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AN ANALOGUE OF THE ROBINSON-SCHENSTED-KNUTH CORRESPONDENCE AND NON-SYMMETRIC CAUCHY KERNELS FOR TRUNCATED STAIRCASES

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ABSTRACT: We prove a restriction of an analogue of the Robinson-Schensted-Knuth correspondence for semi-skyline augmented fillings, due to Mason, to multisets of cells of a staircase possibly truncated by a smaller staircase at the upper left end corner, or at the bottom right end corner. The restriction to be imposed on the pairs of semi-skyline augmented fillings is that the pair of shapes, rearrangements of each other, satisfies an inequality in the Bruhat order, w.r.t. the symmetric group, where one shape is bounded by the reverse of the other. For semi-standard Young tableaux the inequality means that the pair of their right keys is such that one key is bounded by the Schützenberger's evacuation of the other. This bijection is then used to obtain an expansion formula of the non-symmetric Cauchy kernel, over staircases or truncated staircases, in the basis of Demazure characters of type A, and the basis of Demazure atoms. The expansion implies a Lascoux's expansion formula, when specialised to staircases or truncated staircases, and make explicit, in the latter, the Young tableaux in the Demazure crystal by interpreting Demazure operators via elementary bubble sorting operators acting on weak compositions.

Keywords: Non-symmetric Cauchy kernels, Demazure character, key polynomial, Demazure operator, semi-skyline augmented filling, RSK analogue, crystal.
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1. Introduction

Given the general Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and its quantum group $U_q(\mathfrak{gl}_n)$, finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ are also classified by the highest weight. Let λ be a dominant integral weight (i.e. a partition), and $V(\lambda)$ the integrable representation with highest weight λ, and u_λ the highest weight vector. For a given permutation w in the symmetric group $\mathfrak{S}_n$, minimum for the Bruhat order in its class modulo the stabiliser of λ, the Demazure module is defined to be $V_w(\lambda) := U_q(\mathfrak{g})^{>0}.u_{w\lambda}$, and the Demazure character is the character of $V_w(\lambda)$. Kashiwara [12, 13] has associated with λ a crystal $\mathfrak{B}^\lambda$, which can be realised in

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type A as a coloured directed graph whose vertices are all semi-standard Young tableaux (SSYTs) of shape λ with entries $\leq n$, and the edges are coloured with a colour i, for each pair of crystal operators f_i, e_i, such that there exists a coloured i-arrow from the vertex P to P' if and only if $f_i(P) = P'$, equivalently, $e_i(P') = P$, for $i = 1, \ldots, n-1$. Littelmann [23] conjectured and Kashiwara [14] proved that the intersection of a crystal basis of V_λ with $V_w(\lambda)$ is a crystal basis for $V_w(\lambda)$. The resulting subset $\mathfrak{B}_{w\lambda} \subseteq \mathfrak{B}^\lambda$ is called Demazure crystal, and the Demazure character corresponding to λ and w, is the sum of the weight monomials of the SSYTs in the Demazure crystal $\mathfrak{B}_{w\lambda}$.

Demazure characters (or key polynomials) are also defined through Demazure operators (or isobaric divided differences). They were introduced by Demazure [3] for all Weyl groups and were studied combinatorially, in the case of $\mathfrak{S}_n$, by Lascoux and Schützenberger [18] who produce a crystal structure. We assume throughout $\mathbb{N}$ as the set of nonnegative integers. The action of the simple transpositions $s_i \in \mathfrak{S}_n$ on weak compositions in $\mathbb{N}^n$, by permuting the entries i and $i+1$, induces an action of $\mathfrak{S}_n$ on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by considering weak compositions α as exponents of monomials $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ [21], and defining $s_i x^\alpha := x^{s_i \alpha}$ as the simple transposition of x_i and x_{i+1} in the monomial x^α. If $f \in \mathbb{Z}[x_1, \ldots, x_n]$, $s_i f$ indicates the result of the action of s_i in each monomial of f. For $i = 1, \ldots, n-1$, one defines the linear operators π_i, $\widehat{\pi}_i$ on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\pi_i f = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}, \qquad \widehat{\pi}_i f = (\pi_i - 1) f = \pi_i f - f, \tag{1}$$

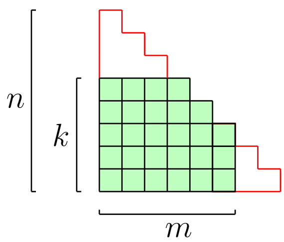
where 1 is the identity operator on $\mathbb{Z}[x_1, \ldots, x_n]$. These operators are called isobaric divided differences [21], and the first is the Demazure operator [3] for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.

The 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$, a deformation of the group algebra of $\mathfrak{S}_n$, can be faithfully realized either by its action on $\mathbb{Z}[x_1, \ldots, x_n]$ via isobaric divided difference operators $\{\pi_i : 1 \leq i < n\}$ or $\{\hat{\pi}_i : 1 \leq i < n\}$, or by the action on weak compositions in $\mathbb{N}^n$ via the elementary bubble sort operators, *i.e.* π_i is viewed as the operator which sorts the entries in positions i and $i+1$ in weakly increasing order. They are used to generate two kinds of key polynomials [18, 30], the Demazure characters [3, 11], and the Demazure atoms [25]. For $\alpha \in \mathbb{N}^n$, the key polynomial κ_α (resp. $\hat{\kappa}_\alpha$) is $\kappa_\alpha(x) = \hat{\kappa}_\alpha(x) = x^\alpha$, if α is a partition. Otherwise, $\kappa_\alpha(x) = \pi_i \kappa_{s_i\alpha}(x)$ (resp. $\hat{\kappa}_\alpha(x) = \hat{\pi}_i \hat{\kappa}_{s_i\alpha}(x)$), if $\alpha_{i+1} > \alpha_i$. The key polynomials lift the Schur polynomials when $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 \leq \ldots \leq \alpha_n$, and then $\kappa_\alpha = s_{(\alpha_n, \ldots, \alpha_1)}$. It should be noticed that the action of Demazure operators on key polynomials κ_α is described by the action of the elementary bubble sort operators on the weak composition α, $\pi_i \kappa_\alpha = \kappa_{s_i\alpha}$, if $\alpha_i > \alpha_{i+1}$, otherwise, $\pi_i \kappa_\alpha = \kappa_\alpha$ [30]. (One could also describe the action of the operator $\hat{\pi}_i$ on Demazure atom $\widehat{\kappa}_\alpha$ but we shall not need it.) Both key polynomials $\{\kappa_\alpha : \alpha \in \mathbb{N}^n\}$, and $\{\hat{\kappa}_\alpha : \alpha \in \mathbb{N}^n\}$ form linear $\mathbb{Z}$-bases for $\mathbb{Z}[x_1, \ldots, x_n]$. As the π_i's and $\hat{\pi}_i$'s both generate the 0-Hecke algebra and the change of bases is directly related with the Bruhat order on permutations, Demazure characters decompose into Demazure atoms in a sum over the Bruhat order [19, 29].

If $w' \leq w$ in the Bruhat order on the classes modulo the stabiliser of λ, $\mathfrak{B}_{w'\lambda} \subseteq \mathfrak{B}_{w\lambda}$. Setting $\widehat{\mathfrak{B}}_{w\lambda} := \mathfrak{B}_{w\lambda} \setminus \bigcup_{w' < w} \mathfrak{B}_{w'\lambda}$, one has the decomposition $\mathfrak{B}_{w\lambda} = \bigcup_{w' \leq w} \widehat{\mathfrak{B}}_{w'\lambda}$ [23]. A key tableau is a SSYT whose content is a rearrangement of the shape. Each component $\widehat{\mathfrak{B}}_{w\lambda}$ has exactly one key of shape λ and content $w\lambda$, $key(w\lambda)$. Lascoux and Schützenberger

[18] have characterised $\widehat{\mathfrak{B}}_{w\lambda}$ as the set of those SSYTs whose right key is the unique key tableau in $\widehat{\mathfrak{B}}_{w\lambda}$, and defined the Demazure atom (or standard basis) $\widehat{\kappa}_{w\lambda}(x)$ to be the sum of the weight monomials over $\widehat{\mathfrak{B}}_{w\lambda}$. As the sum of the weight monomials over all the crystal $\mathfrak{B}^\lambda$ gives the Schur polynomial s_λ, the Demazure atoms form a decomposition of the Schur functions. Specialising the combinatorial formula for the nonsymmetric Macdonald polynomials $E_\gamma(x;q;t)$ given in [7], by setting $q=t=0$, implies that $E_\gamma(x;0;0)$ is the sum of the weight monomials of all semi-skyline augmented fillings (SSAF) of shape γ which are fillings of diagrams of weak compositions with positive integers, weakly decreasing upwards along columns, and the rows satisfy inversion conditions. These polynomials are also a decomposition of the Schur polynomial s_λ, with $\gamma^+=\lambda$ the decreasing rearrangement of γ. Semi-skyline augmented fillings are in bijection with semi-standard Young tableaux such that the content is preserved and the right key of the SSYT is the unique key with content the shape of the SSAF [26]. Therefore, the Demazure atom $\widehat{\kappa}_\gamma(x)$ and $E_\gamma(x;0;0)$ are equal [25].

Mason shows [26] that semi-skyline augmented fillings also satisfy a variation of the Robinson-Schensted-Knuth algorithm which commutes with RSK and retains its symmetry. Semi-standard Young tableaux of shape λ and entries $\leq n$ decompose into subsets according to the right key. We see this RSK analogue as a refinement of the ordinary RSK where the right keys are provided. We consider the following Ferrers diagram, in the French convention, $\lambda=(m^{n-m+1},m-1,\dots,n-k+1)$, $1\leq m\leq n$, $1\leq k\leq n$, $n+1\leq m+k$, shown in green colour below,



Theorem 4, in Section 4, exhibits a bijection between multisets of cells of λ and pairs of SSAFs whose shapes satisfy an inequality in the Bruhat order in the symmetric group $\mathfrak{S}_\mathfrak{n}$ such that one shape is bounded by the reverse of the other. In particular, if $m+k=n+1$ then λ is a rectangle and it reduces to the ordinary RSK correspondence. We then use this bijection, in Section 6, to give an expansion of the non-symmetric Cauchy kernel $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}$, where the product is over all the cells (i,j) of λ in French convention. The expansion is obtained in two steps: firstly, the bijection provides an expansion as a sum of products of Demazure atoms and generating functions of SSYTs over the intersection of two Demazure crystals; secondly, interpreting the action of Demazure operators on key polynomials via the action of sorting operators on weak compositions, we compute the Demazure crystal resulting from those intersections, and, therefore, the key polynomials having those generating functions. More precisely, one obtains the general formula

$$\prod_{\substack{(i,j)\in\lambda\\ k\leq m}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y), \tag{2}$$

where ω is the longest permutation of $\mathfrak{S}_k$, and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ is such that for each $i = k, \dots, 1$, the entry α_i is the maximum element among the last $\min\{i, n-m+1\}$ entries of $\omega\mu$ after deleting α_j, for $i < j \leq k$. The Demazure crystal $\mathfrak{B}_{(0^{m-k},\alpha)}$ consists of all semi-standard Young tableaux with entries $\leq m$, shape $(\mu^+, 0^{m-k})$, and right key bounded by $key(0^{m-k}, \alpha)$. If $m < k$, the formula is symmetrical, swapping in (2) x with y, and k with m.

If λ is a rectangle, $\alpha = \mu^+$ and the classical expansion of the symmetric Cauchy kernel in the basis of Schur polynomials is recovered; and if λ is the staircases of length n, $\alpha = \omega\mu$, and the Lascoux's expansion in Theorem 6 of [20], and in [5], of the non symmetric Cauchy kernel in the basis of Demazure characters and the basis of Demazure atoms, is also recovered. The proofs, given by Lascoux, for the latter expansion, use double crystal graphs in [20], and, in [5], with Fu, is based on algebraic properties of isobaric divided differences. For truncated staircases, the expansion (2) implies Lascoux's formula in Theorem 7 of [20], and makes explicit the SSYTs of the Demazure crystal.

Our paper is organised in six sections. In Section 2 we first recall the definitions of compositions, Young tableaux, and key tableaux, then the Bruhat orders of the symmetric group $\mathfrak{S}_n$, and its parabolic quotients, with their several characterizations on weak compositions. In Section 3 we review the necessary terminology and theory of SSAFs, in particular, the variation of Schensted insertion and the RSK analogue for SSAFs where some useful properties for the result to be stated in next section are stressed. Then, in Section 4, we give our main result, Theorem 4, and an illustration of the bijection described in this theorem is provided. Section 5 is devoted to the necessary theory of crystal graphs in type A in connection with the combinatorial descriptions of Demazure operators and the two families of key polynomials. Finally, in the last section, we apply the bijection described in Theorem 4 to obtain expansions of Cauchy kernels over staircases and truncated stair cases. It is shown that the formulas provided by these bijective proofs coincide with Lascoux's formulas and, in addition, in the truncated case, they make explicit the Young tableaux.

