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FUNCTION SPACES OF POLYANALYTIC FUNCTIONS

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ABSTRACT: This is both an introduction and a review of some of the recent developments on Fock and Bergman spaces of polyanalytic functions. The study of polyanalytic functions is a classic topic in complex analysis. However, thanks to the interdisciplinar transference of knowledge promoted within the activities of HCAA network, it has benefited from a cross-fertilization with ideas from signal analysis, quantum physics and random matrices. It is the main purpose of this survey to provide a brief introduction to those ideas and to describe some of the results of the mentioned cross-fertilization. We will put some emphasis on the connections to Gabor and wavelet analysis and in the applications of coorbit and localization theory. The departure point of our investigations is a thought experiment related to a classical problem in the Theory of Signals, the one of multiplexing: sending several signals simultaneously using a single channel.

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1. Introduction

1.1. Definition of a polyanalytic function. Among the most widely studied mathematical objects are the solutions of the Cauchy-Riemann equation

$$\partial_{\overline{z}} F(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right) F(x+i\xi) = 0,$$

known as *analytic functions.* The properties of analytic functions are so remarkable, that, at a first encounter, they are often perceived as "magic". However, the analiticity restriction is so strong that it created a prejudice against non-analytic functions, which are often perceived as unstructured and bad behaved objects and therefore not worthy of further study. But there are non-analytic functions with significant structure and with properties reminiscent of those satisfied by analytic functions.

Such nice non-analytic functions are called *polyanalytic functions.*

A function $F(z,\overline{z})$, defined on a subset of $\mathbb{C}$, and satisfying the generalized Cauchy-Riemann equations

$$(\partial_{\overline{z}})^n F(z,\overline{z}) = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)^n F(x+i\xi, x-i\xi) = 0, \tag{1.1}$$

is said to be *polyanalytic of order* $n-1$. A polyanalytic function $F(z,\overline{z})$ of order $n-1$ is a polynomial of order $n-1$ in $\overline{z}$, with analytic functions $\{\varphi_k(z)\}_{k=0}^{n-1}$ as coefficients:

$$F(z,\overline{z}) = \sum_{k=0}^{n-1} \overline{z}^k \varphi_k(z). \tag{1.2}$$

One can easily be convinced that some fundamental properties of analytic functions cease to be true for polyanalytic functions. For instance, a simple polyanalytic function of order 1 is

$$F(z,\overline{z}) = 1 - |z|^2 = 1 - z\overline{z}.$$

Since

$$\partial_{\overline{z}} F(z,\overline{z}) = -z \text{ and } \left(\partial_{\overline{z}}\right)^2 F(z,\overline{z}) = 0,$$

the function $F(z,\overline{z})$ is not analytic in z, but is *polyanalytic of order one*. This simple example already highlights one of the reasons why the properties of polyanalytic functions can be different of those enjoyed by analytic functions: they can vanish on closed curves without vanishing identically, while analytic functions can not even vanish on a accumulation set of the complex plane! Still, many properties of analytic functions have found an extension to polyanalytic functions, often in a nontrivial form.

1.2. What are polyanalytic functions good for? Imagine some application of analytic functions. By definition, they allow to represent the objects of our application as a function of z (because the function is analytic). We may want to represent the object to obtain a nice theory, we may want to store the information contained in the object and send it to someone. Whatever we want to do, we will always end up with a representation involving powers of z (because the functions are analytic). Not that bad, since we have an infinite number of them. However, several applications of Mathematics, like Quantum Mechanics and Signal Analysis, require infinite dimensions for their theoretical formulation. And when we build a model in the complex plane using analytic functions, all the powers of z are taken.

What if we want to build several models simultaneously for the same plane? Introducing an extra complex variable will bring us the complications related to the study of analytic functions in several complex variables. If $\mathbb{C}$ is not enough for some models, $\mathbb{C}^2$ may seem to much to handle if we want to keep the mathematical problems within a tangible range. One is tempted to ask if there is something in between, but it may seem hard to believe that it is possible to "store" more information in a complex plane without introducing an extra independent variable.

Enter the world of polyanalytic functions!

We are now allowed to use powers of z and $\overline{z}$. This introduces an enormous flexibility. Consider the Hilbert space $\mathcal{L}_2(\mathbb{C})$ of all measurable functions equiped with the norm

$$\|F\|_{\mathcal{L}_2(\mathbb{C})}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} d\mu(z). \tag{1.3}$$

It is relatively easy to observe, using integration by parts (see formula (2.3) below) that, given an analytic function $F(z) \in \mathcal{L}_2(\mathbb{C})$, the function

$$F'(z) - \pi\overline{z}F(z)$$

is orthogonal to $F(z)$. We can create several subspaces of $\mathcal{L}_2(\mathbb{C})$ by multiplying elements of the Fock space of analytic functions by a power of $\overline{z}$. If we consider the sum of all such spaces we obtain the whole $\mathcal{L}_2(\mathbb{C})$. We can do even better: by proper combination of the powers of z and $\overline{z}$ we can obtain an orthogonal decomposition of $\mathcal{L}_2(\mathbb{C})$! This fact, first observed by Vasilevski [71] is due to the following: the polynomials

$$e_{k,j}(z,\overline{z}) = e^{\pi|z|^2} (\partial_z)^k \left[e^{-\pi|z|^2} z^j\right]$$

are orthogonal in both the index j and k and they span the whole space $\mathcal{L}_2(\mathbb{C})$ of square integrable functions in the plane weighted by a gaussian $e^{-\pi|z|^2}$. For every j, we have thus a "copy" of the space of analytic functions which is orthogonal to any of the other copies. It is a remarkable fact that every polyanalytic function of order n can be expressed as a combination of the polynomials $\{e_{k,j}(z,\overline{z})\}_{k<n,j\in\mathbb{N}}$. Thus, we can work simultaneously in n planes keeping the number of degrees of freedom of each one intact. We will see in this paper how this fact can be put in good use, notably in the analysis of the higher Landau levels and in the multiplexing of signals (analysis of several signals simultaneously).

1.3. Some historical remarks. Polyanalytic functions were for the first time considered in [49] by the Russian mathematician G. V. Kolossov (1867-1935) in connection with his research on elasticity. This line of research has been developed by his student Muskhelishvili and the applications of polyanalytic functions to problems in elasticity are well documented in his book [56].

Polyanalytic Function Theory has been investigated intensively, notably by the Russian school led by Balk [15]. More recently the subject gained a renewed interest within operator theory and some interesting properties of the function spaces whose elements are polyanalytic functions have been derived [16], [17], [71], [72]. A new characterization of polyanalytic functions has been obtained by Agranovsky [9]. Our investigations in the topic were originally motivated by applications in signal analysis, in particular by the results of Gröchenig and Lyubarskii on Gabor frames with Hermite functions [43],[44] but soon it was clear that Hilbert spaces of polyanalytic functions lie at the heart of several interesting mathematical topics. Remarkably, they provide explicit representation formulas for the functions in the eigenspaces of the Euclidean Laplacian with a magnetic field, the so called *Landau levels*. The historically-conscious reader may recognize in Polyanalytic Function Theory the ecletic aroma emblematic of the Mathematics oriented to Signal Analysis and Quantum Mechanics, an aroma which was particulary notorious in the body of Mathematics which became known as the Bell papers of the 60's (see the review [67]) and, perhaps even stronger, in the advent of wavelets and coherent states [23], [10].

1.4. Outline. Our purpose is to highlight the connections between different topics and we have organized the paper as follows. We start with a section on the Hilbert space theory of polyanalytic Fock spaces. This includes the description of the "Theoretical Multiplexing Device" which is the basic signal analytic model for our viewpoint. The third section explains how the topic connects to time frequency analysis, more precisely, to the theory of Gabor frames with Hermite functions. We quote some applications in Quantum Physics in section 4, namely the interpretation of the so-called true polyanalytic Fock spaces as the eigenspaces of the Euclidean Landau Hamiltonian with a constant magnetic field. In section 5 we take a close look to the reproducing kernels and some asymptotic results recently obtained in the study of random matrices. Then we make a review of the basis facts of the L^p

theory of polyanalytic Fock spaces. The last section is devoted to hyperbolic analogues of the theory.

2. Fock spaces of polyanalytic functions

2.1. The orthogonal decomposition and the polyanalytic hierarchy. Recall that $\mathcal{L}_2(\mathbb{C})$ denotes the Hilbert space of all measurable functions equiped with the norm

$$\|F\|^2_{\mathcal{L}_2(\mathbb{C})} = \int_{\mathbb{C}} |F(z)|^2 \, e^{-\pi|z|^2} \, d\mu(z),$$

where $d\mu(z)$ stands for area measure on $\mathbb{C}$. If we require the elements of the space to be analytic, we are lead to the Fock space $\mathcal{F}_2(\mathbb{C})$. Polyanalytic Fock spaces $\mathbf{F}_2^n(\mathbb{C})$ arise in an analogous manner, by requiring its elements to be polyanalytic of order $n-1$. They seem to have been first considered by Balk [15, pag. 170] and, more recently, by Vasilevski [71], who obtained the following decompositions in terms of spaces $\mathcal{F}_2^n(\mathbb{C})$ which he called *true poly-Fock spaces*:

$$\mathbf{F}_2^n(\mathbb{C}) = \mathcal{F}_2^1(\mathbb{C}) \oplus ... \oplus \mathcal{F}_2^n(\mathbb{C}). \tag{2.1}$$

$$\mathcal{L}_2(\mathbb{C}) = \bigoplus_{n=1}^{\infty} \mathcal{F}_2^n(\mathbb{C}).$$

We will use the following definition of $\mathcal{F}_2^n(\mathbb{C})$ which is equivalent to the one given by Vasilevski: a function F belongs to the *true polyanalytic Fock space* $\mathcal{F}_2^{n+1}(\mathbb{C})$ if $\|F\|_{\mathcal{L}_2(\mathbb{C})} < \infty$ and there exists an entire function H such that

$$F(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\partial_z\right)^n \left[e^{-\pi|z|^2} H(z)\right]. \tag{2.2}$$

