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ASYMPTOTIC LIMITS FOR THE DOUBLY NONLINEAR EQUATION

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ABSTRACT: This article is concerned with the asymptotic limits of the solutions of the homogeneous Dirichlet problem associated to a doubly nonlinear evolution equation of the form $u_t = \Delta_p u^m + g$, in a bounded domain, as the parameters p and m tend to infinity. We will address the limits in p and m separately and in sequence, eventually completing a convergence diagram for this problem. We prove, under additional assumptions on the domain and initial data, that the equation satisfied at the limit is independent of the order in which we take the limits in p and m.

KEYWORDS: doubly nonlinear equation, asymptotic limit, singular limit.
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1.Introduction

In this paper, we study the asymptotic limits of nonnegative solutions of the following boundary value problem associated to the doubly nonlinear equation (DNE)

$$\begin{cases} u_t = \Delta_p u^m + g & \text{in } \Omega_T := (0,T)\times\Omega \\ u^m = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega \\ u(0) = u_0, & \end{cases} \tag{1.1}$$

where Δ_p is the p-Laplacian. Both the initial datum u_0 and the source term g are integrable and u_0 is nonnegative. To analyze the limit when $p\to\infty$, we take Ω to be any bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. To evaluate the limit as m goes to infinity, we further assume that Ω is a ball of radius R, which we denote by $B(0,R)$.

The equation in (1.1) is a doubly nonlinear parabolic equation with a double degeneracy in the slow diffusion case $m(p-1)>1$. It arises as a model in several physical contexts, for example in the study of non-Newtonian fluids [27], turbulent flow of a gas in porous media ([28]) and glaciology ([16],[24]).

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Since the early eighties, extensive work has been done for the asymptotic limit of initial-value problems associated to the porous medium equation (PME), corresponding to the case $p = 2$,

$$u_t = \Delta u^m \ (m > 1),$$

as m tends to infinity ([8], [13], [15], [33]), as well as for the p-Laplace equation (PLE), corresponding to $m = 1$,

$$u_t = \Delta_p u \ (p > 2),$$

as p goes to infinity ([3], [22]).

However, very few references appear in the literature on the asymptotic limits for the doubly nonlinear equation, when both $m \neq 1$ and $p \neq 2$. As far as we know, only some studies have been done in the case that p is fixed and m goes to infinity, specially for the associated Cauchy problem ([12], [23] and [25]). In [23] it was proved, under very strong geometric assumptions on the initial data, that the nonnegative solutions of the Cauchy problem associated to the doubly nonlinear equation in the real line with $g \equiv 0$ converge to a function, which is one on an interval, determined by the initial data, and equal to the initial data outside that interval. It was conjectured in [12] and [25] by Bénilan and Igbida that the solutions of problem (1.1) with $g \equiv 0$ and any open domain $\Omega \subseteq \mathbb{R}^N$ converge to a function that solves a generalized mesa problem with plateau of height one. Precisely, the function equals one on a set, which is characterized as the noncoincidence set of a variational inequality involving the p-Laplacian, and equals the initial data outside that set. The authors also proved that the conjecture holds in $\mathbb{R}^N$ for radial, nondecreasing initial data. The main contribution of this paper is to shed some light into the complete picture by generalizing some of the results known for the prototype equations (PME) and (PLE) to the (DNE).

We start with some properties of the doubly nonlinear equation within nonlinear semigroup theory, using mainly the results in [7] and [25]. Let us define the nonlinear operator $A_{p,m}$ in $L^1(\Omega)$ by

$$\begin{gathered} A_{p,m}u = -\Delta_p u^m, \\ \mathrm{D}\left(A_{p,m}\right) = \{ \ u \in L^\infty(\Omega);\ u^m \in W_0^{1,p}(\Omega) \text{ and } \Delta_p u^m \in L^1(\Omega) \ \}, \end{gathered} \tag{1.2}$$

where r^m denotes $|r|^{m-1}r$ for all $r \in \mathbb{R}$. Then problem (1.1) can be recasted as

$$\begin{cases} u_t + A_{p,m}u = g \quad \text{on } (0,T) \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Let us now consider the functional $\Phi_p : L^2(\Omega) \to [0,+\infty]$ defined by

$$\Phi_p(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p dx & \text{if } u \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

The functional Φ_p is convex, proper, lower semicontinuous (l.s.c), $\Phi_p(0) = 0$ and for all $h \in H_0$, where H_0 is given by

$$H_0 := \{h \in C^1(\mathbb{R}) |\ h(0) = 0 \text{ and } 0 \le h' \le 1\}, \tag{1.5}$$

we have

$$\Phi_p(w + h(\hat{w} - w)) + \Phi_p(\hat{w} - h(\hat{w} - w)) \le \Phi_p(w) + \Phi_p(\hat{w}),\ \forall w, \hat{w} \in L^2(\Omega). \tag{1.6}$$

As already noted in [25], $v = A_{p,m}(u)$ in $\mathscr{D}'(\Omega)$ if and only if

$$\begin{cases} u \in L^\infty(\Omega),\ u^m \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \\ \Phi_p(\eta) \ge \Phi_p(u^m) + \displaystyle\int_\Omega v(\eta - u^m)dx,\ \forall \eta \in L^\infty(\Omega). \end{cases} \tag{1.7}$$

Then, by [7], it is known that $A_{p,m}$ is a T-accretive operator in $L^1(\Omega)$, which satisfies $R(I + \lambda A_{p,m}) \supseteq L^\infty(\Omega)$ for all $\lambda > 0$ and its domain $D(A_{p,m})$ is dense in $L^1(\Omega)$. Furthermore, the following condition is satisfied for all $\lambda > 0$ and $f \in L^\infty(\Omega)$

$$(I + \lambda A_{p,m})^{-1} f \ll f, \tag{1.8}$$

where, for all $u, v \in L^1(\Omega)$, we write $u \ll v$ if and only if

$$\int_\Omega j(u)dx \le \int_\Omega j(v)dx \ \text{ for all } \ j : \mathbb{R} \to [0,\infty] \text{ convex, l.s.c and } j(0) = 0.$$

Hence, problem (1.1) admits a unique solution in the mild sense, as defined in [19] and [25], which we will denote by $u_{p,m}$. We refer to [6], [10] and [31] for more on mild solutions and general results regarding non-linear semigroup theory. Moreover, if $g \equiv 0$ and $m(p-1) > 1$, then $\overline{A}_{p,m}$ (the closure of $A_{p,m}$ in $L^1(\Omega)$) generates a nonlinear semigroup of contractions in $L^1(\Omega)$ denoted by $S_{p,m}(t)$, and it can be proved that the mild solution of (1.1) satisfies the standard notion of weak solution ([12], see also [25] for details).

Let us now define $A_{\infty,\infty}$ in the following way

$$v \in A_{\infty,\infty}(u) \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega),\ \exists w \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}) \text{ with } w \in \tilde{\mathbb{K}}, \\ u \in \operatorname{sign}(w) \text{ and } 0 \ge \displaystyle\int_\Omega v(\xi - w)dx,\ \ \forall \xi \in \tilde{\mathbb{K}}. \end{cases} \tag{1.9}$$

with

$$\tilde{\mathbb{K}} := \{\xi \in L^1(\Omega) |\ |\nabla \xi| \leq 1 \text{ a.e.}\}, \tag{1.10}$$

where

$$\operatorname{sign}(r) = \begin{cases} -1 & \text{if } r < 0, \\ [-1, +1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases} \tag{1.11}$$

We will show later that the operator $A_{p,m}$ converges in the resolvent sense to the operator $A_{\infty,\infty}$ as p and m tend to infinity, independently of whichever limit we take first. To be precise, we will prove the following.

