Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–29

FINITE DIFFERENCE SCHEMES FOR NONLINEAR COMPLEX REACTION-DIFFUSION PROCESSES: STABILITY ANALYSIS

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Abstract: In this paper we establish the stability condition of a general class of finite difference schemes applied to nonlinear complex reaction-diffusion equations. We consider the numerical solution of both implicit and semi-implicit discretizations. To illustrate the theoretical results we present some numerical examples computed with a semi-implicit scheme applied to a nonlinear equation.

Keywords: finite differences, complex reaction diffusion, stability.

1. Introduction

Complex diffusion is a commonly used denoising procedure in image processing [6]. In particular, nonlinear complex diffusion proved to be a numerically well conditioned technique that has been successfully applied in medical imaging despeckling [3]. The stability condition for finite difference methods applied to the linear diffusion equation has been investigated extensively and it is widely documented in literature (see e.g. [11, 13]). A stability result for the linear complex case was derived in [5].

The stability properties of a class of finite difference schemes for the nonlinear complex diffusion equation, were studied in [1], where only the explicit and implicit scheme where studied and no reaction term was considered. In this paper we extend those results for nonlinear complex reaction-diffusion equations, considering discretizations also with a semi-implicit finite difference scheme, in addition to the explicit and implicit schemes. Applications of interest include diffusion processes which are commonly used in image processing, as for example in noise removal, inpainting, stereo vision or optical

*The research of the first and second authors was partially supported by project UTAustin/MAT/0066/2008 and by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011. The work of the third author was partially supported by FCT project PTDC/SAU-ENB/111139/2009 and by the program FEDER/COMPETE FCOMP-01-0124-FEDER-015712. The authors were also supported by FCT (PTDC/SAU-ENB/119132/2010) and program COMPETE (QREN -FCOMP-01-0124-FEDER-021014).
Received July 18, 2013.

flow (see e.g. [3, 4, 6, 7, 8, 9, 14, 15, 16]). Complex difusion with reactive term appears also in the well-known Schrödinger equation, though conservative numerical methods are usually used instead of the finite difference approach [10, 12].

Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 1$, with boundary $\Gamma = \partial\Omega$. Typically Ω is the cartesian product of open intervals in $\mathbb{R}$, i.e.,

$$\Omega = \prod_{j=1}^{d}(a_j, b_j), \tag{1}$$

with $a_j, b_j \in \mathbb{R}$. Let $Q = \Omega\times(0,T]$, with $T > 0$, and $v : \bar{Q} = \bar{\Omega} \times [0,T] \longrightarrow \mathbb{C}$. We consider a reaction diffusion process with a non-constant complex coefficient $D(x,t,v) = D_R(x,t,v) + iD_I(x,t,v)$ and non-constant complex reaction term $F(x,t,v) = F_R(x,t,v) + iF_I(x,t,v)$, where $D_R(x,t,v)$, $D_I(x,t,v)$, $F_R(x,t,v)$, $F_I(x,t,v)$ are real functions dependent on v. We need to assume that

$$D_R(x,t,v) \geq 0, \qquad (x,t) \in \bar{Q}, \tag{2}$$

and that there exists a constant $L > 0$ such that

$$|D(x,t,v)| \leq L, \qquad (x,t) \in \bar{Q}. \tag{3}$$

These inequalities (2) and (3) can easily be shown to hold for the diffusion coefficient in [3] and [6].

We define the initial boundary value problem for the unknown complex function u

$$\frac{\partial u}{\partial t}(x,t) = \nabla \cdot (D(x,t,u)\nabla u(x,t)) + F(x,t,u), \quad (x,t) \in Q, \tag{4}$$

under the initial condition

$$u(x,0) = u^0(x), \quad x \in \overline{\Omega}, \tag{5}$$

and with either the Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \Gamma, \quad t \in [0,T], \tag{6}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \Gamma, \quad t \in [0,T], \tag{7}$$

where $\dfrac{\partial u}{\partial \nu}$ denotes the derivative in the direction of the exterior normal to Γ.

For the reaction term we will consider the following decomposition

$$F(x,t,v) = F_0(x,t) + F_L(x,t)v + F_{NL}(x,t,v), \tag{8}$$

with $F_0(x,t) = F_{0R}(x,t) + iF_{0I}(x,t)$, $F_L(x,t) = F_{LR}(x,t) + iF_{LI}(x,t)$ and $F_{NL}(x,t,v) = F_{NLR}(x,t,v) + iF_{NLI}(x,t,v)$, where $F_{0R}(x,t)$, $F_{0I}(x,t)$, $F_{LR}(x,t)$, $F_{LI}(x,t)$, $F_{NLR}(x,t,v)$ and $F_{NLI}(x,t,v)$ are real functions. For the nonlinear term, we consider that there exists a complex function χ such that

$$F_{NL}(x,t,v) = F_{NL}(x,t,0) + J(x,t,v)v, \tag{9}$$

with

$$J(x,t,v) = F'_{NL}(x,t,v) + \chi(v), \tag{10}$$

and $|\chi(r)| \longrightarrow 0$ as $|r| \longrightarrow 0$, being F'_{NL} the Fréchet derivative of F_{NL} with respect to the third component.

Expression (4) involves both Schrödinger type equations and parabolic equations and includes the possibility of having a source term, a linear reaction term, a nonlinear reaction term or none of them (see (8)).

The paper is organized as follows: in Section 2 we describe the implicit and semi-implicit numerical methods simultaneously by embedding them into a two-parameter family of finite difference schemes. In Section 3 we derive a stability result of the numerical methods considered in the previous section. In the last section some numerical experiments are shown to confirm the theoretical analysis.

2. Numerical method

Let us construct a mesh on $\overline{Q}$. Let h_k denotes the mesh-size in the kth spatial coordinate direction, such that $h_k = (b_k - a_k)/N_k$, $k = 1, \ldots, d$, with $N_k \geq 2$ an integer. The set of points

$$x_j = (a_1 + j_1 h_1, \ldots, a_d + j_d h_d), \quad 0 \leq j_k \leq N_k, k = 1, ..., d,$$

defines a space grid that we denote by $\overline{\Omega}_h$. For the temporal interval we consider the mesh

$$0 = t^0 < t^1 < \cdots < t^{M-1} < t^M = T,$$

where $M \geq 1$ is an integer and $\Delta t^m = t^{m+1} - t^m$, $m = 0, \ldots, M-1$. Let $h = \max h_k$ and $\Delta t = \max \Delta t^m$. We denote by $\overline{Q}_h^{\Delta t}$ the mesh in $\overline{Q}$ defined by the cartesian product of the space grid $\overline{\Omega}_h$ and a grid in the temporal domain. Let $Q_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap Q$ and $\Gamma_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap \Gamma \times [0,T]$.