With this definition it is easy to verify that the spaces $\mathcal{F}_2^n(\mathbb{C})$ are orthogonal using Green´s formula:

$$\int_{\mathbf{D}_r} f(z)\partial_{\overline{z}}\overline{g(z)}dz = -\int_{\mathbf{D}_r} \partial_{\overline{z}} f(z)\overline{g(z)}dz + \frac{1}{i}\int_{\delta\mathbf{D}_r} f(z)\overline{g(z)}dz. \tag{2.3}$$

and its higher order version obtained by iterating (2.3):

$$\begin{aligned}\int_{\mathbf{D}_r} f(z)\left(\frac{d}{d\overline{z}}\right)^n\overline{g(z)}dz &= (-1)^n\int_{\mathbf{D}_r}\left(\frac{d}{d\overline{z}}\right)^n f(z)\overline{g(z)}dz \\ &+\frac{1}{i}\sum_{j=0}^{n-1}(-1)^j\int_{\delta\mathbf{D}_r}\left(\frac{d}{d\overline{z}}\right)^j f(z)\left(\frac{d}{d\overline{z}}\right)^{n-j-1}\overline{g(z)}dz\end{aligned} \tag{2.4}$$

A visual image of the polyanalytic hierarchy is the following decomposition of $\mathcal{L}_2(\mathbb{C})$ (an orthogonal decomposition in the spaces $\{\mathcal{F}_2^n(\mathbb{C})\}_{k=1}^{\infty}$ and a union of the nested spaces $\{\mathbf{F}_2^n(\mathbb{C})\}_{k=1}^{\infty}$).

$$\begin{array}{ccc} \mathcal{F}_2^1(\mathbb{C}) & =\mathbf{F}_2^1(\mathbb{C}) & =\mathcal{F}_2(\mathbb{C}) \\ \mathcal{F}_2^1(\mathbb{C})\oplus\mathcal{F}_2^2(\mathbb{C}) & =\mathbf{F}_2^2(\mathbb{C}) & \\ .. & & \\ ... & & \\ \mathcal{F}_2^1(\mathbb{C})\oplus .....\oplus\mathcal{F}_2^n(\mathbb{C}) & =\mathbf{F}_2^n(\mathbb{C}) & \\ .... & & \\ \mathcal{F}_2^1(\mathbb{C})\oplus .....\mathcal{F}_2^n(\mathbb{C})\oplus\mathcal{F}_2^{n+1}(\mathbb{C})\oplus ... & =\bigoplus_{n=1}^{\infty}\mathcal{F}_2^n(\mathbb{C}) & =\mathcal{L}_2(\mathbb{C}) \end{array}$$

2.2. Reproducing kernels of the polyanalytic Fock spaces. The reproducing kernels of the polyanalytic Fock spaces have been computed using several different methods: invariance properties of the Landau Laplacian [13], composition of unitary operators [71], Gabor transforms with Hermite functions [2], and the expansion in the kernel basis functions [34]. Nice formulas are obtained using the Laguerre polynomials

$$L_k^{\alpha}(x)=\sum_{i=0}^{k}(-1)^i\binom{k+\alpha}{k-i}\frac{x^i}{i!}.$$

The reproducing kernel of the space $\mathcal{F}_2^n(\mathbb{C})$, $\mathcal{K}^n(z,w)$, can be written as

$$\mathcal{K}^n(z,w)=\pi L_{n-1}^0(\pi\left|z-w\right|^2)e^{\pi z\overline{w}}.$$

This gives the explicit formula for the orthogonal projection P^n required at the step (5) of our theoretical multiplexing device in the next section:

$$(P^nF)(w)=\int_{\mathbb{C}}F(z)\pi L_{n-1}^0(\pi\left|z-w\right|^2)e^{\pi z(\overline{w}-\overline{z})}d\mu(z). \tag{2.5}$$

The reproducing kernel of the space $\mathbf{F}_2^n(\mathbb{C})$ is denoted by $\mathbf{K}^n(z,w)$. Using the formula $\sum_{k=0}^{n-1} L_k^{\alpha} = L_{n-1}^{\alpha+1}$, (2.1) gives

$$\mathbf{K}^n(z,w) = \pi L_{n-1}^1(\pi \left|z-w\right|^2) e^{\pi z \overline{w}}.$$

2.3. A thought experiment: multiplexing of signals. A classical problem in the Theory of Signals is the one of Multiplexing, that is, transmitting several signals over a single channel in such a way that it is possible to recover the original signal at the receiver [14]. We will use the multiplexing idea as a thought experiment providing intuition about our ideas, later to be developed rigorously. We suggest the reading of [48] for considerations regarding the role of these kind of experiences, ubiquous in Theoretical Physics, in the modern mathematical landscape.

The center of our attention is now the orthogonal decomposition (2.1). Assume that we can somehow (we will do it in the next section) construct a map $\mathcal{B}^n$ sending an arbitrary $f \in L^2(\mathbb{R})$ to the space $\mathcal{F}_2^n(\mathbb{C})$. Then we can, at least theoretically, proceed as follows.

(1) Given n signals $f_1, \ldots, f_n$, with finite energy ($f_k \in L^2(\mathbb{R})$ for every k), process each individual signal by evaluating $\mathcal{B}^k f_k$. This encodes each signal into one of the n orthogonal spaces $\mathcal{F}^1(\mathbb{C}), \ldots, \mathcal{F}^n(\mathbb{C})$.
(2) Construct a new signal $F = \mathbf{Bf} = \mathcal{B}^1 f_1 + \ldots + \mathcal{B}^n f_n$ as a superposition of the n processed signals.
(3) Sample, transmit, or process F.
(4) Let P^k denote the orthogonal projection from $\mathbf{F}_2^n(\mathbb{C})$ onto $\mathcal{F}^k(\mathbb{C})$, then $P^k(F) = \mathcal{B}^k f_k$ by virtue of (2.1).
(5) Finally, after inverting each of the transforms $\mathcal{B}^k$, we recover each component f_k in its original form.

The combination of n independent signals into a single signal $\mathbf{B}^n\mathbf{f}$ and the subsequent processing provides our multiplexing device. With two signals this can be outlined in the following scheme.

$$\begin{array}{ccccccccc}
f_1 & \to & \mathcal{B} f_1 & & & & \mathcal{B} f_1 & \to & f_1 \\
 & & & \searrow & & P^1 \nearrow & & & \\
 & & & & \mathcal{B} f_1 + \mathcal{B}^2 f_2 = \mathbf{Bf} & & & & \\
 & & & \nearrow & & P^2 \searrow & & & \\
f_2 & \to & \mathcal{B}^2 f_2 & & & & \mathcal{B}^2 f_2 & \to & f_2
\end{array}$$

We believe that the above device has pratical applications, but we will pursue another direction in our reasoning: we will use the above scheme as a source of mathematical ideas-with some poetic license, we may say that we apply signal analysis to mathematics.

2.4. The polyanalytic Bargmann transform. The construction of the map $\mathcal{B}^k$ above can be done as follows. To map the first signal $f_1 \in L^2(\mathbb{R})$ to the space $\mathcal{F}_2^1(\mathbb{C}) = \mathcal{F}_2(\mathbb{C})$ we can of course use the good old Bargmann transform $\mathcal{B}$, where

$$\mathcal{B}f(z) = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi z^2 - \frac{\pi}{2}t^2} dt.$$

The remaining signals are mapped using

$$\mathcal{B}^{k+1} f(z) = \left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\partial_z\right)^k \left[e^{-\pi|z|^2} \mathcal{B}f(z)\right]$$

It can be proved that $\mathcal{B}^k : L^2(\mathbb{R}) \to \mathcal{F}_2^k(\mathbb{C})$ is a Hilbert space isomorphism, by observing that the Hermite functions are mapped to the orthogonal basis $\{e_{k,n}(z, \overline{z}) : n \geq 0\}$ of $\mathcal{F}_2^k(\mathbb{C})$, where

$$e_{k,n}(z) = \left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\partial_z\right)^k \left[e^{-\pi|z|^2} e_n(z)\right] \tag{2.6}$$

and

$$e_n(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} z^n$$

is the orthogonal monomial basis of the Fock space.

We can now define a transform $\mathbf{B}^n : L^2(\mathbb{R}, \mathbb{C}^n) \to \mathbf{F}^n(\mathbb{C})$ by mapping each vector $\mathbf{f} = (f_1, ..., f_n) \in L^2(\mathbb{R}, \mathbb{C}^n)$ to the following polyanalytic function of order n:

$$\mathbf{B}^n \mathbf{f} = \mathcal{B}^1 f_1 + ... + \mathcal{B}^n f_n\,. \tag{2.7}$$

This map is again a Hilbert space isomorphism and is called the *polyanalytic Bargmann transform* [1].

2.5. A polyanalytic Weierstrass function. Our construction of the previous section is done at a purely theoretical level, since we have no way of soring and processing a continuous signal. However, we can construct a discrete counterpart of the theory. Following [3], an analogue of the Whittaker-Shannon-Kotel´nikov sampling theorem can be constructed using a polyanalytic version of the Weierstrass sigma function. Let σ be the Weierstrass sigma function corresponding to Λ defined by

$$\sigma_{\Lambda}(z)=z\prod_{\lambda\in\Lambda\setminus\{0\}}\left(1-\frac{z}{\lambda}\right)e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}\,,$$

To simplify our notations we will write the results in terms of the square lattice, $\Lambda=\alpha(\mathbb{Z}+i\mathbb{Z})$ consisting of the points $\lambda=\alpha l+i\alpha m$, $k,m\in\mathbb{Z}$, but most of what we will say is also true for general lattices. To write down our explicit sampling formulas, the following polyanalytic extension of the Weierstrass sigma function is required:

$$S_{\Lambda}^{n+1}(z)=\left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}}e^{\pi|z|^2}\left(\partial_z\right)^n\left[e^{-\pi|z|^2}\frac{(\sigma_{\Lambda}(z))^{n+1}}{n!z}\right].$$

Clearly, $S_{\Lambda}^{1}(z)=\sigma_{\Lambda}(z)/z$. Let $\sigma_{\Lambda^{\circ}}(z)$ be the Weierstrass sigma function associated to the adjoint lattice $\Lambda^{\circ}=\alpha^{-1}(\mathbb{Z}+i\mathbb{Z})$ of Λ and consider the corresponding polyanalytic Weierstrass function $S_{\Lambda^0}^{n}(z)$. With this terminology we have:

Theorem 1. [3]. *If $\alpha^2<\frac{1}{n+1}$, every $F\in\mathcal{F}_2^{n+1}(\mathbb{C})$ can be written as:*

$$F(z)=\sum_{\lambda\in\alpha(\mathbb{Z}+i\mathbb{Z})}F(\lambda)e^{\pi\overline{\lambda}z-\pi|\lambda|^2}S_{\Lambda^0}^{n+1}(z), \tag{2.8}$$

