Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–30

FINITE DIFFERENCE SCHEMES FOR NONLINEAR COMPLEX REACTION-DIFFUSION PROCESSES: CONVERGENCE ANALYSIS

ADÉRITO ARAÚJO, SÍLVIA BARBEIRO AND PEDRO SERRANHO

ABSTRACT: This paper is devoted to the proof of the convergence properties of a class of finite difference schemes applied to nonlinear complex reaction-diffusion equations. We investigate the accuracy of the numerical solution considering implicit and semi-implicit discretizations. To illustrate the theoretical results we present some numerical examples computed with a semi-implicit scheme applied to a nonlinear equation.

KEYWORDS: finite differences, complex reaction-diffusion, convergence.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^d$, $d \in \{1, 2\}$, with boundary $\Gamma = \partial\Omega$. Typically Ω is the cartesian product of open intervals in $\mathbb{R}$, i.e.,

$$\Omega = \prod_{j=1}^{d}(a_j, b_j), \tag{1}$$

with $a_j, b_j \in \mathbb{R}$. Let $Q = \Omega\times(0, T]$, with $T > 0$, and $v : \bar{Q} = \bar{\Omega} \times [0, T] \longrightarrow \mathbb{C}$. We consider a reaction-diffusion process with a non-constant complex coefficient $D(x, t, v) = D_R(x, t, v) + iD_I(x, t, v)$ and non-constant complex reaction term $F(x, t, v) = F_R(x, t, v) + iF_I(x, t, v)$, where $D_R(x, t, v)$, $D_I(x, t, v)$, $F_R(x, t, v)$, $F_I(x, t, v)$ are real functions dependent on v. We need to assume that

$$D_R(x, t, v) \geq \xi > 0, \qquad (x, t) \in \bar{Q}, \tag{2}$$

and that there exists a constant $L > 0$ such that

$$|D(x, t, v)| \leq L, \qquad (x, t) \in \bar{Q}. \tag{3}$$

*The research of the first and second authors was partially supported by project UTAustin/MAT/0066/2008 and by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011. The work of the third author was partially supported by FCT project PTDC/SAU-ENB/111139/2009 and by the program FEDER/COMPETE FCOMP-01-0124-FEDER-015712. The authors were also supported by FCT (PTDC/SAU-ENB/119132/2010) and program COMPETE (QREN -FCOMP-01-0124-FEDER-021014).

These inequalities (2) and (3) can easily be shown to hold for the diffusion coefficient in [2], [6] and [14].

We define the initial boundary value problem for the unknown complex function $u = u_R + iu_I$

$$\frac{\partial u}{\partial t}(x,t) = \nabla \cdot (D(x,t,u)\nabla u(x,t)) + F(x,t,u), \quad (x,t) \in Q, \tag{4}$$

under the initial condition

$$u(x,0) = u^0(x), \quad x \in \overline{\Omega}, \tag{5}$$

with either the Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \Gamma, \quad t \in [0,T], \tag{6}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \Gamma, \quad t \in [0,T], \tag{7}$$

where $\dfrac{\partial u}{\partial \nu}$ denotes the derivative in the direction of the exterior normal to Γ. For the reaction term we will consider the following decomposition

$$F(x,t,v) = F_L(x,t,v) + F_{NL}(x,t,v), \tag{8}$$

where F_L is a linear operator with respect to v,

$$F_L(x,t,v) = f(x,t) + A(x,t)v,$$

satisfying

$$|A(x,t)| \leq A_{max} \quad \forall (x,t) \in \Omega \times (0,T]. \tag{9}$$

The present paper focuses on deriving convergence results for a class of finite difference schemes for (4)-(5), with (6) or (7), in one and two dimensions. Diffusion processes are commonly used in image processing, as for example in noise removal, inpainting, stereo vision or optical flow (see e.g. [6, 7, 14, 15, 16, 17, 18, 19, 21]) and in particular nonlinear complex diffusion proved to be successfully applied in medical imaging despeckling and denoising ([12, 17]). Although diverse numerical schemes have been implemented to approximately solve the resulting mathematical model, yet, no formal mathematical analysis has been carried out in order to gather the properties of approximate solutions such as error estimates and rates of convergence.

In [2] the authors studied the stability of a one parameter class of finite difference schemes for the non linear complex diffusion equation. Both explicit and implicit schemes were considered by changing the values of the parameter. In [3] the authors analyzed the stability of implicit and semi-implicit finite difference schemes for nonlinear complex reaction-diffusion processes. In image denoising, the stability proof in [2] is important for the cases where the definition of the used image is fixed. However, in the cases where it is possible to increase the definition of the image from previous acquired ones, it is also important to establish convergence results for the filtering process.

The main goal of the present paper is to derive convergence results for a class of implicit and semi-implicit finite difference schemes for the nonlinear complex diffusion equation with a reaction term.

The paper is organized as follows: in Section 2 we describe the implicit and semi-implicit numerical methods simultaneously by embedding them into a one-parameter family of finite difference schemes. The core section of this paper is Section 3, where the rigorous proof of convergence of the semi-implicit and implicit discretization is presented. In the last section some numerical experiments are shown to confirm the theoretical analysis. The paper ends with an appendix where we prove some technical lemmata.

2. Numerical method

Let us construct a mesh on $\overline{Q}$, starting with the case $d = 2$. Let h_k denotes the mesh-size in the kth spatial coordinate direction, such that $h_k = (b_k - a_k)/N_k$, $k = 1, 2$, with $N_k \geq 2$ an integer. The set of points

$$x_j = (a_1 + j_1 h_1, a_2 + j_2 h_2),\ 0 \leq j_1 \leq N_1, 0 \leq j_2 \leq N_2,$$

defines a space grid that we denote by $\overline{\Omega}_h$. For the temporal interval we consider the mesh

$$0 = t^0 < t^1 < \cdots < t^{M-1} < t^M = T,$$

where $M \geq 1$ is an integer and $\Delta t^m = t^{m+1} - t^m$, $m = 0, \ldots, M-1$. Let $h = \max\{h_1, h_2\}$ and $\Delta t = \max \Delta t^m$. We denote by $\overline{Q}_h^{\Delta t}$ the mesh in $\overline{Q}$ defined by the cartesian product of the space grid $\overline{\Omega}_h$ and a grid in the temporal domain. Let $Q_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap Q$ and $\Gamma_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap \Gamma \times [0, T]$.

We associate the coordinate $(j, m) = (j_1, j_2, m)$ to the point $(x_j, t^m) \in \overline{Q}_h^{\Delta t}$ and we denote by V_j^m the value of a mesh function V, defined on $\overline{Q}_h^{\Delta t}$, at the point (x_j, t^m). We define the forward and backward finite differences with respect to (x_j, t^m) in the kth spatial direction by

$$\delta_k^+ V_j^m = \frac{V_{j+e_k}^m - V_j^m}{h_k}, \quad \delta_k^- V_j^m = \frac{V_j^m - V_{j-e_k}^m}{h_k}, \quad k = 1, 2,$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

If $d = 1$, these definitions are simplified for the case of one spatial coordinate instead of two.

On $\overline{Q}_h^{\Delta t}$ we approximate (4)–(5) by the one-parameter family of finite difference schemes

$$\frac{U_j^{m+1} - U_j^m}{\Delta t^m} = \sum_{k=1}^{d} \delta_k^+ (D_{j-(1/2)e_k}^{m,\mu} \delta_k^- U_j^{m+1}) + F_j^{m,\mu} \quad \text{in} \quad \tilde{Q}_h^{\Delta t}, \tag{10}$$

with

$$U_j^0 = u0(x_j) \quad \text{in} \quad \overline{\Omega}_h, \tag{11}$$

and either

$$U_j^m = 0 \quad \text{in} \quad \Gamma_h^{\Delta t}, \tag{12}$$

in the case of homogeneous Dirichlet boundary conditions (6), or

$$\sum_{k=1}^{d} \left(\delta_k^+ U_j^m + \delta_k^- U_j^m \right) \nu_k = 0 \quad \text{in} \quad \Gamma_h^{\Delta t}, \tag{13}$$

in the case of homogeneous Neumann boundary conditions (7), where

$$U_j^{m+\mu} = \mu U_j^{m+1} + (1-\mu) U_j^m, \quad \mu \in \{0, 1\}, \tag{14}$$

and U_j^m represents the approximation of $u(x_j, t^m)$,

$$D_{j-(1/2)e_k}^{m,\mu} = \frac{D(x_j, t^{m+1}, U_j^{m+\mu}) + D(x_{j-e_k}, t^{m+1}, U_{j-e_k}^{m+\mu})}{2}, \quad \mu \in \{0, 1\}$$

and

$$F_j^{m,\mu} = F_{Lj}^{m+1} + F_{NLj}^{m,\mu} = F_L(x_j, t^{m+1}, U^{m+1}) + F_{NL}(x_j, t^{m+1}, U^{m+\mu}).$$

We use the notation $\tilde{Q}_h^{\Delta t}$ for the set $Q_h^{\Delta t}$ or $\overline{Q}_h^{\Delta t}$, respectively, in the case of Dirichlet or Neumann boundary conditions, and ν_k represents the kth component of the normal vector ν.

Note that the cases $\mu = 0$ and $\mu = 1$ correspond to a semi-implicit and implicit discretization, respectively. In the semi-implicit case (semi-implicit Euler method), the diffusion coefficient and the non-linear part of the reaction term are treated explicitly.

3. Convergence

The main result of this paper is Theorem 1. Estimates for the difference between the pointwise restriction of the exact solution on the discretization nodes and the finite difference solution are proved.

To provide a proper functional setting, we need to define spaces involving time-dependent functions ([11]). Let X denote a Banach space with norm $\|.\|_X$. In what follows, X is shorthand for any of the usual Sobolev spaces $W^{s,p}(\Omega)$ (which we also denote by $H^s(\Omega)$ in the case $p=2$) or the Banach space $L^\infty(\Omega)$. The space $L^\infty(0,T;X)$ consists of all measurable functions $v:[0,T]\to X$ with

$$\|v\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0\le t\le T} \|v(t)\|_X < \infty.$$

In what follows, $\|\cdot\|_h$ will denote the discrete L^2 norm, which will be specified later in this section.

Theorem 1. *Let the solution u of (4)-(5), with (6) or (7), lie in $L^\infty(0,T;$ $H^3(\Omega))$, where Ω is a one-dimensional or a two-dimensional domain defined by (1). Let us assume that D and F_{NL} are Lipschitz continuous with respect to the third component, with Lipschitz constant C_D and C_F, respectively, in the sense that*

$$|D(x,t,v) - D(x,t,w)| \le C_D|v(x,t) - w(x,t)| \qquad \forall (x,t)\in\bar{Q}, \tag{15}$$

$$|F_{NL}(x,t,v) - F_{NL}(x,t,w)| \le C_F|v(x,t) - w(x,t)| \qquad \forall (x,t)\in\bar{Q}, \tag{16}$$

and that $D(.,t^{m+1},u(.,t^{m+\mu}))\in H^2(\Omega)$, $F_L(.,t^{m+1},u(.,t^{m+1}))\in H^2(\Omega)$, $F_{NL}(.,t^{m+1},u(.,t^{m+\mu}))\in H^2(\Omega)$, $m=0,\dots,M-1$. If (2), (3) and (9) hold, and $\dfrac{\partial u}{\partial t}\in L^\infty(0,T;H^2(\Omega))$, $\dfrac{\partial^2 u}{\partial t^2}\in L^\infty(0,T;\,H^1(\Omega))$, then the numerical solution U of (10)-(11), with (12) or (13), satisfies the following error estimate

$$\|R_h u - U\|_h \le \mathcal{O}(h^2) + \mathcal{O}(\Delta t), \tag{17}$$

where $R_h u$ denotes the pointwise restriction of the function u to the space grid $\overline{\Omega}_h$.

We will prove the convergence for both uni and bi-dimensional cases. In what follows C denotes a generic positive constant.