We associate the coordinate $(j,m)=(j_1,\dots,j_d,m)$ to the point $(x_j,t^m)\in\overline{Q}_h^{\Delta t}$ and we denote by V_j^m the value of a mesh function V, defined on $\overline{Q}_h^{\Delta t}$, at the point (x_j,t^m). We define the forward and backward finite differences with respect to (x_j,t^m) in the kth spatial direction by

$$\delta_k^+ V_j^m = \frac{V_{j+e_k}^m - V_j^m}{h_k}, \quad \delta_k^- V_j^m = \frac{V_j^m - V_{j-e_k}^m}{h_k},$$

where e_k denotes the kth element of the natural basis in $\mathbb{R}^d$.

On $\overline{Q}_h^{\Delta t}$ we approximate (4)–(5) by the two-parameter family of finite difference schemes

$$\frac{U_j^{m+1}-U_j^m}{\Delta t^m} = \sum_{k=1}^{d} \delta_k^+ (D_{j-(1/2)e_k}^{m,\mu,\theta} \delta_k^- U_j^{m+\theta}) + F_j^{m,\mu,\theta} \quad \text{in} \quad \tilde{Q}_h^{\Delta t}, \tag{11}$$

with

$$U_j^0 = u0(x_j) \quad \text{in} \quad \overline{\Omega}_h, \tag{12}$$

and either

$$U_j^m = 0 \quad \text{in} \quad \Gamma_h^{\Delta t}, \tag{13}$$

in the case of homogeneous Dirichlet boundary conditions (6), or

$$\sum_{k=1}^{d} \left(\delta_k^+ U_j^m + \delta_k^- U_j^m\right) \nu_k = 0 \quad \text{in} \quad \Gamma_h^{\Delta t}, \tag{14}$$

in the case of homogeneous Neumann boundary conditions (7), where

$$U_j^{m+\mu\theta} = \mu\theta U_j^{m+1} + (1-\mu\theta) U_j^m \tag{15}$$

and U_j^m represents the approximation of $u(x_j,t^m)$,

$$D_{j-(1/2)e_k}^{m,\mu,\theta} = \frac{D(x_j,t^{m+\theta},U_j^{m+\mu\theta}) + D(x_{j-e_k},t^{m+\theta},U_{j-e_k}^{m+\mu\theta})}{2}$$

and

$$F_j^{m,\mu,\theta} = F_0(x_j,t^{m+\theta}) + F_L(x_j,t^{m+\theta})U^{m+\theta} + F_{NL}(x_j,t^{m+\theta},U^{m+\mu\theta}),$$

$\mu\in\{0,1\},\theta\in[0,1]$.

We use the notation $\tilde{Q}_h^{\Delta t}$ for the set $Q_h^{\Delta t}$ or $\overline{Q}_h^{\Delta t}$, respectively, in the case of Dirichlet or Neumann boundary conditions, and ν_k represents the kth component of the normal vector ν.

Note that, when $\mu = 1$, the cases $\theta = 0$, $\theta = \frac{1}{2}$ and $\theta = 1$ correspond, respectively, to the explicit Euler, Crank-Nicolson an implicit Euler schemes. When $\mu = 0$, we have the semi-implicit case (semi-implicit Euler method when $\theta = 1$), that is, the diffusion coefficient and the non-linear part of the reaction term are treated explicitly.

In this paper we will consider two cases: the case when $\mu = 1$, which corresponds to the usual θ-method, and the case where $\mu = 0$ and $\theta = 1$, i.e. the semi-implicit Euler scheme. For all cases we suppose that

$$F_{LR}(x_j, t^{m+1}) \leq F_{LRmax} \tag{16}$$

and

$$J_R(x_j, t^{m+1}, U_j^{m+\theta}) \leq J_{Rmax}, \tag{17}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, where $J_R(x, t, v)$ is the real part of $J(x, t, v)$ given by (10). For $\mu = 1$ and $\theta \in [0, \frac{1}{2})$ or $\mu = 0$ and $\theta = 1$ we also consider

$$J_I(x_j, t^{m+1}, U_j^{m+\theta}) \leq J_{Imax}, \tag{18}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, where $J_I(x, t, v)$ is the imaginary part of $J(x, t, v)$ given by (10). In addition, for $\mu = 1$ and $\theta \in [0, \frac{1}{2})$ we also need to assume that

$$F_{LI}(x_j, t^{m+1}) \leq F_{LImax}, \tag{19}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. We need the notation

$$|F_{Lmax}|^2 = F_{LRmax}^2 + F_{IRmax}^2, \qquad |J_{max}|^2 = J_{Rmax}^2 + J_{Imax}^2 \tag{20}$$

In what follows, $\|\cdot\|_h$ will denote the discrete L^2 norm, which will be specified in the next section.

3. Stability

In this section we derive the continuous dependence of the numerical solution on the initial data and on the right-hand side.

3.1. Implicit and explicit case. Let us first consider the case where $\mu = 1$. In this case we have the usual θ-method.

Theorem 1. *Let U_j^m be the numerical solution of (4)–(5), with homogeneous Dirichlet (6) or Neumann (7) boundary conditions, given by (11)–(12) with (13) or (14), respectively. Let us consider $\mu = 1$ and suppose that (16) and (17) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$.*

If $\theta \in [\frac{1}{2}, 1]$ the method is stable under the condition

$$0 < \zeta \le 1 - 4\theta^2 \Delta t^m K_\epsilon, \quad ,\zeta \in \mathbb{R}^+, \tag{21}$$

with, for all $\epsilon \neq 0$,

$$K_\epsilon = F_{LRmax} + J_{Rmax} + \epsilon^2, \tag{22}$$

If $\theta \in [0, \frac{1}{2})$ then the method is stable under the condition (21) with, for all $\epsilon \neq 0$,

$$\begin{aligned} K_\epsilon &= F_{LRmax} + J_{Rmax} + \epsilon^2 + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2) \\ &\quad \times \left((1+\epsilon^2)|F_{Lmax}|^2 + (1+\epsilon^{-2})|J_{max}|^2\right), \end{aligned} \tag{23}$$

and

$$1 - \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^2)\frac{4}{h^2} \max_{x_j \in \bar{\Omega}_h} \frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}} \ge 0, \tag{24}$$

provided that (18) and (19) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, $|D_j^{m,1,\theta}|$ is bounded and

$$0 < \xi \le D_{Rj}^{m,1,\theta} \qquad \forall j, m. \tag{25}$$

Proof. To prove this result we will consider the unidimensional case and Neumann boundary conditions. For higher dimension or Dirichlet boundary conditions, the proof follows the same steps.