2. Weak compositions, key tableaux and Bruhat orders on $\mathfrak{S}_n$

2.1. Young tableaux and key tableaux. Let $\mathbb{N}$ denote the set of non-negative integers. Fix a positive integer n, and define $[n]$ as the set $\{1, \dots, n\}$. A weak composition $\gamma = (\gamma_1 \dots, \gamma_n)$ is a vector in $\mathbb{N}^n$. If $\gamma_i = \cdots = \gamma_{i+k-1}$, for some $k \geq 1$, then we also write $\gamma = (\gamma_1 \dots, \gamma_{i-1}, \gamma_i^k, \gamma_{i+k}, \dots, \gamma_n)$. We often concatenate weak compositions $\alpha \in \mathbb{N}^r$ and $\beta \in \mathbb{N}^s$, with $r + s = n$, to form the weak composition $(\alpha, \beta) = (\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s) \in \mathbb{N}^n$. A weak composition γ whose entries are in weakly decreasing order, that is, $\gamma_1 \geq \cdots \geq \gamma_n$, is said to be a partition. Every weak composition γ determines a unique partition γ^+ obtained by arranging the entries of γ in weakly decreasing order. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is identified with its Young diagram (or Ferrers shape) $dg(\lambda)$ in French convention, an array of left-justified cells with λ_i cells in row i from the bottom, for $1 \leq i \leq n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$. The number $\ell(\lambda)$ of rows in the Young diagram $dg(\lambda)$ with a positive number of cells is said to be the length of the partition λ. A filling of shape λ, in the alphabet $[n]$, is a map $T : dg(\lambda) \to [n]$. A semi-standard Young tableau (SSYT) T of shape λ, in the alphabet $[n]$, is a filling of $dg(\lambda)$ which is weakly increasing in each row from left to right and strictly increasing up in each column. Let SSYT_n denote the set of all semi-standard Young tableaux with entries $\leq n$. The column word of $T \in \mathrm{SSYT}_n$ is the word, over the alphabet $[n]$, which consists of the entries of each column, read top to bottom and left to right. The content or weight of $T \in \mathrm{SSYT}_n$ is the content or weight of its column word which is the weak composition $c(T) = (\alpha_1, \dots, \alpha_n)$ such that T has α_i cells with entry i.

A key tableau is a semi-standard Young tableau such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection [30] between weak compositions in $\mathbb{N}^n$ and keys in the alphabet $[n]$ given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the key such that for all j, the first γ_j columns contain the letter j. Any key tableau is of the form $key(\gamma)$ with γ its content and γ^+ the shape. In particular, when $\gamma = \gamma^+$ it is also called Yamanouchi tableau of shape γ.

2.2. Bruhat orders on $\mathfrak{S}_n$. The symmetric group $\mathfrak{S}_n$ is generated by the simple transpositions $s_i = (i\ i+1)$, $1 \leq i < n$, which satisfy the Coxeter relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i, \ \text{ for } \ |i-j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{3}$$

Given $\sigma \in \mathfrak{S}_n$, let $\sigma = s_{i_N} \cdots s_{i_1}$ be a decomposition of σ into simple transpositions. When the number N in a such decomposition is minimised we say that we have a reduced decomposition of σ, and N is called its length $\ell(\sigma)$. In this case, we say that the sequence of indices $(i_N, \dots, i_1)$ is a reduced word for σ. The unique element of maximal length in $\mathfrak{S}_n$ is denoted by ω. It is a well known fact that any two reduced decompositions for σ are connected by a sequence of the last two Coxeter relations (3), called, respectively, commutation and braid relations.

Given θ and σ in $\mathfrak{S}_n$, σ covers θ in the (strong) Bruhat order if there is a transposition $t = (i\ j)$, $1 \leq i,\ j \leq n$, such that $t\theta = \sigma$ and their lengths differ in one unity [2]. It can be shown that in this case there is also a transposition t' such that $\theta t' = \sigma$. The Bruhat order in $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ which is the transitive closure of the relations

$$\theta < t\theta, \text{ if } \ell(\theta) < \ell(t\theta), \ (t \text{ transposition}, \theta \in \mathfrak{S}_n). \tag{4}$$

We recall the subword property of the (strong) Bruhat order in a Coxeter group W.

Theorem 1. [2] *Let θ, σ in $\mathfrak{S}_n$ and $(i_N, \dots, i_1)$ a reduced word for σ, then $\theta \leq \sigma$ if and only if there exists a subsequence of $(i_N, \dots, i_1)$ which is a reduced word for θ.*

Notice that the maximal length element ω is the maximal element of the Bruhat order, $\sigma \leq \omega$, for any $\sigma \in \mathfrak{S}_n$, and it satisfies $\omega^2 = 1$. Besides, its left and right translations $\sigma \to \omega\sigma$ and $\sigma \to \sigma\omega$ are anti automorphisms for the Bruhat order.

The action of $\mathfrak{S}_n$ in $\mathbb{N}^n$ is defined by the left action of a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n$, written in one-line notation, on a vector $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathbb{N}^n$, that is, $\sigma^{-1}(\gamma_1,\gamma_2,\dots,\gamma_n) := (\gamma_{\sigma_1}, \gamma_{\sigma_2}, \dots, \gamma_{\sigma_n})$, each component γ_i ends up at position σ_i in the sequence permutated by σ. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition, and $\mathfrak{S}_n\lambda$ the $\mathfrak{S}_n$-orbit of λ. The stabiliser of λ under the action of $\mathfrak{S}_n$ is $stab_\lambda := \{\sigma \in \mathfrak{S}_n : \sigma\lambda = \lambda\}$. Given $\sigma \in \mathfrak{S}_n$, the class of σ modulo the stabiliser of λ, is the (left) coset $\sigma stab_\lambda$. Two permutations σ and θ in $\mathfrak{S}_n$ are said to be in the same class modulo the stabiliser of λ if their cosets are equal or, equivalently, $\theta^{-1}\sigma \in stab_\lambda$. Each coset in $\mathfrak{S}_n/stab_\lambda$, the set of left cosets of $stab_\lambda$ in $\mathfrak{S}_n$, has a unique shortest permutation called a minimal length coset representative for $\mathfrak{S}_n/stab_\lambda$ [2].

The $\mathfrak{S}_n$-orbit of λ is therefore in bijection with the set of cosets of $\mathfrak{S}_n$ modulo the stabiliser of λ, that is, with the set of minimal length coset representatives of $\mathfrak{S}_n/stab_\lambda$, whose cardinality is given by $n!/|stab_\lambda|$ [1, 2, 10]. It is known [33, 34] that the restriction of the Bruhat ordering from $\mathfrak{S}_n$ to the set of minimal length coset representatives for $\mathfrak{S}_n/stab_\lambda$ can be converted to an ordering of $\mathfrak{S}_n\lambda$ by taking the transitive closure of the relations

$$\gamma < t\gamma, \text{ when } \gamma_i > \gamma_j,\ i < j, \text{ and } t \text{ the transposition } (i\, j). \tag{5}$$

If α_1 and α_2 are in the $\mathfrak{S}_n$-orbit of λ, and σ_1 and σ_2 are the shortest length coset representatives of $\mathfrak{S}_n/stab_\lambda$ such that $\sigma_1\lambda = \alpha_1$, $\sigma_2\lambda = \alpha_2$, then we write $\alpha_1 \leq \alpha_2$ in the sense of (5) which is equivalent to $\sigma_1 \leq \sigma_2$ in the Bruhat order. Henceforth, we say that (5) defines the Bruhat order on the weak compositions in the $\mathfrak{S}_n$-orbit of λ.

If we replace, in (4), t with the simple transposition s_i, which translates into $\theta^{-1}(i+1) < \theta^{-1}(i)$, the transitive closure of a such relations defines the left weak Bruhat order on $\mathfrak{S}_n$. Its restriction to the set of minimal length coset representatives of $\mathfrak{S}_n/stab_\lambda$ is then converted to an ordering in $\mathfrak{S}_n\lambda$ by replacing, in (5), t with s_i. Consider now the elementary bubble sorting operation π_i, $1 \leq i < n$, on words $\gamma_1\gamma_2\cdots\gamma_n$, with repetitions of letters, of length n (or weak compositions in $\mathbb{N}^n$), which sorts the letters in positions i and $i+1$ in weakly increasing order, that is, it swaps γ_i and γ_{i+1} if $\gamma_i > \gamma_{i+1}$, or fixes $\gamma_1\gamma_2\cdots\gamma_n$ otherwise. Define the partial order on $\mathfrak{S}_n\lambda$ by taking the transitive closure of the relations $\gamma < \pi_i(\gamma)$ when $\gamma_i > \gamma_{i+1}$, ($\gamma \in \mathfrak{S}_n\lambda$ and $1 \leq i < n$). It coincides with the induced left weak Bruhat ordering of $\mathfrak{S}_n\lambda$. The bottom of the Hasse diagram of the partial order defined by (5) is the partition λ and the top is the reverse of λ.

It can be proved that the elementary bubble sorting operations π_i, $1 \leq i < n$, satisfy the relations

$$\pi_i^2 = \pi_i,\ \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}, \text{ and } \pi_i\pi_j = \pi_j\pi_i, \text{ for } |i-j| > 1. \tag{6}$$

The minimal length coset representatives for $\mathfrak{S}_n/stab_\lambda$ are characterized in [1, 2, 10]. However, we recall here a construction of that set, due to Lascoux, in [20], where the notion of key tableau is used. This allows to convert the tableau criterion for the Bruhat order in $\mathfrak{S}_n$ to a tableau criterion in the induced Bruhat order (5) in the orbit $\mathfrak{S}_n\lambda$. The bijection between staircase keys of shape $(n, \dots, 1)$ and permutations in $\mathfrak{S}_n$ gives a tableau criterion for the Bruhat order [4, 24]. If σ is a permutation in $\mathfrak{S}_n$, its key tableau, $key(\sigma(n, \dots, 1))$, is the semi-standard Young tableau with shape $(n, \dots, 1)$, in which the i^{th} column consists of the $n-i+1$ integers $\sigma(1), \dots, \sigma(n-i+1)$, placed in increasing order from bottom to top. Reciprocally any staircase key may be obtained in this way by defining the following permutation: first write the element of the right most column of the key then the new element that appears in the column next to the last, and so on. We have therefore the well-known tableau criterion for the Bruhat order in $\mathfrak{S}_n$.

Proposition 1. [24] *Let σ, $\beta \in \mathfrak{S}_n$, we have $\sigma \leq \beta$ if and only if $key(\sigma(n, \dots, 1)) \leq key(\beta(n, \dots, 1))$ for the entrywise comparison.*

In [20], Lascoux constructs the shortest element in the coset $\sigma stab_\lambda$ such that $\sigma\lambda = \gamma \in \mathbb{N}^n$ using the key tableau of γ as follows: firstly, add the complete column $[n \dots 1]$ as the left most column of $key(\gamma)$, if γ has an entry equal to zero, secondly, write the elements of the right most column of $key(\gamma)$ in increasing order then the new elements that appear in the column next to the last in increasing order and so on until the first column. The resulting word is the desired permutation in $\mathfrak{S}_n$.

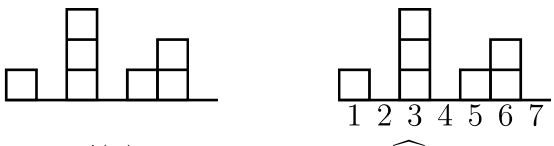
Example 1. *Let* $\gamma = (1,3,0,1)$ *and* $key(\gamma) = \begin{array}{lll} 4 & & \\ 2 & & \\ 1 & 2 & 2 \end{array}$. *First add the complete column* $[4,3,2,1]$, *to get* $\begin{array}{llll} 4 & & & \\ 3 & 4 & & \\ 2 & 2 & & \\ 1 & 1 & 2 & 2 \end{array}$. *Hence,* $\sigma = 2143$ *is the shortest permutation in the coset* $\sigma\, stab_{3110}$*, where* $stab_{3110} =< s_2 >$.

Theorem 2. *Let* α_1 *and* α_2 *be in the* $\mathfrak{S}_n$*-orbit of* λ*. Then* $\alpha_1 \leq \alpha_2$ *if and only if* $key(\alpha_1) \leq key(\alpha_2)$.

