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ON TOEPLITZ OPERATORS AND LOCALIZATION OPERATORS

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ABSTRACT: This note is a contribution to a problem of Lewis Coburn concerning the relation between Toeplitz operators and Gabor-Daubechies localization operators. We will show that, for any *localization operator* with a *general window* $w \in \mathcal{F}_2(\mathbb{C})$ (the Fock space of analytic functions square-integrable on the complex plane), there exists a differential operator of infinite order D, with constant coefficients explicitly determined by w, such that *the localization operator with symbol f coincides with the Toeplitz operator with symbol Df*. This extends results of Coburn, Lo and Engliš, who obtained similar results in the case where w is a polynomial window. Our technique of proof combines their methods with a direct sum decomposition in true polyanalytic Fock spaces. Thus, *polyanalytic functions are used as a tool to prove a theorem about analytic functions.*

1. Introduction

Let $\mathcal{L}_p(\mathbb{C})$ $(1 \leq p < \infty)$ be the weighted L^p-space of functions with norm

$$\|u\|_{\mathcal{L}_p(\mathbb{C})} = \left(\int_{\mathbb{C}} |u(z)|^p e^{-\frac{\pi}{2}p|z|^2} dz \right)^{\frac{1}{p}},$$

dz being the area measure on $\mathbb{C}$ and $\mathcal{L}_\infty(\mathbb{C})$ the space of measurable functions on $\mathbb{C}$ such that $|u(z)|e^{-\frac{\pi}{2}|z|^2}$ is bounded, endowed with the norm

$$\|u\|_{\mathcal{L}_\infty(\mathbb{C})} = \sup_{z\in\mathbb{C}} |u(z)|e^{-\frac{\pi}{2}|z|^2}.$$

Let H be a Hilbert space contained in $\mathcal{L}_2(\mathbb{C})$, with reproducing kernel $K(\zeta, z)$. For $f \in L^\infty(\mathbb{C})$, the *Toeplitz operator* Toep_f with symbol $f(z)$ is the operator

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acting on H defined by

$$(\mathrm{Toep}_f g)(\zeta) = \int_{\mathbb{C}} f(z)g(z)K(\zeta, z)e^{-\pi|z|^2}dz, \forall\ g \in H.$$

Denote by $\mathcal{F}_p(\mathbb{C})$ the space of entire functions with membership in $\mathcal{L}_p(\mathbb{C})$. The Weyl operator

$$W_z w(\zeta) = e^{\pi\overline{z}\zeta - \frac{\pi}{2}|z|^2} w(\zeta - z) \tag{1.1}$$

acts unitarily on $\mathcal{L}_2(\mathbb{C})$. If $w \in \mathcal{L}_2(\mathbb{C})$ and $f \in L^\infty(\mathbb{C})$, the *Gabor-Daubechies localization operator* $L_f^{(w)}$, with "window" w and "symbol" f, is the operator acting on $\mathcal{F}_2(\mathbb{C})$ defined, in the weak sense, by

$$\left\langle L_f^{(w)} u, v \right\rangle = \int_{\mathbb{C}} f(z) \langle u, W_z w\rangle_{d\mu}\ \langle W_z w, v\rangle_{d\mu}\ dz, \forall\ u, v \in \mathcal{F}_2(\mathbb{C}).$$

A considerable research activity on localization operators has been motivated by the article of Daubechies [5]. See also [9], [4]. An interesting connection between localization and Toeplitz operators has been observed by Coburn [3].

In this note we will investigate further the relation between localization and Toeplitz operators. Our main contribution is simply outlined in the following paragraph.

Denote by $\mathcal{F}_p(\mathbb{C})$ the space of entire functions with membership in $\mathcal{L}_p(\mathbb{C})$. We will show that it is possible to write a localization operator with a window $w \in \mathcal{F}_2(\mathbb{C})$ as

$$L_f^{(w)} = \mathrm{Toep}_{D(w)f}$$

where, in the right hand side, we have a Toeplitz operator whose symbol is obtained from the symbol of the localization operator by the action of a differential operator $D(w)$, whose coefficients are constants explicitly determined by w.