Theorem 1.1. *Let the domain Ω be a ball in $\mathbb{R}^N$ and $f \in L^\infty(\Omega)$, be radial and non-negative. Then, we have, for all $\lambda > 0$,*

$$L^1 - \lim_{p\to\infty} \lim_{m\to\infty} (I + \lambda A_{p,m})^{-1} f = L^1 - \lim_{m\to\infty} \lim_{p\to\infty} (I + \lambda A_{p,m})^{-1} f = (I + \lambda A_{\infty,\infty})^{-1} f.$$

Thus, by means provided by the nonlinear semigroup theory, we will obtain the convergence diagram for the solutions $u_{p,m}$ of problem (1.1). We will eventually need to restrict to the case in which the domain Ω is a ball to obtain the complete convergence diagram for the solutions. We will also require the initial data (which will be denoted by u_{0_m} to emphasize its dependence on the parameter m) to be radial, non-negative, and such that $u_{0_m}^m \in \tilde{\mathbb{K}}$ and $\|u_{0_m}\|_\infty \leq 1$.

The operators to which $A_{p,m}$ will converge, in the resolvent sense, as p and m tend to infinity, will be denoted by $A_{\infty,m}$ and $A_{p,\infty}$, respectively, and will be defined in the sequel. Under the conditions on the initial data u_{0_m} already stated, we prove that the solutions of problem (1.1) reformulated as the abstract evolution problem (1.3) converge as p and m tend to inifinity to the solutions of the Dirichlet problem associated to the operators $A_{\infty,m}$ and $A_{p,\infty}$ (see theorem 2.4 and theorem 3.7 below). Finally, the convergence of the solutions of these problems to the solutions of the Dirichlet problem associated to $A_{\infty,\infty}$ will be proved in theorem 4.2 and theorem 4.4. Therefore, we show that the equation it satisfies in the limit is independent of the order in which we take the limits in p and m.

The results can be summarized in the convergence diagram below.

The paper is organized as follows. In section 2, we examine the asymptotic limit of problem (1.1) as p tends to infinity and we do a similar study in secion 3 for the asymptotic limit with respect to the parameter m. In section 4, we conclude by proving that the solutions of the problems obtained in section 2 and 3 both converge to the same limit problem.

$$
\begin{tikzcd}
\boxed{\begin{array}{c} (u_{p,m})_t + A_{p,m} u_{p,m} = g \\ u_{p,m}(0) = u_{0_m} \end{array}} \arrow[r, "p \to \infty"] \arrow[d, "m \to \infty"'] & \boxed{\begin{array}{c} (u_m)_t + A_{\infty,m} u_m \ni g \\ u_m(0) = u_{0_m} \end{array}} \arrow[d, "m \to \infty"] \\
\boxed{\begin{array}{c} (u_p)_t + A_{p,\infty} u_p \ni g \\ u_p(0) = u_{0_\infty} \end{array}} \arrow[r, "p \to \infty"'] & \boxed{\begin{array}{c} u_t + A_{\infty,\infty} u \ni g \\ u(0) = u_{0_\infty} \end{array}}
\end{tikzcd}
$$

FIGURE 1. Complete convergence diagram

2.Limit of solutions as $p \to \infty$

Since when $m = 1$, the equation in (1.1) reduces to (PLE), we will first recall some known results for the asymptotic limit as $p \to \infty$ of the parabolic p-Laplace equation. The motivation, in this case, has been mainly the physical significance of the evolution problem obtained at the limit, when $p \to \infty$, for example, a sandpile model ([3], [22], [32]), a Bean's critical-state model for type II superconductivity ([5], [35] and [36]) and river networks ([32], see also [18]). We refer to [14], [30] and [29] for the limiting behaviour of the variable exponent p-Laplacian and to [1] and [2] for the limit as $p \to \infty$ of the nonlocal analogous of the p-Laplace equation.

Let us briefly recall some results for the Cauchy problem associated to the parabolic p-Laplace equation

$$
\begin{cases} (u_p)_t - \operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = g & \text{in } (0,T) \times \mathbb{R}^N \\ u_p = u_0 & \text{on } \{t = 0\} \times \mathbb{R}^N, \end{cases} \tag{2.1}
$$

which was investigated in [3] (see also [22]), where $N + 1 \leq p < \infty$, u_0 is a Lipschitz function with compact support, satisfying

$$
\|\nabla u_0\|_{L^\infty(\mathbb{R}^N)} \leq 1, \tag{2.2}
$$

and the function g is smooth, with compact support in $[0,T] \times \mathbb{R}^N$ for each $T > 0$.

The following reinterpretation of the p-parabolic problem in (2.1)

$$\begin{cases} g-(u_p)_t = \partial I_p(u_p) & \text{a.e. } t>0 \\ u_p = u_0 & t=0 \end{cases}$$

was used, where ∂I_p denotes the single-valued subdifferential of the functional I_p defined as

$$I_p(v) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathbb{R}^N} |\nabla v|^p dx & \text{if } v \in L^2(\mathbb{R}^N),\, |\nabla v| \in L^p(\mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

The passage to the limit under the conditions listed above was completely solved in [3] and the main results are summarized in the following proposition.

Proposition 2.1. [3, Theorem 3.2] *Consider the Cauchy problem for the parabolic p-Laplacian in (2.1), with conditions on the initial value u_0 and source term g as explained above. Then we can extract a subsequence $\{p_i\}$, p_i tending to infinity, and a limit u such that, for each $T>0$,*

$$\begin{cases} u_{p_i} \to u \;\; \text{a.e. and in } L^2((0,T)\times\mathbb{R}^N) \\ \nabla u_{p_i} \rightharpoonup \nabla u, \;\; (u_{p_i})_t \rightharpoonup u_t \text{ weakly in } L^2((0,T)\times\mathbb{R}^N), \end{cases} \tag{2.4}$$

and the limit function u satisfies

$$\begin{cases} g-u_t \in \partial I_\infty(u) & \text{a.e. } t>0 \\ u=u_0 & t=0, \end{cases} \tag{2.5}$$

where ∂I_∞ is the subdifferential of the convex functional

$$I_\infty(v) = \begin{cases} 0 & \text{if } v\in\mathbb{K}, \\ +\infty & \text{otherwise,} \end{cases}$$

for

$$\mathbb{K} = \{w \in L^2(\mathbb{R}^N) |\; |\nabla w| \le 1 \text{ a.e.}\}.$$

To analyze the asymptotic behaviour with respect to the parameter p, the main ingredient necessary to pass to the limit will be the convergence of the operator $A_{p,m}$, as p tends to infinity. We observe that, by (1.7), the operator $A_{p,m}$ continues to "act as a subdifferential" even when $m \neq 1$. It seems reasonable then, that when p goes to infinity, the operator $A_{\infty,m}$, obtained at the limit, will also "act as a subdifferential" of an indicator function of a convex set $\tilde{\mathbb{K}}$.