We rewrite (11)–(12), (14) as a system by separating the real and imaginary parts, U_R and U_I, respectively, of the main variable $U = (U_0, \ldots, U_N)$. We shall then study the convergence of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \ldots, N$, $m = 0, \ldots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x^+(D_{Rj^-}^{m+\theta}\delta_x^- U_{Rj}^{m+\theta}) - \delta_x^+(D_{Ij^-}^{m+\theta}\delta_x^- U_{Ij}^{m+\theta}) + F_{Rj}^{m+\theta}, \\ \qquad\qquad\qquad j = 0, \ldots, N, \quad m = 0, \ldots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x^+(D_{Ij^-}^{m+\theta}\delta_x^- U_{Rj}^{m+\theta}) + \delta_x^+(D_{Rj^-}^{m+\theta}\delta_x^- U_{Ij}^{m+\theta}) + F_{Ij}^{m+\theta}, \\ \qquad\qquad\qquad j = 0, \ldots, N, \quad m = 0, \ldots, M-1, \end{cases} \tag{26}$$

with initial condition

$$U_{Rj}^0 = u_R^0(x_j), \quad U_{Ij}^0 = u_I^0(x_j), \quad j = 0, \ldots, N,$$

and homogeneous Neumann boundary conditions

$$U^m_{R-1} = U^m_{R1},\ U^m_{RN-1} = U^m_{RN+1}, U^m_{I-1} = U^m_{I1},\ U^m_{IN-1} = U^m_{IN+1}, \quad m = 0, \dots, M, \tag{27}$$

where

$$D^{m+\theta}_{j^-} = D^{m,1,\theta}_{j^-} = \frac{D(x_{j-1}, t^{m+\theta}, U^{m+\theta}_{j-1}) + D(x_j, t^{m+\theta}, U^{m+\theta}_j)}{2}, \tag{28}$$

$j = 1, \dots, N,\ m = 0, \dots, M$, and

$$F^{m+\theta}_j = F^{m,1,\theta}_j = F(x_j, t^{m+\theta}, U^{m+\theta}_j) = F^{m+\theta}_{Rj} + iF^{m+\theta}_{Ij},$$

$j = 0, \dots, N$, $m = 0, \dots, M-1$. In (26) and (27) we need the extra points $x_{-1} = x_0 - h$ and $x_{N+1} = x_N + h$ and we define $D^{m+\theta}_{0^-} = D^{m+\theta}_{1^-}$, $D^{m+\theta}_{(N+1)^-} = D^{m+\theta}_{N^-}$.

We consider the discrete L^2 inner products

$$(U, V)_h = \frac{h}{2} U_0 \overline{V}_0 + \sum_{j=1}^{N-1} h U_j \overline{V}_j + \frac{h}{2} U_N \overline{V}_N \tag{29}$$

and

$$(U, V)_{h^*} = \sum_{j=1}^{N} h U_j \overline{V}_j, \tag{30}$$

and their corresponding norms

$$\|U\|_h = (U, U)_h^{1/2} \quad \text{and} \quad \|U\|_{h^*} = (U, U)_{h^*}^{1/2}. \tag{31}$$

Multiplying both members of the first and second equations of (26) by, respectively, $U^{m+\theta}_R$ and $U^{m+\theta}_I$, according to the discrete inner product $(\cdot, \cdot)_h$ and using summation by parts we obtain

$$\begin{aligned}\left(\frac{U^{m+1}_R - U^m_R}{\Delta t^m}, U^{m+\theta}_R\right)_h + \left(\frac{U^{m+1}_I - U^m_I}{\Delta t^m}, U^{m+\theta}_I\right)_h &+ \|(D^{m+\theta}_{R^-})^{1/2} \delta^-_x U^{m+\theta}\|^2_{h^*} \\ &= \left(F^{m+\theta}_R, U^{m+\theta}_R\right)_h + \left(F^{m+\theta}_I, U^{m+\theta}_I\right)_h.\end{aligned}$$

Since we can write

$$U^{m+\theta} = \Delta t^m \left(\theta - \frac{1}{2}\right) \frac{U^{m+1} - U^m}{\Delta t^m} + \frac{U^{m+1} + U^m}{2}, \tag{32}$$

we get

$$\Delta t^m \left(\theta - \frac{1}{2}\right) \left\| \frac{U^{m+1} - U^m}{\Delta t^m} \right\|_h^2 + \frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^-}^{m+\theta})^{1/2} \delta_x^- U^{m+\theta}\|_{h^*}^2$$
$$= \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h .$$

If $\theta \in [\frac{1}{2}, 1]$ we immediately obtain that

$$\begin{aligned} \frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} &+ \|(D_{R^-}^{m+\theta})^{1/2} \delta_x^- U^{m+\theta}\|_{h^*}^2 \\ &\leq \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h . \end{aligned} \tag{33}$$

Let us now look to the right-hand side of (33). Considering the decomposition (8)–(9) we can write

$$\begin{aligned} &\left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h \\ &\qquad = (F_R(\cdot, t^{m+\theta}, 0), U_R^{m+\theta})_h + (F_I(\cdot, t^{m+\theta}, 0), U_I^{m+\theta})_h \\ &\qquad + (F_{LR}(\cdot, t^{m+\theta}) U_R^{m+\theta}, U_R^{m+\theta})_h + (F_{LR}(\cdot, t^{m+\theta}) U_I^{m+\theta}, U_I^{m+\theta})_h \\ &\qquad + (J_R(\cdot, t^{m+\theta}, U^{m+\theta}) U_R^{m+\theta}, U_R^{m+\theta})_h \\ &\qquad + (J_R(\cdot, t^{m+\theta}, U^{m+\theta}) U_I^{m+\theta}, U_I^{m+\theta})_h. \end{aligned}$$

Since,

$$(J_R(\cdot, t^{m+\theta}, U^{m+\theta}) U_R^{m+\theta}, U_R^{m+\theta})_h \leq J_{Rmax} \|U_R^{m+\theta}\|_h^2$$

and, with the necessary modifications, we obtain a correspondent inequality for $(J_R(\cdot, t^{m+1}, U^{m+\theta}) U_I^{m+\theta}, U_I^{m+\theta})_h$, using Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h \\ &\qquad \leq \|F_R(\cdot, t^{m+\theta}, 0)\|_h \|U_R^{m+\theta}\|_h + \|F_I(\cdot, t^{m+\theta}, 0)\|_h \|U_I^{m+\theta}\|_h \\ &\qquad + F_{LRmax} \|U^{m+\theta}\|_h^2 + J_{Rmax} \|U^{m+\theta}\|_h^2 \end{aligned}$$

which leads to

$$\begin{aligned} &\left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h \\ &\qquad \leq \frac{1}{4\epsilon^2} \|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2 \|U^{m+\theta}\|_h^2 \\ &\qquad + \left(F_{LRmax} + J_{Rmax}\right) \|U^{m+\theta}\|_h^2, \end{aligned}$$

where $\epsilon \neq 0$. Then, from (33),

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^-}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2$$
$$\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\,\|U^{m+\theta}\|_h^2 \quad (34)$$

and so

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m}$$
$$\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\,\|U^{m+\theta}\|_h^2. \quad (35)$$

