}.$$  (10)

Eq. (6) has the equivalent forms:

$$D(gf) = Dg E_1 f + D f E_2 g,$$  (11)

$$D(gf) = Dg E_2 f + D f E_1 g.$$
Also, one has two equivalent forms for (7):

\[ D(g/f) = \frac{Dg E_1 f - Df E_1 g}{E_1 f E_2 f}, \]  
\[ D(g/f) = \frac{Dg E_2 f - Df E_2 g}{E_1 f E_2 f}. \]  

### 2.2. Laguerre-Hahn orthogonal polynomials and auxiliary results.

We shall consider formal orthogonal polynomials related to a (formal) Stieltjes function defined by

\[ S(x) = \sum_{n=0}^{\infty} u_n x^{-n-1} \]  

where \((u_n)\), the sequence of moments, is such that \(\det \begin{bmatrix} u_i & j = 0\end{bmatrix}_{i,j=0}^n \neq 0\), \(n \geq 0\), \(u_0 = 1\). The orthogonal polynomials related to \(S\), \(P_n, n \geq 0\), are the diagonal Padé denominators of (14), thus the numerator polynomial (of degree \(n-1\)), henceforth denoted by \(P_{n-1}^{(1)}\), and the denominator \(P_n\) (of degree \(n\)) are determined through

\[ S(x) - P_{n-1}^{(1)}(x)/P_n(x) = O(x^{-2n-1}), \quad x \to \infty. \]  

Throughout the paper we consider each \(P_n\) monic, and we will denote the sequence of monic polynomials \(\{P_n\}_{n \geq 0}\) by SMOP.

Monic orthogonal polynomials satisfy a three term recurrence relation [20]

\[ P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \]  

with \(P_0(x) = 0\), \(P_1(x) = 1\), and \(\gamma_n \neq 0\), \(n \geq 1\), \(\gamma_0 = u_0 = 1\).

The sequence \(\{P_n^{(1)}\}_{n \geq 0}\), also known as the sequence of associated polynomials of the first kind, satisfies the three term recurrence relation

\[ P_n^{(1)}(x) = (x - \beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x), \quad n = 1, 2, \ldots \]  

with \(P_0^{(1)}(x) = 0\), \(P_1^{(1)}(x) = 1\).

An equivalent form of (15), often encountered in the literature of orthogonal polynomials (see, for example, [21] and its list of references), is given by

\[ q_n = P_n S - P_{n-1}^{(1)}, \quad n \geq 1, \quad q_0 = S, \]  

(17)
where \( q_n, n \geq 0 \), are the so-called functions of the second kind corresponding to \( \{P_n\}_{n \geq 0} \). The sequence \( \{q_n\}_{n \geq 0} \) also satisfies a three term recurrence relation,

\[
q_{n+1}(x) = (x - \beta_n)q_n(x) - \gamma_n q_{n-1}(x), \quad n = 0, 1, 2, \ldots
\]

(18)

with initial conditions \( q_{-1} = 1, q_0(x) = S(x) \).

We will make use of the following result (see [6]).

**Lemma 1.** Let \( \{P_n\}_{n \geq 0} \) be a SMOP and let \( \{P_n^{(1)}\}_{n \geq 0} \) be the sequence of associated polynomials of the first kind. The following holds:

\[
\mathbb{E}_j P_n^{(1)} \mathbb{E}_j P_n - \mathbb{E}_j P_{n+1} \mathbb{E}_j P_n^{(1)} = \prod_{k=0}^{n} \gamma_k, \quad j = 1, 2, \quad n \geq 0.
\]

(19)

Therefore, for each \( j = 1, 2 \), \( \mathbb{E}_j P_n^{(1)} \) and \( \mathbb{E}_j P_{n+1} \) do not share zeroes.

**Proof:** Eq. (19) follows from the application of the operator \( \mathbb{E}_j \), \( j = 1, 2 \), to the identity

\[
P_n^{(1)} P_n - P_{n+1} P_n^{(1)} = \prod_{k=0}^{n} \gamma_k, \quad n \geq 0.
\]

From (19) there follows the statement concerning the zeros. \( \blacksquare \)

**Definition 1.** A SMOP \( \{P_n\}_{n \geq 0} \) related to a Stieltjes function \( S \) (14) is said to be *Laguerre-Hahn* if \( S \) satisfies a Riccati equation

\[
A(x) \mathbb{D}S(x) = B(x) \mathbb{E}_1 S(x) \mathbb{E}_2 S(x) + C(x) \mathbb{M} S(x) + D(x),
\]

(20)

where \( A, B, C, D \) are polynomials in \( x \), \( A \neq 0 \).

If \( B \equiv 0 \), then \( \{P_n\}_{n \geq 0} \) is said to be *semi-classical*.

We will make use of the Theorem that follows.