We know from [11] that, for all $\lambda > 0$ and $f \in L^2(\mathbb{R}^N)$,

$$(I+\lambda\partial I_p)^{-1}f \to (I+\lambda\partial I_\infty)^{-1}f \text{ in } L^2(\mathbb{R}^N), \tag{2.6}$$

when $p \to \infty$.

Let us now define $\Phi_\infty : L^1(\Omega) \to [0,+\infty]$ by

$$\Phi_\infty(u) = \begin{cases} 0 & u \in \tilde{\mathbb{K}} = \{\xi \mid |\nabla\xi| \le 1 \text{ a.e.}\} \\ +\infty & \text{otherwise.} \end{cases}$$

Then, using the same notation as in [9],

$$\partial_{L^1}\Phi_\infty = \left\{ (u,v) \in \tilde{\mathbb{K}} \times L^1(\Omega) \mid \int (u-w)v \ge 0, \text{ for } w \in \tilde{\mathbb{K}} \text{ with } (u-w)v \in L^1(\Omega) \right\}.$$

We would expect $A_{p,m}$ to converge to some operator $A_{\infty,m}$ that acts as $\partial_{L^1}\Phi_\infty(u^m)$.

We will prove that the limit operator $A_{\infty,m}$ behaves as

$$v \in A_{\infty,m}u \iff \begin{cases} u,v \in L^1(\Omega), \ u^m \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}), \\ u^m \in \tilde{\mathbb{K}} \text{ and } 0 \ge \displaystyle\int_\Omega v(\xi - u^m)dx \ \ \forall \xi \in \tilde{\mathbb{K}}, \end{cases} \tag{2.7}$$

where $\tilde{\mathbb{K}}$ is as in (1.10).

For this purpose, let us focus on the elliptic equation associated to the operator $A_{p,m}$, i.e., $z_{p,m} := (I + A_{p,m})^{-1}f$ for $f \in L^\infty(\Omega)$, since we will be interested in the properties of the resolvent operator, $J_\lambda^{A_{p,m}} := (I + \lambda A_{p,m})^{-1}$, to pass to the limit.

By properties of the operator $A_{p,m}$, we see that for every $f \in L^\infty(\Omega)$, $z_{p,m}$ is the unique solution of the problem

$$\begin{cases} z_{p,m} - \Delta_p z_{p,m}^m = f & \text{on } \Omega \\ z_{p,m} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

in the following sense

$$\begin{cases} z_{p,m} \in L^\infty(\Omega), \ z_{p,m}^m \in W_0^{1,p}(\Omega) \text{ and} \\ -\Delta_p z_{p,m}^m = f - z_{p,m} \text{in } \mathscr{D}'(\Omega). \end{cases} \tag{2.9}$$

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $f \in L^\infty(\Omega)$ and $z_{p,m}$ be the solution of (2.8). Then, when $p \to \infty$, we have*

$$z_{p,m} \to z_m \text{ in } L^1(\Omega),$$

and for any $q>1$

$$z_{p,m}^m \rightharpoonup z_m^m \text{ in } W^{1,q}(\Omega).$$

Furthermore, z_m^m satisfies

$$\|\nabla z_m^m\|_\infty \leq 1,$$

and z_m is the unique solution of the problem

$$\begin{cases} z_m \in L^1(\Omega),\ z_m^m \in W^{1,\infty} \cap C_0(\overline{\Omega}) \\ 0 \geq \displaystyle\int_\Omega (f - z_m)(\xi - z_m^m)dx \ \ \forall \xi \in \tilde{\mathbb{K}} \end{cases}$$

where $\tilde{\mathbb{K}}$ is defined in (1.10).

Proof: By the properties of the operator $A_{p,m}$ and the known a priori estimates for the elliptic p-Laplace equation ([26]) the convergence of the sequences $\{z_{p,m}\}$ and $\{z_{p,m}^m\}$ follow. The L^∞-bound for $|\nabla z_m^m|$ is proved as in Lemma 3.2 in [22]. We can then pass to the limit using (1.7) to obtain the result. ■

Corollary 2.3. *For all $f \in L^\infty(\Omega)$ and $\lambda > 0$, we obtain, when $p \to \infty$,*

$$(I + \lambda A_{p,m})^{-1} f \to (I + \lambda A_{\infty,m})^{-1} f \text{ in } L^1(\Omega). \tag{2.10}$$

Remark 2.1. In the case that $g \equiv 0$, then for all $f \in L^1(\Omega)$ such that $f^m \in \tilde{\mathbb{K}}$, we obtain

$$(I + \lambda A_{p,m})^{-1} f \to f \text{ in } L^1(\Omega),$$

for all $\lambda > 0$, as $p \to \infty$.

Hence, if we consider $m > \dfrac{1}{p-1}$, $u_{p,m}$ a solution of (1.1) with $g \equiv 0$ and $u_0^m \in \tilde{\mathbb{K}}$ then, when $p \to \infty$, we have

$$u_{p,m} \to u_0 \text{ in } C([0,T);L^1(\Omega)).$$

The main theorem of this section then follows.

Theorem 2.4. *Consider the problem (1.1), where Ω is a bounded domain in $\mathbb{R}^N$, $u_0 \in L^1(\Omega)$, $u_0^m \in \tilde{\mathbb{K}}$ and $g \in L^1(\Omega_T)$. Then, there exists a subsequence p_i tending to infinity and a function u_m such that, for each $T > 0$,*

$$u_{p_i,m} \to u_m \text{ in } C([0,T];L^1(\Omega)),$$

and u_m is the unique mild solution of

$$\begin{cases} g - (u_m)_t \in A_{\infty,m}(u_m) & \text{in } (0,T] \times \Omega \\ u_m = u_0 & \{t = 0\} \times \Omega, \end{cases} \tag{2.11}$$

where $A_{\infty,m}$ is given by (2.7)-(1.10).

Proof: Since the operator $A_{\infty,m}$ is T-accretive (see [4] for details), then from corollary 2.3 we can deduce the convergence in the resolvent sense of the operator $\overline{A}_{p,m}$ to the m-T-accretive operator $\overline{A}_{\infty,m}$. Besides, by the conditions imposed on the initial data, we have that $u_0 \in \overline{D(A_{\infty,m})}$. Hence, the result follows by a classical result in nonlinear semigroup theory, which states that if each A_k, $k = 1, \cdots, \infty$ is a m-accretive operator in a Banach space X, $u_{0_k} \in \overline{D(A_k)}$ and u_k is the mild solution of

$$\frac{du_k}{dt} + A_k u_k \ni g_k, \;\; u_k(0) = u_{0_k},$$

then $(I + A_k)^{-1} f \to (I + A_\infty)^{-1} f$ for $f \in X$, $u_{0_k} \to u_{0_\infty}$ in X and $g_k \to g_\infty$ in $L^1(0,T;X)$, implies $u_k \to u_\infty$ in $C([0,\infty);X)$ (see, for example, [17] and [21] for statement and references). ■

Remark 2.2. When $m = 1$, the operator $A_{p,m}$ reduces to the p-Laplace operator defined in $L^1(\Omega)$, which restricted to $L^2(\Omega)$ coincides with the subdifferential ∂I_p in bounded domains. The same argument applies to show that $A_{\infty,m}$ coincides with ∂I_∞ when $m = 1$ and restricted to $L^2(\Omega)$. Hence, theorem 2.4 serves as a generalization (in the mild sense) of proposition 2.1.