Using the definition of $U^{m+\theta}$ we get

$$\left(1 - 4\theta^2\Delta t^m K_\epsilon\right)\|U^{m+1}\|_h^2$$
$$\leq \left(1 + 4(1-\theta)^2\Delta t^m K_\epsilon\right)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2,$$

for $m = 0, \ldots, M-1$, with K_ϵ given by (22). If (21) holds we get

$$\|U^{m+1}\|_h^2$$
$$\leq \frac{1 + 4(1-\theta)^2\Delta t^m K_\epsilon}{1 - 4\theta^2\Delta t^m K_\epsilon}\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2(1 - 4\theta^2\Delta t^m K_\epsilon)}\|F(\cdot, t^{m+\theta}, 0)\|_h^2$$
$$\leq (1 + 4(\theta^2 + (1-\theta)^2)\zeta^{-1}\Delta t^m K_\epsilon)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2\zeta}\|F(\cdot, t^{m+\theta}, 0)\|_h^2.$$

Summing through m and using the Discrete Duhamel's Principle (Lemma 4.1 in Appendix B of [5]) we get

$$\|U^k\|_h^2 \leq e^{4(\theta^2 + (1-\theta)^2)\zeta^{-1}K_\epsilon t^k}\left(\|U^0\|_h^2 + \frac{1}{2\epsilon^2\zeta}\sum_{m=0}^{k-1}\|F(\cdot, t^{m+\theta}, 0)\|_h^2\Delta t^m\right),$$

which proves the stability.

We now consider the case where $\theta \in [0, \frac{1}{2})$. In this case we have

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^-}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2$$
$$\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\,\|U^{m+\theta}\|_h^2$$
$$+\Delta t^m\left(\frac{1}{2} - \theta\right)\left\|\frac{U^{m+1} - U^m}{\Delta t^m}\right\|_h^2. \quad (36)$$

Since

$$\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 = \left\|\frac{U_R^{m+1}-U_R^m}{\Delta t^m}\right\|_h^2 + \left\|\frac{U_I^{m+1}-U_I^m}{\Delta t^m}\right\|_h^2 \tag{37}$$

and, following [1], we deduce that

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \ \le\ & (1+\eta_1^2)\frac{4}{h^2}\max_{x_j\in\bar\Omega_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^-}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2\\ &+(1+\eta_1^{-2})\left(\|F_R^{m+\theta}\|_h^2+\|F_I^{m+\theta}\|_h^2\right),\end{aligned}$$

where $\eta_1 \neq 0$. Using (8)–(9) we get

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \ \le\ & (1+\eta_1^2)\frac{4}{h^2}\max_{x_j\in\bar\Omega_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^-}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2\\ &+(1+\eta_1^{-2})(1+\eta_2^{-2})\|F(\cdot,t^{m+\theta},0)\|_h^2\\ &+(1+\eta_1^{-2})(1+\eta_2^2)(1+\eta_3^2)(F_{LRmax}^2+F_{LImax}^2)\|U^{m+\theta}\|_h^2\\ &+(1+\eta_1^{-2})(1+\eta_2^2)(1+\eta_3^{-2})(J_{Rmax}^2+J_{Imax}^2)\|U^{m+\theta}\|_h^2,\end{aligned}$$

where $\eta_2, \eta_3 \neq 0$. Using the definition of $U^{m+\theta}$ and $\eta_1=\eta_2=\eta_3=\epsilon$ we get

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \ \le\ & (1+\epsilon^2)\frac{4}{h^2}\max_{x_j\in\bar\Omega_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^-}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2\\ &+(1+\epsilon^{-2})^2\|F(\cdot,t^{m+\theta},0)\|_h^2\\ &+2\theta^2(1+\epsilon^{-2})(1+\epsilon^2)\Big((1+\epsilon^2)|F_{Lmax}|^2\\ &\qquad +(1+\epsilon^{-2})|J_{max}|^2\Big)\|U^{m+1}\|_h^2\\ &+2(1-\theta)^2(1+\epsilon^{-2})(1+\eta^2)\Big((1+\epsilon^2)|F_{Rmax}|^2\\ &\qquad +(1+\epsilon^{-2})|J_{max}|^2\Big)\|U^m\|_h^2.\end{aligned}$$

Then, considering the previous inequality in (36) and if (24) holds, we get

$$\begin{aligned}\Big(1-4\theta^2\Delta t^m K_\epsilon\Big)\|U^{m+1}\|_h^2 \le \Big(1+4(1-\theta)^2\Delta t^m K_\epsilon\Big)\|U^m\|_h^2\\ +2\Delta t^m\left(\frac{1}{4\epsilon^2}+\Delta t^m\left(\frac{1}{2}-\theta\right)(1+\epsilon^{-2})^2\right)\|F(\cdot,t^{m+1},0)\|_h^2,\end{aligned}$$

for $m = 0, \dots, M-1$, with K_ϵ given by (23). If (21) holds, summing through m and using the Discrete Duhamel's Principle we get

$$\|U^k\|_h^2 \leq e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}K_\epsilon t^k}$$
$$\times \left(\|U^0\|_h^2 + 2\left(\frac{1}{4\epsilon^2} + T\left(\frac{1}{2}-\theta\right)(1+\epsilon^{-2})^2\right) \sum_{m=0}^{k-1} \|F(\cdot, t^{m+\theta}, 0)\|_h^2 \Delta t^m \right),$$

which concludes the proof. ■

Remark 1. *If $F(x,t,0) = 0$, we may prove that, for $\theta \in [\frac{1}{2}, 1]$, if*

$$0 < \zeta \leq 1 - 4\theta^2 \Delta t^m K,$$

for some $\zeta \in \mathbb{R}^+$, with

$$K = F_{LRmax} + J_{Rmax},$$

we get

$$\|U^{m+1}\|_h^2 \leq (1 + 4(\theta^2 + (1-\theta)^2)\zeta^{-1}\Delta t^m K)\|U^m\|_h^2.$$

Summing through m and using the Discrete Duhamel's Principle we get

$$\|U^k\|_h^2 \leq e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}Kt^k}\|U^0\|_h^2.$$

If, in addition, F_{LRmax} and J_{Rmax} are non-positive, the method is unconditionally stable.