Proof: Let σ_1 and σ_2 be the shortest length representatives of $\mathfrak{S}_n/stab_\lambda$ such that $\sigma_1\lambda = \alpha_1$, $\sigma_2\lambda = \alpha_2$. Then, $\alpha_1 \leq \alpha_2$ if and only if $\sigma_1 \leq \sigma_2$ in Bruhat order, and, by Proposition 1, this means $key(\sigma_1(n,\ldots,1)) \leq key(\sigma_2(n,\ldots,1))$. Using the constructions of σ_1 and σ_2 explained above this is equivalent to say that $key(\alpha_1) \leq key(\alpha_2)$. ■

3. Semi-skyline augmented fillings

3.1. Definitions and properties. We follow most of the time the conventions and terminology in [7, 8] and [25, 26]. A weak composition $\gamma = (\gamma_1,\ldots,\gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j boxes in column j, for $1 \leq j \leq n$. Formally, the *column diagram* of γ is the set $dg'(\gamma) = \{(i,j) \in \mathbb{N}^2 : 1 \leq j \leq n, 1 \leq i \leq \gamma_j\}$ where the coordinates are in French convention, i indicates the vertical coordinate, indexing the rows, and j the horizontal coordinate, indexing the columns. (The prime reminds that the components of γ are the columns.) The number of cells in a column is called the height of that column and a cell a in a column diagram is denoted $a = (i,j)$, where i is the row index and j is the column index. The *augmented diagram* of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0,j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells adjoined in row 0. This adjoined row is called the *basement* and it always contains the numbers 1 through n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. For example, column diagram and the augmented diagram for $\gamma = (1,0,3,0,1,2,0)$ are



$dg'(\gamma)$ $\widehat{dg}(\gamma)$

An *augmented filling* F of an augmented diagram $\widehat{dg}(\gamma)$ is a map $F : \widehat{dg}(\gamma) \to [n]$, which can be pictured as an assignment of positive integer entries to the non-basement cells of $\widehat{dg}(\gamma)$. Let $F(i)$ denote the entry in the i^{th} cell of the augmented diagram encountered when F is read across rows from left to right, beginning at the highest row and working down to the bottom row. This ordering of the cells is called the *reading order*. A cell $a = (i,j)$ precedes a cell $b = (i',j')$ in the reading order if either $i' < i$ or $i' = i$ and $j' > j$. The reading word of F is obtained by recording the non-basement entries in reading order. The content of an augmented filling F is the weak composition $c(F) = (\alpha_1, \dots, \alpha_n)$ where α_i is the number of non-basement cells in F with entry i, and n is the number of basement elements.

The standardization of F is the unique augmented filling that one obtains by sending the i^{th} occurrence of j in the reading order to $i + \sum_{m=1}^{j-1} \alpha_m$.

Let $a, b, c \in \widehat{dg}(\gamma)$ three cells situated as follows, $\begin{array}{l}\boxed{a}\cdots\boxed{c}\\ \boxed{b}\end{array}$, where a and c are in the same row, possibly the first row, possibly with cells between them, and the height of the column containing a and b is greater than or equal to the height of the column containing c. Then the triple a, b, c is an inversion triple of type 1 if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a counterclockwise orientation. Similarly, consider three cells $a, b, c \in \widehat{dg}(\gamma)$ situated as follows, $\begin{array}{r}\boxed{b}\\ \boxed{a}\cdots\boxed{c}\end{array}$ where a and c are in the same row (possibly the basement) and the column containing b and c has strictly greater height than the column containing a. The triple a, b, c is an inversion triple of type 2 if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a clockwise orientation.

Define a semi-skyline augmented filling (SSAF) of an augmented diagram $\widehat{dg}(\gamma)$ to be an augmented filling F such that every triple is an inversion triple and columns are weakly decreasing from bottom to top. The shape of the semi-skyline augmented filling is γ and denoted by $sh(F)$.

The picture below is an example of a semi-skyline augmented filling with shape $(1, 0, 3, 2, 0, 1)$, reading word 1321346 and content $(2, 1, 2, 1, 0, 1)$.

$$\begin{array}{cccccc} & & \boxed{1} & & & \\ & & \boxed{3} & \boxed{2} & & \\ \boxed{1} & & \boxed{3} & \boxed{4} & & \boxed{6} \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

The entry of a cell in the first row of a SSAF is equal to the basement element where it sits and, thus, in the first row the cell entries strictly increase from left to the right. For any weak composition γ in $\mathbb{N}^n$, there is at least one SSAF with shape γ, by putting γ_i cells with entries i in the top of the basement element i. For example, the following is a SSAF of shape and content $(1, 1, 3, 2, 0, 1)$,

$$\begin{array}{cccccc} & & \boxed{3} & & & \\ & & \boxed{3} & \boxed{4} & & \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & & \boxed{6} \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

In [26] a sequence of lemmas provide several conditions on triples of cells in a SSAF. In particular, we recall Lemma 2.6 in [26] which characterises completely the relative values of the entries in the cells of a type 2 inversion triple in a SSAF. This property of type 2 inversion triples will be used in the proof of our main theorem. Given a cell a in SSAF F define $F(a)$ to be the entry in a.

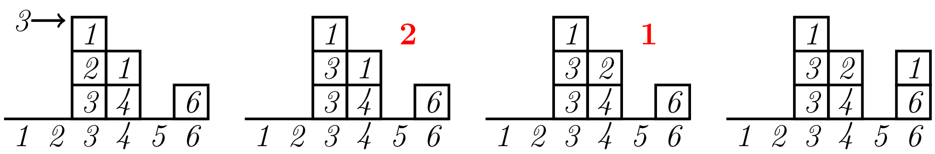
Lemma 1. [26] *If* $\{a, b, c\}$, $\boxed{a}\cdots\begin{array}{|c|}\hline b\\\hline c\\\hline\end{array}$, *is a type* 2 *inversion triple in* F *then* $F(a) < F(b) \leq F(c)$.

3.2. An analogue of Schensted insertion and RSK for SSAFs. The fundamental operation of the Robinson-Schensted-Knuth (RSK) algorithm is Schensted insertion which is a procedure for inserting a positive integer k into a semi-standard Young tableau T. In [26], Mason defines a similar procedure for inserting a positive integer k into a SSAF F, which is used to describe an analogue of the RSK algorithm. If F is a SSAF of shape γ, we set $F := (F(j))$, where $F(j)$ is the entry in the j^{th} cell in reading order, with the cells in the basement included, and j goes from 1 to $n + \sum_{i=1}^{n} \gamma_i$. If $\hat{j}$ is the cell immediately above j and the cell is empty, set $F(\hat{j}) = 0$. The operation $k \to F$, for $k \leq n$, is defined as follows.
Procedure. The insertion $k \to F$:

1. Set $i := 1$, set $x_1 := k$, set $p_0 = \emptyset$, and set $j := 1$.
2. If $F(j) < x_i$ or $F(\hat{j}) \geq x_i$, then increase j by 1 and repeat this step. Otherwise, set $x_{i+1} := F(\hat{j})$ and set $F(\hat{j}) := x_i$. Set $p_i = (b + 1, a)$, where (b, a) is the j^{th} cell in reading order. (This means that the entry x_i "*bumps*" the entry x_{i+1} from the cell p_i.)
3. If $x_{i+1} \neq 0$ then increase i by 1, increase j by 1, and repeat step 2.
4. Set t_k equal to p_i, which is the termination cell, and terminate the algorithm.

The procedure terminates in finitely many steps and the result is a SSAF.

Example 2. *Insertion* 3 *to the SSAF.*



Based on this Schensted insertion analogue, Mason gives a weight preserving and shape rearranging bijection Ψ between SSYTs and SSAFs over the alphabet $[n]$. The bijection Ψ is defined to be the insertion, from right to left, of the column word of a SSYT into the empty SSAF with basement $1, \ldots, n$. Using the shape of $\Psi(T)$, the bijection provides the right key of T, $K_+(T)$, a notion due to Lascoux and Schützenberger [18]. There are now several ways to describe the right key of a tableau [18, 6, 22, 27]. For our purpose we consider the following Mason's theorem as the definition of right key of T.

Theorem 3. (Mason [25]) *Given an arbitrary SSYT T, let γ be the shape of $\Psi(T)$. Then $K_+(T) = key(\gamma)$.*

Let SSAF_n be the set of all semi-skyline augmented fillings with basement $[n]$. Given the partition $\lambda \in \mathbb{N}^n$, let $\mathfrak{B}^\lambda$ denote the set of all semi-standard Young tableaux in SSYT_n of shape λ. The bijection Ψ between SSYT_n and SSAF_n and this theorem decompose $\mathfrak{B}^\lambda$ in a disjoint union of semi-standard Young tableaux w.r.t. to their right keys:

$$\mathfrak{B}^\lambda = \biguplus_{\gamma \in \mathfrak{S}_n \lambda} \{T \in SSYT_n : K_+(T) = key(\gamma)\}.$$

Example 3. *One has* $sh(\Psi(T)) = (2, 0, 4, 3, 1)$,

$$
\underset{T}{\begin{array}{|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 3 & 4 \\
\cline{1-3} 2 & 3 & 4 \\
\hline 1 & 1 & 1 & 3 \\
\hline
\end{array}}
\xrightarrow{\ \Psi\ }
\underset{\Psi(T)}{\begin{array}{ccccc}
 & & 1 & & \\
 & & 3 & 2 & \\
1 & & 3 & 4 & \\
1 & & 3 & 4 & 5 \\
\hline
1 & 2 & 3 & 4 & 5
\end{array}}\ ,\qquad
\underset{K_+(T) = key(2,0,4,3,1)}{\begin{array}{|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 4 & 4 \\
\cline{1-3} 3 & 3 & 4 \\
\hline 1 & 1 & 3 & 3 \\
\hline
\end{array}}
$$

Given the alphabet $[n]$, the RSK algorithm is a bijection between biwords in lexicographic order and pairs of SSYT of the same shape over $[n]$. Equipped with the Schensted insertion anlogue, Mason finds in [26] an analogue Φ of the RSK yielding a pair of SSAFs. This bijection has an advantage over the classical RSK because the pair of SSAFs comes along with the extra pair of right keys.

The two line array $w = \begin{pmatrix} i_1 & i_2 & \cdots & i_l \\ j_1 & j_2 & \cdots & j_l \end{pmatrix}$, $i_r < i_{r+1}$, or $i_r = i_{r+1}$ & $j_r \leq j_{r+1}$, $1 \leq i, j \leq l-1$, with $i_r,\ j_r \in [n]$, is called a biword in lexicographic order over the alphabet $[n]$. The map Φ defines a bijection between the set $\mathbb{A}_n$ of all biwords w in lexicographic order in the alphabet $[n]$, and pairs of SSAFs with shapes in the same $\mathfrak{S}_n$-orbit, and the contents are respectively those of the second and first rows of w.

Procedure. The map $\Phi : \mathbb{A}_n \longrightarrow SSAF_n \times SSAF_n$. Let $w \in \mathbb{A}_n$.

1. Set $r := l$, where l is the number of biletters in w. Let $F = \emptyset = G$, where $\emptyset$ is the empty $SSAF$.
2. Set $F := (j_r \to F)$. Let h_r be the height of the column in $(j_r \to F)$ at which the insertion procedure $(j_r \to F)$ terminates.
3. Place i_r on top of the leftmost column of height $h_r - 1$ in G such that doing so preserves the decreasing property of columns from bottom to top. Set G equal to the resulting figure.
4. If $r - 1 \neq 0$, repeat step 2 for $r := r - 1$. Else terminate the algorithm.

Remark 1. 1. *The entries in the top row of the biword are weakly increasing when read from left to right. Henceforth, if $h_r > 1$, placing i_r on top of the leftmost column of height $h_r - 1$ in G preserves the decreasing property of columns. If $h_r = 1$, the i_r^{th} column of G does not contain an entry from a previous step. It means that number i_r sits on the top of basement i_r.*
2. *Let h be the height of the column in F at which the insertion procedure $(j \to F)$ terminates. Lemma 1 implies that there is no column of height $h+1$ in F to the right.*

Corollary 1. (Mason [25, 26]) *The RSK algorithm commutes with the above analogue Φ. That is, if (P, Q) is the pair of SSYTs produced by RSK algorithm applied to biword w, then $(\Psi(P), \Psi(Q)) = \Phi(w)$, and $K_+(P) = key(sh(\Psi(P)))$, $K_+(Q) = key(sh(\Psi(Q)))$.*

This result is summarised in the following scheme from which, in particular, it is clear the RSK analogue Φ also shares the symmetry of RSK,

$$
\begin{tikzcd}
 & w \arrow[dl, dash, "\mathrm{RSK}"'] \arrow[dr, dash, "\Phi"] & \\
(P,Q) \arrow[rr, dash, "\Psi"'] & & (F,G)
\end{tikzcd}
\qquad
\begin{array}{c}
c(P) = c(F),\ c(Q) = c(G), \\
sh(F)^+ = sh(G)^+ = sh(P) = sh(Q), \\
K_+(P) = key(sh(F)),\ K_+(Q) = key(sh(G)).
\end{array}
$$

4. Main Theorem

We prove a restriction of the bijection Φ to multisets of ordered pairs of positive integers $\{(a_1,b_1),(a_2,b_2),\dots,(a_r,b_r)\}$, $r\ge 0$, where (a_i,b_i) are cell-coordinates of a truncated staircase of length n, such that the staircases of length $n-k$ on the upper left corner, or of length $n-m$ on the bottom right corner, with $1\le m\le n$, $1\le k\le n$ and $k+m\ge n+1$, are erased. The restriction to be imposed on the pairs of SSAFs is that the pair of shapes in $\mathbb{N}^m\times\mathbb{N}^k$, but in a same $\mathfrak{S}_n$-orbit, satisfy an inequality in the Bruhat order, where each shape is bounded by the reverse of the other. Equivalently, pairs of SSYTs whose right keys are such that each one is bounded by the evacuation of the other.

The following lemma gives sufficient conditions to preserve the Bruhat order relation between two weak compositions by adding one box to their column diagrams.