Our methods of proof build on previous work of Coburn [3], Lo [12] and Engliš [6], but contain a technical innovation, based on *polyanalytic functions*.

A polyanalytic function of order n is a polynomial of order $n-1$ in $\overline{z}$ with analytic functions $\{\varphi_k(z)\}_{k=0}^{n-1}$ as coefficients:

$$F(z) = \sum_{k=0}^{n-1} \overline{z}^k \varphi_k(z), \tag{1.2}$$

so that an analytic function is a polyanalytic function of order 1. Alternatively, F satisfies the generalized Cauchy-Riemann equations

$$\left(\partial_{\overline{z}}\right)^n F(z) = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)^n F(x+i\xi) = 0. \tag{1.3}$$

A simple example of a polyanalytic function of order 2 is

$$F(z) = 1 - |z|^2 = 1 - z\overline{z}.$$

Observe that

$$\partial_{\overline{z}} F(z) = -z, \text{ while } \left(\partial_{\overline{z}}\right)^2 F(z) = 0.$$

Polyanalytic functions have applications in signal analysis [1], [2] and determinantal point processes [11].

We would like to emphasize that, in our proofs, *polyanalytic functions are used as a tool to prove a theorem about analytic functions.* Indeed, to extend Coburn-Lo-Engliš theorem from the polynomial case to a general window, we will use an idea of Vasilevski [14], who has shown that arbitrary $\mathcal{L}_2(\mathbb{C})$ functions can be decomposed using a sequence of orthogonal spaces which are "slices" of Fock spaces of polyanalytic functions. Vasilevski coined the expression *true polyanalytic Fock spaces* to quote such slices. We will require a slightly more general version of his result, since an essential step in the proof requires the decomposition of a non-analytic function of $\mathcal{L}_1(\mathbb{C})$ into polyanalytic basis functions. For this purpose we will use the direct sum decomposition of $\mathcal{L}_p(\mathbb{C})$ [2] into L^p-counterparts of *true* polyanalytic Fock spaces. Polyanalytic functions were already implicit in the papers of Coburn [3], Lo [12] and Engliš [6]. There is a time-frequency interpretation of this fact: if one takes a Hermite function as a window in a Gabor transform, the result is, up to a weight, a polyanalytic version of the Bargmann transform [1], [2].

The paper is organized in the following way: section 2 describes the main result and gives a brief account of its forerunners. In section 3 we collect the preliminar material about polyanalytic expansions and Engliš preparatory results which we will use in section 4, where the main result is proved.

2. Main result

For an element $z = x + i\omega$ in $\mathbb{C}$, consider the Cauchy-Riemann operator and its conjugate,

$$\partial_{\overline{z}} = \frac{d}{d\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \omega}\right), \qquad \partial_z = \frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial \omega}\right).$$

It is clear from (1.1) that $W_z 1 = e^{\pi\overline{z}\zeta - \frac{\pi}{2}|z|^2}$ and $\langle F, W_z 1\rangle_{d\mu} = e^{-\frac{\pi}{2}|z|^2}F(z)$, since $e^{\pi\overline{z}\zeta}$ is the reproducing kernel of $\mathcal{F}_2(\mathbb{C})$. This gives, for $w(z) = 1$ and $f \in L^\infty(\mathbb{C})$, the following relation between Toeplitz and localization operators in the *weak sense* for any $g, h \in \mathcal{F}_2(\mathbb{C})$:

$$\langle \mathrm{Toep}_f g, h\rangle_{d\mu} = \langle L_f^{(1)} g, h\rangle_{d\mu}.$$