We first note that, as a result of Taylor expansion about t^{m+1}, if $\dfrac{\partial^2 u}{\partial t^2} \in L^\infty(0,T;\, L^2(\Omega))$ then

$$\frac{u(x,t^{m+1}) - u(x,t^m)}{\Delta t^m} = \frac{\partial u}{\partial t}(x,t^{m+1}) + \Delta t^m \rho_{u,m}(x), \quad \forall x \in \Omega, \tag{18}$$

with

$$\|\rho_{u,m}\|_{L^2(\Omega)} \le C \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^\infty(t^m,t^{m+1};L^2(\Omega))},$$

and, for any sufficiently smooth function $g(t)$,

$$g(t^{m+1}) = g(t^m) + \Delta t^m \rho_{g,m}, \tag{19}$$

with

$$|\rho_{g,m}| \le \|\frac{dg}{dt}\|_{L^\infty(t^m,t^{m+1})}.$$

3.1. Uni-dimensional case. We will consider the unidimensional case and Neumann boundary conditions. For Dirichlet boundary conditions the proof follows the same steps.

We rewrite (10)–(11), (13) as a system by separating the real and imaginary parts, U_R and U_I, respectively, of the main variable $U = (U_0, \ldots, U_N)$. We shall then study the convergence of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \ldots, N$, $m = 0, \ldots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x^+(D_{Rj^-}^{m,\mu}\delta_x^- U_{Rj}^{m+1}) - \delta_x^+(D_{Ij^-}^{m,\mu}\delta_x^- U_{Ij}^{m+1}) + F_{Rj}^{m,\mu}, \\ \qquad\qquad\qquad\qquad j = 0, \ldots, N, \quad m = 0, \ldots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x^+(D_{Ij^-}^{m,\mu}\delta_x^- U_{Rj}^{m+1}) + \delta_x^+(D_{Rj^-}^{m,\mu}\delta_x^- U_{Ij}^{m+1}) + F_{Ij}^{m,\mu}, \\ \qquad\qquad\qquad\qquad j = 0, \ldots, N, \quad m = 0, \ldots, M-1, \end{cases} \tag{20}$$

with initial condition

$$U_{Rj}^0 = u_R^0(x_j), \quad U_{Ij}^0 = u_I^0(x_j), \quad j = 0, \ldots, N,$$

and homogeneous Neumann boundary conditions

$$U_{R-1}^m = U_{R1}^m,\ U_{RN-1}^m = U_{RN+1}^m, U_{I-1}^m = U_{I1}^m,\ U_{IN-1}^m = U_{IN+1}^m, \quad m = 0, \ldots, M.$$

In (20) we need the extra points $x_{-1} = x_0 - h$ and $x_{N+1} = x_N + h$ and we define $D_{0^-}^{m,\mu} = D_{1^-}^{m,\mu}$, $D_{(N+1)^-}^{m,\mu} = D_{N^-}^{m,\mu}$.

We consider the discrete L^2 inner products

$$(U,V)_h = \frac{h}{2}U_0\overline{V}_0 + \sum_{j=1}^{N-1} hU_j\overline{V}_j + \frac{h}{2}U_N\overline{V}_N \tag{21}$$

and

$$(U,V)_{h^*} = \sum_{j=1}^{N} hU_j\overline{V}_j, \tag{22}$$

and their corresponding norms

$$\|U\|_h = (U,U)_h^{1/2} \quad \text{and} \quad \|U\|_{h^*} = (U,U)_{h^*}^{1/2}. \tag{23}$$

Let $E = R_h u - U$, $E_R = R_h u_R - U_R$ and $E_I = R_h u_I - U_I$. Multiplying both members of the first and second equations of (20) by E_R^{m+1} and E_I^{m+1}, respectively, according to the discrete inner product (21), using (18) and taking into account the boundary conditions, we obtain

$$\begin{aligned}
&\left(\frac{E_R^{m+1} - E_R^m}{\Delta t^m}, E_R^{m+1}\right)_h + \|(D_{R^-}^{m,\mu})^{1/2}\delta_x^- E_R^{m+1}\|_{h^*}^2 = \\
&\left(R_h\frac{\partial u_R}{\partial t}(t^{m+1}), E_R^{m+1}\right)_h + \Delta t^m \left(\rho_{u_R,m}, E_R^{m+1}\right)_h \\
&+ \left(D_{R^-}^{m,\mu}\delta_x^- R_h u_R^{m+1}, \delta_x^- E_R^{m+1}\right)_{h^*} - \left(D_{I^-}^{m,\mu}\delta_x^- R_h u_I^{m+1}, \delta_x^- E_R^{m+1}\right)_{h^*} \\
&+ \left(D_{I^-}^{m,\mu}\delta_x^- E_I^{m+1}, \delta_x^- E_R^{m+1}\right)_{h^*} - \left(F_R^{m,\mu}, E_R^{m+1}\right)_h
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\left(\frac{E_I^{m+1} - E_I^m}{\Delta t^m}, E_I^{m+1}\right)_h + \|(D_{R^-}^{m,\mu})^{1/2}\delta_x^- E_I^{m+1}\|_{h^*}^2 = \\
&\left(R_h\frac{\partial u_I}{\partial t}(t^{m+1}), E_I^{m+1}\right)_h + \Delta t^m \left(\rho_{u_I,m}, E_I^{m+1}\right)_h \\
&+ \left(D_{I^-}^{m,\mu}\delta_x^- R_h u_R^{m+1}, \delta_x^- E_I^{m+1}\right)_{h^*} + \left(D_{R^-}^{m,\mu}\delta_x^- R_h u_I^{m+1}, \delta_x^- E_I^{m+1}\right)_{h^*} \\
&- \left(D_{I^-}^{m,\mu}\delta_x^- E_R^{m+1}, \delta_x^- E_I^{m+1}\right)_{h^*} - \left(F_I^{m,\mu}, E_I^{m+1}\right)_h.
\end{aligned} \tag{25}$$

Let $I_j = (x_j, x_{j+1})$ and $x_{j+1/2} = x_j + \frac{h}{2}$, $j = 1, \ldots, N-1$.

From (4), we get

$$\begin{aligned}\int_{x_j-1/2}^{x_j+1/2}\frac{\partial u}{\partial t}(x,t^{m+1})\,dx &= \int_{x_j-1/2}^{x_j+1/2}\frac{\partial}{\partial x}(D(x,t^{m+1},u(x,t^{m+1}))\frac{\partial u}{\partial x}\big(x,t^{m+1})\big)\,dx\\ &+\int_{x_j-1/2}^{x_j+1/2}F(x,t^{m+1},u(x,t^{m+1}))\,dx. \qquad (26)\end{aligned}$$

Let us define $x_{-1/2}=x_0$ and $x_{N+1/2}=x_N$. Multiplying (26) by $(E^{m+1})_j$, using integration and a summation by parts and taking into account the boundary conditions, we obtain

$$\begin{aligned}&\sum_{j=0}^{N}\int_{x_j-1/2}^{x_j+1/2}\frac{\partial u_R}{\partial t}(x,t^{m+1})\,dx(E_R^{m+1})_j=\\ &\quad -h\sum_{j=1}^{N}D_R(x_{j-1/2},t^{m+1},u)\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j\\ &\quad +h\sum_{j=1}^{N}D_I(x_{j-1/2},t^{m+1},u)\frac{\partial u_I}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j\\ &\quad +\sum_{j=0}^{N}\int_{x_j-1/2}^{x_j+1/2}F_R(x,t^{m+1},u)\,dx(E_R^{m+1})_j \qquad (27)\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=0}^{N}\int_{x_j-1/2}^{x_j+1/2}\frac{\partial u_I}{\partial t}(x,t^{m+1})\,dx(E_I^{m+1})_j=\\ &\quad -h\sum_{j=1}^{N}D_I(x_{j-1/2},t^{m+1},u)\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_I^{m+1})_j\\ &\quad -h\sum_{j=1}^{N}D_R(x_{j-1/2},t^{m+1},u)\frac{\partial u_I}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_I^{m+1})_j\\ &\quad +\sum_{j=0}^{N}\int_{x_j-1/2}^{x_j+1/2}F_I(x,t^{m+1},u)\,dx(E_I^{m+1})_j. \qquad (28)\end{aligned}$$

In order to provide the desired bounds, we start by deducing that

$$\begin{aligned}\left(\frac{E_R^{m+1}-E_R^m}{\Delta t^m}, E_R^{m+1}\right)_h &= \frac{1}{\Delta t^m}\left(E_R^{m+1}, E_R^{m+1}\right)_h - \frac{1}{\Delta t^m}\left(E_R^m, E_R^{m+1}\right)_h \\ &= \frac{1}{2\Delta t^m}\|E_R^{m+1}\|_h^2 - \frac{1}{2\Delta t^m}\|E_R^m\|_h^2 + \frac{1}{2\Delta t^m}\|E_R^{m+1}-E_R^m\|_h^2.\end{aligned}$$

And then

$$\left(\frac{E_R^{m+1}-E_R^m}{\Delta t^m}, E_R^{m+1}\right)_h \geq \frac{1}{2\Delta t^m}\left(\|E_R^{m+1}\|_h^2 - \|E_R^m\|_h^2\right). \tag{29}$$

Likewise

$$\left(\frac{E_I^{m+1}-E_I^m}{\Delta t^m}, E_I^{m+1}\right)_h \geq \frac{1}{2\Delta t^m}\left(\|E_I^{m+1}\|_h^2 - \|E_I^m\|_h^2\right). \tag{30}$$

In the next bound we take in account that $\|E^0\|_h = 0$ and we use (29) and (30). Subtracting (27) and (28) from (24) and (25), multiplying the result by $2\Delta t^m$ and summing from 0 to $M-1$, we obtain

$$\begin{aligned}&2\left(\min \Delta t^m\right)\sum_{m=0}^{M-1}\left(\|(D_{R^-}^{m,\mu})^{1/2}\delta_x^- E_R^{m+1}\|_{h^*}^2 + \|(D_{R^-}^{m,\mu})^{1/2}\delta_x^- E_I^{m+1}\|_{h^*}^2\right) \\ &+\|E_R^M\|_h^2 + \|E_I^M\|_h^2 \leq 2\Delta t\sum_{m=0}^{M-1}(|T_1|+|T_2|+|T_3|+|T_4|+|T_5|+|T_6|+|T_7| \\ &\qquad\qquad +|T_8|+|T_9|),\end{aligned} \tag{31}$$

where

$$T_1 = \left(R_h\frac{\partial u_R}{\partial t}(t^{m+1}), E_R^{m+1}\right)_h - \sum_{j=0}^{N}\int_{x_j-1/2}^{x_j+1/2}\frac{\partial u_R}{\partial t}(x,t^{m+1})\,dx(E_R^{m+1})_j,$$

$$T_2 = \left(R_h\frac{\partial u_I}{\partial t}(t^{m+1}), E_I^{m+1}\right)_h - \sum_{j=0}^{N}\int_{x_j-1/2}^{x_j+1/2}\frac{\partial u_I}{\partial t}(x,t^{m+1})\,dx(E_I^{m+1})_j,$$

$$T_3 = \Delta t^m\left(\rho_{u_R,m}, E_R^{m+1}\right)_h + \Delta t^m\left(\rho_{u_I,m}, E_I^{m+1}\right)_h,$$

$$
\begin{aligned}
T_4 &= h\sum_{j=1}^{N} D_R(x_{j-1/2},t^{m+1},u)\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j \\
&\quad - \left(D_{R^-}^{m,\mu}\delta_x^- R_h u_R^{m+1}, \delta_x^- E_R^{m+1}\right)_{h^*},
\end{aligned}
$$

$$
\begin{aligned}
T_5 &= -h\sum_{j=1}^{N} D_I(x_{j-1/2},t^{m+1},u)\frac{\partial u_I}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j \\
&\quad + \left(D_{I^-}^{m,\mu}\delta_x^- R_h u_I^{m+1}, \delta_x^- E_R^{m+1}\right)_{h^*},
\end{aligned}
$$

$$
\begin{aligned}
T_6 &= h\sum_{j=1}^{N} D_I(x_{j-1/2},t^{m+1},u)\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_I^{m+1})_j \\
&\quad - \left(D_{I^-}^{m,\mu}\delta_x^- R_h u_R^{m+1}, \delta_x^- E_I^{m+1}\right)_{h^*},
\end{aligned}
$$

$$
\begin{aligned}
T_7 &= h\sum_{j=1}^{N} D_R(x_{j-1/2},t^{m+1},u)\frac{\partial u_I}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_I^{m+1})_j \\
&\quad - \left(D_{R^-}^{m,\mu}\delta_x^- R_h u_I^{m+1}, \delta_x^- E_I^{m+1}\right)_{h^*},
\end{aligned}
$$

$$
T_8 = \left(F_{LR}^{m+1} + F_{NLR}^{m,\mu}, E_R^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_R(x,t^{m+1},u)\,dx(E_R^{m+1})_j,
$$

$$
T_9 = \left(F_{LI}^{m+1} + F_{NLI}^{m,\mu}, E_I^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_I(x,t^{m+1},u)\,dx(E_I^{m+1})_j.
$$

In what follows we will obtain estimates for the terms in the right hand side of (31). Lets start by estimating T_1.