Lemma 2. *Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\dots,\beta_n)$ be in the same $\mathfrak{S}_n$-orbit, with $key(\beta)\le key(\alpha)$. Given $k\in\{1,\dots,n\}$, let $k'\in\{1,\dots,n\}$ be such that $\beta_{k'}$ is the left most entry of β satisfying $\beta_{k'}=\alpha_k$. Then if $\tilde\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k+1,\dots,\alpha_n)$ and $\tilde\beta=(\beta_1,\beta_2,\dots,\beta_{k'}+1,\dots,\beta_n)$, it holds $key(\tilde\beta)\le key(\tilde\alpha)$.*

Proof: Let $k,\ k'\in\{1,\dots,n\}$ as in the lemma, and put $\alpha_k=\beta_{k'}=m\ge 1$. (The proof for $m=0$ is left to the reader. The case of interest for our problem is $m>0$ which is related with the procedure of map Φ.) This means that k appears exactly in the first m columns of $key(\alpha)$, and k' is the smallest number that does not appear in column $m+1$ of $key(\beta)$ but appears exactly in the first m columns. Let t be the row index of the cell with entry k' in column m of $key(\beta)$. Every entry less than k' in column m of $key(\beta)$ appears in column $m+1$ as well, and since in a key tableau each column is contained in the previous one, this implies that the first t rows of columns m and $m+1$ of $key(\tilde\beta)$ are equal. The only difference between $key(\tilde\beta)$ and $key(\beta)$ is in columns $m+1$, from row t to the top. Similarly if z is the row index of the cell with entry k in column $m+1$ of $key(\tilde\alpha)$, the only difference between $key(\tilde\alpha)$ and $key(\alpha)$ is in columns $m+1$ from row z to the top. To obtain column $m+1$ of $key(\tilde\beta)$, shift in the column $m+1$ of $key(\beta)$ all the cells with entries $>k'$ one row up, and add to the position left vacant (of row index t) a new cell with entry k'. The column $m+1$ of $key(\tilde\alpha)$ is obtained similarly, by shifting one row up in the column $m+1$ of $key(\alpha)$ all the cells with entries $>k$ and adding a new cell with entry k in the vacant position.

Put $p:=\min\{t,z\}$ and $q:=\max\{t,z\}$. We divide the columns $m+1$ in each tableau pair $key(\beta),\ key(\tilde\beta)$ and $key(\alpha),\ key(\tilde\alpha)$ into three parts: the first, from row one to row $p-1$; the second, from row p to row q; and the third, from row $q+1$ to the top row. The first parts of column $m+1$ of $key(\tilde\beta)$ and $key(\beta)$ are the same, equivalently, for $key(\tilde\alpha)$ and $key(\alpha)$. The third part of column $m+1$ of $key(\tilde\beta)$ consists of row q plus the third part of $key(\beta)$, equivalently, for $key(\tilde\alpha)$ and $key(\alpha)$. As columns $m+1$ of $key(\beta)$ and $key(\alpha)$ are entrywise comparable, the same happens to the first and third parts of columns $m+1$ in $key(\tilde\beta)$ and $key(\tilde\alpha)$. It remains to analyse the second parts of the pair $key(\tilde\beta),\ key(\tilde\alpha)$ which we split into two cases according to the relative magnitude of p and q.

Case 1. $p=t<q=z$. Let $k'<b_t<\cdots<b_{z-1}$ and $d_t<\cdots<d_{z-1}<k$ be respectively the cell entries of the second parts of columns $m+1$ in the pair $key(\tilde\beta),\ key(\tilde\alpha)$. By construction $k'<b_t\le d_t<d_{t+1}$, $b_i<b_{i+1}\le d_{i+1}$, $t<i<z-2$, and $b_{z-1}\le d_{z-1}<k$, and, therefore, the second parts are also comparable.

Case 2. $p = z \leq q = t$. In this case, the assumption on k' implies that the first q rows of columns m and $m+1$ of $key(\tilde{\beta})$ are equal. On the other hand, since column m of $key(\beta)$ is less or equal than column m of $key(\alpha)$, which is equal to the column m of $key(\tilde{\alpha})$ and in turn is less or equal to column $m+1$ of $key(\tilde{\alpha})$, forces by transitivity that the second part of column $m+1$ of $key(\tilde{\beta})$ is less or equal than the corresponding part of $key(\tilde{\alpha})$. ■

We illustrate the lemma with

Example 4. *Let* $\beta = (3, 2^2, 1, 0^2, 1)$, $\alpha = (2, 0, 3, 0, 1, 2, 1)$, $\tilde{\beta} = (3, 2^3, 0^2, 1)$, *and* $\tilde{\alpha} = (2, 0, 3, 0, 2^2, 1)$,

$$key(\beta) = \begin{array}{lll} 7 & & \\ 4 & & \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} \leq key(\alpha) = \begin{array}{lll} 7 & & \\ 6 & & \\ 5 & 6 & \\ 3 & 3 & \\ 1 & 1 & 3 \end{array} \quad key(\tilde{\beta}) = \begin{array}{lll} 7 & & \\ 4 & 4 & \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} \leq key(\tilde{\alpha}) = \begin{array}{lll} 7 & & \\ 6 & 6 & \\ 5 & 5 & \\ 3 & 3 & \\ 1 & 1 & 3 \end{array}.$$

We are now ready to state and prove the main theorem.

Theorem 4. *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $\Phi(w) = (F, G)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w one has $i + j \leq n + 1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$, where ω is the longest permutation of $\mathfrak{S}_n$. Moreover, if the first row of w is a word in the alphabet $[k]$, with $1 \leq k \leq n$, and the second row is a word in the alphabet $[m]$, with $1 \leq m \leq n$, the shape of G has the last $n - k$ entries equal to zero, and the shape of F the last $n - m$ entries equal to zero.*

Proof: *"Only if part"*. We prove by induction on the number of biletters of w. If w is the empty word then F and G are the empty semi-skyline augmented filling and there is nothing to prove. Let $w' = \begin{pmatrix} i_{p+1} & i_p & \cdots & i_1 \\ j_{p+1} & j_p & \cdots & j_1 \end{pmatrix}$ be a biword in lexicographic order such that $p \geq 0$ and $i_t + j_t \leq n + 1$ for all $1 \leq t \leq p + 1$, and $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ such that $\Phi(w) = (F, G)$. Let $F' := (j_{p+1} \to F)$ and h the height of the column in F' at which the insertion procedure terminates. There are two possibilities for h which the third step of the algorithm procedure of Φ requires to consider.

• $h = 1$. It means j_{p+1} is sited on the top of the basement element j_{p+1} in F and therefore i_{p+1} goes to the top of the basement element i_{p+1} in G. Let G' be the semi-skyline augmented filling obtained after placing i_{p+1} in G. As $i_{p+1} \leq i_t$, for all t, i_{p+1} is the bottom entry of the first column in $key(sh(G'))$ whose remain entries constitute the first column of $key(sh(G)$. Suppose $n + 1 - j_{p+1}$ is added to the row z of the first column in $key(\omega sh(F))$ by shifting all the entries above it one row up. Let $i_{p+1} < a_1 < \cdots < a_z < a_{z+1} < \cdots < a_l$ and $b_1 < b_2 < \cdots < n + 1 - j_{p+1} < b_z < \cdots < b_l$ be respectively the cell entries of the first columns in the pair $key(sh(G'))$, $key(\omega sh(F'))$, where $a_1 < \cdots < a_z < \cdots < a_l$ and $b_1 < \cdots < b_z < \cdots < b_l$ are respectively the cell entries of the first columns in the pair $key(sh(G))$, $key(\omega sh(F))$. If $z = 1$, as $i_{p+1} \leq n + 1 - j_{p+1}$ and $a_i \leq b_i$ for all $1 \leq i \leq l$, then $key(sh(G')) \leq key(\omega sh(F'))$. If $z > 1$, as $i_{p+1} < a_1 \leq b_1 < b_2$, we have $i_{p+1} \leq b_1$ and $a_1 \leq b_2$. Similarly $a_i \leq b_i < b_{i+1}$, and $a_i \leq b_{i+1}$, for all $2 \leq i \leq z - 2$. Moreover

$a_{z-1} \leq b_{z-1} < n+1-j_{p+1}$, therefore $a_{z-1} \leq n+1-j_{p+1}$. Also $a_i \leq b_i$ for all $z \leq i \leq l$. Hence. $key(sh(G')) \leq key(\omega sh(F'))$.

• $h > 1$. Place i_{p+1} on the top of the leftmost column of height $h-1$. This means by Lemma 2 $key(sh(G')) \leq key(\omega sh(F'))$.

"*If part*". We prove the contrapositive statement. If there exists a biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w such that $i+j > n+1$, then at least one entry of $key(sh(G))$ is strictly bigger than the corresponding entry of $key(\omega sh(F))$.

Let $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ be a biword in lexicographic order on the alphabet $[n]$, and $\begin{pmatrix} i_t \\ j_t \end{pmatrix}$ the first biletter in w, from right to left, with $i_t + j_t > n+1$. Set $F_0 = G_0 := \emptyset$, and for $d \geq 1$, let $(F_d, G_d) := \Phi \begin{pmatrix} i_d \cdots i_1 \\ j_d \cdots j_1 \end{pmatrix}$. First apply the map Φ to the biword $\begin{pmatrix} i_{t-1} \cdots i_1 \\ j_{t-1} \cdots j_1 \end{pmatrix}$ to obtain the pair (F_{t-1}, G_{t-1}) of SSAFs whose right keys satisfy, by the "only if part" of the theorem, $key(sh(G_{t-1})) \leq key(\omega sh(F_{t-1}))$. Now insert j_t to F_{t-1}. As $i_k + j_k \leq n+1$, for $1 \leq k \leq t-1$, $i_k + j_k \leq n+1 < i_t + j_t$, and $i_t \leq i_k$, $1 \leq k \leq t-1$, then $j_t > j_k$, $1 \leq k \leq t-1$, and, since w is in lexicographic order, this implies $i_t < i_{t-1}$. Therefore, j_t sits on the top of the basement element j_t in F_{t-1} and i_t sits on the top of the basement element i_t in G_{t-1}. It means that $n+1-j_t$ is added to the first row and first column of $key(\omega sh(F_{t-1}))$ and all entries in this column are shifted one row up. Similarly, i_t is added to the first row and first column of $key(sh(G_{t-1}))$, and all the entries in this column are shifted one row up. As $i_t > n+1-j_t$ then the first columns of $key(sh(G_t))$ and $key(\omega sh(F_t))$ respectively, are not entrywise comparable, and we say that we have a "*problem*" in the key-pair $(key(sh(G_t)), key(\omega sh(F_t)))$. From now on, "*problem*" means $i_t > n+1-j_t$ in some row of a pair of columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$, with $d \geq t$. Let $d \geq t$ and denote by J the column with basement j_t in F_d, and by I the column with basement i_t in G_d. Let $|J|$ and $|I|$ denote, respectively, the height of J and I, and let r_i and k_i denote the number of columns of height $\geq i \geq 1$, respectively, to the right of J and to the left of I. The classification of the "problem" will follow from a sequence of four claims below.

Claim 1: Let (F_d, G_d), with $d \geq t$. Then $k_i \geq r_i \geq 0$, for all $i \geq 1$.

Proof: By induction on $d \geq t$. For $d = t$, one has, $k_i = r_i = 0$, for all $i \geq 1$. Let $d \geq t$, and suppose (F_d, G_d) satisfies $k_i \geq r_i \geq 0$, for all $i \geq 1$. Let us prove for (F_{d+1}, G_{d+1}). If the insertion of j_{d+1} terminates on a column of height l to the left or on the top of J, then $r_i := r_i$, for all i, $k_i := k_i$, for all $i \neq l+1$, and $k_{l+1} := k_{l+1} + 1$, or k_{l+1}. Thus, $k_i \geq r_i$, for all $i \geq 1$. On the other hand, if the insertion of j_{d+1} terminates to the right of J, then in F_d one has $r_l > r_{l+1}$, and two cases have to be considered for placing i_{d+1} in G_d. First, i_{d+1} sits on the left of I and, hence, $k_{l+1} := k_{l+1} + 1 \geq r_{l+1} := r_{l+1} + 1$, $k_i := k_i \geq r_i := r_i$, for $i \neq l+1$. Second, either i_{d+1} sits on the top of I or to the right of I, in both cases, (F_d, G_d) satisfy $k_{l+1} = k_l \geq r_l > r_{l+1}$, and, therefore, $k_{l+1} > r_{l+1}$. This implies for (F_{d+1}, G_{d+1}), $r_{l+1} := r_{l+1} + 1$, and $k_{l+1} := k_{l+1} \geq r_{l+1}$, $k_i := k_i \geq r_i := r_i$, for $i \neq l+1$. ■

Claim 2. Let (F_d, G_d), with $d \geq t$. If $|J| > |I|$, then $k_i > r_i \geq 0$, $i = |I|+1, \ldots, |J|$.