This is equivalent to $L_f^{(1)} g = \mathrm{Toep}_f g$ for any $g \in \mathcal{F}_2(\mathbb{C})$. In case the constant polynomial $e_0(z) = 1$ is replaced by $e_1(z) = \sqrt{\pi}z$ or $e_2(z) = \frac{\pi}{2}z^2$, Coburn's result (cf. [3]) under the change of variable $z \mapsto \sqrt{\pi}z$ gives

$$L_f^{(e_1)} = \mathrm{Toep}_{f+\frac{1}{\pi}\partial_z\partial_{\overline{z}}f}, L_f^{(e_2)} = \mathrm{Toep}_{f+\frac{1}{\pi}\partial_z\partial_{\overline{z}}f+2(\frac{1}{\pi}\partial_z\partial_{\overline{z}})^2 f}.$$

The above relations are satisfied by every symbol $f(z)$ which is either a polynomial in z and $\overline{z}$ or belongs to the algebra $B_a(\mathbb{C})$ of Fourier-Stieltjes transforms with compactly supported measures. Coburn has also shown in [3] that for any *polynomial* $w \in \mathcal{F}_2(\mathbb{C})$ and polynomial f in z and $\overline{z}$, there exists a unique polynomial differential operator D depending on ∂_z, $\partial_{\overline{z}}$ and w such that

$$L_f^{(w)} = \mathrm{Toep}_{D(w)f}. \tag{2.1}$$

Coburn conjectured that the coefficients of the polynomial $D(w)$ were constant and that (2.1) holds for any $f \in B_a(\mathbb{C})$. This conjecture was proved by Lo in [12] when Toep_f acts on analytic polynomials on $\mathbb{C}$. Moreover, Lo extended relation (2.1) to the following class of symbols $E(\mathbb{C})$ which includes $\mathbb{C}[z, \overline{z}]$ and $B_a(\mathbb{C})$:

$$E(\mathbb{C}) = \left\{ f \in C^\infty(\mathbb{C}) \ : \ \forall\, k \in \mathbb{N}_0 \ \exists\, C, \alpha > 0 \ s.t. \ |D^k f(z)| \leq Ce^{\alpha|z|}, z \in \mathbb{C} \right\}.$$

An alternative proof of (2.1) has been obtained by Engliš [6]. Engliš approach works for the whole $\mathcal{F}_2(\mathbb{C})$ and analytic polynomial windows w. It also provides an explicit formula for D. Our main result is the extension of the Coburn-Lo-Engliš theorem to the case of a general window $w \in \mathcal{F}_2(\mathbb{C})$. If the window is not a polynomial, then the differential operator is of infinite order. Recall that the monomials

$$e_k(z) = \left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}} z^k$$

span $\mathcal{F}_p(\mathbb{C})$. Consider also the sequence of polyanalytic orthogonal polynomials defined as

$$e_{j,k}(z) = \left(\frac{\pi^j}{j!}\right)^{\frac{1}{2}} e^{\pi|z|^2} (\partial_z)^j \left[e^{-\pi|z|^2} e_k(z)\right]. \tag{2.2}$$

Let $BC^\infty(\mathbb{C})$ be the space of all $C^\infty(\mathbb{C})$ functions whose partial derivatives are bounded. With this notation, our result is the following statement:

Theorem 1. *For any $w \in \mathcal{F}_2(\mathbb{C})$ and for each symbol $f \in BC^\infty(\mathbb{C})$, the operator $D(w) := D(-\partial_{\overline{z}}, -\partial_z)$ satisfying*

$$L_f^{(w)} = Toep_{D(w)f}$$

is uniquely determined by

$$D(w)f = \sum_{j,k=0}^{\infty} h_{j,k} (-\partial_{\overline{z}})^j (-\partial_z)^k f,$$

where $h_{j,k} = \left(\pi^{j+k} j!k!\right)^{-\frac{1}{2}} \langle |w|^2, e_{j,k}\rangle_{L^2(\mathbb{C},d\mu)}$. Moreover $D(w)f \in L^\infty(\mathbb{C})$ only if $f \in BC^\infty(\mathbb{C})$.

We first present the results from [6], [14] and [2] that will be used in the proof of Theorem 1.