3. The connection to time-frequency analysis

3.1. The Gabor transform. The study of polyanalytic Fock spaces can be significantly enriched via a connection to time-frequency (Gabor) analysis. Recall that the Gabor or Short-time Fourier transform (STFT) of a function or distribution f with respect to a window function g is defined to be

$$V_gf(x,\xi)=\int_{\mathbb{R}}f(t)\overline{g(t-x)}e^{-2\pi i\xi t}dt. \tag{3.1}$$

There is a very important property enjoyed by inner products of this transforms. The following relations are usually called *the orthogonal relations for the short-time Fourier transform.* Let $f_1,f_2,g_1,g_2\in L^2(\mathbb{R})$. Then $V_{g_1}f_1,V_{g_2}f_2\in L^2(\mathbb{R}^2)$ and

$$\langle V_{g_1}f_1,V_{g_2}f_2\rangle_{L^2(\mathbb{R}^2)}=\langle f_1,f_2\rangle_{L^2(\mathbb{R})}\overline{\langle g_1,g_2\rangle_{L^2(\mathbb{R})}}. \tag{3.2}$$

If $\|g\|_{L^2(\mathbb{R})}=1$, the Gabor transform provides an isometry

$$V_g:L^2(\mathbb{R})\to L^2(\mathbb{R}^2),$$

since, if $f, g \in L^2(\mathbb{R}^d)$, then

$$\|V_g f\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} . \tag{3.3}$$

In many applications it is required that the window g is in the Feichtinger algebra S_0. This means that $V_g g \in L^1(\mathbb{R}^2)$. Given a point $\lambda = (\lambda_1, \lambda_2)$ in phase-space $\mathbb{R}^2$, the corresponding time-frequency shift is

$$\pi_\lambda f(t) = e^{2\pi i \lambda_2 t} f(t - \lambda_1), \quad t \in \mathbb{R}.$$

Using this notation, the Gabor transform of a function f with respect to the window g can be written as

$$V_g f(\lambda) = \langle f, \pi_\lambda g \rangle_{L^2(\mathbb{R})} .$$

In analogy to the time-frequency shifts π_λ, there are Bargmann-Fock shifts β_λ defined for functions on $\mathbb{C}$ by

$$\beta_\lambda F(z) = e^{\pi i \lambda_1 \lambda_2} e^{\pi \bar{\lambda} z} F(z - \lambda)\, e^{-\pi |\lambda|^2/2} . \tag{3.4}$$

We observe that the true polyanalytic Bargmann transform intertwines the time-frequency shifts π_λ and the Fock representation β_λ on $\mathcal{F}_2^n(\mathbb{C})$ by a calculation similar to [41, p. 185]:

$$\mathcal{B}^n (\pi_\lambda \gamma)(z) = \beta_\lambda \mathcal{B}^n \gamma(z) , \tag{3.5}$$

for $\gamma \in L^2(\mathbb{R})$. If we choose the Gaussian function $h_0(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$ as a window in (3.1), then a simple calculation shows that the Bargmann transform is related to these special Gabor transforms as follows:

$$\mathcal{B} f(z) = e^{-i\pi x \xi + \pi \frac{|z|^2}{2}} V_{h_0} f(x, -\xi). \tag{3.6}$$

This is a well known fact and the details of the calculation can be found in standart textbooks in time-frequency analysis, see for instance [41, pag. 53].

The key step now is the choice of higher order Hermite functions as windows in (3.1). In Figure 1 one can observe the interesting patterns of the time-frequency concentration when higher Hermite functions are used. We begin by choosing the first Hermite function $h_1(t) = 2^{\frac{1}{4}} 2\pi t e^{-\pi t^2}$. First observe that

$$V_{h_1} f(x, -\xi) = 2\pi V_{h_0}(tf)(x, -\xi) - 2\pi x V_{h_0} f(x, -\xi)$$

and that

$$2\pi \mathcal{B}(tf)(z) = \partial_z B f(z) + \pi z B f(z).$$

Thus, using (3.6)

$$e^{-i\pi x\xi+\pi\frac{|z|^2}{2}}V_{h_1}f(x,-\xi)=\partial_z Bf(z)-\pi\overline{z}Bf(z)=\mathcal{B}^2f(z).$$

With a bit more effort, we can choose the *nth Hermite function*

$$h_n(t)=c_n e^{\pi t^2}\left(\frac{d}{dt}\right)^n\left(e^{-2\pi t^2}\right),$$

where c_n is chosen so that $\|h_n\|_2=1$, as a special window in (3.1).

We find a similar relation between Gabor transforms with Hermite functions and true polyanalytic Bargmann transforms:

$$e^{-i\pi x\xi+\pi\frac{|z|^2}{2}}V_{h_n}f(x,-\xi)=\mathcal{B}^{n+1}f(z). \tag{3.7}$$

This simple observation made in [1] connects polyanalytic Fock spaces with Gabor analysis. The important fact to retain is that the multiplier $e^{-i\pi x\xi+\pi\frac{|z|^2}{2}}$ is the same for every n. This leads us to the next observation. We can define a vector valued Gabor transform

$$\mathbf{V}_{\mathbf{h}_{n-1}}\mathbf{f}(\lambda)=V_{(h_0,...,h_{n-1})}(f_0,...f_{n-1})(\lambda)$$

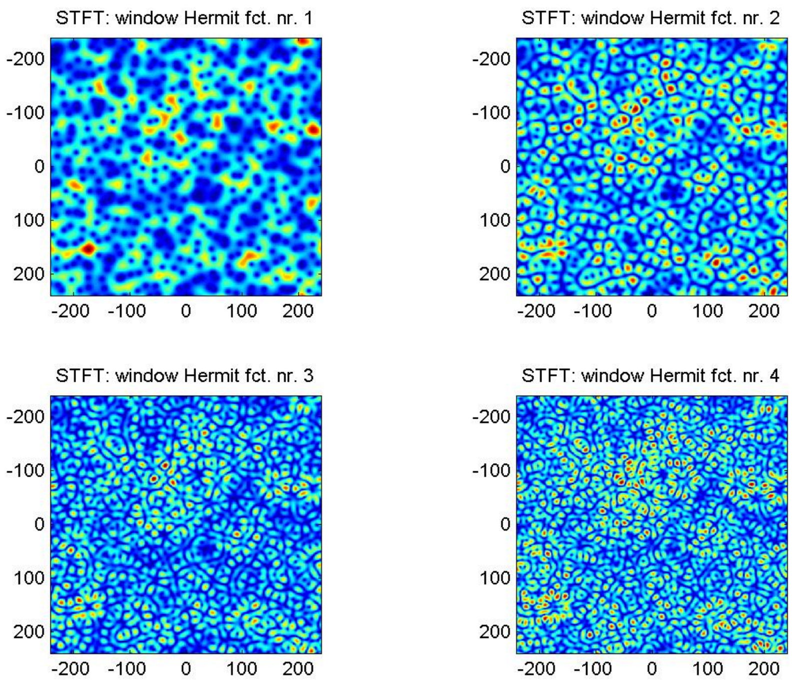
for the purpose of processing simultaneously n signals using a vectorial window constituted by the first n Hermite functions. Since the windows are orthogonal to each other, we can do this by simple superposition (we know that the transformed signals will live in mutually orthogonal function spaces because of the orthogonality conditions (3.2))

$$\mathbf{V}_{\mathbf{h}_{n-1}}\mathbf{f}(\lambda)=\sum_{k=0}^{n-1}V_{h_k}f_k(\lambda).$$

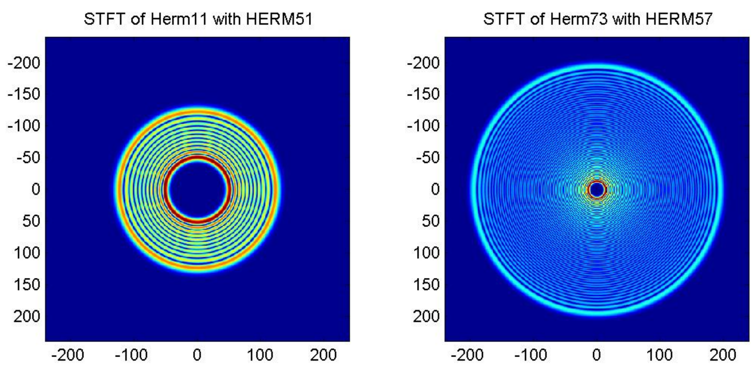
If follows now from (3.7) and from (2.7) that

$$\mathbf{B}^n\mathbf{f}=e^{-i\pi x\xi+\pi\frac{|z|^2}{2}}\mathbf{V}_{\mathbf{h}_{n-1}}\mathbf{f}(\lambda). \tag{3.8}$$

Formula (3.8) is the key for a real variable treatment of polyanalytic Fock spaces. This approach already lead to the proof of results that seemed hopeless using only complex variables. For instance, it was possible to prove that the sampling and interpolation *lattices* of $\mathbf{F}_2^n(\mathbb{C})$ can be characterized by their density [1]. Previously, this result was known only for $n=1$ and the proofs ([52] and [65]) strongly depend on tools like Jensen´s formula, which are not available in the polyanalytic case. The above connection to Gabor analysis solved the problem, by means of a remarkable duality result



(a) Phase space concentrations of a given function with the first four Hermite functions



(b) Basis elements of the Phase space associated with high order Hermite windows

Figure 1. Short-time Fourier transforms with higher order Hermite windows

from time-frequency analysis [63], which turned the polyanalytic problem into a Hermite interpolation (multi-sampling) problem in spaces of analytic functions.

On the other side, the connection between Gabor analysis and polyanalytic functions offers a new technical ammunition to time-frequency analysis. It has already been used in [24, Lemma 1] in a key step of the proof of the main result, where it is shown that a certain time-frequency localization has infinite rank.