First note that $T_1 = (T_{1a} + T_{1b})/2$ with

$$
\begin{aligned}
T_{1a} &= \sum_{j=0}^{N-1}\left[\frac{h}{2}\left(\frac{\partial u_R}{\partial t}(x_j,t^{m+1}) + \frac{\partial u_R}{\partial t}(x_{j+1},t^{m+1})\right)\right. \\
&\quad \left. - \int_{x_j}^{x_{j+1}} \frac{\partial u_R}{\partial t}(x,t^{m+1})dx\right]\left((E_R^{m+1})_{j+1} + (E_R^{m+1})_j\right)
\end{aligned}
$$

and

$$\begin{aligned}
T_{1b} \;=\; & \sum_{j=0}^{N-1} \left[\frac{h}{2} \left(\frac{\partial u_R}{\partial t}(x_{j+1}, t^{m+1}) - \frac{\partial u_R}{\partial t}(x_j, t^{m+1}) \right) \right. \\
& \left. + \int_{x_j}^{x_{j+1/2}} \frac{\partial u_R}{\partial t}(x, t^{m+1}) dx - \int_{x_{j+1/2}}^{x_{j+1}} \frac{\partial u_R}{\partial t}(x, t^{m+1}) dx \right] \\
& \times \left((E_R^{m+1})_{j+1} - (E_R^{m+1})_j \right).
\end{aligned}$$

For T_{1a} we have, from Lemma 1,

$$\begin{aligned}
& \left| \frac{h}{2} \left(\frac{\partial u_R}{\partial t}(x_j, t^{m+1}) + \frac{\partial u_R}{\partial t}(x_{j+1}, t^{m+1}) \right) - \int_{x_j}^{x_{j+1}} \frac{\partial u_R}{\partial t}(x, t^{m+1}) dx \right| \\
& \quad \leq C h^{5/2} \left\| \frac{\partial u_R}{\partial t}(t^{m+1}) \right\|_{H^2(I_j)}
\end{aligned}$$

and so, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|T_{1a}| \;\leq\; & C \left(\sum_{j=0}^{N-1} \left(h^2 \left\| \frac{\partial u_R}{\partial t}(t^{m+1}) \right\|_{H^2(I_j)} \right)^2 \right)^{1/2} \\
& \times \left(\sum_{j=0}^{N-1} \left(h^{1/2} ((E_R^{m+1})_{j+1} + (E_R^{m+1})_j) \right)^2 \right)^{1/2} \\
\leq\; & C \left(\sum_{j=0}^{N-1} h^4 \left\| \frac{\partial u_R}{\partial t}(t^{m+1}) \right\|_{H^2(I_j)}^2 \right)^{1/2} \| E_R^{m+1} \|_h.
\end{aligned}$$

For T_{1b} we have, from Lemma 2 and the Cauchy-Schwarz inequality

$$\begin{aligned} |T_{1b}| &\leq C\left(\sum_{j=0}^{N-1}\left(h^{3/2}\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|_{H^1(I_j)}\right)^2\right)^{1/2} \\ &\quad \times\left(\sum_{j=0}^{N-1}\left((E_R^{m+1})_{j+1}-(E_R^{m+1})_j\right)^2\right)^{1/2} \\ &\leq C\left(\sum_{j=0}^{N-1}h^4\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|^2_{H^1(I_j)}\right)^{1/2}\|\delta_x^- E_R^{m+1}\|_{h^*}. \end{aligned}$$

Consequently

$$\begin{aligned} |T_1| &\leq C\left(\sum_{j=0}^{N-1}h^4\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|^2_{H^2(I_j)}\right)^{1/2}\|E_R^{m+1}\|_h \\ &\quad +C\left(\sum_{j=0}^{N-1}h^4\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|^2_{H^1(I_j)}\right)^{1/2}\|\delta_x^- E_R^{m+1}\|_{h^*}, \end{aligned}$$

and then using the inequality $ab \leq \varepsilon a^2 + \dfrac{1}{4\varepsilon}b^2$ for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we get

$$\begin{aligned} |T_1| &\leq \frac{C}{4}h^4\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|^2_{H^2(\Omega)} + \|E_R^{m+1}\|_h^2 \\ &\quad +\frac{C}{4\epsilon}h^4\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|^2_{H^1(\Omega)} + \epsilon\|\delta_x^- E_R^{m+1}\|^2_{h^*}, \end{aligned} \tag{32}$$

where ϵ is an arbitrary positive constant.

Likewise we obtain an analogous estimate for T_2.

In order to estimate T_3 we start by noticing that

$$
\begin{aligned}
|\left(\rho_{u_R,m}, E_R^{m+1}\right)_h| \;\leq\; & \Bigg|\sum_{j=0}^{N-1} \frac{h}{2}\left(\rho_{u_R,m}(x_j)(E_R^{m+1})_j + \rho_{u_R,m}(x_{j+1})(E_R^{m+1})_{j+1}\right) \\
& - \sum_{j=0}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_{u_R,m}(x)dx(E_R^{m+1})_j\Bigg| \\
& + \left|\sum_{j=0}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_{u_R,m}(x)dx(E_R^{m+1})_j\right|.
\end{aligned}
$$

From Lemma 2 and the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
& \Bigg|\sum_{j=0}^{N-1} \frac{h}{2}\left(\rho_{u_R,m}(x_j)(E_R^{m+1})_j + \rho_{u_R,m}(x_{j+1})(E_R^{m+1})_{j+1}\right) \\
& \qquad - \sum_{j=0}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_{u_R,m}(x)dx(E_R^{m+1})_j\Bigg| \\
& \leq C\left(\sum_{j=0}^{N-1} h^2\|\rho_{u_R,m}\|^2_{H^1(I_j)}\right)^{1/2} \|E_R^{m+1}\|_h.
\end{aligned}
$$

By the Cauchy-Schwarz inequality we get

$$
\left|\sum_{j=0}^{N} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_{u_R,m}(x)dx(E_R^{m+1})_j\right| \leq \|\rho_{u_R,m}\|_{L^2(\Omega)}\|E_R^{m+1}\|_h.
$$

Then, for T_3 we have

$$
\begin{aligned}
|T_3| \;\leq\; & C\Delta t^m\left(\|\rho_{u_R,m}\|_{H^1(\Omega)}\|E_R^{m+1}\|_h + \|\rho_{u_I,m}\|_{H^1(\Omega))}\|E_I^{m+1}\|_h\right), \\
\leq\; & C\frac{(\Delta t^m)^2}{4}\left\|\frac{\partial^2 u_R}{\partial t^2}\right\|^2_{L^\infty(t^m,t^{m+1};H^1(\Omega))} + \|E_R^{m+1}\|_h^2 \\
& + C\frac{(\Delta t^m)^2}{4}\left\|\frac{\partial^2 u_I}{\partial t^2}\right\|^2_{L^\infty(t^m,t^{m+1};H^1(\Omega))} + \|E_I^{m+1}\|_h^2. \qquad (33)
\end{aligned}
$$

We write T_4 in the form

$$
|T_4| = |T_{4a} + T_{4b} + T_{4c} + T_{4d}|,
$$

with

$$T_{4a} = h\sum_{j=1}^{N}\big(D_R(x_{j-1/2},t^{m+1},u(x_{j-1/2},t^{m+1})) - D_R(x_{j-1/2},t^{m+1},u(x_{j-1/2},t^{m+\mu}))\big) \times\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j,$$

$$\begin{aligned} T_{4b} &= h\sum_{j=1}^{N}\left[\left(D_R(x_{j-1/2},t^{m+1},u^\mu) - \frac{D_R(x_{j-1},t^{m+1},u^\mu)+D_R(x_j,t^{m+1},u^\mu)}{2}\right)\right. \\ &\left.\times\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j\right], \end{aligned}$$

$$\begin{aligned} T_{4c} &= h\sum_{j=1}^{N}\left[\left(\frac{D_R(x_{j-1},t^{m+1},u^\mu)+D_R(x_j,t^{m+1},u^\mu)}{2} - D_{Rj^-}^{m,\mu}\right)\right. \\ &\left.\times\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j\right] \end{aligned}$$

and

$$\begin{aligned} T_{4d} &= h\sum_{j=1}^{N} D_{Rj^-}^{m,\mu}\frac{\partial u_R}{\partial x}(x_{j-1/2},t^{m+1})(\delta_x^- E_R^{m+1})_j \\ &- \left(D_{R^-}^{m,\mu}\delta_x^- R_h u_R^{m+1}, \delta_x^- E_R^{m+1}\right)_{h^*}, \end{aligned}$$

where we used the notation

$$u^\mu(x,t^{m+1}) = u(x,t^{m+\mu}). \tag{34}$$

In the case $\mu = 1$,

$$T_{4a} = 0$$

and, in the case $\mu = 0$, by (15) and (19),

$$\begin{aligned}
|T_{4a}| &\leq C_D \Delta t^m \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))} \|u_R(t^{m+1})\|_{W^{1,\infty}(\Omega)} h \sum_{j=1}^{N} (\delta_x^- E_R^{m+1})_j \\
&\leq C \Delta t^m \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))} \|u_R(t^{m+1})\|_{W^{1,\infty}(\Omega)} \|\delta_x^- E_R^{m+1}\|_{h^*} \\
&\leq C \frac{(\Delta t^m)^2}{4\epsilon} \left\| \frac{\partial u}{\partial t} \right\|^2_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))} \|u_R(t^{m+1})\|^2_{W^{1,\infty}(\Omega)} + \epsilon \|\delta_x^- E_R^{m+1}\|^2_{h^*}.
\end{aligned}$$

From the Sobolev imbedding theorem ([1], Theorem 5.4), the norm $\|u_R(t)\|_{W^{1,\infty}(\Omega)}$ is bounded provided that $u_R(t) \in H^2(\Omega)$.