3. Preliminary results

3.1. Direct sum decomposition of $\mathcal{L}_p(\mathbb{C})$. We say that a function F belongs to the *true polyanalytic Fock space* $\mathcal{F}_p^{j+1}(\mathbb{C})$ if $\|F\|_{\mathcal{L}_p(\mathbb{C})} < \infty$ and there exists an entire function H such that

$$F(z) = \left(\frac{\pi^j}{j!}\right)^{\frac{1}{2}} e^{\pi|z|^2} (\partial_z)^j \left[e^{-\pi|z|^2} H(z)\right].$$

The following $\mathcal{L}_p$-extension of Vasilevskii's orthogonal decomposition [14] has been proved in [2, Corollary 1]:

$$\mathcal{L}_p(\mathbb{C}) = \bigoplus_{j=0}^{\infty} \mathcal{F}_p^j(\mathbb{C}) \tag{3.1}$$

It has been shown in [2, Proposition 1] that, for fixed j the linear span of the sequence of polynomials $\{e_{j,k} : k \geq 0\}$ is dense in $\mathcal{F}_p^{j+1}(\mathbb{C})$. Combining this statement with (3.1) it follows that every $F \in \mathcal{L}_p(\mathbb{C})$ can be written in the form

$$F(z) = \sum_{j,k=0}^{\infty} h_{j,k} \; e_{j,k}(z),$$

where $h_{j,k} = \langle F, e_{j,k} \rangle_{d\mu}$ represents the Fourier-Hermite coefficients.

Similar decompositions appeared in the study of middle Hankel operators on Bergman spaces [13].

3.2. Berezin symbols. Let T be a bounded linear operator on $\mathcal{F}_2(\mathbb{C})$ and set

$$K_\zeta(z) = K(\zeta, \zeta)^{-\frac{1}{2}} K(z, \zeta).$$

We define the Berezin symbol $\widetilde{T(\zeta)}$ as $\widetilde{T(\zeta)} = \langle T \; K_\zeta, K_\zeta \rangle_{d\mu}$. In the case of $z = \zeta$, $K(\zeta, \zeta) = e^{\pi|\zeta|^2}$. This gives

$$K_\zeta(z) = W_\zeta e_0(z). \tag{3.2}$$

On the other hand, since $W_z^* = W_{-z}$ is the adjoint of W_z on $\mathcal{L}_2(\mathbb{C})$, it is easy to check the above relations:

$$W_z^* K_\zeta = e^{i\pi \Im(\overline{\zeta} z)} K_{\zeta - z}.$$

The following relations from [6, (17), pg. 6] will be used in the proof of the main result.

Proposition 1. *Let T a bounded linear operator on $\mathcal{F}_2(\mathbb{C})$. Then the following statements hold:*

(1) $\left|\widetilde{T(\zeta)}\right| \leq \|T\|$ *for each* $\zeta \in \mathbb{C}$.
(2) $\langle TF, K_\zeta \rangle_{d\mu} = e^{-\frac{\pi}{2}|\zeta|^2} (T \; F)(\zeta)$ *for any* $F \in \mathcal{F}_2(\mathbb{C})$.
(3) T *is uniquely determined by* $\widetilde{T(\zeta)}$
(4) *For any* $z \in \mathbb{C}$ *we have* $\widetilde{[W_z^* T W_z]}(\zeta) = \widetilde{T(\zeta + z)}$.

The boundedness of the operator $L_f^{(w)}$ follows from the following proposition.