**Theorem 1.** Let \( \{f_n\} \) be a sequence of functions satisfying a three term recurrence relation

\[
f_{n+1}(x) = (x - \beta_n) f_n(x) - \gamma_n f_{n-1}(x), \quad \gamma_n \neq 0, \quad n \geq 0.
\]

(21)

Let \( g_n = f_{n+1}/f_n \) satisfy for all \( n \geq 0 

\[
A_n(x) \mathbb{D}g_n(x) = B_n(x) \mathbb{E}_1 g_n(x) \mathbb{E}_2 g_n(x) + C_n \mathbb{M} g_n(x) + D_n(x),
\]

(22)
with $\mathcal{D}, \mathcal{M}$ the operators defined in (1) and (5), and $A_n, B_n, C_n, D_n$ bounded degree polynomials. Then, for all $n \geq 0$, the following relations hold:

\begin{align}
A_{n+1} &= A_n - \frac{(y_1 - y_2)^2}{2} \frac{D_n}{\gamma_{n+1}}, \\
B_{n+1} &= \frac{D_n}{\gamma_{n+1}}, \\
C_{n+1} &= -C_n - 2\mathcal{M}(x - \beta_{n+1}) \frac{D_n}{\gamma_{n+1}}, \\
D_{n+1} &= A_n + \gamma_{n+1}B_n + \mathcal{M}(x - \beta_{n+1})C_n + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) \frac{D_n}{\gamma_{n+1}}. 
\end{align}

**Proof:** From (21) we get $g_n = (x - \beta_n) - \gamma_n/g_{n-1}$, thus, writing the above equation to $n + 1$,

\begin{equation}
g_{n+1} = (x - \beta_{n+1}) - \gamma_{n+1}/g_n. 
\end{equation}

Applying $\mathcal{D}$ to (27) and using $\mathcal{D}(1/g_n) = -\mathcal{D}g_n/(\mathcal{E}_1 g_n \mathcal{E}_2 g_n)$ (cf. (8)) we get

\begin{equation}
\mathcal{D}g_{n+1} = 1 + \gamma_{n+1} \frac{\mathcal{D}g_n}{\mathcal{E}_1 g_n \mathcal{E}_2 g_n}. 
\end{equation}

Now we multiply the above equation by $A_n$ and use (22), as well as $\mathcal{M}(1/g_n) = \mathcal{M}g_n/(\mathcal{E}_1 g_n \mathcal{E}_2 g_n)$ (cf. (10)), thus obtaining

\begin{equation}
A_n \mathcal{D}g_{n+1} = A_n + \gamma_{n+1}B_n + \gamma_{n+1}C_n \mathcal{M}(1/g_n) + \frac{\gamma_{n+1}D_n}{\mathcal{E}_1 g_n \mathcal{E}_2 g_n}. 
\end{equation}

Note that from (27) we have

\begin{equation}
\mathcal{M}(1/g_n) = \frac{\mathcal{M}(x - \beta_{n+1})}{\gamma_{n+1}} - \frac{\mathcal{M}g_{n+1}}{\gamma_{n+1}}. 
\end{equation}

Also,

\begin{equation}
\frac{\gamma_{n+1}D_n}{\mathcal{E}_1 g_n \mathcal{E}_2 g_n} = \frac{D_n}{\gamma_{n+1}} (y_1 - \beta_{n+1} - \mathcal{E}_1 g_{n+1})(y_2 - \beta_{n+1} - \mathcal{E}_2 g_{n+1}),
\end{equation}

and some computations yield

\begin{equation}
\frac{\gamma_{n+1}D_n}{\mathcal{E}_1 g_n \mathcal{E}_2 g_n} = \frac{D_n}{\gamma_{n+1}} \left( (y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) + (y_1 - y_2)^2 / 2 \mathcal{D}g_{n+1} 
- 2\mathcal{M}(x - \beta_{n+1})\mathcal{M}g_{n+1} + \mathcal{E}_1 g_{n+1} \mathcal{E}_2 g_{n+1} \right). 
\end{equation}
The substitution of (29) and (30) into (28) yields
\[
\left( A_n - 2\frac{(y_1 - y_2)^2}{4} \frac{D_n}{\gamma_{n+1}} \right) \mathcal{D}g_{n+1} = \frac{D_n}{\gamma_{n+1}} \mathcal{E}_1 g_{n+1} \mathcal{E}_2 g_{n+1} \\
+ \left( -C_n - 2\mathcal{M}(x - \beta_{n+1}) \frac{D_n}{\gamma_{n+1}} \right) \mathcal{M} g_{n+1} \\
+ \left( A_n + \gamma_{n+1} B_n + \mathcal{M}(x - \beta_{n+1}) C_n + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) \frac{D_n}{\gamma_{n+1}} \right).
\]

The comparison between the above equation and (22) written to \( n + 1 \) gives us (23)-(26).