3.2. Gabor spaces. Let $\mathcal{G}_g$ denote the subspace of $L^2(\mathbb{R}^2)$ which is the image of $L^2(\mathbb{R})$ under the Gabor transform with the window g,

$$\mathcal{G}_g = \left\{ V_g f : f \in L^2(\mathbb{R}) \right\}.$$

The spaces $\mathcal{G}_g$ are called *model spaces* in [12]. It is well known (see [23]) that Gabor spaces have a reproducing kernel given by

$$k(z, w) = \langle \pi_z g, \pi_w g \rangle_{L^2(\mathbb{R})} \tag{3.9}$$

For instance, if we consider the Gaussian window $g(x) = 2^{\frac{1}{4}} e^{-\pi t^2}$, using the notation $z = x + i\xi$ and $w = u + i\eta$, a calculation (see [41, Lemma 1.5.2]) shows that the reproducing kernel of $\mathcal{G}_g$ is

$$k^0(x, \xi, u, \eta) = e^{\pi i (u+x)(\xi - \eta) - \frac{\pi (u-x)^2 - \pi (\eta - \xi)^2}{2}}.$$

This reproducing kernel can be related with the reproducing kernel of the Fock space:

$$k^0(z, w) = e^{-i\pi(u\eta - x\xi) - \pi \frac{|z|^2 + |w|^2}{2}} e^{\pi \overline{w} z}. \tag{3.10}$$

Similar calculations can be done for $g = h_{n-1}$ and we obtain

$$k^n(z, w) = e^{-i\pi(u\eta - x\xi) - \pi \frac{|z|^2 + |w|^2}{2}} \pi L^0_{n-1}(\pi |z - w|^2) e^{\pi z \overline{w}}$$

for the reproducing kernel of $\mathcal{G}_{h_{n-1}}$. Thus, considering the operator E which maps f to Mf, where

$$M(z) = e^{\pi \frac{|z|^2}{2} - i\pi x \xi},$$

it is clear that E is an isometric isomorphism,

$$E : \mathcal{G}_{h_{n-1}} \to \mathcal{F}^n(\mathbb{C}^d).$$

A similar construction is valid for the vector valued spaces. See [2] for more details.

3.3. Gabor expansions with Hermite functions. To give a more concrete idea of what we are talking about, let us see what Theorem 1 tells about Gabor expansions, more precisely about the required size of the square lattice. From Theorem 1, if $\alpha^2 < \frac{1}{n+1}$, then every $F \in \mathcal{F}_2(\mathbb{C})$ can be written in the form (2.8). Now, applying the inverse Bargmann transform and doing some calculations involving the interwining property between the time-frequency shifts and the Fock shifts (see [3] for the details), one can see that this expansion is exactly equivalent to the Gabor expansion of an $L^2(\mathbb{R})$ function. More precisely, if $\alpha^2 < \frac{1}{n+1}$, every $f \in L^2(\mathbb{R})$ admits the following representation as a Gabor series

$$f(t) = \sum_{l,k\in\mathbb{Z}} c_{k,l} e^{2\pi i\alpha l t} h_n(t-\alpha k), \tag{3.11}$$

with

$$\|c\|_{l^2} \approx \|f\|_{L^2(\mathbb{R})}$$

Stable Gabor expansions of the form (3.11) can be obtained from frame theory. For a countable subset $\Lambda \in \mathbb{R}^2$, one says that the Gabor system $\mathcal{G}(h_n,\Lambda) = \{\pi_\lambda h_n : \lambda \in \Lambda\}$ is a *Gabor frame* or *Weyl-Heisenberg frame* in $L^2(\mathbb{R})$, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\|f\|^2_{L^2(\mathbb{R})} \leq \sum_{\lambda\in\Lambda} \left|\langle f, \pi_\lambda h_n\rangle_{L^2(\mathbb{R})}\right|^2 \leq B\|f\|^2_{L^2(\mathbb{R})}. \tag{3.12}$$

This sort of expansions have been used before for practical purposes, for instance, in image analysis [38]. Their mathematical study ([43], [33], [44], [1], [3], [51]) used a blend of ideas from signal, harmonic and complex analysis, providing this research field with a nice interdiciplinary flavour.

The problem of finding conditions on the set Λ that yield Gabor frames is known as the density of Gabor frames. See [36] for a survey on the topic and the recent paper [45] for the solution of the problem for a large class of windows.

3.4. Fock expansions. Observe that, using the Bargmann transform and the intertwining property (3.5) with $n = 0$, we can write the expansion (3.11) in the Fock space as folows. Writing $\lambda_{l,k} = \alpha k + i\alpha l$ and using (3.6) we can expand every $F \in \mathcal{F}_2(\mathbb{C})$ in the form

$$F(z) = \sum_{l,k\in\mathbb{Z}} c_{k,l} \beta_{\lambda_{l,k}} e_n(z),$$

where β_λ is the Bargmann-Fock shift (3.4) and the system $\{\beta_{\lambda_{l,k}} e_n(z)\}_{l,k\in\mathbb{Z}}$ is *a frame in the Fock space* $\mathcal{F}_2(\mathbb{C})$ of analytic functions. On the other side, if we apply the true polyanalytic Bargmann transform of order $n+1$ together with (3.5) with the same $n+1$, we can use (3.7) and expand every $F \in \mathcal{F}_2^{n+1}(\mathbb{C})$ in the form

$$F(z) = \sum_{l,k\in\mathbb{Z}} c_{k,l} \beta_{\lambda_{l,k}} e_{n,n}(z) \tag{3.13}$$

and the system $\{\beta_{\lambda_{l,k}} e_{n,n}(z)\}_{l,k\in\mathbb{Z}}$ is *a frame in the true polyanalytic space* $\mathcal{F}_2^{n+1}(\mathbb{C})$. To see that this is an expansion of a distiguished sort, recall that using (3.7) and the fact that the reproducing kernel of the Gabor space generated by h_n is given by $\langle \pi_z h_n, \pi_w h_n \rangle_{L^2(\mathbb{R})}$, one can express the reproducing kernel of the true polyanalytic space as

$$e^{-i\pi x\xi + \frac{\pi}{2}|z|^2} \langle \pi_z h_n, \pi_w h_n \rangle_{L^2(\mathbb{R})} = \mathcal{B}^{n+1}(\pi_w h_n)(z) = \beta_w e_{n,n}(z).$$

Thus, (3.13) is an expansion of reproducing kernels and, applying the reproducing formula, the frame inequality for $\{\beta_{\lambda_{l,k}} e_{n,n}(z)\}_{l,k\in\mathbb{Z}}$,

$$A \|F\|^2_{\mathcal{L}_2(\mathbb{C})} \leq \sum_{l,k\in\mathbb{Z}} \left| \left\langle F, \beta_{\lambda_{l,k}} e_{n,n}(z) \right\rangle_{L^2(\mathbb{R})} \right|^2 \leq B \|F\|^2_{\mathcal{L}_2(\mathbb{C})}$$

can be written as

$$A \|F\|^2_{\mathcal{L}_2(\mathbb{C})} \leq \sum_{l,k\in\mathbb{Z}} |F(\lambda_{l,k})|^2 \leq B \|F\|^2_{\mathcal{L}_2(\mathbb{C})}.$$

Thus, the lattice $\{\lambda_{l,k}\}_{l,k\in\mathbb{Z}}$ is a sampling sequence for the true polyanalytic Fock space. Thus, sampling in $\mathcal{F}_2^{n+1}(\mathbb{C})$ is equivalent to (analytic) Fock frames of the form $\{\beta_{\lambda_{l,k}} e_n(z)\}_{l,k\in\mathbb{Z}}$ and both are equivalent to the formulation of Gabor frames with Hermite functions. On the other side, one can construct completely different frames in $\mathcal{F}_2^{n+1}(\mathbb{C})$ by applying $\mathcal{B}^{n+1}$ to the frames $\mathcal{G}(h_m, \Lambda)$ with $m < n$. This means that we have a different representation of the frame $\mathcal{G}(h_m, \Lambda)$ in each space $\mathcal{F}_2^{n+1}(\mathbb{C})$!

An open problem

The first proof of the sufficiency of the condition $\alpha^2 < \frac{1}{n+1}$ for the expansion (3.11) is due to Gröchenig and Lyubarskii [43]. In the same paper, the authors provide some evidence to support the conjecture that the condition may even be sharp (it is known from a general result of Ramanathan and Steger [60] that $\alpha^2 < 1$ is necessary), a statement which would be surprising, since $\alpha^2 < \frac{1}{n+1}$ is exactly the sampling rate necessary and sufficient for the expansion of

n functions using the superframe (the superframe [44] is a vectorial version of frame which has be seen to be equivalent to sampling in the polyanalytic space [1]). The following problem seems to be quite hard.

Problem 1. [43] *Find the exact range of* α *such that* $\mathcal{G}\left(h_n, \alpha(\mathbb{Z}+i\mathbb{Z})\right)$ *is a frame.*

Recently, Lyubarskii and Nes [51] found that $\alpha^2 = \frac{3}{5} > \frac{1}{2}$ is a sufficient condition for the case $n = 1$. They also proved that, if $\alpha^2 = 1 - \frac{1}{j}$, no odd function in the Feichtinger algebra [29] generates a Gabor frame. In [51], supported by their results and by some numerical evidence, the authors formulated a conjecture.

Conjecture 1. [51] *If* $\alpha^2 < 1$ *and* $\alpha^2 \neq 1 - \frac{1}{j}$*, then* $\mathcal{G}\left(h_1, \alpha(\mathbb{Z}+i\mathbb{Z})\right)$ *is a frame.*

3.5. Sampling and Interpolation in $\mathbf{F}_2^n(\mathbb{C})$. We say that a set Λ is a set of sampling for $\mathbf{F}_2^n(\mathbb{C})$ if there exist constants $A, B > 0$ such that, for all $F \in \mathbf{F}_2^n(\mathbb{C})$,

$$A\left\|F\right\|^2_{\mathbf{F}_2^n(\mathbb{C})} \leq \sum_{\lambda\in\Lambda} \left|F(\lambda)\right|^2 e^{-\pi|\lambda|^2} \leq B\left\|F\right\|^2_{\mathbf{F}_2^n(\mathbb{C})}.$$

A set Λ is a set of interpolation for $\mathbf{F}_2^n(\mathbb{C})$ if for every sequence $\{a_{i(\lambda)}\} \in l^2$, we can find a function $F \in \mathbf{F}_2^n(\mathbb{C})$ such that

$$e^{i\pi\lambda_1\lambda_2 - \frac{\pi}{2}|\lambda|^2} F(\lambda) = a_{i(\lambda)},$$

for every $\lambda \in \Lambda$. The sampling and interpolation lattices of $\mathbf{F}_2^n(\mathbb{C})$ can be characterized by their density. For the square lattice the results are as follows.

Theorem 2. *The lattice* $\alpha(\mathbb{Z}+i\mathbb{Z})$ *is a set of sampling for* $\mathbf{F}_2^n(\mathbb{C})$ *if and only if* $\alpha^2 < \frac{1}{n+1}$.