Remark 2. *For $\theta \in [0, \frac{1}{2})$, the following particular cases are easily deduced from the previous theorem.*

(1) *If $F(x,t,0) = 0$, the stability conditions are (21) and (24) with*

$$K_\epsilon = F_{LRmax} + J_{Rmax} + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-1}) \times \Big((1+\epsilon^2)|F_{Lmax}|^2 + (1+\epsilon^{-2})|J_{max}|^2\Big).$$

(2) *If $F_L(x,t) = 0$, the stability conditions are (21) and (24) with*

$$K_\epsilon = J_{Rmax} + \epsilon^2 + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2)|J_{max}|^2.$$

(3) *If $J(x,t,U) = 0$, the stability conditions are (21) and (24) with*

$$K_\epsilon = F_{LRmax} + \epsilon^2 + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2)|F_{Lmax}|^2.$$

Remark 3. *If we consider the Dirichlet boundary conditions then, according to the discrete Poincaré-Friedrichs inequality, there exists a constant $C(\Omega)$, depending on Ω, such that*

$$C(\Omega)\|U^{m+1}\|_h \leq \|\delta_x^- U^{m+1}\|_{h^*}.$$

So, for $\theta \in [\frac{1}{2}, 1]$, if (25) holds, inequality (34) implies

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \xi C(\Omega)\|U^{m+\theta}\|_h^2$$
$$\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\|U^{m+\theta}\|_h^2 \quad (38)$$

Considering $\epsilon^2 = \frac{1}{2}\xi C(\Omega)$, then $\xi C(\Omega) - \epsilon^2 > 0$ and we obtain

$$\begin{aligned}\left(1 - 4\theta^2\Delta t^m K\right)\|U^{m+1}\|_h^2 \ \leq\ & \left(1 + 4(1-\theta)^2\Delta t^m K\right)\|U^m\|_h^2 \\ & + \frac{\Delta t^m}{\xi C(\Omega)}\|F(\cdot, t^{m+1}, 0)\|_h^2,\end{aligned}$$

for $m = 0, \ldots, M-1$, with

$$K = F_{LRmax} + J_{Rmax}.$$

Then, the stability condition is (21) with $K_\epsilon = K$ (does not depend on ϵ).

With the same arguments, for $\theta \in [0, \frac{1}{2})$ and Dirichlet boundary conditions, we may prove that, if (25) holds, the stability conditions are (21) and (24) with

$$\begin{aligned}K_\epsilon = F_{LRmax} + J_{Rmax} + \Delta t^m\left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2) \\ \times\Big((1+\epsilon^2)|F_{Lmax}|^2 + (1+\epsilon^{-2})|J_{max}|^2\Big).\end{aligned}$$

Remark 4. *For $\theta \in [\frac{1}{2}, 1]$, if both F_{LRmax} and J_{Rmax} are non-positive, (25) holds and we consider Dirichlet boundary conditions, the method is unconditionally stable.*

3.2. Semi-Implicit case. Let us now consider the case where $\mu = 0$ and $\theta = 1$, i.e, the semi-implicit Euler method.

Theorem 2. *Let U_j^m be the numerical solution of (4)–(5), with homogeneous Dirichlet (6) or Neumann (7) boundary conditions, given by (11)–(12) with*

(13) or (14), respectively. Let us consider $\mu = 0$, $\theta = 1$ and suppose that (16), (17) and (18) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$.

The numerical method is stable under the condition

$$0 < \zeta \leq 1 - 2\Delta t^m K_\epsilon, \quad ,\zeta \in \mathbb{R}^+, \tag{39}$$

with, for all $\epsilon \neq 0$,

$$K_\epsilon = F_{LRmax} + \frac{1}{2}|J_{max}|^2 + \epsilon^2. \tag{40}$$

Proof. As for the previous theorem, to prove this result we will consider the unidimensional case and Neumann boundary conditions. For higher dimension or Dirichlet boundary conditions, the proof follows the same steps.

We rewrite (11)–(12), (14) as a system by separating the real and imaginary parts, U_R and U_I, respectively, of the main variable. We shall then study the stability of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \dots, N$, $m = 0, \dots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x^+(D_{Rj^-}^{m,0,1}\delta_x^- U_{Rj}^{m+1}) - \delta_x^+(D_{Ij^-}^{m,0,1}\delta_x^- U_{Ij}^{m+1}) + F_{Rj}^{m,0,1} \\ \qquad\qquad\qquad j = 1, \dots, N-1, m = 0, \dots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x^+(D_{Ij^-}^{m,0,1}\delta_x^- U_{Rj}^{m+1}) + \delta_x^+(D_{Rj^-}^{m,0,1}\delta_x^- U_{Ij}^{m+1}) + F_{Ij}^{m,0,1}, \\ \qquad\qquad\qquad j = 1, \dots, N-1, m = 0, \dots, M-1, \end{cases} \tag{41}$$

with initial condition and homogeneous Neumann boundary conditions given as in the previous theorem, where

$$D_{j^-}^{m,0,1} = \frac{D(x_{j-1}, t^{m+1}, U_{j-1}^m) + D(x_j, t^{m+1}, U_j^m)}{2}, \tag{42}$$

$j = 1, \dots, N$, $m = 0, \dots, M$, and

$$F^{m,0,1} = F_0(., t^{m+1}) + F_L(., t^{m+1})U^{m+1} + F_{NL}(., t^{m+1}, U^m) = F_{Rj}^{m,0,1} + iF_{Ij}^{m,0,1},$$

$j = 0, \dots, N$, $m = 0, \dots, M-1$. In (41) we need the extra points $x_{-1} = x_0 - h$ and $x_{N+1} = x_N + h$ and we define $D_{0^-}^{m,0,1} = D_{1^-}^{m,0,1}$, $D_{(N+1)^-}^{m,0,1} = D_{N^-}^{m,0,1}$.

We consider the discrete L^2 inner products defined by (29)–(30) their corresponding norms.

Multiplying both members of the first and second equations of (41) by, respectively, U_R^{m+1} and U_I^{m+1}, according to the discrete inner product $(\cdot,\cdot)_h$, and using summation by parts we obtain, as for (33),

$$\begin{aligned}\frac{\|U^{m+1}\|_h^2-\|U^m\|_h^2}{2\Delta t^m} &+ \|(D_{R^-}^{m,0,1})^{1/2}\delta_x^- U^{m+1}\|_{h^*}^2\\ &\leq \left(F_R^{m,0,1},U_R^{m+1}\right)_h+\left(F_I^{m,0,1},U_I^{m+1}\right)_h. \end{aligned}\tag{43}$$

Let us now look to the right-hand side of (43). Considering (8)–(9) we obtain

$$\begin{aligned}&\left(F_R^{m,0,1},U_R^{m+1}\right)_h+\left(F_I^{m,0,1},U_I^{m+1}\right)_h\\ &\qquad=(F_R(\cdot,t^{m+1},0),U_R^{m+1})_h+(F_I(\cdot,t^{m+1},0),U_I^{m+1})_h\\ &\qquad+(F_{LR}(\cdot,t^{m+1})U_R^{m+1},U_R^{m+1})_h+(F_{LR}(\cdot,t^{m+1})U_I^{m+1},U_I^{m+1})_h\\ &\qquad+(J_R(\cdot,t^{m+1},U^m)U_R^m,U_R^{m+1})_h+(J_R(\cdot,t^{m+1},U^m)U_I^m,U_I^{m+1})_h\\ &\qquad-(J_I(\cdot,t^{m+1},U^m)U_I^m,U_R^{m+1})_h+(J_I(\cdot,t^{m+1},U^m)U_R^m,U_I^{m+1})_h.\end{aligned}$$