3. Characterization theorem

**Theorem 2.** Let \( S \) be a Stieltjes function, let \( \{P_n\}_{n \geq 0} \) be the corresponding SMOP, and let \( \{P_n^{(1)}\}_{n \geq 0}, \{q_n\}_{n \geq 0} \) be the sequence of associated polynomials of the first kind and the sequence of functions of the second kind, respectively. The following statements are equivalent:

(a) \( S \) satisfies the Riccati equation (20),
\[
A \mathcal{D}S = B \mathcal{E}_1 S \mathcal{E}_2 S + C \mathcal{M} S + D,
\]
where \( A, B, C, D \) are polynomials;

(b) \( P_n \) and \( P_n^{(1)} \) satisfy the difference relations for all \( n \geq 1 \),
\[
\begin{align*}
A \mathcal{D}P_n &= l_{n-1} \mathcal{E}_1 P_n - C/2 \mathcal{E}_2 P_n - B \mathcal{E}_2 P_n^{(1)} + \Theta_{n-1} \mathcal{E}_1 P_{n-1}, \\
A \mathcal{D}P_n^{(1)} &= l_{n-1} \mathcal{E}_1 P_n^{(1)} + C/2 \mathcal{E}_2 P_n^{(1)} + D \mathcal{E}_2 P_n + \Theta_{n-1} \mathcal{E}_1 P_{n-2};
\end{align*}
\]

(c) \( q_n \) satisfies for all \( n \geq 0 \),
\[
A \mathcal{D}q_n = l_{n-1} \mathcal{E}_1 q_n + (B \mathcal{E}_1 S + C/2) \mathcal{E}_2 q_n + \Theta_{n-1} \mathcal{E}_1 q_{n-1},
\]
where \( l_n, \Theta_n \) are polynomials of uniformly bounded degrees satisfying the initial conditions \( l_{-1} = C/2, \Theta_{-1} = D \).

The proof of Theorem 2 will be given at the next section.

**Remark.** The characterizations stated in Theorem 1 are not uniquely represented. One can also deduce that the following statements (a), (b), (c) are equivalent:

(a) \( S \) satisfies the Riccati equation (20),
\[
A \mathcal{D}S = B \mathcal{E}_1 S \mathcal{E}_2 S + C \mathcal{M} S + D;
\]
(b) $P_n$ and $P_n^{(1)}$ satisfy the difference relations for all $n \geq 1$,

\[
\begin{align*}
A D P_n &= l_{n-1} E_2 P_n - C/2 E_1 P_n - B E_1 P_{n-1}^{(1)} + \Theta_{n-1} E_2 P_{n-1}, \\
A D P_{n-1}^{(1)} &= l_{n-1} E_2 P_{n-1}^{(1)} + C/2 E_1 P_{n-1}^{(1)} + D E_1 P_n + \Theta_{n-1} E_2 P_{n-2}^{(1)};
\end{align*}
\tag{33}
\]

(c) $q_n$ satisfies for all $n \geq 0$,

\[
A D q_n = l_{n-1} E_2 q_n + (B E_2 S + C/2) E_1 q_n + \Theta_{n-1} E_2 q_{n-1}.
\tag{34}
\]

Therefore, we deduce the result that follows.

**Theorem 3.** Let $S$ be a Stieltjes function satisfying the Riccati equation

\[
A D S = B E_1 S E_2 S + C M S + D,
\]

where $A, B, C, D$ are polynomials. Let $\{P_n\}_{n \geq 0}$ be the SMOP related to $S$, and let $\{P_n^{(1)}\}_{n \geq 0}, \{q_n\}_{n \geq 0}$ be the sequence of associated polynomials of the first kind and the sequence of functions of the second kind, respectively. The following relations hold, for all $n \geq 0$:

\[
\begin{align*}
A D P_{n+1} &= (l_n - C/2) M P_{n+1} - B M P_n^{(1)} + \Theta_n M P_n, \\
A D P_n^{(1)} &= (l_n + C/2) M P_n^{(1)} + D M P_{n+1} + \Theta_n M P_{n-1}^{(1)}, \\
A D q_n &= (l_{n-1} + C/2) M q_n + B (M S M q_n - M (S q_n)) + \Theta_{n-1} M q_{n-1}.
\end{align*}
\tag{35-37}
\]

**Proof:** Sum (31) and (33) to get (35) and (36). Following the same idea, sum (32) and (34) to get (37). \qed

**Remark.** The equations (35)-(37) extend the ones given in [22] for the semi-classical case.

**Corollary 1.** The polynomials $l_n, \Theta_n$ of Theorems 2, 3 satisfy, for all $n \geq 0$,

\[
\begin{align*}
l_{n+1} + l_n + M (x - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}} &= 0, \\
\Theta_{n+1} &= A + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}} \\
&+ \left( \gamma_{n+1} - \frac{(y_1 - y_2)^2}{4} \right) \frac{\Theta_{n-1}}{\gamma_n} + 2 M (x - \beta_{n+1}) l_n,
\end{align*}
\tag{38-39}
\]

with initial conditions $l_{-1} = C/2, \Theta_{-1} = D$. 
Proof: Multiply (32), written to $n+1$, by $E_2 q_n$ and subtract to (32) multiplied by $E_2 q_{n+1}$. Then, multiply the resulting equation by $1/(E_1 q_n E_2 q_n)$, thus obtaining