Theorem 3. *The lattice* $\alpha(\mathbb{Z}+i\mathbb{Z})$ *is a set of interpolation for* $\mathbf{F}_2^n(\mathbb{C})$ *if and only if* $\alpha^2 > \frac{1}{n+1}$.

So far, there is no proof of these results using only complex variables. The proof in [1] is based on the observation that the polyanalytic Bargmann transform is an isometric isomorphism

$$\mathbf{B}^n : \mathcal{H} \to \mathbf{F}^n(\mathbb{C}^d).$$

and the sampling problem can be transformed in a problem about Gabor superframes with Hermite functions. Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, ..., f_{n-1})$ with the inner product with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}} = \sum_{0 \leq k \leq n-1} \langle f_k, g_k \rangle_{L^2(\mathbb{R}^d)} . \tag{3.14}$$

The time-frequency shifts π_λ act coordinate-wise in a obvious way.

The vector valued system $\mathcal{G}(\mathbf{g}, \Lambda) = \{\pi_\lambda \mathbf{g}\}_{(x,w) \in \Lambda}$ is a *Gabor superframe* for $\mathcal{H}$ if there exist constants A and B such that, for every $\mathbf{f} \in \mathcal{H}$,

$$A \left\| \mathbf{f} \right\|_{\mathcal{H}}^2 \leq \sum_{\lambda \in \Lambda} \left| \langle \mathbf{f}, \pi_\lambda \mathbf{g} \rangle_{\mathcal{H}} \right|^2 \leq B \left\| \mathbf{f} \right\|_{\mathcal{H}}^2 . \tag{3.15}$$

Then the above sampling theorem is equivalent to the following statement about Gabor superframes with Hermite functions:

Theorem 4. [44] *Let $h_n = (h_0, ..., h_{n-1})$ be the vector of the first n Hermite functions. Then $G(h_n, \alpha(Z + iZ))$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^n)$, if and only if $\alpha^2 < \frac{1}{n+1}$.*

Superframes were introduced in a more abstract form in [47] and in the context of "multiplexing" in [14]. A structure result of Gabor analysis, the so called *Ron-Shen duality* [63] transforms the superframe problem into a problem about multiple Riesz sequences, which can be further transformed in a problem about multiple interpolation in the Fock space. The solution of the multiple interpolation problem can be obtained as a special case of the results of [20]. The dual of this argument proves the second theorem. The characterization of the lattices yielding Gabor superframes with Hermite functions had been previously obtained by Gröchenig and Lyubarskii in [44], using the Wexler-Rax orthogonality relations and solving the resulting interpolation problem in the Fock space. An approach using group theoretic methods has been considered by Führ [33]. The hard part of the above results is the sufficiency of the condition in Theorem 2. This would follow from the explicit formula in Theorem 1, if a complex variables proof was available (the case $n = 0$ is a simple consequence of the Cauchy formula).

Problem 2. *Find a proof of Theorem 1 without using the structure of Gabor frames (in particular, one using only complex variables).*

4. L^p theory and Modulation spaces

4.1. Banach Fock spaces of polyanalytic functions. The L^p version of the polyanalytic Bargmann-Fock spaces has been introduced in [3]. In its study the link to Gabor analysis has been particulary useful. For $p \in [1,\infty[$ write $\mathcal{L}_p(\mathbb{C})$ to denote the Banach space of all measurable functions equiped with the norm

$$\|F\|_{\mathcal{L}_p(\mathbb{C})} = \Big(\int_{\mathbb{C}} |F(z)|^p \, e^{-\pi p \frac{|z|}{2}^2} \, dz \Big)^{1/p}.$$

For $p = \infty$, we have $\|F\|_{\mathcal{L}_\infty(\mathbb{C})} = \sup_{z\in\mathbb{C}} |F(z)| \, e^{-\pi \frac{|z|^2}{2}}$.

Definition 1. *We say that a function F belongs to the* polyanalytic Fock space $\mathbf{F}_p^{n+1}(\mathbb{C})$, *if $\|F\|_{\mathcal{L}_p(\mathbb{C})} < \infty$ and F is polyanalytic of order n.*

Definition 2. *We say that a function F belongs to the* true polyanalytic Fock space $\mathcal{F}_p^{n+1}(\mathbb{C})$ *if $\|F\|_{\mathcal{L}_p(\mathbb{C})} < \infty$ and there exists an entire function H such that*

$$F(z) = \left(\frac{\pi^n}{n!} \right)^{\frac{1}{2}} e^{\pi |z|^2} \left(\partial_z\right)^n \left[e^{-\pi |z|^2} H(z) \right].$$

Clearly, $\mathcal{F}_p^1(\mathbb{C}) = \mathcal{F}_p(\mathbb{C})$ is the standard Bargmann-Fock space. The space $\mathcal{F}_1^1(\mathbb{C})$ is the Bargmann-Fock image of the Feichtinger algebra.

The orthogonal decomposition (2.1) extends to the p-norm setting. Similar results appear in [61] for the unit disk case. For $1 < p < \infty$:

$$\begin{aligned} \mathbf{F}_p^n(\mathbb{C}) &= \mathcal{F}_p^1(\mathbb{C}) \oplus ... \oplus \mathcal{F}_p^n(\mathbb{C}). \\ \mathcal{L}_p(\mathbb{C}) &= \bigoplus_{n=1}^{\infty} \mathcal{F}_p^n(\mathbb{C}). \end{aligned}$$

This decomposition has been used in [8] as an essential ingredient in the proof of a result relating localization operators to Toeplitz operators.

4.2. Mapping properties of the true polyanalytic Bargmann transform in modulation spaces. For the investigation of the mapping properties of the true polyanalytic Bargmann transform $\mathcal{F}_p^n(\mathbb{C})$ we need the concept of *modulation space.* Following [41], the modulation space $M^p(\mathbb{R})$, $1 \le p \le \infty$, consists of all tempered distributions f such that $V_{h_0} f \in L^p(\mathbb{R}^2)$ equipped with the norm

$$\|f\|_{M^p(\mathbb{R})} = \|V_{h_0} f\|_{L^p(\mathbb{R}^2)}.$$

Modulation spaces were introduced in [28]. The case $p = 1$ is the Feichtinger Algebra, which has been mentioned previously. It would be probably be hard to prove the following statement directly, but the Modulation space theory provides a simple proof.

Theorem 5. *Given $F \in \mathcal{F}_p^n(\mathbb{C})$ there exists $f \in M^p(\mathbb{R})$ such that $F = \mathcal{B}^n f$. Moreover, there exist constants C, D such that:*

$$C \left\| F \right\|_{\mathcal{L}_p(\mathbb{C})} \leq \left\| \mathcal{B} f \right\|_{\mathcal{L}_p(\mathbb{C})} \leq D \left\| F \right\|_{\mathcal{L}_p(\mathbb{C})} . \tag{4.1}$$

The key observation leading to the proof of this result is the following: since the definition of Modulation space is independent of the particular window chosen [41, Proposition 11.3.1], then the norms

$$\left\| f \right\|'_{M^p(\mathbb{R}^2)} = \left\| V_{h_n} f \right\|_{L^p(\mathbb{R}^2)}$$

and

$$\left\| f \right\|_{M^p(\mathbb{R}^2)} = \left\| V_{h_0} f \right\|_{L^p(\mathbb{R}^2)} ,$$

are equivalent. Then, using the relations between the true polyanalytic Bargmann transform and the Gabor transform with Hermite functions (3.7) provide a norm equivalence which can be transfered to the whole $\mathcal{F}_p^n(\mathbb{C})$ due to the mapping properties of the true polyanalytic Bargmann transform [3].

The properties of the polyanalytic projection are kept intact. Indeed, we have the following result.

Proposition 1. *The operator P^n is bounded from $\mathcal{L}_p(\mathbb{C})$ to $\mathcal{F}_p^{n+1}$ for $1 \leq p \leq \infty$. Moreover, if $F \in \mathcal{F}_p^{n+1}$ then $P^n F = F$.*

Combining the above L^p theory of the polyanalytic Bargmann-Fock spaces, a result from localization [42] and coorbit theory [30], with the estimates from [43], provides a far reaching generalization of Corollary 1.

Theorem 6. *Assume that $\Lambda \subseteq \mathbb{R}^2$ is a lattice and $\alpha^2 < \frac{1}{n+1}$*

(i) Then F belongs to the true poly-Fock space $\mathcal{F}_p^{n+1}$, if and only if the sequence with entries $e^{-\pi|\lambda|^2/2}F(\lambda)$ belongs to $\ell^p(\Lambda)$, with the norm equivalence

$$\left\| F \right\|_{\mathcal{F}_p^{n+1}} \asymp \left(\sum_{\lambda \in \Lambda} |F(\lambda)|^p e^{-\pi p |\lambda|^2/2} \right)^{1/p} .$$

(ii) Let $S^n_{\Lambda^0}(z)$ be the polyanalytic Weierstrass function on the adjoint lattice Λ^0. Then every $F \in \mathcal{F}^{n+1}_p(\mathbb{C})$ can be written as

$$F(z) = \sum_{\lambda \in \Lambda} F(\lambda) e^{\pi \overline{\lambda} z - \pi |\lambda|^2} S^n_{\Lambda^0}(z - \lambda)\,. \tag{4.2}$$

The sampling expansion converges in the norm of $\mathcal{F}^{n+1}_p$ for $1 \leq p < \infty$ and pointwise for $p = \infty$.