Remark 2.3. When there is no source term, i.e., $g \equiv 0$, we can study the singular limit of the solutions of problem (1.1) as $p \to \infty$. This is the case when $u_0^m \notin \tilde{\mathbb{K}}$ and we assume u_0 is nonnegative and u_0^m is Lipschitz with

$$||\nabla u_0^m||_{L^\infty(\Omega)} = L > 1.$$

Observe that for $u \in D(A_{p,m})$ and $\lambda > 0$, we have that

$$\lambda u \in D(A_{p,m}) \text{ and } A_{p,m}(\lambda u) = \lambda^\beta A_{p,m}(u),$$

where $\beta = m(p-1)$. Therefore $A_{p,m}$ is a homogeneous operator and we can use a scaling argument which applies to the general setting of abstract nonlinear evolution equations governed by homogeneous accretive operators as in [11]. We consider the natural rescaling, taking into account what has been done for the problem $(DNE)_{p,m}$, when $m = 1$ in [22] and for $p = 2$ in [13], which is the following

$$v_{j,m}(t,x) = t u_{p_j,m}\left(\frac{t^{m(p_j-1)}}{m(p_j-1)}, x\right), \; (0 \le t \le 1). \tag{2.12}$$

Thus, using the convergence of the operators established in corollary 2.3, as well as remark 2.1, the passage to the limit in this case is solved by the methods in [11]. We obtain that there exists a limit function v_m such that, when $p_j \to \infty$,

$$u_{p_j,m}(t) \to v_m(1) \text{ in } L^1(\Omega),$$

and v_m satisfies the following properties:

(i) $v_m(t) = tu_0$ for any $t \in [0,\tau]$ where $\tau = 1/L^{1/m} < 1$,

(ii) v_m is the unique mild solution of the evolution problem

$$\begin{cases} \dfrac{v_m}{t} - (v_m)_t \in A_{\infty,m} v_m & (\tau < t \leq 1) \\ \\ v_m = v_0 = \tau u_0 & (t = \tau). \end{cases} \tag{2.13}$$

3.Limit of solutions as $m \to \infty$

We move on to study the asymptotic limit as m goes to infinity. When $p = 2$, the equation in (1.1) simplifies to the (PME). Hence, we recall the result for the asymptotic limit, as m goes to infinity, of the following Dirichlet porous medium problem

$$\begin{cases} (u_m)_t = \Delta |u_m|^{m-1} u_m & \text{in } (0,\infty)\times\Omega \\ u_m = 0 & \text{on } (0,\infty)\times\partial\Omega \\ u_m(0) = u_0 & \text{on } \Omega, \end{cases} \tag{3.1}$$

where Ω is an open domain of $\mathbb{R}^N$ not necessarily bounded, $m \geq 1$, and $u_0 \in L^1(\Omega)$.

Theorem 3.1. [8, Theorem 2] *Let u_m be the solution of problem (3.1) with initial value $u_0 \in L^1(\Omega)$, $u_0 \geq 0$. Then there exists a time-indepedent limit function $\underline{u_0}$ such that, when $m \to \infty$,*

$$u_m \to \underline{u_0} = u_0 \chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]} \text{ in } L^1(\Omega),$$

uniformly for t in a compact set in $(0,\infty)$, where $\underline{w}$ satisfies

$$\underline{w} \in H^2(\Omega)\cap H_0^1(\Omega),\ \ \underline{w} \geq 0,\ \ 0 \leq \Delta\underline{w} + u_0 \leq 1 \text{ in } \mathscr{D}'(\Omega),\ \ \underline{w}(\Delta\underline{w} + u_0 - 1) = 0,$$

or equivalently $\underline{w}$ is the solution of the mesa problem

$$\underline{u_0}, \underline{w} \in L^1(\Omega)^+,\ \ \operatorname{sign}(\underline{w}) - \Delta\underline{w} \ni u_0 \text{ in } \mathscr{D}'(\Omega),\ \ \underline{u_0} \in \operatorname{sign}(\underline{w}). \tag{3.2}$$

Let us now define the operator $A_{p,\infty}$ as

$$v \in A_{p,\infty}u \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega),\ \exists w \in V,\ u \in \operatorname{sign}(w) \ \text{ a.e. in } \Omega \\ -\Delta_p w = v \ \text{ in } \ \mathscr{D}'(\Omega), \end{cases} \tag{3.3}$$

where V denotes $W_0^{1,p}(\Omega)$ or $W^{1,p}(\mathbb{R}^N)$, depending on whether the domain Ω is bounded or the whole of $\mathbb{R}^N$. It was proved by Bénilan and Igbida in ([12], see [25] for details) that $A_{p,m}$ converges in the resolvent sense to the operator $A_{p,\infty}$ in the case that the domain is the whole of $\mathbb{R}^N$.

Note that since the nonlinearity $|u|^{m-1}u$ converges in the sense of graphs to the multivalued maximal monotone graph ϕ_∞ defined as

$$\phi_\infty(r) = \begin{cases} \emptyset & \text{if } r < -1 \\ (-\infty, 0] & \text{if } r = -1 \\ \{0\} & \text{if } |r| < 1 \\ [0, +\infty) & \text{if } r = 1 \\ \emptyset & \text{if } r > 1, \end{cases} \tag{3.4}$$

and $\phi_\infty(r) = \operatorname{sign}^{-1}(r)$, then it is natural to seek the convergence of the operator $A_{p,m}$ to $A_{p,\infty}$ as defined above.

We will show that the convergence holds for the particular case that the domain Ω is a ball in $\mathbb{R}^N$ of radius R and $f \in L^\infty(\Omega)$, is non-negative and radial. Once again, we will need to examine the properties of the stationary problem associated to the operator $A_{p,m}$, as defined in (2.8).

Remark 3.1. To prove the result in $\mathbb{R}^N$, it was used that $J_\lambda^{A_{p,m}}$ is invariant by translation, together with the L^1- contraction properties of solutions of (2.8), to obtain

$$\|z_{p,m}\|_1 \le \|f\|_1,$$
$$\|z_{p,m}(x+h) - z_{p,m}(x)\|_1 \le \|f(x+h) - f(x)\|_1 \ \ \forall h > 0.$$

Therefore $z_{p,m}$ is relatively compact in $L^1_{loc}(\mathbb{R}^N)$. Since we are interested in working in a bounded domain, we no longer have the translation invariance to make use of, and we need a different compactness result. We emphasize that all the other results in [25], used for the convergence of the operators, apply to general open domains in $\mathbb{R}^N$, not necessarily bounded.