So, using Cauchy-Schwarz inequality, we have

$$(J_R(\cdot,t^{m+1},U^m)U_R^m,U_R^{m+1})_h\leq J_{Rmax}^2\|U_R^{m+1}\|_h\|U_R^m\|_h$$

and so

$$(J_R(\cdot,t^{m+1},U^m)U_R^m,U_R^{m+1})_h\leq\frac{1}{2}\left(J_{Rmax}^2\|U_R^{m+1}\|_h^2+\|U_R^m\|_h^2\right)$$

and, with the necessary modifications, we obtain a correspondent inequality for $(J_R(\cdot,t^{m+1},U^m)U_I^m,U_I^{m+1})_h$. We also have, considering the Cauchy-Schwarz inequality,

$$\begin{aligned}-(J_I(\cdot,t^{m+1},U^m)U_I^m,U_R^{m+1})_h&+(J_I(\cdot,t^{m+1},U^m)U_R^m,U_I^{m+1})_h\\ &\leq\frac{1}{2}\left(J_{Imax}^2\|U^{m+1}\|_h^2+\|U^m\|_h^2\right)\end{aligned}$$

Then, for the right-hand side of (43), we have

$$\begin{aligned}&\left(F_R^{m,0,1},U_R^{m+1}\right)_h+\left(F_I^{m,0,1},U_I^{m+1}\right)_h\\ &\qquad\leq\|F_R(\cdot,t^{m+1},0)\|_h\|U_R^{m+1}\|_h+\|F_I(\cdot,t^{m+1},0)\|_h\|U_I^{m+1}\|_h\\ &\qquad+F_{LRmax}\|U^{m+1}\|_h^2\\ &\qquad+\frac{1}{2}\left(J_{Rmax}^2+J_{Imax}^2\right)\|U^{m+1}\|_h^2+\|U^m\|_h^2\end{aligned}$$

which leads to

$$\begin{aligned}\left(F_R^{m,0,1}, U_R^{m+1}\right)_h + \left(F_I^{m,0,1}, U_I^{m+1}\right)_h& \\ &\leq \frac{1}{4\epsilon^2}\|F(\cdot,t^{m+1},0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\ &+ \left(F_{LRmax} + \frac{1}{2}|J_{max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2,\end{aligned}$$

where $\epsilon \neq 0$. Then, from (43),

$$\begin{aligned}&\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^-}^{m,0,1})^{1/2}\delta_x^- U^{m+1}\|_{h^*}^2 \\ &\leq \frac{1}{4\epsilon^2}\|F(\cdot,t^{m+1},0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\ &+ \left(F_{LRmax} + \frac{1}{2}|J_{max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2,\end{aligned} \tag{44}$$

and so

$$\begin{aligned}\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} &\leq \frac{1}{4\epsilon^2}\|F(\cdot,t^{m+1},0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\ &+ \left(F_{LRmax} + \frac{1}{2}|J_{max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2,\end{aligned} \tag{45}$$

Using the definition of $U^{m+\theta}$ we get

$$(1 - 2\Delta t^m K_\epsilon)\|U^{m+1}\|_h^2 \leq (1 + 2\Delta t^m)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2}\|F(\cdot,t^{m+1},0)\|_h^2, \tag{46}$$

for $m = 0,\ldots,M-1$, with K_ϵ given by (40). If (39) holds, summing through m and using the Discrete Duhamel's Principle we get

$$\|U^k\|_h^2 \leq e^{2(1+K_\epsilon)\zeta^{-1}t^k}\left(\|U^0\|_h^2 + \frac{1}{2\epsilon^2\zeta}\sum_{m=0}^{k-1}\|F(\cdot,t^{m+1},0)\|_h^2\Delta t^m\right),$$

which concludes the proof. ■

Remark 5. *If $F(x,t,0) = 0$, we may prove that if*

$$0 < \zeta \leq 1 - 2\Delta t^m K, \tag{47}$$

for some $\zeta \in \mathbb{R}^+$, with

$$K = F_{LRmax} + \frac{1}{2}|J_{max}|^2, \tag{48}$$

we get

$$\|U^{m+1}\|_h^2 \leq \left(1 + 2\Delta t^m (1+K)\zeta^{-1}\right) \|U^m\|_h^2,$$

for $m = 0, \dots, M-1$, *and so*

$$\|U^k\|_h^2 \leq e^{2(1+K)\zeta^{-1}t^k} \|U^0\|_h^2.$$

If, in addition, $K \leq 0$, *then the method is unconditionally stable.*

Remark 6. *As in Remark 3, if we consider the Dirichlet boundary conditions and (25) holds, the stability condition is (47) with* K *given by (48).*

Remark 7. *If* $F_{NL} \equiv 0$, F_{LRmax} *is non-positive, (25) holds and we consider Dirichlet boundary conditions, the method is unconditionally stable.*

4. Numerical examples

In this section we will illustrate the stability results using appropriate numerical examples.

We start by noting that the stability condition for the explicit method has already been illustrated in [1], though without a reactive term. Since the numerical results are very similar, we will leave the explicit scheme out of this illustration, refering the reader to [1] for details. We will also leave out of this section the illustration of the stability of the implicit scheme, since we expect that the choice of linearization method may further influence the results.

In this way, we will focus the numerical illustrations on the stability of the semi-implicit scheme with Neumann boundary condition, since the stability condition (though similar to the Dirichlet case) is slighty more complex.

In this way, we consider equation (4) with

$$x_1, x_2 \in (0,\pi) \times (0,\pi), \quad t \in (0,T]$$

with initial and Neumann boundary conditions given, respectively, by

$$u(x_1,x_2,0) = \cos(x_1)\cos(x_2)$$

and

$$\frac{\partial u}{\partial \nu}(0,x_2,t) = \frac{\partial u}{\partial \nu}(\pi,x_2,t) = \frac{\partial u}{\partial \nu}(x_1,0,t) = \frac{\partial u}{\partial \nu}(x_1,\pi,t) = 0.$$

Given a constant $A \in \mathbb{C}$, for

$$\begin{aligned} F(x_1,x_2,t,v) =& (A+2i)v + 2v^2 \\ & - \left(\sin^2(x_1)\cos^2(x_2) + \cos^2(x_1)\sin^2(x_2)\right) e^{2At} \end{aligned}$$

and

$$D(x_1, x_2, t, v) = i + v,$$

the exact solution is given by

$$u(x_1, x_2, t) = \cos(x_1)\cos(x_2)e^{At}.$$

We also note that with this choice of reactive term F we have

$$\begin{aligned} F_0(x_1, x_2, t) =& -\left(\sin^2(x_1)\cos^2(x_2) + \cos^2(x_1)\sin^2(x_2)\right) e^{2At}, \\ F_L(x, t) =& A + 2i, \\ F_{NL}(x, t, v) =& 2v^2 \quad (\text{ and } F_{NL}(x, t, 0) = 0), \\ J(x, t, v) =& 2v^2. \end{aligned}$$

We will now consider two different possibilities for the value of the constant A that will induce different behaviours on the solution and therefore on the stability condition.