Remark 1. *If $\alpha^2 > n+1$, then Λ is an interpolating sequence for $\mathcal{F}^{n+1}(\mathbb{C})$. Moreover, the interpolation problem is solved by* [3]

$$F(z) = \sum_{\lambda \in \Lambda} a_\lambda e^{\pi \overline{\lambda} z - \pi |\lambda|^2/2} \, S^n_{\Lambda}(z - \lambda). \tag{4.3}$$

5. The Landau levels and displaced Fock states

In addition to time-frequency analysis, polyanalytic Fock spaces also appear in several topics in quantum physics. We will here describe how the polyanalytic structure shows up in the Landau levels associated with a single particle within an Euclidean plane in the presence of an uniform magnetic field perpendicular to the plane and also

5.1. The Euclidean Laplacian with a magnetic field. Consider a single charged particle moving on a complex plane with an uniform magnetic field perpendicular to the plane. Its motion is described by the Schrödinger operator

$$H_B = -\frac{1}{4}\left((\partial_x + iBy)^2 + (\partial_y - iBx)^2 \right) - \frac{1}{2}$$

acting on $L^2(\mathbb{C})$. Here $B > 0$ is the strength of the magnetic field. Writing

$$\widetilde{\Delta_z} = e^{\frac{B}{2}|z|^2} H_B e^{-\frac{B}{2}|z|^2}$$

we obtain the following Laplacian on $\mathbb{C}$

$$\widetilde{\Delta_z} = -\partial_z \partial_{\overline{z}} + B\overline{z}\partial_{\overline{z}}. \tag{5.1}$$

This Laplacian is a positive and selfadjoint operator in the Hilbert space $\mathcal{L}_2(\mathbb{C})$ and the set $\{n, n \in \mathbb{Z}^+\}$ can be shown to be the pure point spectrum of $\widetilde{\Delta_z}$ in $\mathcal{L}_2(\mathbb{C})$. There are other Laplacians in the literature related to this one [57], [58], [68]. The eigenspaces of $\widetilde{\Delta_z}$, are known as the *Landau levels.* In [13] the authors consider

$$A^2_{n,B}(\mathbb{C}) = \{F \in \mathcal{L}_2(\mathbb{C}) : \widetilde{\Delta_{z,B}} F = nf\},$$

and obtain an orthogonal basis for the spaces $A^2_{n,B}$. When $B=\pi$ we can use the results in [13] (comparing either the orthogonal basis or the reproducing kernels of both spaces) to see that

$$A^2_{m,\pi}(\mathbb{C}) = \mathcal{F}_2^n(\mathbb{C}). \tag{5.2}$$

5.2. Displaced Fock states. This section summarizes work to be developed further in [7] Now, we can define a set of coherent states $\mid z >_n$ for each Landau level n. This can be done by displacing via the representation β_w the vector $\mid 0 >_n$ of $\mathcal{F}_2^n(\mathbb{C})$ with the following wavefunction

$$\langle z \mid 0\rangle_n = e_{n,n}(z).$$

Precisely,

$$|w>_n = \beta_w|0>_n$$

and the wavefunction is given by

$$\langle z \mid w\rangle_n = \beta_w e_{n,n}(z).$$

We will call this the *true polyanalytic representation* of the Landau level coherent states. Now, observing that

$$e^{\pi|z|^2}\partial_z\left[e^{-\pi|z|^2}F(z)\right] = \partial_z F(z) - \pi\overline{z}F(z),$$

we conclude from the unitarity of the true polyanalytic Bargmann transform that the operator

$$(\partial_z - \pi\overline{z}) : \mathcal{F}_2^n(\mathbb{C}) \to \mathcal{F}_2^{n+1}(\mathbb{C})$$

is unitary and that

$$e_{n,n}(z) = (\partial_z - \pi\overline{z})^n\, e_n(z).$$

Now, combining this identity with the intertwining property (3.5) gives

$$\begin{aligned}
\beta_w e_{n,n}(z) &= \beta_w \mathcal{B}^{n+1}(h_n)(z) \\
&= \mathcal{B}^{n+1}(\pi_w h_n)(z) \\
&= (\partial_z - \pi\overline{z})^n\, \mathcal{B}^n(\pi_w h_n)(z) \\
&= (\partial_z - \pi\overline{z})^n\, \beta_w e_n(z).
\end{aligned}$$

Since the operator $(\partial_z - \pi\overline{z})^n$ is unitary $\mathcal{F}_2(\mathbb{C}) \to \mathcal{F}_2^{n+1}(\mathbb{C})$ we have the following equivalent representation of the Landau level coherent states in the *analytic* Fock space

$$\langle z \mid w\rangle_n = \beta_w e_n(z).$$

A similar representation has been obtained by Wünsche [69], using quite different methods. These coherent states are called displaced Fock states, since they are obtained by displacing by a Fock shift an already excited Fock state.

A natural question concerns the completeness properties of these coherent states. More precisely, given a lattice Λ what are the complete discrete subsystems $\{|\lambda >_n, \lambda \in \Lambda\}$ of this system of coherent states? In order to do so, we go back to the section about "Fock frames" where the completeness and basis properties of the above have been shown to be equivalent to those of Gabor frames with Hermite functions and to the results about sampling in the true polyanalytic spaces. Using the true polyanalytic transform, the results about Gabor frames with Hermite function translate to sampling in true polyanalytic Fock spaces as follows.

Proposition 2. *The lattice $\alpha(\mathbb{Z}+i\mathbb{Z})$ is a set of sampling for $\mathcal{F}_2^n(\mathbb{C})$ if and only $\mathcal{G}\left(h_n, \alpha(\mathbb{Z}+i\mathbb{Z})\right)$ is a Gabor frame.*

Thus, we conclude that, in particular, if $\alpha^2 < \frac{1}{n+1}$, the subsystems of states constituted by the lattice $\alpha(\mathbb{Z}+i\mathbb{Z})$ are complete in the Landau levels. Now, take $B=1$ and observe that

$$\widetilde{\Delta_z} = \left(-\partial_z + \overline{z}\right)\left(\partial_{\overline{z}}\right).$$

This suggests us to consider the operators

$$\begin{aligned} \mathfrak{a}^+ &= -\partial_z + \overline{z} \\ \mathfrak{a}^- &= \partial_{\overline{z}}, \end{aligned}$$

which are formally adjoint to each other and satisfy the commutation relations for the quantum mechanic creation and annihilation operators. Vasilevski [71, Theorem 2.9] proved that the operators

$$\begin{aligned} \sqrt{\frac{(k-1)!}{(l-1)!}}\left(\mathfrak{a}^+\right)^{l-k}|_{\mathcal{F}_2^k(\mathbb{C})} &: \mathcal{F}_2^k(\mathbb{C}) \to \mathcal{F}_2^l(\mathbb{C}) \\ \sqrt{\frac{(k-1)!}{(l-1)!}}\mathfrak{a}^{l-k}|_{\mathcal{F}_2^k(\mathbb{C})} &: \mathcal{F}_2^l(\mathbb{C}) \to \mathcal{F}_2^k(\mathbb{C}) \end{aligned}$$

are Hilbert spaces isomorphisms (and one is the inverse of the other). Given our identification (5.2) we conclude that *the operators a^+ and a^- are, respectively, the raising and lowering operators between two different Landau levels.*

For other Quantum Physics applications of Gabor transforms which include as special cases polyanalytic Fock spaces see [19].

Similar considerations lead Wünsche in [69] to derive the following representation for the displaced Fock states $|z,n>$:

$$|z,n>=\frac{(-1)^n}{\sqrt{n!}}\left(-\partial_z+\overline{z}\right)^n|z> . \tag{5.3}$$

In view of our remarks in this section, one realizes that (5.3) is essentially the map $T:\mathcal{F}_2(\mathbb{C})\to\mathcal{F}_2^{n+1}(\mathbb{C})$ such that

$$T:F(z)\to e^{\pi|z|^2}\left(\partial_z\right)^n\left[e^{-\pi|z|^2}F(z)\right].$$

and the displaced Fock states are also true polyanalytic Fock spaces. We can now use Gröchenig and Lyubarskii result to show that if $\alpha^2<\frac{1}{n+1}$ then the subsystem of these coherent states constituted by the square lattice on the plane is overcomplete. From Ramathan and Steeger general result [60], we know that if $\alpha^2>1$ they are not. This can be seen as analogues of Perelomov completeness result [59] in the setting of displaced Fock states.

6. Polyanalytic Ginibre ensembles

The polyanalytic Ginibre ensemble has been introduced by Haimi and Hendenmalm [34]. Its has a physical motivation based on the Landau level interpretation described in the previous section. For the *kth* Landau level, consider the wave functions of the form

$$\psi_{k,j}(z)=e_{k,j}(z)e^{-\pi|z|^2},$$

where

$$e_{k,j}(z)=\left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}}e^{\pi|z|^2}\left(\partial_z\right)^k\left[e^{-\pi|z|^2}e_j(z)\right],\quad j\geq 0.$$

Thus, the wave function of a system consisting of the first n Landau levels with N non-interacting fermions at each Landau level k, with wavefunctions $\psi_{k,j}$ is given by the determinant

$$\Psi_{n,N}=\det[\psi_{k,j}(z_{s,i})]_{nN\times nN},\qquad 1\leq i,j\leq N,\qquad 1\leq k,s\leq n.$$

This can be rewritten as the probability density of a determinantal point process

$$\Psi_{n,N} = \frac{1}{(nN)!} \det[K_{n,N}(z_{s,i}, z_{k,j})]_{nN\times nN}, \qquad 1 \le i,j \le N, \qquad 1 \le k,s \le n,$$

whose correlation kernel is given as

$$K_{n,N}(z,w) = \sum_{k=1}^{n}\sum_{j=1}^{N} \psi_{k,j}(z)\overline{\psi_{k,j}(w)}.$$

To have a glimpse of what we are talking about, we first give a brief description of what is a determinantal point process and then give more details about the polyanalytic Ginibre ensemble.

6.1. Determinantal point processes. Consider an infinite dimensional Hilbert space H with continuous functions having their values in $X \subseteq \mathbb{C}$, with a reproducing kernel $K(z,w)$, that is, for every $f \in H$,

$$f(z) = \langle f, K(z,.)\rangle_H .$$

Suppose that $\{\varphi_j\}$ is a basis of H. Define the *N-dimensional kernel*

$$K^N(z,w) = \sum_{j=0}^{N} \varphi_j(z)\overline{\varphi_j(w)}. \tag{6.1}$$

Using N-dimensional kernels one can describe *determinantal point processes* [18] for N points $(z_1,...,z_N) \in X^N$ using their n-point intensities

$$dP^N(z_1,...,z_n) = \frac{1}{n!} \det\left[K^N(z_i,z_j)\right]_{i,j=1}^{n} d\mu(z_1)...d\mu(z_n).$$

We will be more concerned with the 1-point intensity

$$dP^N(z) = K^N(z,z)d\mu(z),$$

which allows us to evaluate the expected number of points to be found in a certain region if they are distributed according to the determinantal point process.

Using the reproducing kernels of the polyanalytic and true polyanalytic Fock spaces, one can define interesting determinantal point processes which are generalizations of the Ginibre ensemble, a determinantal point process in $\mathbb{C}$ related to the reproducing kernel of a Gabor space associated with a Gaussian window h_0:

$$K_{h_0}(z,w) = e^{-i\pi(u\eta - x\xi) - \pi\frac{|z|^2+|w|^2}{2}} e^{\pi\overline{w}z}.$$

The corresponding polynomial kernel (6.1) is

$$K_{h_0}^N(z,w) = e^{-\pi\frac{|z|^2+|w|^2}{2}} \sum_{j=0}^{N} \frac{(\pi\overline{w}z)^j}{j!}.$$

This is the kernel of the *Ginibre ensemble* [54, Chapter 15].