We will, from this point on in this section, restrict the choice of domain to the ball of radius R and the case that f is non-negative and radial, that is, there exists a function l such that $f(x) = l(|x|)$. Since $J_\lambda^{A_{p,m}}$ is invariant by rotation, then the solution of (2.8) is radial and there exists $v_{p,m}$ such that $z_{p,m}(x) = v_{p,m}(|x|)$ and verifies

$$\begin{cases} v - \dfrac{(r^{N-1}|(v^m)_r|^{p-2}(v^m)_r)_r}{r^{N-1}} = l \quad \text{in } I = (0,R) \\ v(0) = v(R) = 0, \end{cases} \tag{3.5}$$

where we momentarily suppress the subscripts m and p. We are interested in obtaining a smooth approximation for this problem. For this purpose, let us consider as well $\rho \in C_0^\infty(I)$, $\rho \geq 0$, $\displaystyle\int \rho = 1$ and for any function k let us define the convolution

$$\rho_\varepsilon * k(x) = \int \rho_\varepsilon(x-y)k(y)dy, \ \varepsilon > 0,$$

where $\rho_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon$. Then, adapting accordingly the results in [20], which apply for the one dimensional doubly nonlinear diffusion equation in (1.1), to our radial case, we have the following result.

Theorem 3.2. *Let v be the unique solution of (3.5) for $l \geq 0$, $l \in L^\infty(B(0,R))$. For $p > N$, $m > 0$, there exists a smooth approximation $\Psi_\varepsilon(v,b)$ of the nonlinearity $\Psi(b) = r^{N-1}b|b|^{p-2}$ with*

$$\Psi_\varepsilon(v,b) = r^{N-1}b|b|^{p-2} + \frac{n\varepsilon}{m}v^{n-m}b,$$

where $n = (p-1)(m+1) - 1$ and $r \in (0,R)$, such that for $l_\varepsilon = \varepsilon + \rho_\varepsilon * l$ and $v_{0_\varepsilon} = \varepsilon$, the problem

$$\begin{cases} v_\varepsilon - (r^{N-1}|(v_\varepsilon^m)_x|^{p-2}(v_\varepsilon^m)_r)_r - \varepsilon(v_\varepsilon^n)_{rr} = v_\varepsilon - (\Psi_\varepsilon(v_\varepsilon,(v_\varepsilon^m)_r))_r = l_\varepsilon \quad \text{in } I \\ v_\varepsilon(0) = v_\varepsilon(R) = v_{0_\varepsilon}, \end{cases} \tag{3.6}$$

where $I = (0,R)$, has a unique solution $v_\varepsilon \in C^\infty(I)$ satisfying

(i) $0 < \varepsilon < v_\varepsilon < \varepsilon + \|f\|_\infty$.
(ii) *v_ε converges uniformly in compact subsets of I to v.*
(iii) *$(v_\varepsilon^m)_r \to (v^m)_r$ as $\varepsilon \to 0$ a.e. $r \in I$.*

Remark 3.2. If $p < N$, then the previous theorem continues to hold. However, depending on the relationship between p and m, a different approximation operator Ψ_ε would be needed to pass to the limit in ε.

Theorem 3.3. *Let $z_{p,m}$ be a solution of (2.8). If $\Omega = B(0,R)$ and f is radial and non-negative, then the total variation of $z_{p,m}$ is uniformly bounded.*

Proof: As explained above, if f is radial and non-negative then the solution $z_{p,m}$ of (2.8) has a radial representative $v_{p,m}$ which verifies (3.5). Since the total variation of $z_{p,m}$, $\int_{B(0,R)} |\nabla z_{p,m}(x)|dx$, and $\int_0^R |(v_{m,p})_r| r^{N-1} dr$ differ only by a constant which is independent of m then it is enough to prove the uniform bound of the second. Morever, by theorem 3.2, there exists a smooth approximation of the solution $v_{p,m}$ of (3.5), which we will continue to denote by v_ε and satisfies

$$\begin{cases} r^{N-1}v_\varepsilon - (r^{N-1}|(v_\varepsilon^m)_r|^{p-2}(v_\varepsilon^m)_r + \varepsilon \frac{n}{m} v_\varepsilon^{n-m}(v_\varepsilon^m)_r)_r = l_\varepsilon r^{N-1} \ \text{ in } (0,R) \\ v_\varepsilon(0) = v_\varepsilon(R) = v_{0_\varepsilon}. \end{cases} \tag{3.7}$$

We differentiate the equation in (3.7) with respect to r to obtain

$$(r^{N-1}v_\varepsilon)_r - (r^{N-1}|(v_\varepsilon^m)_r|^{p-2}(v_\varepsilon^m)_r + \varepsilon \frac{n}{m} v_\varepsilon^{n-m}(v_\varepsilon^m)_r)_{rr} = (l_\varepsilon r^{N-1})_r.$$

Denote now $b_\varepsilon = (v_\varepsilon^m)_r$ and consider a sequence of functions h_δ which satisfy $h_\delta \in C^\infty(\mathbb{R})$, $h'_\delta \geq 0$, $0 = h_\delta(0) \leq |h_\delta| \leq 1$. We multiply the above equation by $h_\delta(b_\varepsilon)$ and integrate over $(0,R)$ to obtain

$$\begin{aligned} \int_0^R (r^{N-1}v_\varepsilon)_r h_\delta(b_\varepsilon) dr &= \int_0^R (r^{N-1}|b_\varepsilon|^{p-2}b_\varepsilon + \varepsilon \frac{n}{m} v_\varepsilon^{n-m} b_\varepsilon)_{rr} h_\delta(b_\varepsilon) dr \\ &\quad + \int_0^R (l_\varepsilon r^{N-1})_r h_\delta(b_\varepsilon) dr \\ &\leq -\int_0^R (r^{N-1}|b_\varepsilon|^{p-2}b_\varepsilon + \varepsilon \frac{n}{m} v_\varepsilon^{n-m} b_\varepsilon)_r (h_\delta(b_\varepsilon))_r \\ &\quad + \sum_{\partial I} h_\delta(b_\varepsilon)(r^{N-1}|b_\varepsilon|^{p-2}b_\varepsilon + \varepsilon \frac{n}{m} v_\varepsilon^{n-m} b_\varepsilon)_r \\ &\quad + \int_0^R |(l_\varepsilon r^{N-1})_r| \\ &\leq R^{N-1}(v_{0_\varepsilon} - l_\varepsilon(R)) + \int_0^R |(l_\varepsilon r^{N-1})_r| dr. \end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^R (v_\varepsilon)_r r^{N-1} h_\delta(b_\varepsilon)dr &= \int_0^R (v_\varepsilon r^{N-1})_r h_\delta(b_\varepsilon)dr - \int_0^R (r^{N-1})_r v_\varepsilon h_\delta(b_\varepsilon)dr \\
&\leq R^{N-1}(v_{0_\varepsilon} - l_\varepsilon(R)) + \int_0^R |(l_\varepsilon r^{N-1})_r| dr \\
&\quad - \int_0^R (r^{N-1})_r v_\varepsilon h_\delta(b_\varepsilon)dr \\
&\leq R^{N-1}(v_{0_\varepsilon} - l_\varepsilon(R)) + \int_0^R |(l_\varepsilon)_r| r^{N-1} dr + C(R)\|l_\varepsilon\|_\infty.
\end{aligned}$$

Taking h_δ such that $h_\delta(s) \to \operatorname{sign}_0(s)$ as $\delta \to 0$, then

$$\int_0^R |(v_\varepsilon)_r| r^{N-1} dr \leq R^{N-1}(v_{0_\varepsilon} - l_\varepsilon(R)) + \int_0^R |(l_\varepsilon)_r| r^{N-1} dr + C(R)\|l_\varepsilon\|_\infty.$$

Hence, as $\varepsilon \to 0$,

$$\int_0^R |(v_{p,m})_r| r^{N-1} dr \leq \int_0^R |l_r| r^{N-1} dr + C(R)\|l\|_\infty.$$

∎

Once we have established the substitute to the compactness result in $\mathbb{R}^N$ to our particular case, we can use the following set of lemmas to show the convergence of the operators.