In order to estimate T_{4b} we start by noticing that, by Lemma 3, holds

$$\left| D_R(x_{j-1/2}, t^{m+1}, u^\mu) - \frac{D_R(x_{j-1}, t^{m+1}, u^\mu) + D_R(x_j, t^{m+1}, u^\mu)}{2} \right| \\
\leq C h^{3/2} \|D_R(., t^{m+1}, u(., t^{m+\mu}))\|_{H^2(I_{j-1})},$$

and, by the Cauchy-Schwarz inequality, follows the estimate

$$|T_{4b}| \leq C h^2 \|u_R(t^{m+1})\|_{W^{1,\infty}(\Omega)} \|D_R(., t^{m+1}, u(., t^{m+\mu}))\|_{H^2(\Omega)} \|\delta_x^- E_R^{m+1}\|_{h^*}.$$

So we obtain the bound

$$|T_{4b}| \leq C \frac{h^4}{4\epsilon} \|u_R(t^{m+1})\|^2_{W^{1,\infty}(\Omega)} \|D_R(., t^{m+1}, u(., t^{m+\mu}))\|^2_{H^2(\Omega)} + \epsilon \|\delta_x^- E_R^{m+1}\|^2_{h^*}.$$

Using (15) we obtain

$$|T_{4c}| \leq \frac{C}{4\epsilon} \|u_R(t^{m+1})\|^2_{W^{1,\infty}(\Omega)} \|E_R^m\|_h^2 + \epsilon \|\delta_x^- E_R^{m+1}\|^2_{h^*}.$$

By (3) and Lemma 4 we deduce that, for $r \in \{1, 2\}$,

$$\begin{aligned}
|T_{4d}| &\leq C \left(\sum_{j=0}^{N-1} h^{2r} \|u_R(t^{m+1})\|^2_{H^{1+r}(I_j)} \right)^{1/2} \|\delta_x^- E_R^{m+1}\|_{h^*} \\
&\leq C \frac{h^{2r}}{4\epsilon} \|u_R(t^{m+1})\|^2_{H^{1+r}(\Omega)} + \epsilon \|\delta_x^- E_R^{m+1}\|^2_{h^*}.
\end{aligned}$$

The estimates for T_5, T_6 and T_7 are obtained in an analogous way.

We write T_8 in the form

$$|T_8| = |T_{8a} + T_{8b}|,$$

with

$$T_{8a} = \left(F_{LR}^{m+1}, E_R^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_{LR}(x, t^{m+1}, u)\, dx (E_R^{m+1})_j,$$

$$T_{8b} \;=\; \left(F_{NLR}^{m,\mu}, E_R^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_{NLR}(x, t^{m+1}, u)\, dx (E_R^{m+1})_j.$$

Using (9), we have

$$\begin{aligned}
&\left|\left(F_{LR}^{m+1}, E_R^{m+1}\right)_h - \left(R_h F_{LR}(., t^{m+1}, u), E_R^{m+1}\right)_h\right| \\
&\quad = \left|\left(R_h A_R(., t^{m+1}) E_R^{m+1} - R_h A_I(., t^{m+1}) E_I^{m+1}, E_R^{m+1}\right)_h\right| \\
&\quad \le \frac{1}{4} A_{max}^2 \|E^{m+1}\|_h^2 + \|E_R^{m+1}\|_h^2. \qquad (35)
\end{aligned}$$

Using the same type of analysis as for T_1 we get

$$\begin{aligned}
&\left|\left(R_h F_{LR}(., t^{m+1}, u), E_R^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_{LR}(x, t^{m+1}, u)\, dx (E_R^{m+1})_j\right| \\
&\qquad \le \frac{C}{4} h^4 \left\|F_{LR}(., t^{m+1}, u(., t^{m+1}))\right\|_{H^2(\Omega)}^2 + \|E_R^{m+1}\|_h^2 \\
&\qquad + \frac{C}{4\epsilon} h^4 \left\|F_{LR}(., t^{m+1}, u(., t^{m+1}))\right\|_{H^1(\Omega)}^2 + \epsilon \|\delta_x^- E_R^{m+1}\|_{h^*}^2. \qquad (36)
\end{aligned}$$

From (35) and (36) follows

$$\begin{aligned}
|T_{8a}| \;\le\; & \frac{1}{4} A_{max}^2 \|E^{m+1}\|_h^2 + 2\|E_R^{m+1}\|_h^2 \\
& + \frac{C}{4} h^4 \left\|F_{LR}(., t^{m+1}, u(., t^{m+1}))\right\|_{H^2(\Omega)}^2 \\
& + \frac{C}{4\epsilon} h^4 \left\|F_{LR}(., t^{m+1}, u(., t^{m+1}))\right\|_{H^1(\Omega)}^2 + \epsilon \|\delta_x^- E_R^{m+1}\|_{h^*}^2.
\end{aligned}$$

Considering the notation (34), we write T_{8b} in the form

$$\begin{aligned} T_{8b} &= \left(F_{NLR}^{m,\mu}, E_R^{m+1}\right)_h - \left(R_h F_{NLR}(.,t^{m+1},u^\mu), E_R^{m+1}\right)_h \\ &+ \left(R_h F_{NLR}(.,t^{m+1},u^\mu), E_R^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_{NLR}(x,t^{m+1},u^\mu)\, dx (E_R^{m+1})_j \\ &+ \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} \left(F_{NLR}(x,t^{m+1},u^\mu) - F_{NLR}(x,t^{m+1},u)\right)\, dx (E_R^{m+1})_j. \end{aligned}$$

Using (16), we have

$$\begin{aligned} \left|\left(F_{NLR}^{m,\mu}, E_R^{m+1}\right)_h - \left(R_h F_{NLR}(.,t^{m+1},u^\mu), E_R^{m+1}\right)_h\right| &\le C_F \|E^{m+\mu}\|_h \|E_R^{m+1}\|_h \\ &\le \frac{C_F^2}{2}\|E^{m+\mu}\|_h^2 + \frac{1}{2}\|E_R^{m+1}\|_h^2. \end{aligned} \tag{37}$$

Likewise (36) we obtain

$$\begin{aligned} &\left|\left(R_h F_{NLR}(.,t^{m+1},u^\mu), E_R^{m+1}\right)_h - \sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} F_{NLR}(x,t^{m+1},u^\mu)\, dx (E_R^{m+1})_j\right| \\ &\qquad \le \frac{C}{4} h^4 \left\|F_{NLR}(.,t^{m+1},u^\mu)\right\|_{H^2(\Omega)}^2 + \|E_R^{m+1}\|_h^2 \\ &\qquad + \frac{C}{4\epsilon} h^4 \left\|F_{NLR}(.,t^{m+1},u^\mu)\right\|_{H^1(\Omega)}^2 + \epsilon \|\delta_x^- E_R^{m+1}\|_{h^*}^2. \end{aligned} \tag{38}$$

In the case $\mu = 1$,

$$\sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} \left(F_{NLR}(x,t^{m+1},u^\mu) - F_{NLR}(x,t^{m+1},u)\right)\, dx (E_R^{m+1})_j = 0.$$

and, in the case $\mu = 0$, by (16) and (19),

$$\begin{aligned} &\sum_{j=0}^{N} \int_{x_j-1/2}^{x_j+1/2} \left(F_{NLR}(x,t^{m+1},u^\mu) - F_{NLR}(x,t^{m+1},u)\right)\, dx (E_R^{m+1})_j \\ &\qquad \le C_F \Delta t^m \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))} \|E_R^{m+1}\|_h \\ &\qquad \le \frac{C_F^2}{2} (\Delta t^m)^2 \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))}^2 + \frac{1}{2}\|E_R^{m+1}\|_h^2. \end{aligned} \tag{39}$$

From (37), (38) and (39) follows

$$\begin{aligned}|T_{8b}| \;\leq\; & \frac{C_F^2}{2}\|E^{m+\mu}\|_h^2 + 2\|E_R^{m+1}\|_h^2 + \frac{C}{4}h^4 \left\|F_{NLR}(.,t^{m+1},u^{\mu})\right\|_{H^2(\Omega)}^2 \\ & + \frac{C}{4\epsilon}h^4 \left\|F_{NLR}(.,t^{m+1},u\mu)\right\|_{H^1(\Omega)}^2 + \epsilon\|\delta_x^- E_R^{m+1}\|_{h^*}^2 \\ & + \frac{C_F^2}{2}(\Delta t^m)^2 \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))}^2 .\end{aligned}$$

For T_9 we use the same type of analysis as for T_8.

The convergence result stated in Theorem 1 follows from (31) using (2) and the estimates for $T_1, \ldots, T_9$. In fact

$$\begin{aligned}& \|E^M\|_h^2 + 2(\xi \min_{0\leq m\leq M-1} \Delta t^m - 11\epsilon\Delta t) \sum_{m=0}^{M-1} \|\delta_x^- E^{m+1}\|_{h^*}^2 \\ & \leq 2\Delta t \sum_{m=0}^{M-1} \left(\frac{C^2}{4}h^4 \left\|\frac{\partial u}{\partial t}(t^{m+1})\right\|_{H^2(\Omega)}^2 + \frac{C}{4\epsilon}h^4 \left\|\frac{\partial u}{\partial t}(t^{m+1})\right\|_{H^1(\Omega)}^2 \right. \\ & \quad + 2\|E^{m+1}\|_h^2 + C\frac{(\Delta t^m)^2}{4\epsilon} \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^\infty(t^m,t^{m+1};H^1(\Omega))}^2 \\ & \quad + C\frac{(\Delta t^m)^2}{4\epsilon} \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))}^2 \|u(t^{m+1})\|_{W^{1,\infty}(\Omega)}^2 \\ & \quad + C\frac{h^4}{4\epsilon}\|u(t^{m+1})\|_{W^{1,\infty}(\Omega)}^2 \|D(.,t^{m+1},u\mu)\|_{H^2(\Omega)}^2 \\ & \quad + \frac{C}{4\epsilon}\|u(t^{m+1})\|_{W^{1,\infty}(\Omega)}^2 \|E^{m+\mu}\|_h^2 + C\frac{h^{2r}}{4\epsilon}\|u(t^{m+1})\|_{H^{1+r}(\Omega)}^2 \\ & \quad + \frac{1}{2}A_{max}^2\|E^{m+1}\|_h^2 + 4\|E_R^{m+1}\|_h^2 \\ & \quad + \frac{C}{4}h^4 \left\|F_L(.,t^{m+1},u)\right\|_{H^2(\Omega)}^2 + \frac{C}{4\epsilon}h^4 \left\|F_L(.,t^{m+1},u)\right\|_{H^1(\Omega)}^2 \\ & \quad + C_F^2\|E^{m+\mu}\|_h^2 + \frac{C}{4}h^4 \left\|F_{NL}(.,t^{m+1},u^{\mu})\right\|_{H^2(\Omega)}^2 \\ & \quad \left. + \frac{C}{4\epsilon}h^4 \left\|F_{NL}(.,t^{m+1},u^{\mu})\right\|_{H^1(\Omega)}^2 + C_F^2(\Delta t^m)^2 \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(t^m,t^{m+1};L^\infty(\Omega))}^2 \right) .\end{aligned} \tag{40}$$

So, for Δt sufficiently small, we apply the discrete version of Gronwall's lemma (see e.g. [9], [20]) to (40) to obtain the convergence estimate (17).

Remark 1. *In the case of Dirichlet boundary conditions we obtain a similar convergence result using the discrete L^2 inner product*

$$(U, V)_h = \sum_{j=1}^{N-1} hU_j\overline{V}_j.$$

Remark 2. *Under the hypothesis of Theorem 1, if we relax the regularity condition $u \in L^\infty(0, T;\ H^2(\Omega))$ instead of $u \in L^\infty(0, T;\ H^3(\Omega))$, from (40) we get the estimate*

$$\|R_h u - U\|_h \leq \mathcal{O}(h) + \mathcal{O}(\Delta t).$$

3.2. Bi-dimensional case. In this section we consider the two-dimensional case and we obtain analogous convergence estimates. For the proof we follow some arguments taken from [10] and [13] instead of [4] and [5].

Let us consider the numerical method (10)–(11), with $d = 2$, assuming (as for the 1D-case) Neumann boundary conditions. In order to clarify the presentation, we only give detail for the case $\mu = 0$ (semi-implicit Euler method).

We start by introducing some notation related with the space domain.