$$A D \left( \frac{q_{n+1}}{q_n} \right) = l_n E_1 \left( \frac{q_{n+1}}{q_n} \right) - l_{n-1} E_2 \left( \frac{q_{n+1}}{q_n} \right) + \Theta_n - \Theta_{n-1} E_1 \left( \frac{q_{n-1}}{q_n} \right) E_2 \left( \frac{q_{n+1}}{q_n} \right), \quad (40)$$

where we used the property (13). From the recurrence relation for $q_n$ there holds

$$\frac{q_{n-1}}{q_n} = \frac{(x - \beta_n)}{\gamma_n} - \frac{1}{\gamma_n} \frac{q_{n+1}}{q_n},$$

thus

$$E_1 \left( \frac{q_{n-1}}{q_n} \right) = \frac{1}{\gamma_n} E_1(x - \beta_n) - \frac{1}{\gamma_n} E_1 \left( \frac{q_{n+1}}{q_n} \right). \quad (41)$$

The substitution of (41) in (40) yields

$$A D \left( \frac{q_{n+1}}{q_n} \right) = l_n E_1 \left( \frac{q_{n+1}}{q_n} \right) - l_{n-1} E_2 \left( \frac{q_{n+1}}{q_n} \right) + \Theta_n - \frac{\Theta_{n-1}}{\gamma_n} E_1(x - \beta_n) E_2 \left( \frac{q_{n+1}}{q_n} \right) + \frac{\Theta_{n-1}}{\gamma_n} E_1 \left( \frac{q_{n+1}}{q_n} \right) E_2 \left( \frac{q_{n+1}}{q_n} \right). \quad (42)$$

On the other hand, if we proceed as above, but starting with the eq. (32) and using the property (12), we obtain

$$A D \left( \frac{q_{n+1}}{q_n} \right) = l_n E_2 \left( \frac{q_{n+1}}{q_n} \right) - l_{n-1} E_1 \left( \frac{q_{n+1}}{q_n} \right) + \Theta_n - \frac{\Theta_{n-1}}{\gamma_n} E_2(x - \beta_n) E_1 \left( \frac{q_{n+1}}{q_n} \right) + \frac{\Theta_{n-1}}{\gamma_n} E_2 \left( \frac{q_{n+1}}{q_n} \right) E_1 \left( \frac{q_{n+1}}{q_n} \right). \quad (43)$$

From the sum of (42) with (43) there follows

$$A D \left( \frac{q_{n+1}}{q_n} \right) = (l_n - l_{n-1}) M \left( \frac{q_{n+1}}{q_n} \right) + \Theta_n + \frac{\Theta_{n-1}}{\gamma_n} E_1 \left( \frac{q_{n+1}}{q_n} \right) E_2 \left( \frac{q_{n+1}}{q_n} \right) - \frac{\Theta_{n-1}}{2 \gamma_n} \left( E_1(x - \beta_n) E_2 \left( \frac{q_{n+1}}{q_n} \right) + E_2(x - \beta_n) E_1 \left( \frac{q_{n+1}}{q_n} \right) \right). \quad (44)$$
The use of
\[
\mathbb{E}_1(x - \beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \mathbb{E}_2(x - \beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)
= 2\mathbb{M}(x - \beta_n)\mathbb{M}\left(\frac{q_{n+1}}{q_n}\right) - 2\frac{(y_1 - y_2)^2}{4}\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right)
\]
in (44) gives us the Riccati equation for \(g_n = q_{n+1}/q_n\),
\[
A_n \mathbb{D}g_n = B_n \mathbb{E}_1 g_n \mathbb{E}_2 g_n + C_n \mathbb{M} g_n + D_n
\]
with
\[
A_n = A - \frac{(y_1 - y_2)^2}{4} \frac{\Theta_{n-1}}{\gamma_n},
B_n = \frac{\Theta_{n-1}}{\gamma_n},
C_n = l_n - l_{n-1} - \mathbb{M}(x - \beta_n) \frac{\Theta_{n-1}}{\gamma_n},
D_n = \Theta_n.
\]
Now we use Theorem 1. Taking into account the relations (23)-(26) for \(A_n, B_n, C_n, D_n\), there follows, for all \(n \geq 0\),
\[
l_{n+1} - l_{n-1} - \mathbb{M}(x - \beta_n) \frac{\Theta_{n-1}}{\gamma_n} + \mathbb{M}(x - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}} = 0, \quad (45)
\]
\[
\Theta_{n+1} = A + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}}
+ \left(\gamma_{n+1} - \frac{(y_1 - y_2)^2}{4} - \mathbb{M}(x - \beta_n)\mathbb{M}(x - \beta_{n+1})\right) \frac{\Theta_{n-1}}{\gamma_n}
+ \mathbb{M}(x - \beta_{n+1})(l_n - l_{n-1}). \quad (46)
\]
To deduce (38) we write (45) in the equivalent form
\[
M_{n+1} = M_n, \quad n \geq 0 \quad \text{and} \quad M_{n+1} = l_{n+1} + l_n + \mathbb{M}(x - \beta_{n+1}) \frac{\Theta_n}{\gamma_{n+1}},
\]
from which there follows \(M_{n+1} = M_0, \quad n \geq 0\). The use of the initial conditions \(l_0 + l_{-1} + \mathbb{M}(x - \beta_0) \frac{\Theta_{-1}}{\gamma_0} = 0\) yield (38).
Using (38) written to \(n - 1\) in (46) we obtain (39).
4. Proof of Theorem 2

Proof of (a) ⇒ (b).