Lemma 3.4. [25, Lemma 2.7] *If $z_{p,m}$ is the solution of (2.8) then $z_{p,m}^m$ is uniformly bounded in $W_0^{1,p}(\Omega)$.*

Lemma 3.5. [25, Lemma 2.11] *For all $m \geq 1$, let $w_m \in W_0^{1,p}(\Omega)$ and $g_m \in L^1(\Omega)$ be such that*

$$-\Delta_p w_m = g_m \text{ in } \mathscr{D}'(\Omega).$$

If there exists $w_\infty \in W_0^{1,p}(\Omega)$ and $g_\infty \in L^1(\Omega)$ such that, when $m \to \infty$, we have

$$g_m \to g_\infty \text{ in } L^1(\Omega), \tag{3.8}$$

$$w_m \rightharpoonup w_\infty \text{ in } W^{1,p}(\Omega), \tag{3.9}$$

$$g_m w_m \to g_\infty w_\infty \text{ in } L^1(\Omega), \tag{3.10}$$

then

$$-\Delta_p w_\infty = g_\infty \text{ in } \mathscr{D}'(\Omega), \tag{3.11}$$

and furthermore, we have

$$\nabla w_m \to \nabla w_\infty \text{ ,in } (L^p(\Omega))^N, \text{ when } m \to \infty. \tag{3.12}$$

The convergence in the resolvent sense of the operators then follows adapting Theorem 2.1 in [25] as follows.

Lemma 3.6. *Let $\Omega = B(0,R)$ and $f \in L^\infty(\Omega)$, $f \geq 0$ and radial. Then, we have that*

$$(I+\lambda A_{p,m})^{-1} f \to (I+\lambda A_{p,\infty})^{-1} f \text{ in } L^1(\Omega),$$

for all $\lambda > 0$, when $m \to \infty$.

Proof: The proof follows as in theorem 2.1 in [25], using instead of the relative compactness in $L^1(\mathbb{R}^N)$, the uniform bound proved in theorem 3.3. Indeeed, Let $f \in L^\infty(\Omega)$, radial and non-negative, and $z_{p,m}$ be a solution of (2.8), by theorem 3.3, then there exists z_p such that

$$z_{p,m} \to z_p \text{ in } L^1(\Omega), \tag{3.13}$$

and by lemma 3.4 there exists some w_p such that

$$z_{p,m}^m \rightharpoonup w_p \text{ in } W_0^{1,p}. \tag{3.14}$$

Therefore $z_p \in \operatorname{sign}(w_p)$ a.e. in Ω. Since we also have that

$$\|z_{p,m}\|_\infty \leq \|f\|_\infty,$$

then

$$(f - z_{p,m})(z_{p,m})^m \to (f - z_p)w \text{ in } L^1(\Omega)$$

and all the hypothesis of lemma 3.5 are satisfied, from which we obtain

$$-\Delta_p w_p = f - z_p \text{ in } \mathscr{D}'(\Omega)$$

and

$$\nabla z_{p,m}^m \to \nabla w_p \text{ in } L^p(\Omega).$$

Thus, recalling that $z_{p,m} := (I+A_{p,m})^{-1} f$ and denoting $(I+A_{p,\infty})^{-1} f$ by z_p, we see that the result holds. ■

Remark 3.3. By proposition 2.3 and lemma 3.4 in [25] we know that given f such that $||f||_\infty \leq 1$, then when $m \to \infty$, we have

$$(I+\lambda A_{p,m})^{-1} f \to f \text{ in } L^1(\Omega)$$

for all $\lambda > 0$. Hence, if $u_{p,m}$ is a solution of (1.1) with $g \equiv 0$ and $\|u_0\|_\infty \le 1$, then when $m \to \infty$ we have

$$u_{p,m} \to u_0 \text{ in } C([0,T);L^1(\Omega))$$

for all $\lambda > 0$.

The main theorem in this section then reads as follows.

Theorem 3.7. *Let $u_{p,m}$ be the solution of the problem (1.1), where $\Omega = B(0,R)$ and u_0 is radial such that $0 \le u_0 \le 1$ and $g \in L^1(\Omega_T)$. Then, there exists a function u_p such that, when $m \to \infty$, for each $T > 0$,*

$$u_{p,m} \to u_p \text{ in } C([0,T];L^1(\Omega))$$

and u_p is the unique mild solution of

$$\begin{cases} (u_p)_t + A_{p,\infty}(u_p) \ni g & \text{in } \Omega \times [0,T] \\ u_p(0) = u_0, \end{cases} \tag{3.15}$$

where $A_{p,\infty}$ is given by (3.3).

Proof: The proof follows the same reasoning as theorem 2.4, using the accretivity of operator $A_{p,\infty}$, lemma 3.6 and the fact that $u_0 \in \overline{D(A_{p,\infty})}$. ■

Remark 3.4. In the case that the domain Ω is a bounded interval in $\mathbb{R}$ the results of theorem 3.3 and thus of theorem 3.7 continue to hold.

Remark 3.5. Let us consider $u_{p,m}$ the solution of (1.1) with $g \equiv 0$, assuming $\Omega = B(0,R)$ and u_0 is radial, non-negative and $\|u_0\|_\infty = M > 1$. Then, considering the same stretching of the time variable as in remark 3.3

$$v_{p,j}(t,x) = t u_{p,m_j}\left(\frac{t^{m_j(p-1)}}{m_j(p-1)}, x\right) \tag{3.16}$$

and the arguments already stated there, we can show the existence of a singular limit and the equation it satisfies. To be precise, by the results in [11], as well as lemma 3.6 and remark 3.3, we see that there exists a function v_p such that when $m_j \to \infty$,

$$u_{p,m_j} \to v_p(1) \text{ in } L^1(\Omega) \text{ uniformly for } t \text{ in a compact set of } (0,\infty) \tag{3.17}$$

and where v_p is given by

(i) $v_p(t) = t u_0$ for any $t \in [0,b]$, and $b = 1/M$,

(ii) v_p is the unique mild solution of the evolution problem

$$\begin{cases} (v_p)_t + A_{p,\infty} v_p \ni \dfrac{v_p}{t} \quad \text{in } (b,\infty) \\ \\ v_p(b) = b u_0. \end{cases} \tag{3.18}$$

4. Asymptotic behaviour of the limit equations

In this section, we will analyze the asymptotic limits of the mild solutions of the equations obtained in section 2 and section 3, when, respectively, m and p go to infinity. Taking into account the restrictions already imposed on the initial data, as well as the domain, we will be able to study the equations satisfied by these limits. By the results of the previous sections, we have identified the convergence of the operator $A_{p,m}$ as p and m tend to infinity. Hence, it is only left to show that the operators $A_{\infty,m}$ and $A_{p,\infty}$ both converge in the resolvent sense to the operator $A_{\infty,\infty}$ defined in (1.9) and thus the equation that is satisfied at the limit is the same independently of the order of the limits in p and m.