Let $\bar{\Omega}_h = \Omega_h \cup \Gamma_h \cup \vartheta$, where $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\Gamma_h = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$,

$$\Gamma_1 = \left\{x_j \in \bar{\Omega}_h : x_j = (a_1, a_2 + j_2h_2), j_2 = 1, ..., N_2 - 1\right\},$$

$$\Gamma_2 = \left\{x_j \in \bar{\Omega}_h : x_j = (a_1 + j_1h_1, b_2), j_1 = 1, ..., N_1 - 1\right\},$$

$$\Gamma_3 = \left\{x_j \in \bar{\Omega}_h : x_j = (b_1, a_2 + j_2h_2), j_2 = 1, ..., N_2 - 1\right\},$$

$$\Gamma_4 = \left\{x_j \in \bar{\Omega}_h : x_j = (a_1 + j_1h_1, a_2), j_1 = 1, ..., N_1 - 1\right\},$$

and

$$\vartheta = \{\vartheta_1 = (a_1, a_2), \vartheta_2 = (a_1, b_2), \vartheta_3 = (b_1, b_2), \vartheta_4 = (b_1, a_2)\}.$$

Figure 1 illustrates grid in the domain.

For the 2D case, the discrete L^2 inner products are

$$(U, V)_h \ = \ \sum_{x_j \in \Omega_h} h_1h_2U_j\overline{V}_j + \sum_{x_j \in \Gamma_h} \frac{h_1h_2}{2}U_j\overline{V}_j + \sum_{x_j \in \vartheta} \frac{h_1h_2}{4}U_j\overline{V}_j,$$

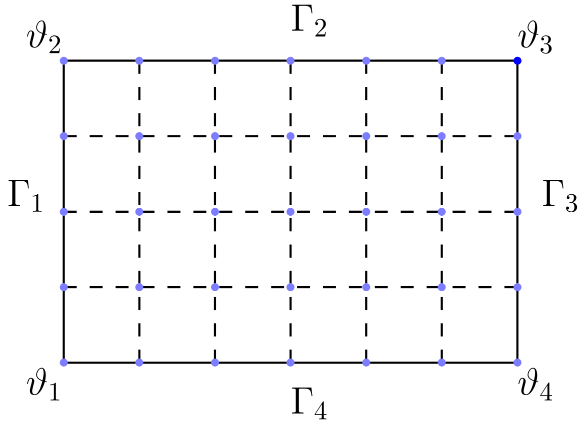

FIGURE 1. Domain $\Omega = (a_1, b_1) \times (a_2, b_2)$ and mesh Ω_h.

$$(U, V)_{h_1^*} = \sum_{x_j \in \Omega_h \cup \Gamma_3} h_1 h_2 U_j \overline{V}_j + \sum_{x_j \in \Gamma_2 \cup \Gamma_4 \cup \vartheta_3 \cup \vartheta_4} \frac{h_1 h_2}{2} U_j \overline{V}_j$$

and

$$(U, V)_{h_2^*} = \sum_{x_j \in \Omega_h \cup \Gamma_2} h_1 h_2 U_j \overline{V}_j + \sum_{x_j \in \Gamma_1 \cup \Gamma_3 \cup \vartheta_2 \cup \vartheta_3} \frac{h_1 h_2}{2} U_j \overline{V}_j,$$

and their correspondent norms are denoted by $\|.\|_h$, $\|.\|_{h_1^*}$ and $\|.\|_{h_2^*}$, respectively.

Using the same notation as in the previous section, the analogous of (24)–(25) become

$$\begin{aligned}
&\left(\frac{E_R^{m+1} - E_R^m}{\Delta t^m}, E_R^{m+1}\right)_h + \sum_{k=1}^{2} \|(D_{R^-}^{m+1})^{1/2} \delta_k^- E_R^{m+1}\|_{h_k^*}^2 = \\
&\qquad \left(R_h \frac{\partial u_R}{\partial t}(t^{m+1}), E_R^{m+1}\right)_h + \Delta t^m \left(\rho_{u_R, m}, E_R^{m+1}\right)_h \\
&\qquad + \sum_{k=1}^{2} \left(D_{R^-}^{m+1} \delta_k^- R_h u_R^{m+1}, \delta_k^- E_R^{m+1}\right)_{h_k^*} - \sum_{k=1}^{2} \left(D_{I^-}^{m+1} \delta_k^- R_h u_I^{m+1}, \delta_k^- E_R^{m+1}\right)_{h_k^*} \\
&\qquad + \sum_{k=1}^{2} \left(D_{I^-}^{m+1} \delta_k^- E_I^{m+1}, \delta_k^- E_R^{m+1}\right)_{h_k^*} - \left(F_R^{m,\mu}, E_R^{m+1}\right)_h \qquad (41)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(\frac{E_I^{m+1}-E_I^m}{\Delta t^m}, E_I^{m+1}\right)_h + \sum_{k=1}^{2} \|(D_{R^-}^{m+1})^{1/2}\delta_k^- E_I^{m+1}\|_{h_k^*}^2 = \\
&\quad \left(R_h\frac{\partial u_I}{\partial t}(t^{m+1}), E_I^{m+1}\right)_h + \Delta t^m \left(\rho_{u_I,m}, E_I^{m+1}\right)_h \\
&\quad + \sum_{k=1}^{2}\left(D_{I^-}^{m+1}\delta_k^- R_h u_R^{m+1}, \delta_k^- E_I^{m+1}\right)_{h_k^*} - \sum_{k=1}^{2}\left(D_{R^-}^{m+1}\delta_k^- R_h u_I^{m+1}, \delta_k^- E_I^{m+1}\right)_{h_k^*} \\
&\quad + \sum_{k=1}^{2}\left(D_{I^-}^{m+1}\delta_k^- E_R^{m+1}, \delta_k^- E_I^{m+1}\right)_{h_k^*} - \left(F_I^{m,\mu}, E_I^{m+1}\right)_h . \qquad (42)
\end{aligned}
$$

With the same arguments as before, we may conclude that

$$
\begin{aligned}
&2\min\Delta t^m \sum_{m=0}^{M-1}\sum_{k=1}^{2}\left(\|(D_{R^-}^{m+1})^{1/2}\delta_k^- E_R^{m+1}\|_{h_k^*}^2 + \|(D_{R^-}^{m+1})^{1/2}\delta_k^- E_I^{m+1}\|_{h_k^*}^2\right) \\
&\quad +\|E_R^M\|_h^2 + \|E_I^M\|_h^2 \le 2\Delta t \sum_{m=0}^{M-1}(|T_1|+|T_2|+|T_3|+|T_4|+|T_5| \\
&\qquad\qquad +|T_6|+|T_7|+|T_8|+|T_9|), \quad (43)
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 \;&=\; \left(R_h\frac{\partial u_R}{\partial t}(t^{m+1}), E_R^{m+1}\right)_h \\
&\quad - \sum_{x_j\in\bar{\Omega}_h}\int_{x_{j-(1/2)e_1}}^{x_{j+(1/2)e_1}}\int_{x_{j-(1/2)e_2}}^{x_{j+(1/2)e_2}} \frac{\partial u_R}{\partial t}(x, t^{m+1})\,dx(E_R^{m+1})_j,
\end{aligned}
$$

$$
\begin{aligned}
T_4 = h_1 \sum_{x_j\in\bar{\Omega}_h\setminus(\Gamma_1\cup\vartheta_1\cup\vartheta_2)} \Big(\int_{x_{j_2-1/2}}^{x_{j_2+1/2}} D_R(x_{j_1-1/2}, x_2, t^{m+1}, u) \frac{\partial u_R}{\partial x_1}(x_{j_1-1/2}, x_2, t^{m+1})\, dx_2 \\
\times (\delta_1^- E_R^{m+1})_j \Big) \\
+ h_2 \sum_{x_j\in\bar{\Omega}_h\setminus(\Gamma_4\cup\vartheta_1\cup\vartheta_4)} \Big(\int_{x_{j_1-1/2}}^{x_{j_1+1/2}} D_R(x_1, x_{j_2-1/2}, t^{m+1}, u) \frac{\partial u_R}{\partial x_2}(x_1, x_{j_2-1/2}, t^{m+1})\, dx_1 \\
\times (\delta_2^- E_R^{m+1})_j \Big) \\
- \Big(D^{m+1,0}_{R^-(1/2)e_1} \delta_1^- R_h u_R^{m+1}, \delta_1^- E_R^{m+1} \Big)_{h_1^*} - \Big(D^{m+1,0}_{R^-(1/2)e_2} \delta_2^- R_h u_R^{m+1}, \delta_2^- E_R^{m+1} \Big)_{h_2^*},
\end{aligned}
$$

and the expressions for $T_2, T_3, T_5 - T_9$ are obtained in the same way as for the 1D case. Here $x = (x_1, x_2)$ and $x_j = (x_{j_1}, x_{j_2})$.

We start by deriving an estimate for T_1. Let $P_1 = (x_{j_1}, x_{j_2})$, $P_2 = (x_{j_1+1}, x_{j_2})$, $P_3 = (x_{j_1+1}, x_{j_2+1})$, $P_4 = (x_{j_1}, x_{j_2+1})$. We will consider the contribution of each rectangle $\square_j = (x_{j_1}, x_{j_1+1}) \times (x_{j_2}, x_{j_2+1})$ which we subdivide in four congruent subrectangles R_1, R_2, R_3, R_4, such that P_i is the common vertex of the region $\square_j$ and R_i, $i = 1, \dots 4$, respectively. The contribution of the region $\square_j$ to T_1, which we represent by $T_1(\square_j)$, is

$$
\begin{aligned}
T_1(\square_j) &= \frac{h_1 h_2}{4} \sum_{i=1}^{4} \frac{\partial u_R}{\partial t}(P_i, t^{m+1}) \left(E_R^{m+1}\right)_{P_i} \\
&\quad - \sum_{i=1}^{4} \int_{R_i} \frac{\partial u_R}{\partial t}(x, t^{m+1})\, dx (E_R^{m+1})_{P_i},
\end{aligned}
$$

where $(E_R^{m+1})_{P_i}$ denotes $E_R(P_i, t^{m+1})$.

Next we will use the following equality

$$
\begin{aligned}
4\sum_{i=1}^{4} c_i d_i &= \sum_{i=1}^{4} c_i \sum_{i=1}^{4} d_i + (c_1 + c_2 - c_3 - c_4)(d_1 + d_2 - d_3 - d_4) \\
&\quad + (c_1 - c_2 + c_3 - c_4)(d_1 - d_2 + d_3 - d_4) \\
&\quad + (c_1 - c_2 - c_3 + c_4)(d_1 - d_2 - d_3 + d_4),
\end{aligned}
$$

with $c_i = \dfrac{h_1 h_2}{4} \dfrac{\partial u_R}{\partial t}(P_i, t^{m+1}) - \displaystyle\int_{\square_i} \frac{\partial u_R}{\partial t}(x, t^{m+1})\, dx$ and $d_i = (E_R^{m+1})_{P_i}$.

We apply this equality to $4T_1(\square_j)$ and study the behavior of the four resulting sums $T_{1a}(\square_j)$, $T_{1b}(\square_j)$, $T_{1c}(\square_j)$ and $T_{1d}(\square_j)$.