If we use $S = \frac{q_n}{P_n} + \frac{P_n^{(1)}}{P_n}$, $n \geq 1$ (cf. (17)), then (20) yields

$$M_n = -A D\left(\frac{P_n^{(1)}}{P_n}\right) + B \mathbb{E}_1\left(\frac{P_n^{(1)}}{P_n}\right) \mathbb{E}_2\left(\frac{P_n^{(1)}}{P_n}\right) + C \mathcal{M}\left(\frac{P_n^{(1)}}{P_n}\right) + D,$$  \hspace{1cm} (47)

where

$$M_n = A D\left(\frac{q_n}{P_n}\right) - B \left[ \mathbb{E}_1\left(\frac{q_n}{P_n}\right) \mathbb{E}_2\left(\frac{q_n}{P_n}\right) + \mathbb{E}_1\left(\frac{q_n}{P_n}\right) \mathbb{E}_2\left(\frac{P_n^{(1)}}{P_n}\right) + \mathbb{E}_1\left(\frac{P_n^{(1)}}{P_n}\right) \mathbb{E}_2\left(\frac{q_n}{P_n}\right) \right] - C \mathcal{M}\left(\frac{q_n}{P_n}\right).$$

By multiplying both hand sides of (47) by $\mathbb{E}_1 P_n \mathbb{E}_2 P_n$ and using the properties (12) and (9), we obtain

$$M_n \mathbb{E}_1 P_n \mathbb{E}_2 P_n = -A D P_n^{(1)} \mathbb{E}_1 P_n + A D P_n \mathbb{E}_1 P_n^{(1)} + B \mathbb{E}_1 P_n^{(1)} \mathbb{E}_2 P_n^{(1)} + C \left( \mathbb{E}_1 P_n^{(1)} \mathbb{E}_2 P_n + \mathbb{E}_1 P_n \mathbb{E}_2 P_n^{(1)} \right) + D \mathbb{E}_1 P_n \mathbb{E}_2 P_n.$$

Now let us write

$$-A D P_n^{(1)} \mathbb{E}_1 P_n + A D P_n \mathbb{E}_1 P_n^{(1)} + B \mathbb{E}_1 P_n^{(1)} \mathbb{E}_2 P_n^{(1)} + C \left( \mathbb{E}_1 P_n^{(1)} \mathbb{E}_2 P_n + \mathbb{E}_1 P_n \mathbb{E}_2 P_n^{(1)} \right) + D \mathbb{E}_1 P_n \mathbb{E}_2 P_n = \hat{\Theta}_{n-1},$$  \hspace{1cm} (48)

where $\hat{\Theta}_{n-1}$ is a bounded degree polynomial, as $q_n(x) = O(x^{-2n-1})$. One has deg($\hat{\Theta}_{n-1}$) = max\{deg($A$) − 2, deg($B$) − 2, deg($C$) − 1\}.

Taking into account $\mathbb{E}_1(P_n^{(1)})\mathbb{E}_1(P_{n-1}) - \mathbb{E}_1(P_n)\mathbb{E}_1(P_{n-2}) = \prod_{k=0}^{n-1} \gamma_k$, $n \geq 1$, (cf. (19)), then (48) can be written as

$$-A D P_n^{(1)} \mathbb{E}_1 P_n + A D P_n \mathbb{E}_1 P_n^{(1)} + B \mathbb{E}_1 P_n^{(1)} \mathbb{E}_2 P_n^{(1)} + C \left( \mathbb{E}_1 P_n^{(1)} \mathbb{E}_2 P_n + \mathbb{E}_1 P_n \mathbb{E}_2 P_n^{(1)} \right) + D \mathbb{E}_1 P_n \mathbb{E}_2 P_n = \Theta_{n-1} \left( \mathbb{E}_1 P_n^{(1)} \mathbb{E}_1 P_{n-1} - \mathbb{E}_1 P_n \mathbb{E}_1 P_{n-2} \right),$$  \hspace{1cm} (49)
where $\Theta_{n-1} = \hat{\Theta}_{n-1}/\prod_{k=0}^{n-1} \gamma_k$.