6.2. The polyanalytic Ginibre ensemble. In [34], a variant of this setting is used in the investigation of the *polyanalytic Ginibre ensemble.* The authors consider the space with reproducing kernel

$$\mathbf{K}_m^n(z,w) = mL_{n-1}^1(m\,|z-w|^2)e^{mz\overline{w}}$$

and the polynomial space

$$Pol_{m,n,N} = span\{z^j\overline{z}^l : 0 \le j \le N-1, 0 \le l \le n-1\}.$$

We remark that using the polyanalytic Hermite polynomials $e_{j,l}(z,\overline{z})$ as defined in (2.6), this is equivalent to writing

$$Pol_{m,n,N} = span\{e_{j,l}(z,\overline{z}) : 0 \le j \le N-1, 0 \le l \le n-1\}.$$

Several interesting asymptotic results are obtained. For instance, denoting the reproducing kernel of $Pol_{m,n,k}$ by $\mathbf{K}_{m,N}^n(z,w)$, it is proved that, if $z,w \in \mathbb{D}$, when $m,N \to \infty$ with $|m-N|$ bounded and $1-|zw| \ge \tau > 0$, then

$$\mathbf{K}_{m,N}^n(z,w) = \mathbf{K}_m^n(z,w) + O(e^{-\frac{1}{2}m\tau^2}e^{m|zw|}).$$

Another result is the blow-up of the 1-point intensity function of the determinantal point process associated with $\mathbf{K}_{m,N}^n(z,w)$. The one point intensity function is $e^{-m|z|^2}\mathbf{K}_{m,N}^n(z,z)$ and its localized version with $z = 1 + m^{-\frac{1}{2}}\xi$ is

$$\mathcal{U}_{m,N,n}(\xi) = \frac{1}{m} e^{-m\left|1+m^{-\frac{1}{2}}\xi\right|^2} \mathbf{K}_{m,N}^n(1+m^{-\frac{1}{2}}\xi, 1+m^{-\frac{1}{2}}\xi)$$

In [34, pag 29] the authors observe that, considered the Hermite polynomials $H_j(t)$ normalized such that

$$e^{tz-\frac{1}{2}z^2} = \sum_{j=0}^{\infty} H_j(t)\frac{z^j}{j!},$$

this can be written as

$$\mathcal{U}_{m,N,n}(\xi) = \sum_{j=0}^{n-1} \frac{1}{r!\sqrt{2\pi}} \int_{-\infty}^{-2Re\xi} H_j(t) e^{-\frac{1}{2}t^2} dt + O(m^{-\frac{1}{2}+\epsilon})$$

Thus, when $m, N \to \infty$ with $|m - N|$ bounded, $\mathcal{U}_{m,N,n}(\xi)$ is essentially determined by the density

$$\rho(t) = \sum_{j=0}^{n-1} \frac{1}{j!\sqrt{2\pi}} H_j(t)^2 e^{-\frac{1}{2}t^2},$$

which is the one point intensity in the Gaussian Unitary Ensemble. Thus, if we make the polyanaliticity degree increase, we are led to the Wigner semicircle law (see [54]):

$$\mathcal{U}_{m,N,n}(\xi) \approx \frac{2n}{\pi} \int_{-1}^{-n^{-\frac{1}{2}} Re\xi} \sqrt{1-\tau^2} d\tau.$$

Something similar happens to the Berezin measure

$$dB_{m,n,N}^{(z)}(w) = \frac{\left|\mathbf{K}_{m,N}^{n}(z,w)\right|^2}{\mathbf{K}_{m,N}^{n}(z,z)} e^{-m|z|^2} dz;$$

while the asymptotics of $dB_{m,n,N}^{(z)}(w)$ are similar to the case $n = 1$, defining the blow-up Berezin density at 1 by $\hat{B}_{m,n,N}^{(1)}(w) = m^{-1} B_{m,n,N}^{(1)}(1 + m^{-1/2}w)$, we have, as $m, N \to +\infty$ with $N = m + O(1)$, the following asymptotics, with uniform control on compact sets:

$$\hat{B}_{m,n,N}^{(1)}(w) = \frac{1}{\pi n} \left| \sum_{j=0}^{n-1} \frac{1}{j!} \int_{-\infty}^{-w} H_j(t+w) H_j(t-\overline{w}) e^{-\frac{1}{2}t^2} dt \right|^2 + O(m^{-\frac{1}{2}+\epsilon}).$$

Remark 2. *Comparing this set up with section 4.1, one recognizes the parameter m as the strengh of the magnetic field B. Therefore, the physical interpretation of the above limit $m, N \to \infty$ consists of increasing the strength of the magnetic field and simultaneously the number of independent states in the system. In the analytic case this has been done for more general weights* [11].

Remark 3. *A version of the polyanalytic Ginibre ensemble allowing for more general weights in the corresponding Fock space has been recently considered* *in* [35], *providing a polyanalytic setting similar to the one considered in* [11] *for analytic functions.*

7. Hyperbolic analogues: wavelets and Bergman spaces

Most of what we have seen about polyanalytic Fock spaces has an analogue in the hyperbolic setting. However, the hyperbolic setting presents several difficulties and the topic is far from being understood. We will outline some of what is known and what one would expect applying "time-frequency intuition" to this setting. Some facts contained on the material which is currently under investigation in [6] will be included.

7.1. Wavelets and Laguerre functions. For every $x \in \mathbb{R}$ and $s \in \mathbb{R}^+$, let $z = x + is \in \mathbb{C}^+$ and define

$$\pi_z g(t) = s^{-\frac{1}{2}} g(s^{-1}(t-x)).$$

For a vector $\mathbf{g} = (g_1, ..., g_n)$ such that the Fourier transforms of any two functions g_i and g_j are orthogonal in $L^2(\mathbb{R}^+, t^{-1})$, define π_z pointwisely as

$$\pi_z \mathbf{g} = (\pi_z g_1, ..., \pi_z g_n).$$

Let $\mathcal{H} = H^2(\mathbb{C}^+, \mathbb{C}^n)$ be the inner product space whose vector components belong to $H^2(\mathbb{C}^+)$, the standard Hardy space of the upper half-plane, equipped with the natural inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}} = f_1 g_1 + ... + f_n g_n.$$

We say that the vector valued system $\mathcal{W}(\mathbf{g}, \Lambda)$ is a *wavelet superframe* for $\mathcal{H}$ if there exist constants A and B such that, for every $\mathbf{f} \in \mathcal{H}$,

$$A \left\| \mathbf{f} \right\|_{\mathcal{H}}^2 \leq \sum_{\lambda \in \Lambda} \left| \langle \mathbf{f}, \pi_\lambda \mathbf{g} \rangle_{\mathcal{H}} \right|^2 \leq B \left\| \mathbf{f} \right\|_{\mathcal{H}}^2 . \tag{7.1}$$

The orthogonality conditions imposed on the entries of the vector $\mathbf{g}$ allow us to recover the original definitions of superframes [14]. Indeed, it serves the same original motivation for the introduction of superframes in signal analysis: a tool for the *multiplexing of signals.* We consider wavelet superframes with analyzing wavelets $(\frac{\Phi_0^\alpha}{c_{\Phi_0^\alpha}}, ..., \frac{\Phi_n^\alpha}{c_{\Phi_n^\alpha}})$, where $c_{\Phi_n^\alpha}^2 = \frac{\Gamma(n+\alpha+1)}{n!}$ is the admissibility constant of the vector component Φ_n^α defined via the Fourier transforms as

$$\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{1}{2}} l_n^\alpha(2t), \quad \text{with} \quad l_n^\alpha(t) = t^{\frac{\alpha}{2}} e^{-\frac{t}{2}} \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{t^k}{k!}. \tag{7.2}$$

The functions Φ_n^α above have been chosen as the substitutes of the Hermite functions used by Gröchenig and Lyubarskii to construct Gabor superframes [44]. As we will see, our choice is well justified by physical, operator theory and function theory arguments. We start with the remark that the functions Φ_n^α provide an orthogonal basis for all the $g \in H^2(\mathbb{C}^+)$ satisfying the admissibility condition

$$\|\mathcal{F}g\|^2_{L^2(\mathbb{R}^+,t^{-1})} < \infty,$$

and we note in passing that, according to the results in [25], such admissible functions constitute a Bergman space.

7.2. Wavelet frames with Laguerre functions. The fundamental question about wavelet frames is, given a wavelet g, to characterize the sets of points Λ such that $\mathcal{W}(g,\Lambda)$ is a wavelet frame (and the corresponding problem for the superframes defined above). This problem is much more difficult that the corresponding one for Gabor frames. The only characterization known so far is about the case $n = 0$ in (7.2) because the problem can be reduced to the density of sampling in the Bergman spaces, which has been completely understood by Seip in [66]. An important research problem is to understand how Seip's results extend to the whole family $\{\Phi_n^\alpha\}$. The only thing known to the present date is a necessary condition obtained in [4] in terms of a convenient set of points for discretization known as the "hyperbolic lattice" $\Gamma(a,b) = \{a^m bk, a^m\}_{k,m\in\mathbb{Z}}$:

Theorem 7. *If $\mathcal{W}(\Phi_n^{2\alpha-1},\Gamma(a,b))$ is a wavelet frame for $H^2(\mathbb{C}^+)$, then*

$$b \log a < 2\pi \frac{n+1}{\alpha}.$$

As far as our knowledge goes, and despite 30 years of intensive research in the field of wavelets, it seems that the above paragraph completely describes the state of art in the topic. Beyond [66] and [4] nothing seems to be known.

One of the limitations of wavelet theory is the absence of duality theorems like those used in [44] and [1] in order to obtain precise conditions on the lattice generating Gabor superframes with Hermite functions. For this reason, there are no known analogue results for Wavelet superframes with Laguerre functions. Still, a lot can be said. The existence of wavelet frames with windows $\mathbf{\Phi}_n^\alpha$ follows from coorbit theory along the lines of [32], where the case $n = 0$ is explained in detail.

Using the "time-frequency intuition" provided by the results in section 3 of this survey, one would expect the following to be true.