Using the Lemma 5, we obtain

$$\left|\frac{h_1h_2}{4}\sum_{i=1}^{4}\frac{\partial u_R}{\partial t}(P_i,t^{m+1})-\int_{\square_j}\frac{\partial u_R}{\partial t}(x,t^{m+1})\,dx\right| \\ \leq C(h_1^2+h_2^2)\max_{s_1+s_2=2}\left\|\frac{\partial^3 u_R}{\partial t\partial x_1^{s_1}\partial x_2^{s_2}}(t^{m+1})\right\|_{L^1(\square_j)},$$

where $s_1, s_2 \in \{0,1,2\}$, and then

$$|T_{1a}(\square_j)| \leq C(h_1^2+h_2^2)\max_{s_1+s_2=2}\left\|\frac{\partial^3 u_R}{\partial t\partial x_1^{s_1}\partial x_2^{s_2}}(t^{m+1})\right\|_{L^1(\square_j)}\sum_{i=1}^{4}|(E_R^{m+1})_{P_i}|.$$

We can write $T_{1b}(\square_j)$ in the form

$$(c_1+c_2-c_3-c_4)(d_1+d_2-d_3-d_4)= \\ (c_1+c_2-c_3-c_4)h_2(-(\delta_2^- E_R^{m+1})_{P_4}-(\delta_2^- E_R^{m+1})_{P_3}),$$

and we obtain

$$|T_{1b}(\square_j)| \leq |c_1+c_2-c_3-c_4|h_2\left(|(\delta_2^- E_R^{m+1})_{P_4}|+|(\delta_2^- E_R^{m+1})_{P_3})|\right).$$

Using Lemma 6, we get

$$|c_i| \leq C(h_1+h_2)\max_{s=1,2}\left\|\frac{\partial^2 u_R}{\partial t\partial x_s}(t^{m+1})\right\|_{L^1(\square_j)}, \quad i=1,2,3,4,$$

and then

$$\begin{aligned}|T_{1b}(\square_j)| \;&\leq\; C(h_1^2+h_2^2)\max_{s=1,2}\left\|\frac{\partial^2 u_R}{\partial t\partial x_s}(t^{m+1})\right\|_{L^1(\square_j)} \\ &\quad\times\left(|(\delta_2^- E_R^{m+1})_{P_4}|+|(\delta_2^- E_R^{m+1})_{P_3})|\right).\end{aligned}$$

The other sums, $T_{1c}(\square_j)$ and $T_{1d}(\square_j)$, can be bounded in the same way as $T_{1b}(\square_j)$.

Summing the contribution of all the rectangles in the domain, we obtain

$$\begin{aligned}|T_1| \;&\leq\; C(h_1^2+h_2^2)\left(\sum_{\square_j\subset\bar\Omega}\left\|\frac{\partial u_R}{\partial t}(t^{m+1})\right\|_{H^2(\square_j)}^2\right)^{1/2} \\ &\quad\times\left(\|E_R^{m+1}\|_h+\|\delta_1^- E_R^{m+1}\|_{h_1^*}+\|\delta_2^- E_R^{m+1}\|_{h_2^*}\right).\end{aligned}$$

Let us now obtain an estimate for T_4. As for the 1D case, we split T_4 in several terms

$$|T_4| = |T_{4_a} + T_{4b_1} + T_{4b_2} + T_{4c_1} + T_{4dc_2} + T_{4d_1} + T_{4d_2}|,$$

where T_{4_a} has the natural correspondence to the same quantity in the 1D case, T_{4b_1} and T_{4b_2} have natural correspondence to T_{4b} in the 1D case with respect to the space variables x_1 and x_2, respectively. Analogously, we define T_{4c_1}, T_{4dc_2}, T_{4d_1} and T_{4d_2}. For most of this terms the analysis is very similar to the 1D case. As before we can use the Sobolev imbedding theorem ([1]) to conclude that the norm $\|u(t)\|_{W^{1,\infty}(\Omega)}$ is bounded. In order to use this argument we need the assumption $u(t) \in H^3(\Omega)$.

We will only derive in detail the bound for T_{4d_1},

$$\begin{aligned}
T_{4d_1} &= h_1 \sum_{x_j \in \bar{\Omega}_h \setminus (\Gamma_1 \cup \vartheta_1 \cup \vartheta_2)} \Big(\int_{x_{j_2-1/2}}^{x_{j_2+1/2}} D^{m+1,0}_{Rj^-(1/2)e_1} \frac{\partial u_R}{\partial x_1}(x_{j_1-1/2}, x_2, t^{m+1})\, dx_2 (\delta_1^- E_R^{m+1})_j \Big) \\
&\quad - \Big(D^{m+1,0}_{R^-} \delta_1^- R_h u_R^{m+1}, \delta_1^- E_R^{m+1} \Big)_{h_1^*}.
\end{aligned}$$

In order to estimate T_{4d_1} we consider $T_{4d_1} = T_{4d_{1a}} + T_{4d_{1b}}$ with

$$\begin{aligned}
T_{4d_{1a}} = h_1 &\sum_{x_j \in \bar{\Omega}_h \setminus (\Gamma_1 \cup \vartheta_1 \cup \vartheta_2)} \Big(\int_{x_{j_2-1/2}}^{x_{j_2+1/2}} D^{m+1,0}_{Rj^-(1/2)e_1} \frac{\partial u_R}{\partial x_1}(x_{j_1-1/2}, x_2, t^{m+1})\, dx_2 (\delta_1^- E_R^{m+1})_j \Big) \\
&- h_1 \sum_{x_j \in \bar{\Omega}_h \setminus (\Gamma_1 \cup \vartheta_1 \cup \vartheta_2)} \Big(\int_{x_{j_2-1/2}}^{x_{j_2+1/2}} D^{m+1,0}_{Rj^-(1/2)e_1} \delta_1^- u_R^{m+1}(x_{j_1}, x_2)\, dx_2 (\delta_1^- E_R^{m+1})_j \Big),
\end{aligned}$$

and

$$\begin{aligned}
T_{4d_{1b}} &= h_1 \sum_{x_j \in \bar{\Omega}_h \setminus (\Gamma_1 \cup \vartheta_1 \cup \vartheta_2)} \Big(\int_{x_{j_2-1/2}}^{x_{j_2+1/2}} D^{m+1,0}_{Rj^-(1/2)e_1} \delta_1^- u_R^{m+1}(x_{j_1}, x_2)\, dx_2 (\delta_1^- E_R^{m+1})_j \Big) \\
&\quad - \Big(D^{m+1,0}_{R^-(1/2)e_1} \delta_1^- u_R^{m+1}, \delta_1^- E_R^{m+1} \Big)_{h_1^*}.
\end{aligned}$$

Applying Lemma 4 we obtain

$$\left| \frac{\partial u_R}{\partial x_1}(x_{j_1-1/2}, x_2, t^{m+1}) - \delta_1^- u_R^{m+1}(x_{j_1}, x_2) \right| \le C h_1 \left\| \frac{\partial^3 u_R}{\partial x_1^3}(., x_2, t^{m+1}) \right\|_{L^1(I_j)}.$$

Summing all the terms, applying the Cauchy-Schwarz inequality and considering (3) we obtain

$$|T_{4d_{1a}}| \;\le\; C h_1^2 \|u_R(t^{m+1})\|_{H^3(\Omega)} \|\delta_1^- E_R^{m+1}\|_{h_1^*}.$$

and then

$$|T_{4d_{1a}}| \ \leq \ C\frac{h_1^4}{4\epsilon}\|u_R(t^{m+1})\|^2_{H^3(\Omega)} + \epsilon\|\delta_1^- E_R^{m+1}\|^2_{h_1^*}.$$

Let us now estimate $T_{4d_{1b}}$. We start by noticing that

$$h_1\delta_1^- u_R^{m+1}(x_{j_1}, x_2) = \int_{x_{j_1-1}}^{x_{j_1}} \frac{\partial u_R}{\partial x_1}(x_1, x_2, t^{m+1})\, dx_1,$$

and then

$$\begin{aligned} T_{4d_{1b}} \ = \ & \sum_{x_j\in\bar{\Omega}_h\backslash(\Gamma_1\cup\vartheta_1\cup\vartheta_2)} \Big(\int_{x_{j_1-1}}^{x_{j_1}}\int_{x_{j_2-1/2}}^{x_{j_2+1/2}} D^{m+1,0}_{Rj^-(1/2)e_1}\frac{\partial u_R}{\partial x_1}(x_1, x_2, t^{m+1})\, dx_1\, dx_2 \\ & \times(\delta_1^- E_R^{m+1})_j\Big) \\ & - \sum_{x_j\in\Omega_h\cup\Gamma_3}\Big(\int_{x_{j_1-1}}^{x_{j_1}} h_2 D^{m+1,0}_{Rj^-(1/2)e_1}\frac{\partial u_R}{\partial x_1}(x_1, x_{j_2}, t^{m+1})\, dx_1(\delta_1^- E_R^{m+1})_j\Big) \\ & - \sum_{x_j\in\Gamma_2\cup\Gamma_4\cup\vartheta_3\cup\vartheta_4}\Big(\int_{x_{j_1-1}}^{x_{j_1}} \frac{h_2}{2} D^{m+1,0}_{Rj^-(1/2)e_1}\frac{\partial u_R}{\partial x_1}(x_1, x_{j_2}, t^{m+1})\, dx_1(\delta_1^- E_R^{m+1})_j\Big). \end{aligned}$$

In the same way as for T_1 in the previous section (see (32)) and taking (3) into account, we have

$$\begin{aligned} |T_{4d_{1b}}| \ \leq \ & C\frac{h_2^4}{4}\|u_R(t^{m+1})\|^2_{H^3(\Omega)} + \|E_R^{m+1}\|^2_h \\ & + C\frac{h_2^4}{\epsilon}\|u_R(t^{m+1})\|^2_{H^2(\Omega)} + \epsilon\|\delta_2^- E_R^{m+1}\|^2_{h_2^*}. \end{aligned}$$

Considering all the contributions, we obtain the proof of Theorem 1 for the 2D case.

4. Numerical results

In this section, we will illustrate the theoretical results for convergence for the semi-implicit method (that is, $m = 1$ and $\mu = 0$), for both the Dirichlet and Neumann boundary conditions.

4.1. Dirichlet case. Let us consider the equation

$$\frac{\partial u}{\partial t} = \nabla\cdot(D\nabla u) + f, \quad x_1, x_2 \in (0,\pi)\times(0,\pi), t\in(0,1],$$

with initial and Dirichlet boundary conditions given, respectively, by

$$u(x_1, x_2, 0) = \sin(x_1)\sin(x_2)$$

and

$$u(0, x_2, t) = u(\pi, x_2, t) = u(x_1, 0, t) = u(x_1, \pi, t) = 0.$$

Given two constants $A, B \in \mathbb{C}$, for

$$\begin{aligned} f(x_1, x_2, t) =&(A + 2B)\sin(x_1)\sin(x_2)e^{At} + \\ &\left[2\sin^2(x_1)\sin^2(x_2) - \cos^2(x_1)\sin^2(x_2) - \sin^2(x_1)\cos^2(x_2)\right] e^{2At} \end{aligned}$$

and

$$D(x_1, x_2, t, u) = B + u,$$

the exact solution is given by

$$u(x_1, x_2, t) = \sin(x_1)\sin(x_2)e^{At}.$$

For the following, we will consider

$$A = -2 - 2i, \quad B = 1 + 1i.$$

4.1.1. *Order 1 in time.* In this section we will consider constant spatial spacements $h_1 = h_2$ and step in time Δt. Moreover, we will successively half the spatial spacements h_1, h_2 and step in time Δt in order to illustrate the linear numerical order of convergence in time.

One gets the approximations $U^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$ for

$$u(\pi/2, \pi/2, T) = -0.05632 - 0.12306i$$

on the central point $(\pi/2, \pi/2)$ of the spatial domain at the final time $T = 1$ given in Table 1. We note that

$$E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}} = u(\pi/2, \pi/2, T) - U^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}.$$

Moreover, the order of convergence p can be approximated by

$$p \approx \log_2(|E_n|/|E_{n+1}|), \tag{44}$$

where E_n and E_{n+1} are the errors considering

$$\Delta t = \frac{1}{2^n}, \quad h_1 = h_2 = \frac{\pi}{2^{n+1}}, \quad n = 0, 1, \ldots$$

TABLE 1. Aproximation, error and numerical estimate on the order of convergence p for the Dirichlet Case, obtained by halfing the step in time and the spatial spacement.