4.1. Case 1: $\mathbf{F_{LR} \leq 0}$. For $A = -1 + i$, we have that $F_{LR} = -1 < 0$. We will now consider the upper bound (46) (taking $\epsilon = 1$) and compare it with the actual norm $\|U^m\|_h^2$. We also note that if the time step Δt is such that there exists no $\xi > 0$ so that (39) is satisfied, then no theoretical upper bound is known and the numerical solution might become unbounded in time (even in cases where the solution is bounded).

The numerical results are shown in figures 1 and 2. It can be seen that for smaller steps in time, the ratio stays bounded by the theoretical upper bound. For higher time steps (namely for time steps that do not satisfy the stability condition), there is no theoretical upper bound and the norm of the approximation increases rapidly.

4.2. Case 2: $\mathbf{F_{LR} > 0}$. For $A = 0.1 + i$, we have that $F_{LR} = 0.1 > 0$. In this way, the condition (39) is harder to satisfy, since now F_{LRmax} is positive.

Again we compare the theoretical the upper bound (46) and the actual norm $\|U^m\|_h^2$.

The numerical results are shown in figures 3 and 4. It can be seen that though in some cases the theoretical bound increases, the numerical results might stay bounded. Similarly to the previous case, for higher steps in time, the approximation's norm increases rapidly.

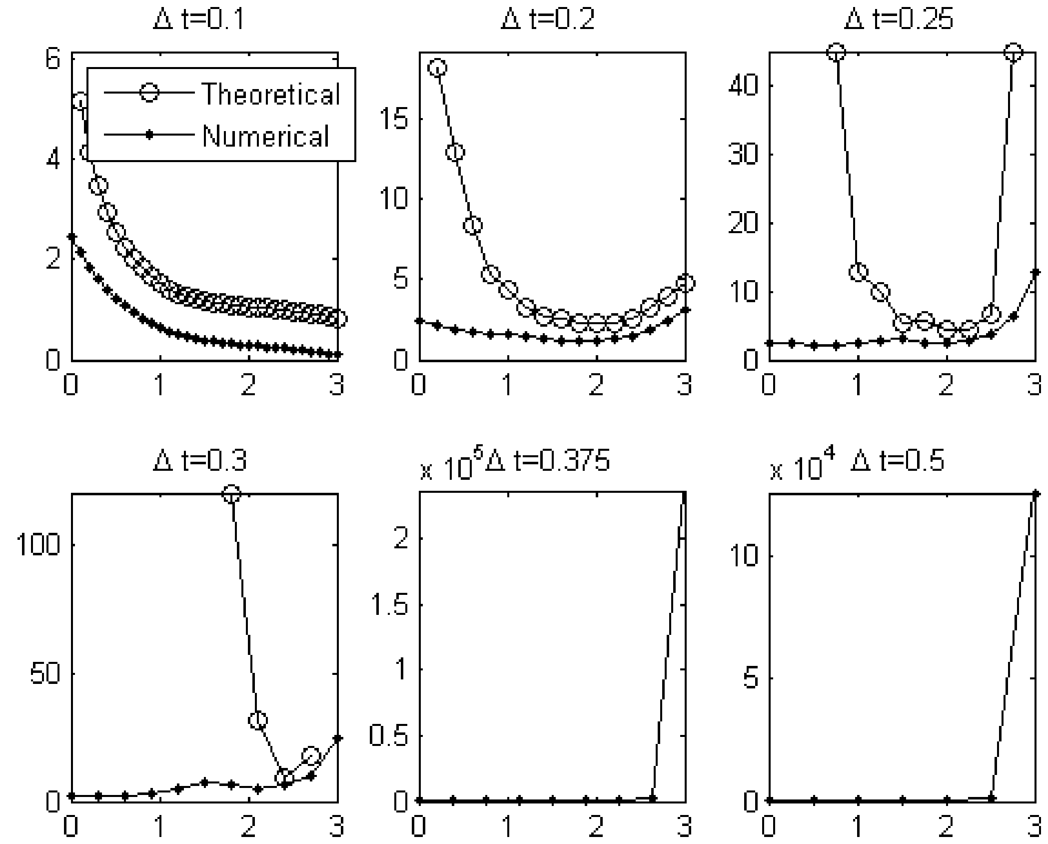

FIGURE 1. Case 1: Evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (46) for several time steps Δt. No plot on the theoretical upper bound, means there exists no ξ that satifies (39).

5. Conclusions

In this paper we have established the stability conditions for finite difference schemes in the context of complex diffusion with reactive terms. In this way we have extended a previous stability result [1] to the semi-implicit scheme and to the presence of reactive terms in complex diffusion.

In this way we have shown that both the implicit and semi-implicit schemes are stable under some conditions on the time step. We note that at a fixed time, there is always a small enough time step for which the method is stable, since the stability condition is an upper bound for the time step.

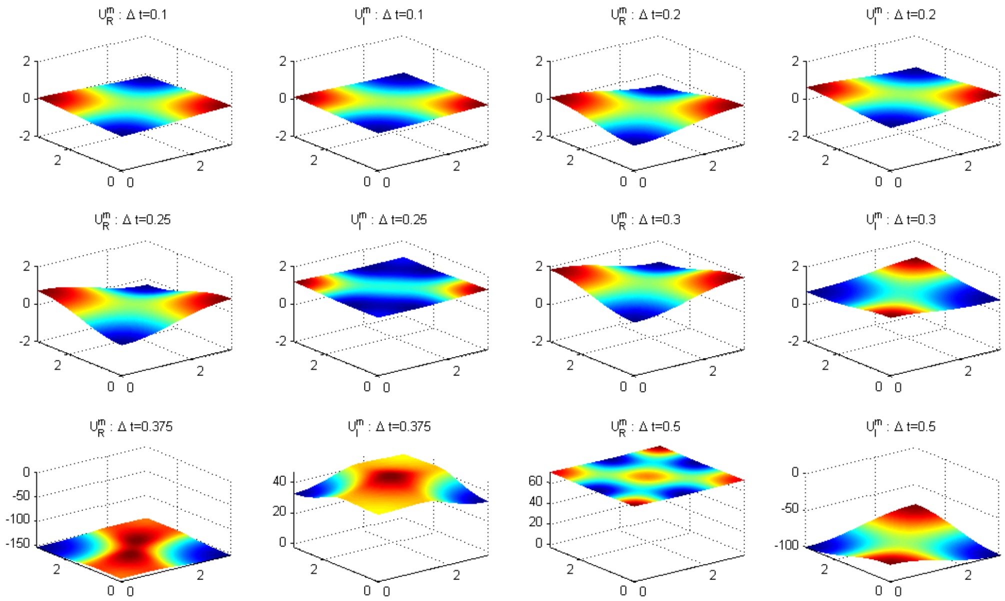

FIGURE 2. Case 1: Real and imaginary parts of the approximation U^m for the final time $T = 3$ for several time steps Δt.