Proof: Since, for $d = t$, it holds $|I| = |J|$, there is a $d > t$ where for the first time one has $|J| = |I| + 1$. We assume that, for some $d > t$, one has (F_d, G_d) with $|J| - |I| \geq 1$. Then, either (F_{d-1}, G_{d-1}) has $|I| = |J|$ or $|J| > |I|$. In the first case, it means that the insertion of j_d has terminated on the top of J and the cell i_d sits on the left of I on a column of height $|J| = |I|$, otherwise, it would sit on the top of I. Then, by the previous claim, $k_{|J|+1} := k_{|J|+1} + 1 > r_{|J|+1} := r_{|J|+1}$. In the second case, we suppose that, (F_{d-1}, G_{d-1}) satisfies $k_i > r_i \geq 0$, for $i = |I| + 1, \ldots, |J|$. Put $z := |I|$ and $h := |J|$. Let us prove for (F_d, G_d), when $|J| > |I|$. If the insertion of j_d terminates in a column of height l $(\neq h - 1)$ to the left of J then $r_i := r_i$, for all $i \geq 1$, $k_{l+1} := k_{l+1} + 1$, or k_{l+1} and $k_i := k_i$, for $i \neq l + 1$, and $z \leq |I| < |J| = h$, $|J| - |I| \geq 1$. Therefore, $k_i > r_i \geq 0$, for $i = |I| + 1, \ldots, |J|$. If the insertion terminates on the top of J, then $|J| = h + 1$, $|I| = z$, $r_i := r_i$, for all $i \geq 1$, $k_i := k_i$, for $i = z + 1, \ldots, h$, and $k_{h+1} := k_{h+1} + 1 > r_{h+1}$ or $k_{h+1} := k_h > r_h \geq r_{h+1}$. Again $k_i > r_i$, for $i = |I| + 1, \ldots, |J| = h + 1$. Finally, if the insertion terminates to the right of J, $|J| = h$ and three cases for the height l have to be considered. When $l < z$, or $l \geq h$, $r_i := r_i < k_i := k_i$, for $i = z + 1, \ldots, |J|$; when $l = z$, then either $|I| = z$ and $k_{z+1} := k_{z+1} + 1 > r_{z+1} := r_{z+1} + 1$, $k_i := k_i > r_i := r_i$, $z < i \leq |J|$, or $z + 1 = |I| \leq |J|$ and $k_i := k_i > r_i := r_i$, $i = z + 2, \ldots, |J|$; and when $z < l < h$, then $|I| = z$ and $r_i := r_i$, $i \neq l + 1$, and either $k_{l+1} := k_{l+1} + 1 > r_{l+1} := r_{l+1} + 1$ or $k_{l+1} = k_l > r_l \geq r_{l+1} := r_{l+1} + 1$. Henceforth $k_i > r_i$, for $i = |I| + 1, \ldots, |J|$. ∎

Claim 3: Let (F_d, G_d), with $d \geq t$, be such that, for some $s \geq 1$, one has $|I|, |J| \geq s$ and $k_s = r_s > 0$. Then, for (F_{d+1}, G_{d+1}) there exists also a $s \geq 1$ with the same properties.

Proof: Observe that, from the previous claim, $k_{s+1} = r_{s+1}$ and $|J| \geq s + 1$ only if $|I| \geq s + 1$. If the insertion of j_{d+1} terminates on the top of a column of height $l \neq s - 1$, then still $|I|, |J| \geq s$ and $k_s = r_s > 0$. It remains to analyse when $l = s - 1$ which means that the insertion of j_{d+1} either terminates to the left or to the right of J. In the first case, (F_d, G_d) satisfies $|J| \geq s + 1$ (using Remark 1), $r_s = r_{s+1}$, and, therefore, $k_s \geq k_{s+1} \geq r_{s+1} = r_s = k_s \geq k_{s+1}$. It implies for (F_{d+1}, G_{d+1}) that $k_{s+1} = r_{s+1} > 0$, $|J|, |I| \geq s + 1$, and thus the claim is true for $s + 1$. In the second case, (F_d, G_d) satisfies $k_{s-1} \geq r_{s-1} > r_s = k_s$ and thus $k_{s-1} > k_s$. Thereby the cell i_{d+1} sits to the left of I and $r_s := r_s + 1 = k_s := k_s + 1$, with $|I|, |J| \geq s$. The claim is true for s. ∎

Next claim describes the pair (F_d, G_d) of SSAFs, for $d \geq t$, when it does not fit the conditions of Claim 3.

Claim 4. Let (F_d, G_d), with $d \geq t$, be a pair of SSAFs such that, for all $i = 1, \ldots, \min\{|I|, |J|\}$, $k_i = r_i > 0$ never holds. Then, $|J| \leq |I|$ and, there is $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$.

Proof: We show by induction on $d \geq t$ that (F_d, G_d) either satisfy the conditions of the Claim 3 or, otherwise, $|J| \leq |I|$ and, there is $1 \leq f \leq |J|$, such that $r_i < k_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. For $d = t$, we have $|I| = |J| = 1$, and $k_i = r_i = 0$, $i \geq 1$. Put $f := 1$. Let (F_d, G_d), with $d \geq t$. If (F_d, G_d) fits the conditions of Claim 3, then (F_{d+1}, G_{d+1}) does it as well. Otherwise, assume for (F_d, G_d), $|J| \leq |I|$, and, there exists $1 \leq f \leq |J|$, such that $r_i < k_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. We show next that (F_{d+1}, G_{d+1}) either fits the conditions of the previous Claim 3, or, otherwise, it is as described in the present claim. If the insertion of j_{d+1} terminates to the left of J, and i_{d+1} sits on the top or to the right of I, still $|I| \geq |J|$ and there is nothing to prove. If i_{d+1} sits on the top of a column of height l, to the left of I, then, since $k_f = 0$, one has $l < f$, and two cases have to be considered. When $l = f - 1$, it implies $|I| \geq |J| \geq f + 1$, $r_f = 0$ and $k_f := 1$, and (F_{d+1}, G_{d+1}) satisfies the claim for $f + 1$; in the case of $l < f - 1$, $r_{l+1} < k_{l+1} := k_{l+1} + 1$ and still, for the same f, $k_i > r_i$, $1 \leq i < f$, $k_i = r_i = 0$, $i \geq f$. If the insertion of j_{d+1} terminates on the top of J, since $k_{|J|} = 0$ and $|I| \geq |J|$, then i_{d+1} either sits on the top of I when $|I| = |J|$, and still for the same f, $k_i > r_i$, $1 \leq i < f$, $k_i = r_i = 0$, $i \geq f$, or sits to the right of I, when $|I| > |J|$, and still $|I| \geq |J| + 1$, and, for the same f, $k_i > r_i$, $1 \leq i < f$, $k_i = r_i = 0$, $i \geq f$. If the insertion of j_{d+1} terminates to the right of J on the top of a column of height $l < f$ (recall that $r_f = 0$), then, since $|I| > f$, i_{d+1} either sits on the left of I or to the right of I. In the first case, if $l = f - 1$, one has $r_f := r_f + 1 = k_f := k_f + 1 = 1$, and, therefore, we are in the conditions of Claim 3, with $s = f < |J| \leq |I|$; if $l < f - 1$, still $r_{l+1} := r_{l+1} + 1 < k_{l+1} := k_{l+1} + 1$, so $k_i > r_i$, for $1 \leq i < f$ and $r_i = k_i = 0$, for $i \geq f$. In the second case, it means $k_{l+1} = k_l > r_l \geq r_{l+1} := r_{l+1} + 1$ and hence $k_{l+1} > r_{l+1} := r_{l+1} + 1$, with $l + 1 < f$. Similarly, $k_i > r_i$, for $1 \leq i < f$ and $k_i = r_i = 0$, $i \geq f$. ∎

Classification of the "problem": For any $d \geq t$, either there exists $s \geq 1$ such that $|J|, |I| \geq s$, $r_s = k_s > 0$; or $1 \leq |J| \leq |I|$, and there exists $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. In the first case, one has a "problem" in the $(r_s + 1)^{th}$ rows of the s^{th} columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$. In the second case, one has a problem in the bottom of the $|J|^{th}$ columns.

Finally, if the second row of w is over the alphabet $[m]$, there is no cell on the top of the basement of F greater than m. Therefore, the shape of F has the last $n - m$ entries equal to zero and thus its decreasing rearrangement is a partition of length $\leq m$. Using the symmetry of Φ, the other case is similar. ∎

Remark 2. *1. Given $\nu \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^n$ such that $\beta \leq \omega\nu$, there exists always a pair (F, G) of SSAFs with shapes ν and β respectively.*

2. If the rows in w are swapped, one obtains the biword $\tilde{w}$ such that $\Phi(\tilde{w}) = (G, F)$ with $key(sh(F)) \leq key(\omega sh(G))$.

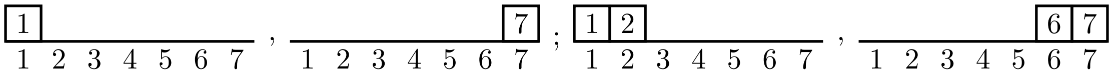
3. Note that $sh(F) \leq \omega sh(G)$ is equivalent to $sh(G) \leq \omega sh(F)$.

Using the bijection Ψ between $SSYT$ and $SSAF$ and the Schützenberger's evacuation on semi-standard tableaux [31, 32], one has,

Corollary 2. *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $w \xrightarrow{RSK} (P, Q)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w we have $1 \leq j \leq m \leq n$, $1 \leq i \leq k \leq n$, and $i+j \leq n+1$ if and only if Q has entries $\leq k$, P has entries $\leq m$, and $K_+(Q) \leq evac(K_+(P))$, where "evac" denotes evacuation.*

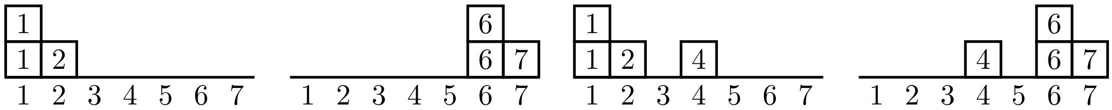
Two examples are given to illustrate Theorem 4.

Example 5. (1) Given $w = \begin{pmatrix} 4 & 6 & 6 & 7 \\ 4 & 1 & 2 & 1 \end{pmatrix}$, $\Phi(w)$ and the key-pair $key(sh(G)) \leq key(\omega sh(F))$ are calculated.



$sh(F_1) = (1, 0^6)$ $\quad sh(G_1) = (0^6, 1)$ $\quad sh(F_2) = (1^2, 0^5)$ $\quad sh(G_2) = (0^5, 1^2)$

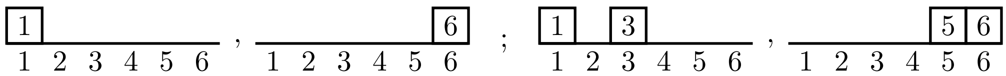
$key(sh(G_1)) = 7 = key(\omega sh(F_1))$ $\quad key(sh(G_2)) = 67 = key(\omega sh(F_2))$



$sh(F_3) = (2, 1, 0^5)$ $\quad sh(G_3) = (0^5, 2, 1)$ $\quad sh(F_4) = (2, 1, 0, 1, 0^3)$ $\quad sh(G_4) = (0^3, 1, 0, 2, 1)$

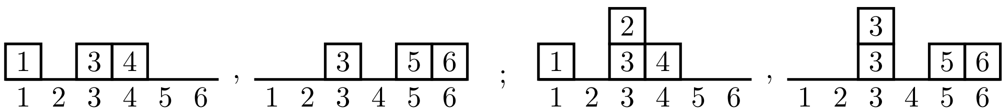
$$key(sh(G_3)) = \begin{array}{cc} 7 & \\ 6 & 6 \end{array} \leq \begin{array}{cc} 7 & \\ 6 & 7 \end{array} = key(\omega sh(F_3)) \qquad key(sh(G_4)) = \begin{array}{cc} 7 & \\ 6 & \\ 4 & 6 \end{array} \leq \begin{array}{cc} 7 & \\ 6 & \\ 4 & 7 \end{array} = key(\omega sh(F_4))$$

(2) Let $w = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 & 6 \\ 6 & 3 & 2 & 4 & 3 & 1 \end{pmatrix}$, with $n = 6$, $i_2 = 5 > 6 + 1 - 3$. We calculate $\Phi(w)$ whose key-pair $key(sh(G)), key(\omega sh(F))$ is not entrywise comparable.