We follow the same line of ideas as in previous sections and study the stationary problem associated to $A_{\infty,m}$. Denoting $z_m := (I + A_{\infty,m})^{-1} f$, by theorem 2.2, the problem has a solution in the sense that

$$\begin{cases} z_m \in L^1(\Omega),\ z_m^m \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}), \\ |\nabla z_m^m| \le 1 \text{ a.e.},\ 0 \ge \displaystyle\int_\Omega (f - z_m)(\xi - z_m^m)dx,\ \forall \xi \in \tilde{\mathbb{K}}, \end{cases} \tag{4.1}$$

and we next prove the convergence in the resolvent sense of the operator $A_{\infty,m}$ to $A_{\infty,\infty}$.

Lemma 4.1. *Let Ω be a bounded domain. Then, for all $f \in L^\infty(\Omega)$ and $\lambda > 0$, we have,*

$$(I + \lambda A_{\infty,m})^{-1} f \to (I + \lambda A_{\infty,\infty})^{-1} f \text{ in } L^1(\Omega),$$

when $m \to \infty$.

Proof: We would like to show that there exists a unique function $\underline{z}$ such that, when $m \to \infty$,

$$z_m \to \underline{z} \text{ in } L^1(\Omega) \tag{4.2}$$

and

$$\underline{z} \in \text{sign}(w) \text{ is such that } 0 \ge \int_\Omega (f - \underline{z})(\xi - w)dx,\ \forall \xi \in \tilde{\mathbb{K}}. \tag{4.3}$$

Since z_m^m is Lipschitz continuous, taking $y \in \partial\Omega$, we have

$$|z_m^m(x)| \le |z_m^m(x) - z_m^m(y)| + |z_m^m(y)| \le \|\nabla z_m^m\|_\infty |x-y| \le \operatorname{diam}(\Omega).$$

Therefore, there exists a subsequence $\{m_i\}$ such that, for some w,

$$z_{m_i}^{m_i} \to w, \tag{4.4}$$

and

$$\nabla z_{m_i}^{m_i} \overset{*}{\rightharpoonup} \nabla w \text{ in } L^\infty(\Omega : \mathbb{R}^N). \tag{4.5}$$

Moreover

$$\|\nabla w\|_\infty \le \liminf_{m_i\to\infty} \|\nabla z_{m_i}^{m_i}\|_\infty \le 1. \tag{4.6}$$

Recall now that, by property (1.8), for $f \in L^\infty(\Omega)$ and $z_{p,m}$ a solution of (2.8), we have

$$\|z_{p,m}\|_r \le \|f\|_r \text{ for any } 1 \le r \le \infty,$$

and taking $p \to \infty$, it continues to hold that

$$\|z_m\|_r \le \|f\|_r \text{ for } 1 \le r \le \infty, \tag{4.7}$$

We will now use the Frechét-Kolmogorov's theorem to prove the relative compactness in $L^1(\Omega)$ of $\{z_m,\ m > 1\}$. According to this result, by (4.7), it would be enough to prove that for every $y \in \mathbb{R}^N$ small enough and $\Omega' \subset\subset \Omega$, the following holds

$$\sup_m \|z_m(x+y) - z_m(x)\|_{L^1(\Omega')} \le \|f(x+y) - f(x)\|_{L^1(\Omega')}. \tag{4.8}$$

To this end, let us consider the equation in (4.1). Then, for all $\Omega' \subset\subset \Omega$, ξ_1, $\xi_2 \in \tilde{\mathbb{K}}$ with $\operatorname{supp}\xi_i \subset \Omega'$ and $y \in \mathbb{R}^N$ such that $|y| < \operatorname{dist}(\operatorname{supp}\xi_i, \partial\Omega)$, $i = 1,2$, we obtain

$$0 \ge \int_{\Omega'} (f(x) - z_m(x))(\xi_1(x) - z_m^m(x))dx \tag{4.9}$$

and

$$0 \ge \int_{\Omega'} (f(x+y) - z_m(x+y))(\xi_2(x) - z_m^m(x+y))dx. \tag{4.10}$$

Let us take a sequence of functions $h_\delta \in C^\infty(\Omega)$, $0 \le h'_\delta \le 1$, $0 = h_\delta(0) \le |h_\delta| \le 1$ and the following choices for ξ_1 and ξ_2

$$\xi_1(x) = h_\delta(z_m^m(x+y) - z_m^m(x)) + z_m^m(x),$$

and

$$\xi_2(x) = -h_\delta(z_m^m(x+y) - z_m^m(x)) + z_m^m(x+y).$$

Adding (4.9) and (4.10), we have

$$\begin{aligned}\int_{\Omega'} & h_\delta(z_m^m(x+y)-z_m^m(x))(z_m(x+y)-z_m(x))dx \\ &\leq \int_{\Omega'}(f(x+y)-f(x))h_\delta(z_m^m(x+y)-z_m^m(x))dx \\ &\leq \int_{\Omega'}|f(x+y)-f(x)|dx.\end{aligned}$$

Taking h_δ such that $h_\delta(r)\to \mathrm{sign}_0(r)$ as $\delta\to 0$,

$$\int_{\Omega'}|z_m(x+y)-z_m(x)|dx \leq \int_{\Omega'}|f(x+y)-f(x)|dx$$

and (4.8) is satisfied. By (4.2) and (4.4), it then follows that $\underline{z}\in \mathrm{sign}(w)$. Recall, by (4.7), that

$$\|z_m\|_\infty \leq \|f\|_\infty.$$

Therefore, using also (4.4), taking the limit as m tends to infinity in (4.1), we get

$$0\geq \lim_{m\to\infty}\int_\Omega (f-z_m)(\xi-z_m^m)dx = \int_\Omega (f-\underline{z})(\xi-w)dx.$$

To prove uniqueness, let us suppose that there exist two solutions, that is, $z_i\in \mathrm{sign}(w_i)$, $i=1,2$, which satisfy

$$0\geq \int_\Omega (f-z_1)(\xi-w_1)dx \tag{4.11}$$

and

$$0\geq \int_\Omega (f-z_2)(\xi-w_2)dx. \tag{4.12}$$

Substituting $\xi=w_2$ and $\xi=w_1$ respectively in (4.11) and (4.12), since $w\in\tilde{\mathbb{K}}$, by (4.6), we obtain

$$0\geq \int_\Omega (z_1-z_1)(w_1-w_2)$$

and therefore the solution is unique.

■

We are now ready to obtain the regular limit of the solutions u_m and the equation it satisfies, under the additional condition that the initial data which we will denote by u_{0_m} satisfies $\|u_{0_m}\|_\infty \leq 1$. The following theorem holds.

Theorem 4.2. *Consider the problem in (2.11), where u_{0_m} and g satisfy the same conditions as in theorem 2.4, as well as $\|u_{0_m}\|_\infty \leq 1$. Then, there exists a subsequence m_i tending to infinity, and a unique function $\underline{u}$ such that, for each $T > 0$,*

$$u_{m_i} \to \underline{u} \text{ in } C([0,T];L^1(\Omega)),$$

with

$$u_{0_m} \to u_{0_\infty} \text{ in } L^1(\Omega),$$

and $\underline{u}$ is the unique mild solution of

$$\begin{cases} \underline{u}_t + A_{\infty,\infty}(\underline{u}) \ni g \quad \text{in } \Omega \times (0,T] \\ \underline{u}(0) = u_{0_\infty}, \end{cases} \tag{4.13}$$

where $A_{\infty,\infty}$ is given by (1.9).