As usual, for the explicit scheme, a stability condition that relates the magnitude of the time step and the spatial spacement needs to be satisfied.

Finally we have illustrated the theoretical results with numerical examples, to show cases of stability and unstability.

Parallel work [2] establishes a convergence result for these finite different schemes in the context of complex diffusion with reactive terms.

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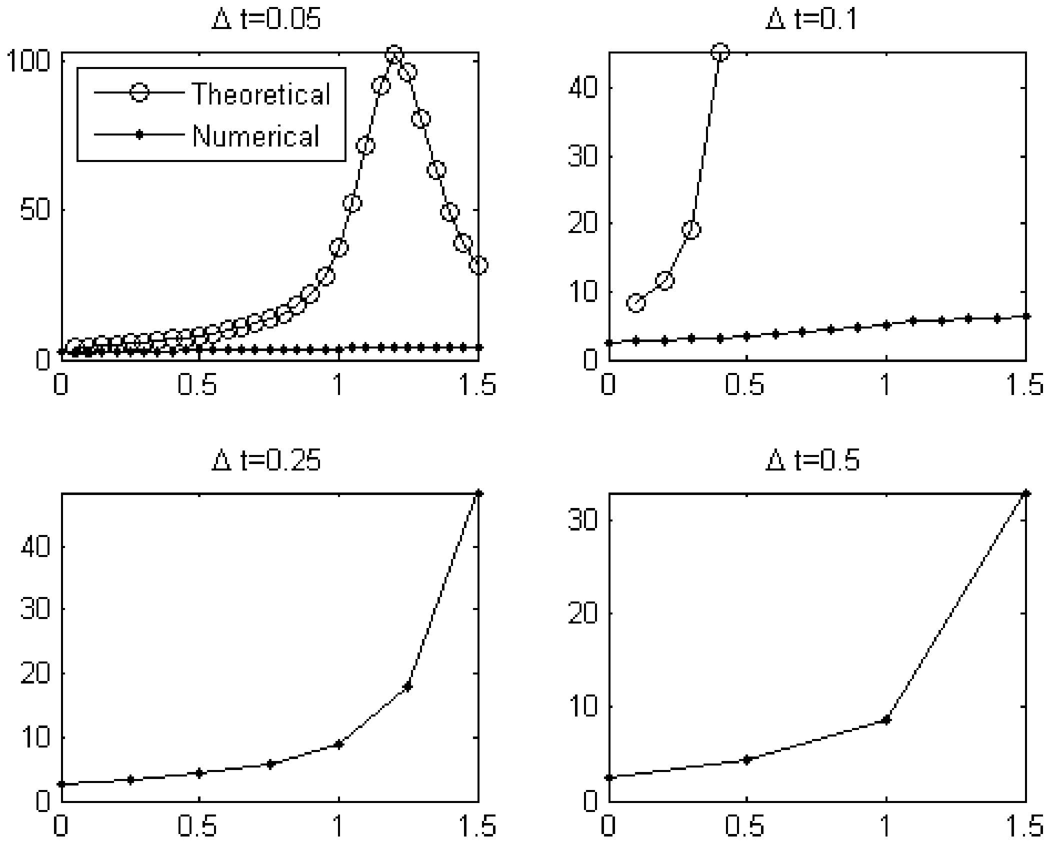

FIGURE 3. Case 2: Evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (46) for several time steps Δt. No plot on the theoretical upper bound, means there exists no ξ that satisfies (39).

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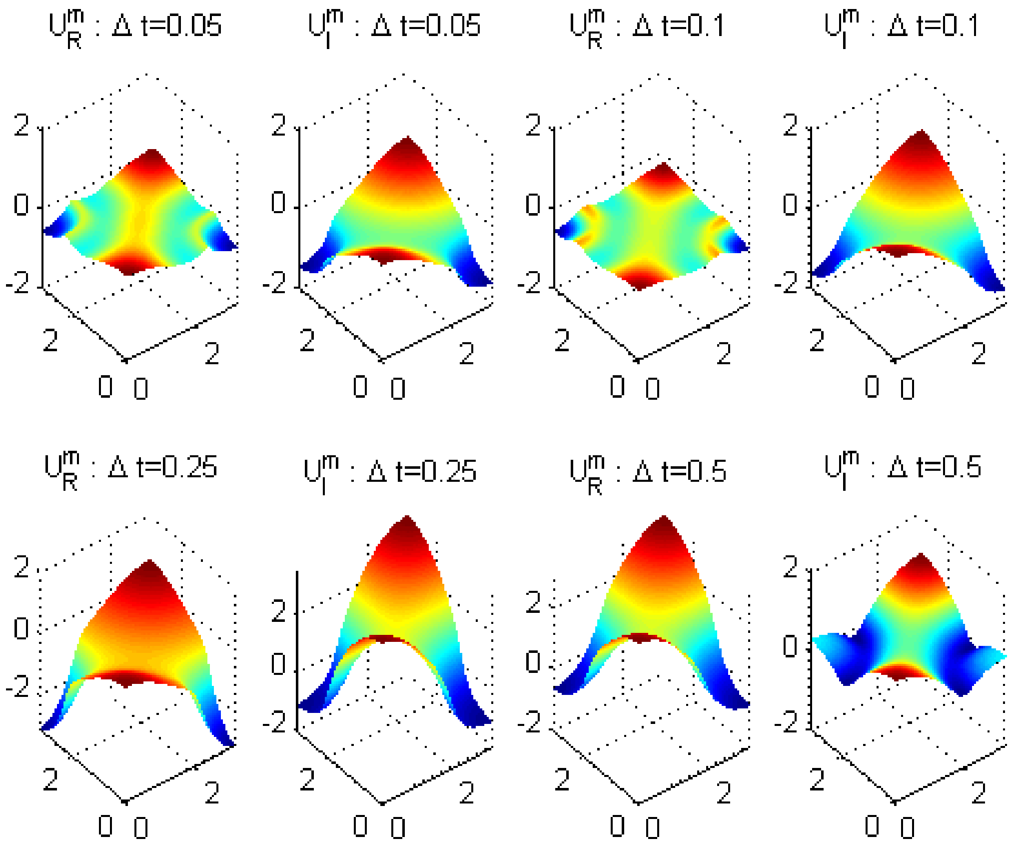

FIGURE 4. Case 2: Real and imaginary parts of the approximation U^m for final time $T = 1.5$ for several time steps Δt.

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