$h_1 = h_2$	Δt	$U^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$	$E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$	$\left\|E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}\right\|$	p
$\pi/2$	1	0.23565-0.10324i	0.29197+0.01982i	0.29264	0.79414
$\pi/4$	1/2	0.11239-0.11884i	0.16871+0.00422i	0.16876	0.76446
$\pi/8$	1/4	0.04300-0.12548i	0.09932-0.00242i	0.09935	0.83094
$\pi/16$	1/8	-0.00056-0.12628i	0.05576-0.00322i	0.05585	0.89736
$\pi/32$	1/16	-0.02642-0.12535i	0.02990-0.00229i	0.02998	0.94333
$\pi/64$	1/32	-0.04078-0.12440i	0.01553-0.00134i	0.01559	0.97023
$\pi/128$	1/64	-0.04839-0.12378i	0.00793-0.00072i	0.00796	0.98474
$\pi/256$	1/128	-0.05232-0.12343i	0.00400-0.00037i	0.00402	-

TABLE 2. L^2-discrete norm of the error and numerical estimate on the order of convergence p for the Dirichlet Case, obtained by halfing the step in time and the spatial spacement.

$h_1 = h_2$	Δt	$\|u(.,.,T) - U^M\|_h$	p
$\pi/2$	1	0.45968	0.78001
$\pi/4$	1/2	0.26770	0.81094
$\pi/8$	1/4	0.15259	0.85691
$\pi/16$	1/8	0.08425	0.90866
$\pi/32$	1/16	0.04488	0.94786
$\pi/64$	1/32	0.02327	0.97208
$\pi/128$	1/64	0.01186	0.98554
$\pi/256$	1/128	0.00599	-

Similar results are obtained for the numerical convergence using the L^2 discrete norm of the error $\|u(.,.,T) - U^M\|_h$, as presented in Table 2.

As expected, the numerical orders of convergence tend to 1.

4.1.2. *Order 2 in space.* In this section we will again consider constant spatial spacements $h_1 = h_2$ and step in time Δt. Moreover, we will successively half the spatial spacements h_1, h_2 while we will successively divide by 4 the step in time Δt in order to illustrate the quadratic numerical order of convergence in space.

The results are shown in Table 3 for pointwise convergence and in Table 4 for the error measured with the L^2 discrete norm. Note that for the numerical approximation of the order of convergence p in (44), the error E_n is obtained

TABLE 3. Aproximation, error and numerical estimate on the order of convergence p for the Dirichlet Case, obtained by halfing the spatial spacement and dividing by 4 the step in time.

$h_1 = h_2$	Δt	$U^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$	$E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$	$\left\|E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}\right\|$	p
$\pi/2$	1	0.05703-0.16807i	0.11335-0.04501i	0.12196	1.62845
$\pi/4$	1/4	-0.02090-0.14042i	0.03542-0.01736i	0.03945	1.89815
$\pi/8$	1/16	-0.04714-0.12833i	0.00918-0.00527i	0.01058	1.97857
$\pi/16$	1/64	-0.05402-0.12444i	0.00230-0.00138i	0.00269	1.99486
$\pi/32$	1/256	-0.05574-0.12341i	0.00058-0.00035i	0.00067	1.99873
$\pi/64$	1/1024	-0.05618-0.12315i	0.00014-0.00009i	0.00017	-

TABLE 4. L^2 discrete norm of the error and numerical estimate on the order of convergence p for the Dirichlet Case, obtained by halfing the spatial spacement and dividing by 4 the step in time.

$h_1 = h_2$	Δt	$\|u(.,.,T) - U^M\|_h$	p
$\pi/2$	1	0.19157	1.65693
$\pi/4$	1/4	0.06075	1.92837
$\pi/8$	1/16	0.01596	1.98124
$\pi/16$	1/64	0.00404	1.99522
$\pi/32$	1/256	0.00101	1.99880
$\pi/64$	1/1024	0.00025	-

considering

$$\Delta t = \frac{1}{2^{2n}}, \quad h_1 = h_2 = \frac{\pi}{2^{n+1}}, \quad n = 0, 1, \ldots$$

As expected, the numerical order of convergence tends to 2.

4.2. Neumann case. Let us consider the equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D\nabla u) + f, \quad x_1, x_2 \in (0,\pi) \times (0,\pi), t \in (0,1],$$

with initial and Neumann boundary conditions given, respectively, by

$$u(x_1, x_2, 0) = \cos(x_1)\cos(x_2)$$

and

$$\frac{\partial u}{\partial \nu}(0, x_2, t) = \frac{\partial u}{\partial \nu}(\pi, x_2, t) = \frac{\partial u}{\partial \nu}(x_1, 0, t) = \frac{\partial u}{\partial \nu}(x_1, \pi, t) = 0.$$

TABLE 5. Aproximation, error and numerical estimate on the order of convergence p for the Neumann Case, obtained by halfing the step in time and the spatial spacement.

$h_1 = h_2$	Δt	$E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$	$\left\lVert E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}} \right\rVert$	p
$\pi/2$	1	-0.01196-0.01235i	0.01719	-0.06301
$\pi/4$	1/2	0.00019-0.01796i	0.01796	0.43921
$\pi/8$	1/4	-0.00303-0.01289i	0.01324	0.69251
$\pi/16$	1/8	-0.00423-0.00702i	0.00820	0.79918
$\pi/32$	1/16	-0.00329-0.00337i	0.00471	0.87788
$\pi/64$	1/32	-0.00203-0.00157i	0.00256	0.93158
$\pi/128$	1/64	-0.00112-0.00074i	0.00134	0.96365
$\pi/256$	1/128	-0.00059-0.00036i	0.00069	-

Again, given two constants $A, B \in \mathbb{C}$, for

$$\begin{aligned} f(x_1, x_2, t) =&(A + 2B)\cos(x_1)\cos(x_2)e^{At} + \\ &\left[2\cos^2(x_1)\cos^2(x_2) - \sin^2(x_1)\cos^2(x_2) - \cos^2(x_1)\sin^2(x_2)\right] e^{2At} \end{aligned}$$

and

$$D(x_1, x_2, t, u) = B + u,$$

the exact solution is given by

$$u(x_1, x_2, t) = \cos(x_1)\cos(x_2)e^{At}.$$

Again, we will consider

$$A = -2 - 2i, \quad B = 1 + 1i.$$

4.2.1. *Order 1 in time.* In this section we will consider constant spatial spacements $h_1 = h_2$ and step in time Δt. Moreover, we will half the spatial spacements h_1, h_2 and step in time Δt in order to illustrate the linear numerical order of convergence in time.

The results are shown in Table 5 for pointwise convergence. Similar results are obtained for the numerical convergence using the discrete norm, as presented in Table 6.

As expected, the numerical orders of convergence converge to 1.

TABLE 6. L^2 discrete norm of the error and numerical estimate on the order of convergence p for the Neumann Case, obtained by halfing the step in time and the spatial spacement.

$h_1 = h_2$	Δt	$\|u(.,.,T) - U^M\|_h$	p
$\pi/2$	1/1	1.01585	0.41741
$\pi/4$	1/2	0.76064	0.75393
$\pi/8$	1/4	0.45105	0.87441
$\pi/16$	1/8	0.24604	0.92853
$\pi/32$	1/16	0.12927	0.95927
$\pi/64$	1/32	0.06648	0.97777
$\pi/128$	1/64	0.03376	0.98835
$\pi/256$	1/128	0.01702	-

TABLE 7. Aproximation, error and numerical estimate on the order of convergence p for the Neumann Case, obtained by halfing the spatial spacement and dividing by 4 the step in time.

$h_1 = h_2$	Δt	$E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}$	$\left\|E^M_{\frac{N_1+1}{2},\frac{N_2+1}{2}}\right\|$	p
$\pi/2$	1	-0.00724-0.01385i	0.01563	0.73198
$\pi/4$	1/4	-0.00477-0.00811i	0.00941	1.66505
$\pi/8$	1/16	-0.00216-0.00203i	0.00297	1.93101
$\pi/16$	1/64	-0.00061-0.00049i	0.00078	1.98676
$\pi/32$	1/256	-0.00015-0.00012i	0.00020	1.99801
$\pi/64$	1/1024	-0.00004-0.00003i	0.00005	-

4.2.2. *Order 2 in space.* In this section we will consider constant spatial spacements $h_1 = h_2$ and step in time Δt. Moreover, we will half the spatial spacements h_1, h_2 while we will divid by 4 the step in time Δt in order to illustrate the quadratic numerical order of convergence in space.

The results are shown in Table 7 for pointwise convergence and in Table 8 for the discrete norm.

As expected, the numerical order of convergence tends to 2.

Appendix A. Technical lemmata

The following lemmata are technical tools needed to derive the convergence estimates. They are a consequence of the Bramble-Hilbert Lemma (see *e.g.* [8]).

TABLE 8. L^2 discrete norm of the error and numerical estimate on the order of convergence p for the Neumann Case, obtained by halfing the spatial spacement and dividing by 4 the step in time.

$h_1 = h_2$	Δt	$\|u(.,.,T) - U^M\|_h$	p
$\pi/2$	1	0.80983	1.43702
$\pi/4$	1/4	0.29909	1.84755
$\pi/8$	1/16	0.08311	1.96560
$\pi/16$	1/64	0.02128	1.99216
$\pi/32$	1/256	0.00535	1.99788
$\pi/64$	1/1024	0.00134	-

A.1. One-dimensional case.

Lemma 1. *For $v \in H^r(I_j)$, $r \in \{1,2\}$, the following estimates hold*

$$\left| \frac{h}{2}\left(v(x_j) + v(x_{j+1})\right) - \int_{x_j}^{x_{j+1}} v(x)dx \right| \leq C h^{r+1/2} \|v^{(r)}\|_{L^2(I_j)} \tag{45}$$

and

$$\left| h v(x_{j+1/2}) - \int_{x_j}^{x_{j+1}} v(x)dx \right| \leq C h^{r+1/2} \|v^{(r)}\|_{L^2(I_j)}. \tag{46}$$

Proof: Let the function w be defined by

$$w(\xi) = v(x_j + \xi h), \quad \xi \in [0,1]. \tag{47}$$

Then

$$\begin{aligned} \left| \frac{h}{2}\big(v(x_j) + v(x_{j+1})\big) - \int_{x_j}^{x_{j+1}} v(x)\,dx \right| &= \left| h\Big(\frac{w(0)+w(1)}{2} - \int_0^1 w(\xi)\,d\xi\Big) \right| \\ &= h|\lambda(w)|, \end{aligned}$$

with

$$\lambda(g) = \frac{g(0)+g(1)}{2} - \int_0^1 g(\xi)\,d\xi, \qquad g \in W^{r,1}((0,1)).$$

This functional is bounded in $W^{r,1}((0,1))$ and vanish for g being a polynomial of degree less or equal to 1. Thus, the Bramble-Hilbert Lemma gives the existence of a positive constant C such that

$$\lambda(g) \leq C|g|_{W^{r,1}((0,1))} = C\|g^{(r)}\|_{L^1((0,1))}.$$

Taking $g = w$ and using the Cauchy-Schwarz inequality

$$h|\lambda(w)| \leq Ch\|w^{(r)}\|_{L^1((0,1))} \leq Ch^r\|v^{(r)}\|_{L^1(I_j)} \leq Ch^{r+1/2}\|v^{(r)}\|_{L^2(I_j)}$$

and (45) holds. For (46) the proof is analogous with

$$\lambda(g) = g\left(\frac{1}{2}\right) - \int_0^1 g(\xi)\, d\xi.$$

■

Lemma 2. *For $v \in H^1(I_j)$ the following estimates hold*

$$\left|\frac{h}{2}v(x_j) - \int_{x_j}^{x_{j+1/2}} v(x)dx\right| \leq Ch^{3/2}\|v'\|_{L^2((x_j,x_{j+1/2}))}, \tag{48}$$

and

$$\left|\frac{h}{2}v(x_j) - \int_{x_{j-1/2}}^{x_j} v(x)dx\right| \leq Ch^{3/2}\|v'\|_{L^2((x_{j-1/2},x_j))}. \tag{49}$$