Proposition 2. *For any $w \in \mathcal{F}_2(\mathbb{C})$ and $f \in L^\infty(\mathbb{C})$, the operator $L_f^{(w)}$ satisfies the boundeness condition:*

$$\left\| L_f^{(w)} \right\| \leq \|f\|_{L^\infty(\mathbb{C})} \|w\|^2_{\mathcal{L}_2(\mathbb{C})}.$$

4. Proof of the main result

Let $w \in \mathcal{F}_2(\mathbb{C})$ and consider

$$k(z) = |w(z)|^2 e^{-\pi|z|^2}$$

and

$$h(z) = e^{-\pi|z|^2}.$$

Using the same arguments as in [6, pg. 6-7], in terms of the standard convolution on $\mathbb{C}$:

$$(g * f)(\zeta) = \int_{\mathbb{C}} g(z) f(\zeta - z) dz,$$

we can we see that

$$\widetilde{L_f^{(w)}}(\zeta) = (k * f)(\zeta) \tag{4.1}$$

$$\widetilde{\mathrm{Toep}}_f(\zeta) = (h * f)(\zeta). \tag{4.2}$$

Now, if $w(z) \in \mathcal{F}_2(\mathbb{C})$, then $|w|^2 \in \mathcal{L}_1(\mathbb{C})$. Thus, we can use the results in section 3.1 with $p = 1$ and find Fourier-Hermite coefficients $h_{j,k} = \langle |w|^2, e_{j,k} \rangle_{d\mu}$ such that

$$|w(z)|^2 = \sum_{j,k=0}^{\infty} h_{j,k} e_{j,k}(z) = H(z, \overline{z}). \tag{4.3}$$

Now let $D(\overline{z}, z)$ be the (uniformly convergent on compact sets) series in z and $\overline{z}$:

$$D(\overline{z}, z) = \sum_{j,k=0}^{\infty} \left(\pi^{j+k} j! k!\right)^{-\frac{1}{2}} h_{j,k}\, \overline{z}^j z^k. \tag{4.4}$$

From Proposition 2 and statements (1), (2) and (4) of Proposition 1, for each $f \in L^\infty(\mathbb{C})$ the Berezin symbols $\widetilde{L_f^{(w)}}(\zeta)$ and moreover, $\widetilde{\mathrm{Toep}}_f(\zeta)$, are invariant under translations. Then, for each $w \in \mathcal{F}_2(\mathbb{C})$, the function $k(z) =$

$|w(z)|^2 e^{-\pi|z|^2}$ in (4.1) *is uniquely determined.* Thus, from Proposition 1 (3), the proof of $L_f^{(w)} = \mathrm{Toep}_{D(w)f}$ amounts to showing that

$$\widetilde{L_f^{(w)}}(\zeta) = \widetilde{\mathrm{Toep}_{D(w)f}}(\zeta).$$

By (4.1) and (4.2) this consists of verifying, for $|w(z)|^2 = H(z,\overline{z})$, the following identity:

$$H(.,\overline{.})e^{-\pi|.|^2} * f(.) = e^{-\pi|.|^2} * D(w)f,$$

or, using the notation (4.4),

$$\int_{\mathbb{C}} D\left(-\partial_{\overline{z}}, -\partial_z\right) f(\zeta - z)e^{-\pi|z|^2}dz = \int_{\mathbb{C}} f(\zeta - z)H(z,\overline{z})e^{-\pi|z|^2}dz. \tag{4.5}$$

Let us prove (4.5). First, a direct computation shows that

$$(-\partial_z)^j \left(-\partial_{\overline{z}}\right)^k \left(e^{-\pi|z|^2}\right) = (\pi^{j+k} j!k!)^{\frac{1}{2}} e^{-\pi|z|^2} e_{j,k}(z).$$

Then, we make repeated use of Green´s formula on the disk $\mathbf{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$, valid for $f, g \in C^1(\mathbf{D}_r)$:

$$\int_{\mathbf{D}_r} f(z)\partial_z\overline{g(z)}dz = -\int_{\mathbf{D}_r} \partial_z f(z)\overline{g(z)}dz + \frac{1}{i}\int_{\delta\mathbf{D}_r} f(z)\overline{g(z)}dz, \tag{4.6}$$

$$\int_{\mathbf{D}_r} f(z)\partial_{\overline{z}}\overline{g(z)}dz = -\int_{\mathbf{D}_r} \partial_{\overline{z}} f(z)\overline{g(z)}dz + \frac{1}{i}\int_{\delta\mathbf{D}_r} f(z)\overline{g(z)}dz. \tag{4.7}$$