Now, let us write (49) as

\[
\begin{align*}
\{ A \mathbb{D}P_n + C/2 \mathbb{E}_2 P_n + B \mathbb{E}_2 P_{n-1}^{(1)} - \Theta_{n-1} \mathbb{E}_1 P_{n-1} \} & \mathbb{E}_1 P_{n-1}^{(1)} \\
= \{ A \mathbb{D}P_{n-1}^{(1)} - C/2 \mathbb{E}_2 P_{n-1}^{(1)} - D \mathbb{E}_2 P_n - \Theta_{n-1} \mathbb{E}_1 P_{n-2}^{(1)} \} \mathbb{E}_1 P_n. \quad (50)
\end{align*}
\]

Since $\mathbb{E}_1 P_{n-1}^{(1)}$ and $\mathbb{E}_1 P_n$ do not have common zeroes, for all $n \geq 1$, then there exists a polynomial, $l_{n-1}$, such that

\[
\begin{align*}
\{ A \mathbb{D}P_n + C/2 \mathbb{E}_2 P_n + B \mathbb{E}_2 P_{n-1}^{(1)} - \Theta_{n-1} \mathbb{E}_1 P_{n-1} \} & = l_{n-1} \mathbb{E}_1 P_n, \\
\{ A \mathbb{D}P_{n-1}^{(1)} - C/2 \mathbb{E}_2 P_{n-1}^{(1)} - D \mathbb{E}_2 P_n - \Theta_{n-1} \mathbb{E}_1 P_{n-2}^{(1)} \} & = l_{n-1} \mathbb{E}_1 P_{n-1}^{(1)},
\end{align*}
\]

that is, we get (31).

**Proof of** (b) $\Rightarrow$ (a).

Let us define $\psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}$. From the three term recurrence relation for $\{P_n\}$ and $\{P_n^{(1)}\}$, there follows that $\psi_n$ satisfies

\[
\psi_n = (x - \beta_n)\psi_{n-1} - \gamma_n \psi_{n-2}, \quad n \geq 1, \quad \psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \psi_0 = \begin{bmatrix} x - \beta_0 \\ 1 \end{bmatrix}. \quad (51)
\]

With the notation $\mathbb{D}\psi_n = \begin{bmatrix} \mathbb{D}P_{n+1} \\ \mathbb{D}P_n^{(1)} \end{bmatrix}$, $\mathbb{E}_j\psi_n = \begin{bmatrix} \mathbb{E}_j P_{n+1} \\ \mathbb{E}_j P_n^{(1)} \end{bmatrix}$, $j = 1, 2$, (31) reads as

\[
A \mathbb{D}\psi_{n-1} = l_{n-1} \mathbb{E}_1 \psi_{n-1} + C \mathbb{E}_2 \psi_{n-1} + \Theta_{n-1} \mathbb{E}_1 \psi_{n-2}, \quad (52)
\]

where $C = \begin{bmatrix} -C/2 & -B \\ D & C/2 \end{bmatrix}$.

In turn, (52) reads as

\[
A \begin{bmatrix} \psi_{n-1}(y_2) - \psi_{n-1}(y_1) \\ y_2 - y_1 \end{bmatrix} = l_{n-1} \psi_{n-1}(y_1) + C \psi_{n-1}(y_2) + \Theta_{n-1} \psi_{n-2}(y_1),
\]

that is,

\[
A_n \psi_{n-1}(y_1) + B \psi_{n-1}(y_2) = C_n \psi_{n-2}(y_1) \quad (53)
\]

with

\[
A_n = (-\frac{A}{y_2 - y_1} - l_{n-1})I, \quad B = \frac{A}{y_2 - y_1}I - C, \quad C_n = \Theta_{n-1}I,
\]

and $I$ denoting the identity matrix of order 2.
Taking \( n + 1 \) in (53) and using the recurrence relation (51) we get
\[
\tilde{A}_n \psi_{n-1}(y_1) + \tilde{B}_n \psi_{n-1}(y_2) = \tilde{C}_n \psi_{n-2}(y_1) + \tilde{D}_n \psi_{n-2}(y_2),
\] (54)
with
\[
\tilde{A}_n = (y_1 - \beta_n) A_{n+1} - E_{n+1}, \quad \tilde{B}_n = (y_2 - \beta_n) B, \quad \tilde{C}_n = \gamma_n A_{n+1}, \quad \tilde{D}_n = \gamma_n B.
\]
Now, we gather (53) and (54) in the system
\[
\mathcal{E}_n \begin{bmatrix} \psi_{n-1}(y_1) \\ \psi_{n-1}(y_2) \end{bmatrix} = \mathcal{F}_n \begin{bmatrix} \psi_{n-2}(y_1) \\ \psi_{n-2}(y_2) \end{bmatrix},
\] (55)
where \( \mathcal{E}_n \) and \( \mathcal{F}_n \) are the block matrices
\[
\mathcal{E}_n = \begin{bmatrix} A_n & B_n \\ \tilde{A}_n & \tilde{B}_n \end{bmatrix}, \quad \mathcal{F}_n = \begin{bmatrix} C_n & 0_{2 \times 2} \\ \tilde{C}_n & \tilde{D}_n \end{bmatrix}.
\]
Note that \( \mathcal{E}_n \) is invertible,
\[
\mathcal{E}_n^{-1} = \gamma_n^{-1} \begin{bmatrix} (y_2 - \beta_n) I & -I \\ -B^{-1} A_n & B^{-1} A_n \end{bmatrix}.
\] (56)
From (55) there follows
\[
\begin{bmatrix} \psi_{n-1}(y_1) \\ \psi_{n-1}(y_2) \end{bmatrix} = \mathcal{G}_n \begin{bmatrix} \psi_{n-2}(y_1) \\ \psi_{n-2}(y_2) \end{bmatrix}, \quad \mathcal{G}_n = \mathcal{E}_n^{-1} \mathcal{F}_n,
\] (57)
being \( \mathcal{G}_n \) an invertible matrix as is a product of invertible matrices.