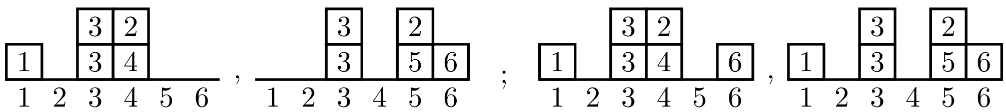
$sh(F_1) = (1, 0^5)$ $\quad sh(G_1) = (0^5, 1)$ $\quad sh(F_2) = (1, 0, 1, 0^3)$ $\quad sh(G_2) = (0^4, 1^2)$

$$key(sh(G_1)) = 6 = key(\omega sh(F_1)) \qquad key(sh(G_2)) = \begin{array}{c} 6 \\ 5 \end{array} \not\leq \begin{array}{c} 6 \\ 4 \end{array} = key(\omega sh(F_2))$$



$sh(F_3) = (1, 0, 1^2, 0^2)$ $\quad sh(G_3) = (0^2, 1, 0, 1^2)$ $\quad sh(F_4) = (1, 0, 2, 1, 0^2)$ $\quad sh(G_4) = (0^2, 2, 0, 1^2)$

$$key(sh(G_3)) = \begin{array}{c} 6 \\ 5 \\ 3 \end{array} \not\leq \begin{array}{c} 6 \\ 4 \\ 3 \end{array} = key(\omega sh(F_3)) \qquad key(sh(G_4)) = \begin{array}{cc} 6 & \\ 5 & \\ 3 & 3 \end{array} \not\leq \begin{array}{cc} 6 & \\ 4 & \\ 3 & 4 \end{array} = key(\omega sh(F_4))$$



$sh(F_5) = (1, 0, 2^2, 0^2)$ $\quad sh(G_5) = (0^2, 2, 0, 2, 1)$ $\quad sh(F_6) = (1, 0, 2^2, 0, 1)$ $\quad sh(G_6) = (1, 0, 2, 0, 2, 1)$

$$key(sh(G_5)) = \begin{array}{cc} 6 & \\ 5 & 5 \\ 3 & 3 \end{array} \not\leq \begin{array}{cc} 6 & \\ 4 & 4 \\ 3 & 3 \end{array} = key(\omega sh(F_5)) \qquad key(sh(G_6)) = \begin{array}{cc} 6 & \\ 5 & \\ 3 & 5 \\ 1 & 3 \end{array} \not\leq \begin{array}{cc} 6 & \\ 4 & \\ 3 & 4 \\ 1 & 3 \end{array} = key(\omega sh(F_6))$$

5. Isobaric divided differences and crystal graphs

5.1. Isobaric divided differences, and the generators of the 0-Hecke algebra. Isobaric divided difference operators π_i and $\widehat{\pi}_i$, $1 \le i < n$, (1) have an equivalent definition

$$\pi_i(x_i^a x_{i+1}^b m) = \begin{cases} x_i^a x_{i+1}^b m + (\sum_{j=1}^{a-b} x_i^{a-j} x_{i+1}^{b+j})m, & \text{if } a > b, \\ x_i^a x_{i+1}^b m, & \text{if } a = b, \\ x_i^a x_{i+1}^b m - (\sum_{j=0}^{b-a-1} x_i^{a+j} x_{i+1}^{b-j})m, & \text{if } a < b, \end{cases} \tag{7}$$

and

$$\widehat{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (\sum_{j=1}^{a-b} x_i^{a-j} x_{i+1}^{b+j})m, & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -(\sum_{j=0}^{b-a-1} x_i^{a+j} x_{i+1}^{b-j})m, & \text{if } a < b, \end{cases} \tag{8}$$

where m is a monomial not containing x_i nor x_{i+1}. It follows from the definition that $\pi_i(f) = f$ and $\widehat{\pi}_i(f) = 0$ if and only if $s_i f = f$. They both satisfy the commutation and the braid relations (3) of $\mathfrak{S}_n$, $\pi_i\pi_j = \pi_j\pi_i$, $\hat{\pi}_i\hat{\pi}_j = \hat{\pi}_j\hat{\pi}_i$ for $|i-j| > 1$, and $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$, $\hat{\pi}_i\hat{\pi}_{i+1}\hat{\pi}_i = \hat{\pi}_{i+1}\hat{\pi}_i\hat{\pi}_{i+1}$, and this guarantees that, for any permutation $\sigma \in \mathfrak{S}_n$, there exists a well defined isobaric divided difference $\pi_\sigma := \pi_{i_N} \cdots \pi_{i_2}\pi_{i_1}$ and $\hat{\pi}_\sigma := \hat{\pi}_{i_N} \cdots \hat{\pi}_{i_2}\hat{\pi}_{i_1}$, where $(i_N, \dots, i_2, i_1)$ is any reduced word for $\mathfrak{S}_n$. In addition, they satisfy the quadratic relations $\pi_i^2 = \pi_i$, $\hat{\pi}_i^2 = -\hat{\pi}_i$.

The 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$, a deformation of the group algebra of $\mathfrak{S}_n$, is an associative $\mathbb{C}$-algebra generated by $T_1, \dots, T_{n-1}$ satisfying the commutation and the braid relations of the symmetric group $\mathfrak{S}_n$, and the quadratic relation $T_i^2 = T_i$ for $1 \le i < n$. Setting $\widehat{T}_i := T_i - 1$, for $1 \le i < n$, one obtains another set of generators of the 0-Hecke algebra $H_n(0)$. The sets $\{T_\sigma, \sigma \in \mathfrak{S}_n\}$ and $\{\hat{T}_\sigma, \sigma \in \mathfrak{S}_n\}$ are both linear basis for $H_n(0)$, where $T_\sigma = T_{i_N} \cdots T_{i_2}T_{i_1}$ and $\hat{T}_\sigma := \hat{T}_{i_N} \cdots \hat{T}_{i_2}\hat{T}_{i_1}$, for any reduced expression $s_{i_N} \cdots s_{i_2}s_{i_1}$ in $\mathfrak{S}_n$ [2]. Since Demazure operators (7) or bubble sort operators (6) satisfy the same relations as T_i, and similarly for isobaric divided difference operators (8) and $\widehat{T}_i$, the 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$ may be viewed as an algebra of operators realised either by isobaric divided differences π_i or $\widehat{\pi}_i$, or by bubble sort operators π_i, swapping entries i and $i+1$ in a weak composition α, if $\alpha_i > \alpha_{i+1}$, and doing nothing, otherwise. Therefore, the sets of the two families of isobaric divided differences $\{\pi_\sigma, \sigma \in \mathfrak{S}_n\}$ and $\{\hat{\pi}_\sigma, \sigma \in \mathfrak{S}_n\}$ are both linear basis for $H_n(0)$, and from the relation between them, $\widehat{\pi}_i = \pi_i - 1$, the change of basis from the first to the second is given by a sum over the Bruhat order in $\mathfrak{S}_n$, precisely [19, 29],

$$\pi_\sigma = \sum_{\theta \le \sigma} \hat{\pi}_\theta. \tag{9}$$

5.2. Demazure characters, Demazure atoms and sorting operators. Let $\lambda \in \mathbb{N}^n$ be a partition and α a weak composition in the $\mathfrak{S}_n$-orbit of λ. If $s_i \in stab_\lambda$ then $\pi_i(x^\lambda) = x^\lambda$ and hence $\hat{\pi}_i(x^\lambda) = 0$. Write $\alpha = \sigma\lambda$, where $\sigma \in \mathfrak{S}_n$ is a minimal length coset representative of $\mathfrak{S}_n/stab_\lambda$. The key polynomial [18, 30] or Demazure character [3, 11] in type A corresponding to the dominant weight λ and permutation σ, is the polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ indexed by the weak composition $\alpha \in \mathbb{N}^n$, defined by

$$\kappa_\alpha(x) := \pi_\sigma x^\lambda, \tag{10}$$

and the standard basis [18] or Demazure atom [25] is defined similarly,

$$\widehat{\kappa}_\alpha(x) := \widehat{\pi}_\sigma x^\lambda. \tag{11}$$

Due to (9), the identity (11) consists of all monomials in κ_α which do not appear in κ_β for any $\beta < \alpha$. Thereby, key polynomials (10) can be decomposed into non intersecting pieces [21],

$$\kappa_\alpha(x) = \sum_{\beta \le \alpha} \widehat{\kappa}_\beta(x), \tag{12}$$

where the ordering is induced by the Bruhat order in the $\mathfrak{S}_n$-orbit of λ.

Key polynomials $\{\kappa_\alpha(x) : \alpha \in \mathbb{N}^n\}$ and Demazure atoms $\{\hat{\kappa}_\alpha(x) : \alpha \in \mathbb{N}^n\}$ form linear $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, \ldots, x_n]$. The change of basis from the first to the second is expressed in (12). The operators π_i act on key polynomials κ_α via elementary bubble sorting operators on the entries of the weak composition α [30],

$$\pi_i \kappa_\alpha = \begin{cases} \kappa_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ \kappa_\alpha & \text{if } \alpha_i \le \alpha_{i+1} \end{cases}. \tag{13}$$

This allows a recursive definition of key polynomials [21]. For $\alpha \in \mathbb{N}^n$, the key polynomial κ_α is $\kappa_\alpha(x) = x^\alpha$, if α is a partition. Otherwise, $\kappa_\alpha(x) = \pi_i \kappa_{s_i\alpha}(x)$, where $\alpha_{i+1} > \alpha_i$, for some i.

5.3. Crystals and combinatorial descriptions of Demazure characters and Demazure atoms. In [18] Lascoux and Schützenberger have given a combinatorial version for Demazure operators π_i and $\hat{\pi}_i$ in terms of crystal (or coplactic) operators f_i, e_i to produce a crystal graph on $\mathfrak{B}^\lambda$, the set of semi-standard Young tableaux with entries $\le n$ and shape λ [12, 13, 35].

A semi-standard Young tableau can be uniquely recovered from its column word. To describe the action of the crystal operators f_i and e_i, $1 \le i < n$, on $T \in \mathfrak{B}^\lambda$, change all i, in the column word of T, to right parentheses ")" and $i+1$ to left parentheses "(". Ignore all other entries and match the parentheses in the usual manner to construct a subword $)^r\ (^s$ of unmatched parentheses. If there is no unmatched right parentheses, that is, $r = 0$, then f_i is not defined in T and put $f_i(T) = 0$; if there is no unmatched left parentheses, that is, $s = 0$, then e_i is not defined in T, and put $e_i(T) = 0$. Otherwise, either $r > 0$ and replace the rightmost unmatched right parenthesis by a left parenthesis to construct $)^{r-1}\ (^{s+1}$, or $s > 0$ and replace the leftmost unmatched left parenthesis by a right parenthesis to construct $)^{r+1}\ (^{s-1}$. Next, in either cases, convert the parentheses back to i and $i+1$ and recover the ignored entries. The resulting word defines the semi-standard Young tableau $f_i(T)$ or $e_i(T)$. For convenience, we extend f_i and e_i to $\mathfrak{B}^\lambda \cup \{0\}$ by setting them to map 0 to 0.

Kashiwara and Nakashima [13, 15] have given to $\mathfrak{B}^\lambda$ a $U_q(\mathfrak{gl}_n)$- crystal structure. We view crystals as special graphs. The crystal graph on $\mathfrak{B}^\lambda$ is a coloured directed graph whose vertices are the elements of $\mathfrak{B}^\lambda$, and the edges are coloured with a colour i, for each pair of crystal operators f_i, e_i, such that there exists a coloured i-arrow from the vertex T to T' if and only if $f_i(T) = T'$, equivalently, $e_i(T') = T$. We refer to [16, 9] for details. Start with the Yamanouchi tableau $Y := key(\lambda)$ and apply all the crystal operators f_i's until each unmatched i has been converted to $i+1$, for $1 \le i < n$ [13, 16] (see Example 6). The resulting set is $\mathfrak{B}^\lambda$ whose elements index basis vectors for the representation of the quantum group $U_q(\mathfrak{gl}_n)$ with highest weight λ. From the definition of this graph, in each vertex there is at most one incident arrow of colour i, and at most one outgoing arrow of colour i. Hence, the crystal $\mathfrak{B}^\lambda$ is the disjoint union of connected components of colour i, $P_1 \to \cdots \to P_k$, called i-strings, of lengths $k-1 \geq 0$, for any i, $1 \leq i < n$. A SSYT P_1 satisfying $e_i(P_1) = 0$ and is said to be the head of the i-string, and, in the case of $f_i(P_k) = 0$, P_k is called the end of the i-string.

Given α in the $\mathfrak{S}_n$-orbit of λ, the Demazure crystal $\mathfrak{B}_\alpha$ is viewed as a certain subgraph of the crystal $\mathfrak{B}^\lambda$ which can be defined inductively [14, 23] by

$$\mathfrak{B}_\alpha = \{f_i^k(T) : T \in \mathfrak{B}_{s_i\alpha}, k \geq 0, e_i(T) = 0\} \setminus \{0\}, \qquad \text{if } \alpha_{i+1} > \alpha_i. \tag{14}$$

(Note that, if α is the reverse of λ, one has $\mathfrak{B}_{\omega\lambda} = \mathfrak{B}^\lambda$, and if $\alpha = \lambda$, one has $\mathfrak{B}_\lambda = \{Y\}$.) The vertices of this subgraph index basis vectors of the Demazure module $V_\sigma(\lambda)$ where σ is a minimal length coset representative modulo the stabiliser of λ, such that $\sigma\lambda = \alpha$. In fact $\mathfrak{B}_\alpha$ (14) is well defined, it does not depend on the reduced expression for σ. More generally, write $\alpha = s_{i_N} \ldots s_{i_2} s_{i_1}\lambda$, with $(i_N, \ldots, i_2, i_1)$ a reduced word, then apply the crystal operator f_{i_1} to the Y until each unmatched i_1 has been converted to $i_1 + 1$, then apply similarly f_{i_2} to each of the previous Young tableaux until each unmatched i_2 has been converted to $i_2 + 1$, and continue this procedure with $f_{i_3}, \ldots, f_{i_N}$. Therefore, $\mathfrak{B}_\alpha = \{f_{i_N}^{m_N} \ldots f_{i_1}^{m_1}(Y) : m_k \geq 0\} \setminus \{0\}$.