Proof: Let

$$\lambda_1(g) = \frac{1}{2}g(0) - \int_0^{1/2} g(\xi)\, d\xi, \quad g \in W^{1,1}((0,1)),$$

and

$$\lambda_2(g) = \frac{1}{2}g(1) - \int_{1/2}^1 g(\xi)\, d\xi, \quad g \in W^{1,1}((0,1)).$$

Both functionals are bounded in $W^{1,1}((0,1))$ and vanish for polynomials of degree zero. Taking $g = w$, with w defined by (47), by the Bramble-Hilbert Lemma there exists of a positive constant C such that (48) and (49) hold. ■

Lemma 3. *For $v \in H^r(I_j)$, $r \in \{1,2\}$, the following estimates hold*

$$\left|\frac{v(x_j) + v(x_{j+1})}{2} - v(x_{j+1/2})\right| \leq Ch^{r-1/2}\|v^{(r)}\|_{L^2(I_j)}. \tag{50}$$

Proof: Let the function w be defined by (47). Then

$$\left|\frac{v(x_j) + v(x_{j+1})}{2} - v(x_{j+1/2})\right| = \left|\frac{w(0) + w(1)}{2} - w\left(\frac{1}{2}\right)\right| = |\lambda(w)|,$$

with

$$\lambda(g) = \frac{g(0) + g(1)}{2} - g\left(\frac{1}{2}\right), \qquad g \in W^{r,1}((0,1)).$$

This functional is bounded in $W^{r,1}((0,1))$ and vanish for polynomials of degree less or equal to 1. Thus, the Bramble-Hilbert Lemma gives the existence of a positive constant C such that

$$\lambda(g) \leq C|g|_{W^{r,1}((0,1))} = C\|g^{(r)}\|_{L^1((0,1))}.$$

Taking $g = w$ and using the Cauchy-Schwarz inequality

$$|\lambda(w)| \leq C\|w^{(r)}\|_{L^1((0,1))} \leq Ch^{r-1}\|v^{(r)}\|_{L^1(I_j)} \leq Ch^{r-1/2}\|v^{(r)}\|_{L^2(I_j)}.$$

■

Lemma 4. *For $v \in H^{1+r}(I_j)$, $r \in \{1,2\}$, the following estimates holds*

$$\left|v'(x_{j-1/2}) - \delta_x^- v(x_j)\right| \leq Ch^{r-1}\|v^{(r+1)}\|_{L^1(I_j)} \leq Ch^{r-1/2}\|v^{(r+1)}\|_{L^2(I_j)}. \quad (51)$$

Proof: Let w be defined by (47). Then

$$v'(x_{j-1/2}) - \delta_x^- v(x_j) = \frac{1}{h}\left[w'\left(\frac{1}{2}\right) - w(1) + w(0)\right] = \frac{1}{h}\lambda(w), \quad (52)$$

with

$$\lambda(g) = g'\left(\frac{1}{2}\right) - g(1) + g(0), \quad g \in W^{r,1}((0,1)).$$

This functional is bounded in $W^{s,1}((0,1))$, with $s \in \{2,3\}$, and vanish for polynomials of degree less or equal to 2. The Bramble-Hilbert Lemma gives the existence of a positive constant C such that

$$|\lambda(g)| \leq C|g|_{W^{r,1}((0,1))} = C\|g^{(r)}\|_{L^1((0,1))}.$$

Taking $g = w$, and using Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\lambda(w)| &\leq C\|w^{(s)}\|_{L^1((0,1))} = Ch^{s-1}\int_{x_j}^{x_{j+1}} |v^{(s)}(x)|\, dx \\ &\leq Ch^{s-1/2}\left(\int_{x_j}^{x_{j+1}} |v^{(s)}(x)|^2\, dx\right)^{1/2}. \end{aligned}$$

From (52),

$$\left|v'(x_{j-1/2}) - \delta_x^- v(x_j)\right| \leq Ch^{r-1}\|v^{(r+1)}\|_{L^1(I_j)} \leq Ch^{r-1/2}\|v^{(r+1)}\|_{L^2(I_j)}.$$

■

The lemmas above were taken from [4] and [5].

A.2. Bi-dimensional case. Let $\square_j = (x_{j_1}, x_{j_1+1}) \times (x_{j_2}, x_{j_2+1})$, $P_1 = (x_{j_1}, x_{j_2})$, $P_2 = (x_{j_1+1}, x_{j_2})$, $P_3 = (x_{j_1+1}, x_{j_2+1})$ and $P_4 = (x_{j_1}, x_{j_2+1})$.

Lemma 5. *For $v \in H^2(\square_j)$, the following estimate holds*

$$\left| \frac{h_1 h_2}{4} \sum_{i=1}^{4} v(P_i) - \int_{\square_j} v(x)\, dx \right| \leq C(h_1^2 + h_2^2) \max_{s_1+s_2=2} \left\| \frac{\partial^2 v}{\partial x_1^{s_1} \partial x_2^{s_2}} \right\|_{L^1(\square_j)}, \quad (53)$$

$s_1, s_2 \in \{0, 1, 2\}$.

Proof: Let the function w be defined by

$$w(\xi, \eta) = v(x_{j_1} + \xi h_1, x_{j_2} + \eta h_2), \quad (\xi, \eta) \in [0,1] \times [0,1]. \quad (54)$$

Then

$$\frac{h_1 h_2}{4} \sum_{i=1}^{4} v(P_i) - \int_{\square_j} v(x)\, dx = h_1 h_2 \lambda(w)$$

with

$$\lambda(g) = \frac{g(0,0) + g(1,0) + g(0,1) + g(1,1)}{4} - \int_0^1 \int_0^1 g(\xi, \eta)\, d\xi\, d\eta,$$

$g \in W^{2,1}((0,1) \times (0,1))$. This functional is bounded in $W^{2,1}((0,1) \times (0,1))$ and vanishes for polynomials (in ξ and η) of degree 1. By Bramble-Hilbert Lemma the estimate

$$|\lambda(g)| \leq C |g|_{W^{2,1}((0,1)\times(0,1))}$$

holds and we obtain the bound (53). ∎

Lemma 6. *For $v \in H^1(\square_j)$, the following estimate holds*

$$\left| h_1 h_2\, v(P_i) - \int_{\square_j} v(x)\, dx \right| \leq C(h_1 + h_2) \max_{s=1,2} \left\| \frac{\partial v}{\partial x_s} \right\|_{L^1(\square_j)}, \quad (55)$$

$i = 1, 2, 3, 4$.

Proof: Let the function w be defined by (54). Then

$$h_1 h_2\, v(P_1) - \int_{\square_j} v(x)\, dx = h_1 h_2 \lambda(w)$$

with

$$\lambda(g) = g(0,0) - \int_0^1 \int_0^1 g(\xi, \eta)\, d\xi\, d\eta,$$

$g \in W^{1,1}((0,1)\times(0,1))$. This functional is bounded and vanishes for polynomials of degree zero. By Bramble-Hilbert Lemma we obtain the estimate

$$|\lambda(g)| \leq C|g|_{W^{1,1}((0,1)\times(0,1))}$$

which leads to the bound (55).

The proof using the points P_2, P_3 and P_4 follows the same steps. ■

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, Boston, MA, 1975.

[2] A. Araújo, S. Barbeiro, and P. Serranho. Stability of finite difference schemes for complex diffusion processes. *SIAM J. Numer. Anal.*, 50 (3):1284–1296, 2012.

[3] A. Araújo, S. Barbeiro, and P. Serranho. Finite difference schemes for nonlinear complex reaction-diffusion processes: stability analysis. *DMUC Report 13-29*, 2013.

[4] S. Barbeiro. *Normas Duais Discretas em Problemas Elípticos, Ph.D. Thesis.* Universidade de Coimbra, 2005.

[5] S. Barbeiro, J. Ferreira, and R. Grigorieff. Supraconvergence of a finite difference scheme for solutions in $H^s(0,L)$. *IMA J. Numer. Anal.*, 25:797–811, 2005.

[6] R. Bernardes, C. Maduro, P. Serranho, A. Araújo, S. Barbeiro, and J. Cunha-Vaz. Improved adaptive complex diffusion despeckling filter. *Optics Express*, 18 (23):24048–24059, 2010.

[7] T. Brox, A. Bruhn, N. Papenberg, and J. Weickert. *Computer Vision - ECCV 2004. Lecture Notes in Computer Science*, volume 3024, chapter High accuracy optical flow estimation based on a theory for warping, pages 25–36. Springer, Berlin, 2004.

[8] T. Dupont and R. Scott. Polynomial approximation of functions in sobolev spaces. *Math. Comp.*, 34 (2):441–463, 1980.

[9] E. Emmrich. Stability and error of the variable two-step bdf for semilinear parabolic problems. *J. Appl. Math. Comput.*, 19 (1):33–35, 2005.

[10] E. Emmrich and R. D. Grigorieff. Supraconvergence of a finite difference scheme for elliptic third kind boundary value problems in fractional order sobolev spaces. *Comput. Methods Appl. Math.*, 6 (2):154–177, 2006.

[11] L. C. Evans. *Partial Differential Equations.* Graduate Studies in Mathematics, 19. American Mathematics Society, Providence, RI, second edition, 2010.

[12] D. Fernández, H. Salinas, and C. Puliafito. Automated detection of retinal layer structures on optical coherence tomography images. *Optics Express*, 13 (25):10200–10216, 2005.

[13] J. Ferreira and R. D. Grigorieff. Supraconvergence and supercloseness of a scheme for elliptic equations on non-uniform grids. *Numerical Functional Analysis and Optimization*, 27 (5-6):539–564, 2006.

[14] G. Gilboa, N. Sochen, and Y. Zeevi. Image enhancement and denoising by complex diffusion processes. *IEEE Trans Pattern Anal Mach Intell*, 26 (8):1020–1036, 2004.

[15] H. Grossauer and O. Scherzer. *Scale Space Methods in Computer Vision, Lecture Notes in Computer Science*, volume 2695, chapter Using the Complex Ginzburg-Landau Equation for Digital Inpainting in 2D and 3D, pages 225–236. Springer, 2003.

[16] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE Trans Pattern Anal Mach Intell*, 12 (7):629–639, 1990.

[17] H. Salinas and D. Fernández. Comparison of PDE-based nonlinear diffusion approaches for image enhancement and denoising in optical coherence tomography. *IEEE Trans. Med. Imaging*, 26 (6):761–771, 2007.

[18] J. Weickert. Anisotropic diffusion filters for image processing based quality control. *Proc. 7th Eur. Conf. Mathematics in Industry*, 1252:355–362, 1994.

[19] J. Weickert. A review of nonlinear diffusion filtering. In B. M. ter Haar Romeny, L. Florack, J. J. Koenderink, and M. A. Viergever, editors, *Scale-Space*, volume 1252 of *Lecture Notes in Computer Science*, pages 3–28. Springer, 1997.

[20] J. Wloka. *Partial differential equations.* Cambridge University Press, 1987.

[21] H. Zimmer, A. Bruhn, L. Valgaerts, M. Breuß, J. Weickert, B. Rosenhahn, and H.-P. Seidel. *Vision, Modeling, and Visualization*, chapter PDE-based anisotropic disparity-driven stereo vision, pages 263–272. AKA Heidelberg, 2008.

Adérito Araújo
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

E-mail address: alma@mat.uc.pt
URL: http://www.mat.uc.pt/~alma

Sílvia Barbeiro
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

E-mail address: silvia@mat.uc.pt
URL: http://www.mat.uc.pt/~silvia

Pedro Serranho
Mathematics Section, Department of Science and Technology, Open University, Campus Tagus- Park, Av. Dr. Jacques Delors, 2740-122 Porto Salvo, Oeiras, Portugal and IBILI, Faculty of Medicine, University of Coimbra, 3000-548 Coimbra, Portugal

E-mail address: Pedro.Serranho@uab.pt