Proof: Observe that since $u_{0_m}^m \in \tilde{\mathbb{K}}$ and $\|u_{0_m}\|_\infty \leq 1$, then as $m \to \infty$, we obtain that there exists a u_{0_∞} such that

$$u_{0_m} \to u_{0_\infty} \text{ in } L^1(\Omega),$$

with $u_{0_\infty} \in \overline{D(A_{\infty,\infty})}$. Hence, using also lemma 4.1 and the fact that $A_{\infty,\infty}$ is T-accretive, the result follows by the same classical result in nonlinear semigroup theory as in theorem 2.4. ∎

We do the same analysis for the elliptic problem associated to the operator $A_{p,\infty}$. Denoting $z_p := (I + A_{p,\infty})^{-1} f$ for all $f \in L^\infty(\Omega)$, this problem has a solution in the following sense:

$$\begin{cases} z_p \in L^1(\Omega), \ \exists w_p \in W_0^{1,p}(\Omega), \ z_p \in \operatorname{sign}(w_p) \text{ a.e. in } \Omega \\ -\Delta_p w_p = f - z_p \text{ in } \mathscr{D}'(\Omega). \end{cases} \tag{4.14}$$

Observe that given what we know about the behaviour of the p-Laplace operator when $p \to \infty$ (see (2.6)), it is natural to seek the convergence in the resolvent sense of the operator $A_{p,\infty}$ to the operator $A_{\infty,\infty}$ as defined in (1.9).

Lemma 4.3. *For all $f \in L^\infty(\Omega)$ and $\lambda > 0$, we have*

$$(I + \lambda A_{p,\infty})^{-1} f \to (I + \lambda A_{\infty,\infty})^{-1} f \text{ in } L^1(\Omega),$$

when $p \to \infty$.

Proof: Recall that by property (1.8), for $f \in L^\infty(\Omega)$ and $z_{p,m}$ a solution of (2.8), we have

$$\|z_{p,m}\|_r \leq \|f\|_r \text{ for any } 1 \leq r \leq \infty,$$

and taking $m \to \infty$, it continues to hold

$$\|z_p\|_r \leq \|f\|_r \text{ for } 1 \leq r \leq \infty, \tag{4.15}$$

for $z_p = (I + A_{p,\infty})^{-1} f$. On the other hand, we see that there exists a w_p which is a solution of the equation in (4.14) and therefore satisfies

$$\int_\Omega |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi dx = \int_\Omega (f - z_p) \varphi dx \;\; \forall \varphi \in \mathscr{D}(\Omega).$$

By density, we can consider $\varphi = w_p$ in the previous expression to obtain

$$\begin{aligned} \int_\Omega |\nabla w_p|^p dx &\leq \|f - z_p\|_\infty \|w_p\|_1 \\ &\leq C \|f - z_p\|_\infty \|\nabla w_p\|_1 \\ &\leq 2C \|f\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla w_p|^p dx \right)^{\frac{1}{p}} |\Omega|^{1 - \frac{1}{p}}. \end{aligned}$$

The second inequality is due to Poincaré's inequality, with $p = 1$, and for the third we use (4.15) and Hölder's inequality. We have as well, by Hölder's inequality,

$$\|\nabla w_p\|_q \leq \|\nabla w_p\|_p |\Omega|^{\frac{1}{q} - \frac{1}{p}},$$

for any $p > q$ and we obtain that $\{w_p\}$ is uniformly bounded in $W_0^{1,q}(\Omega)$ for any $q > 1$. Hence, there exists a subsequence $\{p_i\}$ and a function w such that, when $p_i \to \infty$,

$$w_{p_i} \rightharpoonup w \text{ in } W^{1,q}(\Omega), \text{ for any } q > 1.$$

Thus, passing as necessary to yet another subsequence and relabelling, we deduce

$$\begin{cases} w_{p_i} \to w \text{ in } L^q(\Omega), \\ w_{p_i} \to w \text{ a.e.} \end{cases} \tag{4.16}$$

By the bound in (4.15),we have that there exists a function z such that, for q' the conjugate of q, when $p_i \to \infty$,

$$z_{p_i} \rightharpoonup z \text{ in } L^{q'}(\Omega).$$

Recalling that $z_p \in \operatorname{sign}(w_p)$, then by [34, Lemma $A.2$], it continues to hold in the limit that $z \in \operatorname{sign}(w)$. Moreover,

$$z_{p_i} \to z \text{ a.e.}$$

This together with (4.15) gives us the following strong convergence

$$z_{p_i} \to z \text{ in } L^1(\Omega). \tag{4.17}$$

Besides, by the equation in (4.14), we also have that, for all $\xi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$,

$$\frac{1}{p}\int_\Omega |\nabla \xi|^p dx \geq \frac{1}{p}\int_\Omega |\nabla w_p|^p dx + \int_\Omega (f - z_p)(\xi - w_p)dx \geq \int_\Omega (f - z_p)(\xi - w_p)dx.$$

Taking $\xi \in \tilde{\mathbb{K}}$, assuming by approximation that ξ has compact support, we have by (4.16) and (4.15),

$$0 \geq \lim_{p_i \to \infty} \int_\Omega (f - z_{p_i})(\xi - w_{p_i})dx = \int_\Omega (f - z)(\xi - w)dx.$$

Moreover, we have that,

$$\|\nabla w\|_\infty \leq 1. \tag{4.18}$$

Indeed, just as in Lemma 3.2 in [22], taking $\eta > 0$ and denoting

$$A_\eta = \{x \in \Omega \mid |\nabla w| \geq 1 + \eta\},$$

then

$$(1+\eta)|A_\eta| \leq \int_{A_\eta} |\nabla w| dx \leq \liminf_{p \to \infty} \left(\int_\Omega |\nabla w_p|^p dx \right)^{1/p} |A_\eta|^{1-1/p} \leq |A_\eta|$$

and $|A_\eta| = 0$, showing that (4.18) holds. Uniqueness follows as in lemma 4.1. ∎

Therefore, by the results of corollary 2.3, lemmas 3.6, 4.1 and 4.3, the claims in theorem 1.1 are satisfied. Moreover, we can now obtain the regular limit of the solutions u_p and therefore show the final link in the convergence diagram (see fig. 1).

Theorem 4.4. *Let $\Omega = B(0,R)$ and consider the problem (3.15), where $A_{p,\infty}$ is defined in (3.3), $g \in L^1(\Omega_T)$, and u_0 is radial, non-negative and such that $\|u_0\|_\infty \leq 1$. Then, there exists a function u such that, for each $T > 0$,*

$$u_p \to u \text{ in } C([0,T]; L^1(\Omega))$$

and u is the unique mild solution of

$$\begin{cases} u_t + A_{\infty,\infty} u \ni g & \text{in } \Omega \times [0,T] \\ u(0) = u_0, \end{cases} \tag{4.19}$$

where $A_{\infty,\infty}$ is defined in (1.9).

Proof: The result follows by the same arguments as in theorem 4.2, using lemma 4.3 and the fact that $u_0 \in \overline{D(A_{\infty,\infty})}$. ■

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