Let $T \in \mathfrak{B}^\lambda$, and $f_{s_i}(T) := \{f_i^m(T) : m \geq 0\} \setminus \{0\}$. (If $f_i(T) = 0$, $f_{s_i}(T) = \{T\}$.) If P is the head of an i-string $S \subseteq \mathfrak{B}^\lambda$, $S = f_{s_i}(P)$, and the Demazure operator π_i sends the head of an i-string to the sum of all elements of the string [14],

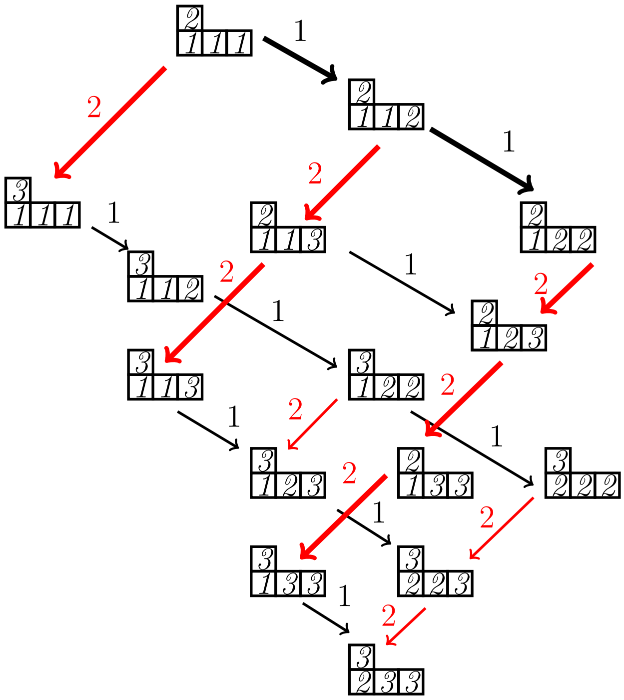
$$\pi_i(x^P) = \sum_{T \in S} x^T, \text{ and } \pi_i(\sum_{T \in S} x^T) = \pi_i(x^P). \tag{15}$$

If $\beta \leq \alpha$, then $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$. Let $s_i\alpha < \alpha$. For any i-string $S \subseteq \mathfrak{B}^\lambda$, either $\mathfrak{B}_{s_i\alpha} \cap S = \mathfrak{B}_\alpha \cap S$ is empty, or $\mathfrak{B}_{s_i\alpha} \cap S = \mathfrak{B}_\alpha \cap S = S$, or $\mathfrak{B}_{s_i\alpha} \cap S$ is only the head of S in which case $S \subseteq \mathfrak{B}_\alpha$. From these string properties, and (15), and since $\mathfrak{B}^\lambda$ is the disjoint union of i-strings, for any $1 \leq i < n$,

$$\sum_{T \in \mathfrak{B}_\alpha} x^T = \pi_i(\sum_{T \in \mathfrak{B}_{s_i\alpha}} x^T). \tag{16}$$

Henceforth, the key polynomial $\kappa_\alpha = \pi_i \kappa_{s_i\alpha}$, for $s_i\alpha < \alpha$, and $\kappa_\alpha(x) = \pi_{i_N} \cdots \pi_{i_1} x^\lambda$ for any reduced word $(i_N, \ldots, i_1)$ such that $s_{i_N} \cdots s_{i_1}\lambda = \alpha$.

Example 6. *The Demazure crystal $\mathfrak{B}_{s_2s_1\lambda}$ with $\lambda = (3, 1, 0)$ is shown with thick edges while the rest of the crystal graph $\mathfrak{B}^\lambda$ is shown with thinner lines. The 1 and 2-strings are represented in black and red colours respectively.*



$\kappa_{103}(x_1, x_2, x_3) = \pi_2\pi_1(x^{310}) = \pi_2(x^{220} + x^{130} + x^{310}) = x^{220} + x^{130} + x^{310} + x^{301} + x^{211} + x^{202} + x^{121} + x^{112} + x^{103}$

Set $\widehat{\mathfrak{B}}_\alpha := \mathfrak{B}_\alpha \setminus \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. Then $\mathfrak{B}_\alpha = \bigcup_{\beta\leq\alpha} \widehat{\mathfrak{B}}_\beta$. In Example 6, with $\alpha = (103) = s_2s_1(3,1,0)$, the top component $\widehat{\mathfrak{B}}_{s_2s_1(3,1,0)} = \mathfrak{B}_{s_2s_1(3,1,0)} \setminus (\mathfrak{B}_{s_1(3,1,0)} \cup \mathfrak{B}_{s_2(3,1,0)})$ consists of the two lowerst thick red strings, starting in the thick black string, minus their heads. The action of the Demazure operator $\hat{\pi}_i$ on an i-string S is the same as π_i minus the head P of S, and, thus, $\hat{\pi}_i(x^P) = 0$ if $S = \{P\}$, and $\hat{\pi}_i(x^P) = \sum_{T\in f_{s_i}(P)\setminus\{P\}}(x^T)$. From the string property, one still has, $\sum_{T\in\widehat{\mathfrak{B}}_\alpha} x^T = \hat{\pi}_i(\sum_{T\in\widehat{\mathfrak{B}}_{s_i\alpha}} x^T)$. Henceforth, the key polynomial $\hat{\kappa}_\alpha = \hat{\pi}_i\hat{\kappa}_{s_i\alpha}$, for $s_i\alpha < \alpha$, and $\hat{\kappa}_\alpha(x) = \hat{\pi}_{i_N}\cdots\hat{\pi}_{i_1}x^\lambda$ with $s_{i_N}\cdots s_{i_1}$ a minimal representative modulo the stabiliser of α. For instance, $\hat{\kappa}_{(103)}(x) = \sum_{T\in\widehat{\mathfrak{B}}_{(1,0,3)}} x^T = \hat{\pi}_2\hat{\pi}_1(x^{310}) = \hat{\pi}_2(x^{220}) + \hat{\pi}_2(x^{130}) = x^{211} + x^{202} + x^{121} + x^{112} + x^{103}$. Lascoux and Schützenberger have characterised the SSYTs in $\widehat{\mathfrak{B}}_\alpha$ [18] as those whose right key is $key(\alpha)$, precisely the unique key tableau in $\widehat{\mathfrak{B}}_\alpha$. The Demazure crystal $\mathfrak{B}_\alpha$ consists of all Young tableaux in $\mathfrak{B}^\lambda$ with right key bounded by $key(\alpha)$.

Theorem 5. (Lascoux, Schützenberger [18]) *The Demazure atom $\widehat{\kappa}_{\sigma\lambda}(x) = \hat{\pi}_\sigma(x^\lambda)$ is the sum of the weight polynomials of all SSYTs with entries $\leq n$ whose right key is equal to $key(\sigma\lambda)$, with σ a minimal length coset representative modulo the stabiliser of λ.*

We may put together the three combinatorial interpretations

$$\hat{\kappa}_\alpha(x) = \sum_{T\in\widehat{\mathfrak{B}}_\alpha} x^T = \sum_{K_+(T)=key(\alpha)} x^T = \sum_{sh(F)=\alpha} x^F,$$

$$\kappa_\alpha(x) = \sum_{T\in\mathfrak{B}_\alpha} x^T = \sum_{K_+(T)\leq key(\alpha)} x^T = \sum_{sh(F)\leq\alpha} x^F.$$

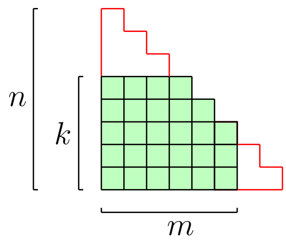
As the sum of the weight monomials over all crystal graph $\mathfrak{B}^\lambda$ gives the Schur polynomial s_λ, each SSYT of shape λ appears in precisely one such polynomial, henceforth, the Demazure atoms form a decomposition of the Schur functions.

6. Expansions of Cauchy kernels over truncated staircases

6.1. Cauchy kernel and Lascoux's non-symmetric Cauchy kernel expansions. Given $n \in \mathbb{N}$, let m and k be fixed positive integers where $1 \leq m \leq n$, $1 \leq k \leq n$. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two sequences of indeterminates. The well-known Cauchy identity expresses the product $\prod_{i=1}^k \prod_{j=1}^m (1 - x_i y_j)^{-1}$ as a sum of products of Schur polynomials s_{μ^+} in $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_m)$,

$$\prod_{(i,j)\in(m^k)} (1 - x_i y_j)^{-1} = \prod_{i=1}^{k} \prod_{j=1}^{m} (1 - x_i y_j)^{-1} = \sum_{\mu^+} s_{\mu^+}(x_1, \dots, x_k) s_{\mu^+}(y_1, \dots, y_m), \tag{17}$$

over all partitions μ^+ of length $\leq \min\{k, m\}$. Using either the RSK correspondence [17] or the Φ correspondence, the Cauchy formula (17) can be interpreted as a bijection between monomials on the left hand side and pairs of SSYTs or SSAFs on the right. Now we replace in the Cauchy kernel the rectangle (m^k) by the truncated staircase $\lambda = (m^{n-m+1}, m - 1, \dots, n - k + 1)$, with $1 \leq m \leq n$, $1 \leq k \leq n$, and $n + 1 \leq m + k$, as depicted in the green diagram below



If $n + 1 = m + k$, we recover the rectangle shape (m^k). When $m = n = k$, one has the staircase partition $\lambda = (n, n - 1, \dots, 2, 1)$. That is, the cells (i, j) in the NW-SE diagonal of the square diagram (n^n) and below it, and thus $(i, j) \in \lambda$ if and only if $i + j \leq n + 1$.

Lascoux has given the following expansion for the non-symmetric Cauchy kernel over staircases, using double crystal graphs in [20], and also in [5], based on algebraic properties of Demazure operators,

$$\prod_{\substack{i+j\leq n+1\\ 1\leq i,j\leq n}} (1 - x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x) \kappa_{\omega\nu}(y), \tag{18}$$

where $\hat{\kappa}$ and κ are the two families of key polynomials in x and y respectively, and ω is the longest permutation of $\mathfrak{S}_n$.

In [20], Lascoux extends (18) to an expansion of $\prod_{(i,j)\in\lambda}(1 - x_i y_j)^{-1}$, over any Ferrers shape λ, by considering $\rho(\lambda) = (t, t - 1, \dots, 1)$, the biggest staircase contained in λ, and a pair of permutations $\sigma(\lambda, NW)$ and $\sigma(\lambda, SE)$ encoding the cells in a NW and SE parts of the diagram λ/ρ, consisting of the cells in λ not in ρ. To define such a pair, one takes an arbitrary cell in the staircase $(t + 1, t, \dots, 1)$ which does not belong to λ. The SW-NE diagonal passing through this cell cuts the diagram of λ/ρ, into two pieces that are called respectively the North-West part and the South-East part of λ/ρ. Fill now each cell of row $r \geq 2$ of the North-West part with the number $r-1$. Similarly, fill each cell of column $c \geq 2$ of the South-East part with the number $c-1$. Reading the columns of the North-West part, from right to left, top to bottom, and interpreting r as the simple transposition s_r, gives a reduced decomposition of the permutation $\sigma(\lambda, NW)$; similarly, reading rows, from right to left, and from top to bottom, of the South-East part, gives the permutation $\sigma(\lambda, SE)$. This pair of permutations depend indeed on the choice of the cell cutting the diagram λ/ρ. Clearly, if this construction is applied to the conjugate Ferrers shape $\overline{\lambda}$, then $\rho(\lambda) = \rho(\overline{\lambda})$ and $\sigma(\overline{\lambda}, NW) = \sigma(\lambda, SE)$ and $\sigma(\overline{\lambda}, SE) = \sigma(\lambda, NW)$.

Theorem 6. (Lascoux, Theorem 7 in [20]) *Let λ be a partition in $\mathbb{N}^n$, $\rho(\lambda) = (t, t-1, \ldots, 1)$ the maximal staircase contained in the diagram of λ, and $\sigma(\lambda, NW)$, $\sigma(\lambda, SE)$ the two permutations obtained by cutting the diagram of λ/ρ as explained above. Then*

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^t} (\pi_{\sigma(\lambda,NW)} \widehat{\kappa}_\mu(x))(\pi_{\sigma(\lambda,SE)} \kappa_{\omega\mu}(y)). \tag{19}$$

For our truncated staircases λ the formula (19) translates to

$$\prod_{\substack{(i,j)\in\lambda \\ k\leq m}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)(\pi_{\sigma(\lambda,SE)} \kappa_{\omega\mu}(y)); \tag{20}$$

$$\prod_{\substack{(i,j)\in\lambda \\ m\leq k}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} (\pi_{\sigma(\lambda,NW)} \widehat{\kappa}_\mu(x))\kappa_{\omega\mu}(y). \tag{21}$$

Indeed (21) is just (20), with x and y swapped, followed by a change from the basis of Demazure characters to the basis of Demazure atoms (12)

$$\prod_{\substack{(i,j)\in\lambda \\ m\leq k}} (1 - x_i y_j)^{-1} = \prod_{\substack{(j,i)\in\overline{\lambda} \\ m\leq k}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\overline{\lambda},SE)}\kappa_{\omega\mu}(x)$$

$$\begin{aligned}
&= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x) &&= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)} \sum_{\beta\leq\omega\mu} \widehat{\kappa}_\beta(x) \\
&= \sum_{\mu\in\mathbb{N}^m} \sum_{\substack{\beta\in\mathbb{N}^m \\ \beta\leq\omega\mu}} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) &&= \sum_{\beta\in\mathbb{N}^m} \sum_{\substack{\mu\in\mathbb{N}^m \\ \mu\leq\omega\beta}} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) \\
&= \sum_{\beta\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) \sum_{\substack{\mu\in\mathbb{N}^m \\ \mu\leq\omega\beta}} \widehat{\kappa}_\mu(y) &&= \sum_{\beta\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x)\kappa_{\omega\beta}(y).
\end{aligned}$$

Next we give a bijective proof of (20) and compute the Demazure character $\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y)$ by making explicit the Young tableaux in the Demazure crystal.