where the line integral over the circle $\delta\mathbf{D}_r$ is oriented counterclockwise. The result is

$$\int_{\mathbf{D}_r} \left(-\partial_{\overline{z}}\right)^k (-\partial_z)^j f(\zeta - z)e^{-\pi|z|^2}dz = (\pi^{j+k} j!k!)^{\frac{1}{2}} \int_{\mathbf{D}_r} f(\zeta - z)e_{j,k}(z)e^{-\pi|z|^2}dz, \tag{4.8}$$

since the assumption $f \in BC^\infty(\mathbb{C})$ assures that all the line integrals appearing after the application of Green´s formula are zero. Since $D(\overline{z}, z)$ converges uniformly on $|z| \leq r$, we can use (4.4) and (4.8), interchange the sum with the integral, and use (4.3) to get

$$\int_{\mathbf{D}_r} D\left(-\partial_{\overline{z}}, -\partial_z\right) f(\zeta - z)e^{-\pi|z|^2}dz = \int_{\mathbf{D}_r} f(\zeta - z)H(z,\overline{z})e^{-\pi|z|^2}dz.$$

Letting $r \to \infty$ we obtain (4.5).

Remark 1. *Our methods of proof apply also for windows* $w(z) \in \mathcal{F}_p(\mathbb{C})$, $p \geq 2$. *However, we believe that the investigation of the case* $1 \leq p \leq 2$ *may be an interesting problem, since it includes the Feichtinger algebra* [7] *(corresponding to* $p = 1$*) which is of particular interest in several applications* [8].

References

[1] L.D. Abreu, *Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions*, Appl. Comp. Harm. Anal. 29 (2010) 287–302.

[2] L. D. Abreu, K. Gröchenig, *Banach Gabor frames with Hermite functions: polyanalytic spaces from the Heisenberg group*, Appl. Anal., 91 (2012), 1981-1997.

[3] L.A. Coburn, *The Bargmann isometry and Gabor-Daubechies wavelet localization operators*, Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), 169–178, Oper. Theory Adv. Appl., 129, Birkhäuser, Basel, 2001.

[4] E. Cordero and K. Gröchenig, *Time-frequency analysis of Localization operators*, J. Funct. Anal., 205 (2003), 107–131.

[5] I. Daubechies, *Time-frequency localization operators: A geometric phase space approach*, IEEE Trans. Inform. Theory, 34 (1988) 605–612.

[6] M. Engliš, *Toeplitz Operators and Localization Operators*, Trans. Amer. Math Soc. 361 (2009) 1039–1052.

[7] H. G. Feichtinger, *On a new Segal algebra*. Monatsh. Math. 92 (1981), no. 4, 269–289.

[8] H. G. Feichtinger, G. A Zimmermann, *A Banach space of test functions for Gabor analysis.* Gabor analysis and algorithms, 123–170, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.

[9] H. G. Feichtinger, K. Nowak *A Szegő-type theorem for Gabor-Toeplitz localization operators.* Michigan Math. J. Volume 49, Issue 1 (2001), 13-21.

[10] G.B. Folland Harmonic Analysis in Phase Space, Princeton University Press, Princeton, New Jersey, 1989.

[11] A. Haimi, H. Hendenmalm, *The polyanalytic Ginibre ensembles.* preprint arXiv:1106.2975.

[12] M-L. Lo, The Bargmann Transform and Windowed Fourier Transform, Integr. Equ. Oper. Theory, 27 (2007), 397–412.

[13] L. Peng, R. Rochberg and Z. Wu, *Orthogonal polynomials and middle Hankel operators on Bergman spaces.* Studia Math. 102 (1992), 57–75.

[14] N. L. Vasilevski, *Poly-Fock spaces,* Differential operators and related topics, Vol. I (Odessa, 1997), 371–386, Oper. Theory Adv. Appl., 117, Birkhäuser, Basel, (2000).

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