Take \( n + 1 \) in (57). On the one hand we have
\[
\begin{bmatrix} \psi_n(y_1) \\ \psi_n(y_2) \end{bmatrix} = \mathcal{G}_{n+1} \begin{bmatrix} \psi_{n-1}(y_1) \\ \psi_{n-1}(y_2) \end{bmatrix},
\] (58)
and, on the other hand, using the three term recurrence relation (51), we have
\[
\begin{bmatrix} \psi_n(y_1) \\ \psi_n(y_2) \end{bmatrix} = \begin{bmatrix} (y_1 - \beta_n) I & 0 \\ 0 & (y_2 - \beta_n) I \end{bmatrix} \begin{bmatrix} \psi_{n-2}(y_1) \\ \psi_{n-2}(y_2) \end{bmatrix} - \gamma_n \begin{bmatrix} \psi_{n-1}(y_1) \\ \psi_{n-1}(y_2) \end{bmatrix},
\]
thus,
\[
\begin{bmatrix} \psi_n(y_1) \\ \psi_n(y_2) \end{bmatrix} = \left( \begin{bmatrix} (y_1 - \beta_n) I & 0 \\ 0 & (y_2 - \beta_n) I \end{bmatrix} - \gamma_n \mathcal{G}_n^{-1} \right) \begin{bmatrix} \psi_{n-1}(y_1) \\ \psi_{n-1}(y_2) \end{bmatrix}.
\] (59)
Consequently, (58) and (59) yield
\[
\mathcal{G}_{n+1} = \begin{bmatrix} (y_1 - \beta_n) I & 0 \\ 0 & (y_2 - \beta_n) I \end{bmatrix} - \gamma_n \mathcal{G}_n^{-1}.
\] (60)
Let us compute $\mathcal{G}^{-1}_n$. Taking into account (56), we obtain

$$\mathcal{G}_n = \begin{bmatrix} X_n I & Y_n B \\ U_n B^{-1} & V_n I \end{bmatrix},$$

where $X_n, Y_n, U_n, V_n$ are the functions given by

$$X_n = \alpha_n \left( (y_2 - \beta_n) \Theta_{n-1} + \gamma_n \left( \frac{A}{y_2 - y_1} + l_n \right) \right),$$

$$Y_n = -\gamma_n \alpha_n,$$

$$U_n = \alpha_n \left( \frac{A}{y_2 - y_1} + l_n \right) \left[ \gamma_n \left( \frac{A}{y_2 - y_1} + l_{n-1} \right) + (y_1 - \beta_n) \Theta_{n-1} \right] + \Theta_n \Theta_{n-1},$$

$$V_n = -\gamma_n \alpha_n \left( \frac{A}{y_2 - y_1} + l_{n-1} \right),$$

with $\alpha_n = \frac{\gamma_{n-1}}{\gamma_n \Theta_{n-2}}$. Therefore, it turns out that

$$\mathcal{G}^{-1}_n = \frac{1}{\delta_n} \begin{bmatrix} V_n I & -Y_n B \\ -U_n B^{-1} & X_n I \end{bmatrix},$$

where $\delta_n$ is the function given by $\delta_n = X_n V_n - Y_n U_n$.

Taking into account (61) and (62), (60) reads

$$X_{n+1} = (y_1 - \beta_n) - \gamma_n V_n / \delta_n,$$

$$Y_{n+1} = \gamma_n Y_n / \delta_n,$$

$$U_{n+1} = \gamma_n U_n / \delta_n,$$

$$V_{n+1} = (y_2 - \beta_n) - \gamma_n X_n / \delta_n.$$

From (63)-(66) there follows that $\delta_{n+1} = X_{n+1} V_{n+1} - Y_{n+1} U_{n+1}$ is given by

$$\delta_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n) - \gamma_n ((y_1 - \beta_n) X_n + (y_2 - \beta_n) V_n) \frac{1}{\delta_n} + \frac{\gamma_n^2}{\delta_n}.$$