Conjecture 2. $\mathcal{W}(\mathbf{\Phi}_n^{2\alpha-1}, \Gamma(a,b))$ *is a wavelet superframe for* $\mathcal{H}$ *if and only if*

$$b \log a < \frac{2\pi}{n+\alpha}. \tag{7.3}$$

The case $n = 0$ follows from the results in [66]. This conjecture is well beyond the existing tools, since the duality between Riesz basis and frames has no known extension to wavelet frames. See the chapter about wavelets in [41]. We have reasonable expectations on the possibility that the connection to polyanalytic Bergman spaces to be described in the next sections may be put in good use for the investigation of this conjecture.

7.3. The hyperbolic Landau levels. The most notable fact about the wavelet transforms associated with Φ_n^α is that they provide a phase space representation for the higher hyperbolic Landau levels with a constant magnetic field introduced in Physics by Comtet ([26], see also [53]). We will now briefly sketch this connection. Let H_B denote the Landau Hamiltonian of a charged particle with a uniform magnetic field on $\mathbb{C}^+$, with magnetic length proportional to $|B|$,

$$H_B = s^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right) - 2iBs \frac{\partial}{\partial x}.$$

It is well known that, if $|B| > 1/2$, the spectrum of the operator H_B consists of both a continuous and a *finite* discrete part. The choice of the functions Φ_n^α as analyzing wavelets yields the eigenspaces associated with the discrete part. Denote by $E_{B,m}(\mathbb{C}^+)$ the subspace of $L^2(\mathbb{C}^+)$ defined as

$$E_{B,n} = \{f : L^2(\mathbb{C}^+) : H_B\psi = e_n\psi\},$$

with $e_n = (|B|-n)(|B|-n-1)$. Zouhair Mouayn has introduced a system of coherent states [55] for the operator H_B. Once we overcome the differences in the notation and parameters, we can identify Mouayn's coherent states [55] with our wavelets and conclude that

$$E_{B,n} = W_{\Phi_n^{2(B-n)-1}} \left(H^2 \left(\mathbb{C}^+ \right) \right).$$

Thus, wavelet frames with the functions Φ_n^α are discrete subsystems of coherent states attached to the hyperbolic Landau levels. The spectral analysis of the operator H_B is also important in number theory, since the solutions of H_B satisfying an automorphy condition are half-integral weight Maass forms, [62], [27]. Maass forms have become proeminent in modern number theory in part thanks to the striking relations to Ramanujan's Mock theta functions (see [21]) As we will see below, the spaces $E_{B,n}$ are, up to a multiplier isomorphism, *true polyanalytic Bergman spaces.*

7.4. Bergman spaces of polyanalytic functions. The Hardy space $H^2(\mathbb{C}^+)$, is constituted by the analytic functions on the upper half plane such that

$$\sup_{0<s<\infty} \int_{-\infty}^{\infty} |f(z)|^2 \, dx < \infty.$$

Let $\mathcal{L}^2_\alpha(\mathbb{C}^+)$ be the space of square-integrable functions in $\mathbb{C}^+$ with respect to $d\mu^+_\alpha(z) = s^\alpha d\mu^+(z)$, where $d\mu^+(z)$ is the standard area measure in $\mathbb{C}^+$. The *weighted analytic Bergman space*, $A_\alpha(\mathbb{C}^+)$, is constituted by $\mathcal{L}^2_\alpha(\mathbb{C}^+)$ functions analytic in $\mathbb{C}^+$. We will also require the space $\mathbf{A}^n_\alpha(\mathbb{C}^+)$, which is the *polyanalytic Bergman space* consisting of all functions in $\mathcal{L}^2_\alpha(\mathbb{C}^+)$ which can be written in the form

$$F(z) = \sum_{p=0}^{n-1} \overline{z}^p F_p(z), \tag{7.4}$$

with $F_0(z), ..., F_{n-1}(z)$ analytic on $\mathbb{C}^+$. There is also a decomposition in true polyanalytic spaces $\mathcal{A}^k_\alpha(\mathbb{C}^+)$, also due to Vasilevski (see the original paper in [70] and the book [72]):

$$\mathcal{L}^2_\alpha(\mathbb{C}^+) = \bigoplus_{n=0}^{\infty} \mathcal{A}^n_\alpha(\mathbb{C}^+).$$

The true polyanalytic spaces $\mathcal{A}^k_\alpha(\mathbb{C}^+)$ can be defined as

$$\mathcal{A}^k_\alpha(\mathbb{C}^+) = \mathbf{A}^k_\alpha(\mathbb{C}^+) \ominus \mathbf{A}^{k-1}_\alpha(\mathbb{C}^+)$$

such that

$$\mathbf{A}^n_\alpha(\mathbb{C}^+) = \mathcal{A}^1_\alpha(\mathbb{C}^+) \oplus ... \oplus \mathcal{A}^n_\alpha(\mathbb{C}^+).$$

The *Bergman transform* of order α is the wavelet transform with a Poisson wavelet times a weight:

$$Ber_\alpha \, f(z) = s^{-\frac{\alpha}{2}-1} W_{\overline{\Phi^\alpha_0}} f(z) = \int_0^\infty t^{\frac{\alpha+1}{2}} \mathcal{F}f(t) e^{izt} dt. \tag{7.5}$$

It is an isomorphism $Ber_\alpha : H^2(\mathbb{C}^+) \to A_\alpha(\mathbb{C}^+)$ and plays the role of the Bargmann transform in the hyperbolic case. The *true* polyanalytic Bergman transform for $f \in H^2(\mathbb{C}^+)$ is given by the formula

$$Ber_\alpha^{n+1} f = \frac{(2i)^n}{n!} s^{-\alpha} \left(\partial_z\right)^n \left[s^{\alpha+n}(Ber_\alpha f)(z)\right]. \tag{7.6}$$

It is an isomorphism $Ber_\alpha : H^2(\mathbb{C}^+) \to A_\alpha(\mathbb{C}^+)$. This has been proved in [5] for the case $\alpha = 1$ and it can be shown for general α by an argument involving special functions. Then, we have the connection to wavelets provided by the formula

$$Ber_\alpha^{n+1} f(z) = s^{-\frac{\alpha}{2}-1} W_{\Phi_n^\alpha} f(z).$$

It is an isomorphism $Ber_\alpha^{n+1} : H^2(\mathbb{C}^+) \to A_\alpha^{n+1}(\mathbb{C}^+)$ and we identify the true polyanalytic Bergman spaces as the eigenspaces associated to the hyperbolic Landau levels. A "thought experience" concerning multiplexing of signals in this context leads to similar constructions to those we found in the Fock/Gabor case. Rephrasing the result about wavelet frames in terms of sampling sequences for polyanalytic Bergman spaces yields the following consequence of Theorem 6:

Corollary 1. *The sequence* $\{a^m bk + a^m i\}$ *is a sampling sequence for* $\mathrm{A}_{2\alpha-1}^{n+1}(\mathbb{C}^+)$ *if and only if* $b \log a < 2\pi \frac{n+1}{\alpha}$.

An in equivalence to the Conjecture 2, we have the following.

Conjecture 3. *The sequence* $\{a^m bk + a^m i\}$ *is a sampling sequence for* $\mathbf{A}_{2\alpha-1}^{n+1}(\mathbb{C}^+)$ *if and only if* $b \log a < \frac{2\pi}{n+\alpha}$.

Much of the results about polyanalytic Fock spaces are likely to find analogues in the hyperbolic setting, but the methods of proof can be quite different. The topic is currently under investigation.

7.5. An orthogonal decomposition of $\mathcal{L}_\alpha^2(\mathbb{D})$. A standart Cayley transform provides and isomorphism between $\mathcal{L}_\alpha^2(\mathbb{C}^+)$ and $\mathcal{L}_\alpha^2(\mathbb{D})$. This motivates the study of certain polyanalytic spaces in the unit disc which, althought not having a direct connection to wavelets, have been recently considered for their intrinsic mathematical content [61] [22]. Let $\mathcal{L}_\alpha^2(\mathbb{D})$ be the space of square-integrable functions in the unit disc, with respect to $dA_\alpha(w) = (1-|w|)^\alpha dA(w)$, where $dA(w)$ is the standart area measure in the unit disc. Denote by $\mathbf{A}_\alpha^n(\mathbb{D})$ the *polyanalytic Bergman space* consisting of all functions in $\mathcal{L}^2_\alpha(\mathbb{D})$ satisfying the higher order Cauchy-Riemann equation. The spaces $\mathbf{A}^n_\alpha(\mathbb{D})$ can be decomposed in a direct sum of *true polyanalytic Bergman spaces* [61]:

$$\mathbf{A}^n(\mathbb{D}) = \mathcal{A}^1_\alpha(\mathbb{D}) \oplus ... \oplus \mathcal{A}^n_\alpha(\mathbb{D}). \tag{7.7}$$

The space $\mathcal{A}^1_\alpha(\mathbb{D}) = A_\alpha(\mathbb{D}) = \mathbf{A}^1_\alpha(\mathbb{D})$ is the Bergman space of analytic functions in the unit disc. The spaces $\mathcal{A}^{n+1}_\alpha(\mathbb{D})$ are constituted by functions in $\mathcal{L}^2_\alpha(\mathbb{D})$ which can be written in the form

$$F(z) = \left(1 - |z|^2\right)^{-\alpha} \left(\frac{d}{dz}\right)^n \left[\left(1 - |z|^2\right)^{\alpha+n} f(z)\right],$$

for some $f(z) \in A_\alpha(\mathbb{D})$. In companion to (7.7) they provide an orthogonal decomposition for the whole $\mathcal{L}^2_\alpha$ space:

$$\mathcal{L}^2_\alpha(\mathbb{D}) = \bigoplus_{n=0}^{\infty} \mathcal{A}^n_\alpha(\mathbb{D}).$$

Remark 4. *The interested reader can verify that, althought the correspondence*

$$F \to \left(\frac{1}{1-w}\right)^{\alpha+1} F\left(i\frac{w+1}{1-w}\right)$$

provides an unitary mapping between the spaces $\mathbf{A}^n_\alpha(\mathbb{C}^+)$ and the spaces $\mathbf{A}^n_\alpha(\mathbb{D})$, it does not provide an unitary mapping between the spaces $\mathcal{A}^n_\alpha(\mathbb{D})$ and $\mathcal{A}^n_\alpha(\mathbb{C}^+)$. This means that the "true" spaces are different if we move from the unit circle to the upper half plane and need to be studied separately.

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