Now we proceed analogously with Magnus [14, 15]. Write $\delta_n = \mu_n / \mu_{n-1}$. Then, we obtain

$$\mu_n X_{n+1} = (y_1 - \beta_n) \mu_n - \gamma_n \mu_{n-1} V_n,$$

$$\mu_n V_{n+1} = (y_2 - \beta_n) \mu_n - \gamma_n \mu_{n-1} X_n,$$

$$\mu_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n) \mu_n - \gamma_n \mu_{n-1} ((y_1 - \beta_n) X_n + (y_2 - \beta_n) V_n) + \gamma_n^2 \mu_{n-1}.$$
The change of variables
\[ \hat{X}_{n+1} = \mu_n X_{n+1}, \quad \hat{V}_{n+1} = \mu_n V_{n+1} \]
yields the relations
\[
\begin{align*}
\hat{X}_{n+1} &= (y_1 - \beta_n)\mu_n - \gamma_n \hat{V}_n, \\
\hat{V}_{n+1} &= (y_2 - \beta_n)\mu_n - \gamma_n \hat{X}_n,
\end{align*}
\]
\[ \mu_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n)\mu_n - \gamma_n \left((y_1 - \beta_n)\hat{X}_n + (y_2 - \beta_n)\hat{V}_n\right) + \gamma^2_n \mu_{n-1}. \]

Remark that the above recurrence relations for \( \hat{X}_n, \hat{V}_n \) and \( \mu_n \) are precisely the recurrence relations satisfied by the products of solutions of the three term recurrence relation (16) at \( y_1 \) and \( y_2 \). Indeed, if
\[ \xi_{n+1} = (y_1 - \beta_n)\xi_n - \gamma_n \xi_{n-1}, \quad \eta_{n+1} = (y_2 - \beta_n)\eta_n - \gamma_n \eta_{n-1}, \]
then the above recurrence relation for \( \hat{X}_n, \hat{V}_n, \mu_n \) is precisely the relation for \( \xi_{n-1}, \xi_n, \eta_{n-1}, \eta_n \), respectively. Taking into account that a basis of the three term recurrence relation \( \tau_{n+1} = (x - \beta_n)\tau_n - \gamma_n \tau_{n-1} \) is constituted by \{\( P_n \)\} and \{\( q_n \)\}, (cf. (16) and (18)), the following must hold: \( \xi_n \) must be a combination of \( P_n(y_1) \) and \( q_n(y_1) \), and \( \eta_n \) must be a combination of \( P_n(y_2) \) and \( q_n(y_2) \). Thus, there are four choices to be considered:

\[ (i) \xi_n = P_n(y_1), \quad \eta_n = P_n(y_2), \]
\[ (ii) \xi_n = P_n(y_1), \quad \eta_n = q_n(y_2), \]
\[ (iii) \xi_n = q_n(y_1), \quad \eta_n = P_n(y_2), \]
\[ (iv) \xi_n = q_n(y_1), \quad \eta_n = q_n(y_2). \]

Therefore, we obtain
\[ \mu_n = \alpha P_n(y_1)P_n(y_2) + \beta P_n(y_1)q_n(y_2) + \gamma q_n(y_1)P_n(y_2) + \delta q_n(y_1)q_n(y_2). \quad (67) \]

Taking \( n = 0 \) in (67) we obtain
\[ \mu_0 = \alpha + \beta q_0(y_2) + \gamma q_0(y_1) + \delta q_0(y_1)q_0(y_1), \]
and such a relation is \( A\tilde{D}S = B\tilde{E}_1S\tilde{E}_2S + C\tilde{M}S + D, \) with
\[ A = \frac{(\gamma - \beta)}{2}(y_2 - y_1), \quad B = \delta, \quad C = \gamma + \beta, \quad D = \alpha - \mu_0. \]

Proof of (a) \( \Rightarrow \) (c).
Note that \( A\tilde{D}S = B\tilde{E}_1S\tilde{E}_2S + C\tilde{M}S + D \) is
\[ A\tilde{D}q_n = l_{n-1}\tilde{E}_1q_n + (B\tilde{E}_1S + C/2)\tilde{E}_2q_n + \Theta_{n-1}\tilde{E}_1q_{n-1} \]
with \( n = 0 \), since \( q_{-1} = 1, q_0 = S, l_{-1} = C/2, \Theta_{-1} = D \).

Let us now deduce the above difference equation for \( n \geq 1 \).

Applying \( A \mathcal{D} \) to \( q_n = P_n S - P_n^{(1)} \), \( n \geq 1 \) (cf. (17)), and using the property (11) we obtain

\[
A \mathcal{D} q_n = A \mathcal{D} P_n E_1 S + A \mathcal{D} S E_2 P_n - A \mathcal{D} P_n^{(1)}.
\]

Using the equations (31) as well as (20) in the above equation, we obtain

\[
A \mathcal{D} q_n = l_{n-1} E_1 (P_n S - P_n^{(1)}) + (B E_1 S + C/2) E_2 (P_n S - P_n^{(1)})
+ \Theta_{n-1} E_1 (P_{n-1} S - P_{n-2}^{(1)}),
\]

thus (32) follows.

**Proof of (c) \( \Rightarrow \) (a).**

Take \( n = 0 \) in (32).

**References**


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