6.2. Our expansions. We now use the bijection in Theorem 4 to give an expansion of the non-symmetric Cauchy kernel for the shape $\lambda = (m^{n-m+1}, m-1, \dots, n-k+1)$, where $1 \le m \le n$, $1 \le k \le n$, and $n+1 \le m+k$, which includes, in particular, the rectangle (17), the staircase (18), and implies the truncated staircases (20).

The generating function for the multisets of ordered pairs of positive integers $\{(a_1, b_1), (a_2, b_2), \dots, (a_r, b_r)\}$, $r \ge 0$, where $(a_i, b_i) \in \lambda$, that is, $a_i + b_i \le n+1$, $1 \le a_i \le k$, $1 \le b_i \le m$, $1 \le i \le r$, weighted by the contents $((\alpha, 0^{n-k});\ (\delta, 0^{n-m})) \in \mathbb{N}^k \times \mathbb{N}^m$, with α_j the number of i's such that $a_i = j$, and δ_j the number of i's such that $b_i = j$, is

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\substack{\{(a_i,b_i)\}_{i=1}^r \\ r\ge 0}} x_{a_1} y_{b_1} \cdots x_{a_r} y_{b_r} = \sum_{\substack{\{(a_i,b_i)\}_{i=1}^r \\ r\ge 0}} x^\alpha y^\delta.$$

Each multiset $\{(a_1, b_1), (a_2, b_2), \dots, (a_r, b_r)\}$, $r \ge 0$, and, hence, each monomial $x_{a_1}\, y_{b_1} \cdots x_{a_r} y_{b_r}$, $r \ge 0$, is in one-to-one correspondence with the lexicographically ordered biword $\left(\begin{smallmatrix} a_r & \cdots & a_1 \\ b_r & \cdots & b_1 \end{smallmatrix}\right)$ in the product of alphabets $[k] \times [m]$, which is bijectively mapped by Φ into the pair (F, G) of SSAFs such that G has entries in $\{a_1, \dots, a_r\}$, F has entries in $\{b_1, \dots, b_r\}$, and their shapes $sh(G) = \mu \in \mathbb{N}^k$, and $sh(F) = \beta \in \mathbb{N}^m$, in a same $\mathfrak{S}_n$-orbit, satisfy $(\beta, 0^{n-m}) \le (0^{n-k}, \omega\mu)$ with ω the longest permutation in $\mathfrak{S}_k$. (For $r = 0$, put $F = G = \emptyset$.) Thereby, $x_{a_1} y_{b_1} \cdots x_{a_r} y_{b_r} = y^F x^G$, for all $r \ge 0$, and we may write

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} \;=\; \sum_{\mu\in\mathbb{N}^k} \sum_{\substack{F,G\,\in SSAF_n \\ sh(F)=\beta\in\mathbb{N}^m,\ sh(G)=\mu \\ (\beta,0^{n-m})\le(0^{n-k},\omega\mu)}} y^F x^G. \tag{22}$$

Assume $k \le m$. Since $(\mu, 0^{n-k})$, $(\beta, 0^{n-m})$ are in a same $\mathfrak{S}_n$-orbit, $(\mu^+, 0^{m-k}) = \beta^+ \in \mathbb{N}^m$.

$$= \sum_{\mu\in\mathbb{N}^k} \left(\sum_{\substack{G\in SSAF_n \\ sh(G)=\mu}} x^G \right) \left(\sum_{\substack{\beta\in\mathbb{N}^m \\ (\beta,0^{n-m})\le(0^{n-k},\omega\mu)}} \sum_{\substack{F\in SSAF_n \\ sh(F)=\beta}} y^F \right) \tag{23}$$

$$= \sum_{\mu\in\mathbb{N}^k} \left(\sum_{\substack{Q\in SSYT_n \\ sh(Q)=\mu^+ \\ K_+(Q)=key(\mu)}} x^Q \right) \left(\sum_{\substack{\beta\in\mathbb{N}^m \\ (\beta,0^{n-m})\le(0^{n-k}\omega\mu)}} \sum_{\substack{P\in SSYT_n \\ sh(P)=\mu^+ \\ K_+(P)=key(\beta)}} y^P \right)$$

$$= \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x) \sum_{\substack{P\in\mathfrak{B}_{(0^{n-k},\omega\mu)} \\ entries\,\le m}} y^P. \tag{24}$$

Let $\nu = (\mu, 0^{n-k})$. Recall that $\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})} = \mathfrak{B}^{(\mu^+,0^{m-k})}$, with ω the longest permutation of $\mathfrak{S}_k$, is the crystal graph consisting of all SSYTs with shape $(\mu^+, 0^{m-k})$ and entries less or equal than m. Henceforth, one has

$$\sum_{\substack{P\in\mathfrak{B}_{\omega\nu} \\ entries\,\le m}} y^P = \sum_{P\in\mathfrak{B}_{\omega\nu}\cap\,\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}} y^P, \tag{25}$$

the weight polynomial of all SSYTs in the $\mathfrak{B}_{\omega\nu}$ with entries less or equal than m, equivalently, of all SSYTs with entries $\leq m$ and shape μ^+ whose right key is bounded by $key(0^{n-k}, \omega\mu)$. It is also equivalent to all SSAFs such that the shape has zeros in the last $n-m$ entries, and is bounded by $\omega\nu$. Next, we determine the Demazure crystal $\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})} = \mathfrak{B}_{\omega\nu} \cap \mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}$ where $\alpha \in \mathbb{N}^k$. This shows that (25) is a key polynomial and describes its indexing weak composition.

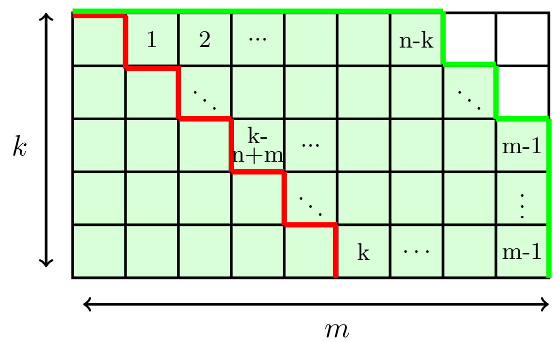
Lemma 3. *Let $\gamma \in \mathbb{N}^n$ such that $\gamma^+ = (\eta, 0^{n-m})$ is a partition of length $\leq m \leq n$. Consider the positive integer sequence $1 \leq i_M, \dots, i_1 < n$ (not necessarily a reduced word of $\mathfrak{S}_n$) such that $\kappa_\gamma(y) = \pi_{i_M} \cdots \pi_{i_1} y^{(\eta,0^{n-m})}$. If $j_s, \dots, j_1$ is the subsequence consisting of elements $\leq m$, it holds*

$$\sum_{\substack{P\in\mathfrak{B}_\gamma \\ entries \leq m}} y^P = \sum_{P\in\mathfrak{B}_\gamma\cap\mathfrak{B}_{(\omega\eta,0^{n-m})}} y^P = \pi_{i_M} \cdots \tilde{\pi}_{j_s} \cdots \tilde{\pi}_{j_1} \cdots \pi_{i_1} y^{(\eta,0^{n-m})}, \tag{26}$$

where the tilde means omission, and ω is the longest permutation of $\mathfrak{S}_m$.

Proof: Notice that $\mathfrak{B}_\gamma \cap \mathfrak{B}_{(\omega\eta,0^{n-m})} = \mathfrak{B}_\gamma \cap \mathfrak{B}^\eta$. If $n = m$ or γ has the last $n-m$ entries equal to zero, then $\gamma \leq (\omega\eta, 0^{n-m})$, $\mathfrak{B}_\gamma \subseteq \mathfrak{B}^\eta$, and $1 \leq i_M, \dots, i_1 < m$. Henceforth, $\sum_{\substack{Q\in\mathfrak{B}_\gamma \\ entries \leq m}} y^P = \sum_{P\in\mathfrak{B}_\gamma} y^P = \kappa_\gamma(y)$. Otherwise, $\mathfrak{B}_\gamma \cap \mathfrak{B}^\eta$ is obtained from $\mathfrak{B}_\gamma$ deleting all the vertices consisting of all SSYTs with entries $> m$, and, therefore, all i-edges incident on them (either getting in or out), in particular, those with $i \geq m$. From the combinatorial interpretation of Demazure operators π_i, (14), (16), this means we are deleting in $\pi_{i_M} \cdots \pi_{i_2} \pi_{i_1} y^{(\eta,0^{n-m})}$ the action of the Demazure operators π_i for $i \geq m$, and, thanks to (13), one still has a key polynomial, precisely, (26). ■

We now calculate the indexing weak composition of the key polynomial (26) in the case $\eta = (\mu^+, 0^{m-k})$ and $\gamma = \omega\nu$, and, therefore, the key polynomial (25). For $\lambda = (m^{n-m+1}, m-1, \dots, n-k+1)$, where $1 \leq k \leq m \leq n$, and $n-k \leq m-1$, one has the shape



and $\sigma(\lambda, SE) = \prod_{i=1}^{k-(n-m)-1} (s_{i+n-k-1} \dots s_i) \prod_{i=0}^{n-m} (s_{m-1} \dots s_{k-(n-m)+i})$.

Proposition 2. *Let $1 \leq k \leq m \leq n$, and $n-m+1 \leq k$. Given $\mu \in \mathbb{N}^k$, let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ such that for each $i = k, \dots, 1$, the entry α_i is the maximum element* *among the last* $\min\{i, n-m+1\}$ *entries of* $\omega\mu$ *after deleting* α_j*, for* $i < j \leq k$*. Then*

$$\begin{aligned}\sum_{\substack{P\in\mathfrak{B}_{\omega\nu}\\ \text{entries} \leq m}} y^P &= \sum_{P\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}} y^P = \sum_{P\in\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})}} y^P \\ &= \pi_{\sigma(\lambda,SE)}\kappa_{(\omega\mu,0^{n-k})}(y) = \kappa_{(0^{m-k},\alpha,0^{n-m})}(y). \qquad (27)\end{aligned}$$

In particular, when $m = n$*, then* $\alpha = \omega\mu$*; and when* $m + k = n + 1$*,* $\alpha = \omega\mu^+$ *and* $\kappa_{(0^{m-k},\omega\mu^+,0^{n-m})}(y) = s_{(\mu^+,0^{m-k})}(y_1, \ldots, y_m)$ *is a Schur function.*

Proof: Recalling the action of Demazure operators π_i on key polynomials as bubble sorting operators on their indexing weak compositions (13), and since $\omega\nu = (0^{n-k}, \omega\mu)$, one may write,

$$\begin{aligned}\kappa_{\omega\nu}(y) &= \prod_{i=1}^{k-(n-m)-1} (\pi_{i+n-k-1}\ldots\pi_i) \\ &\bullet \prod_{i=0}^{n-m} (\pi_{m-1+i}\ldots\pi_{k-(n-m)+i})\kappa_{(\omega\mu,0^{n-k})}(y). \qquad (28)\end{aligned}$$

From Lemma 3, with $\eta = (\mu^+, 0^{m-k})$, $\gamma = \omega\nu$, omitting in (28) the operators with indices $\geq m$, one has

$$\begin{aligned}\sum_{\substack{P\in\mathfrak{B}_{\omega\nu}\\ \text{entries} \leq m}} y^P &= \sum_{P\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}^{(\mu^+,0^{m-k})}} y^P = \pi_{\sigma(\lambda,SE)}\kappa_{(\omega\mu,0^{n-k})}(y) \\ &= \prod_{i=1}^{k-(n-m)-1} (\pi_{i+n-k-1}\ldots\pi_i) \qquad (29) \\ &\bullet \prod_{i=0}^{n-m} (\pi_{m-1}\ldots\pi_{k-(n-m)+i})\kappa_{(\omega\mu,0^{n-k})}(y) \qquad (30) \\ &= \kappa_{(0^{m-k},\alpha,0^{n-m})}(y) \qquad (31)\end{aligned}$$

The Demazure operators in (30) act as bubble sorting operators on the weak composition $(\omega\mu, 0^{n-k})$, shifting $m-k$ times to the right the last $n-m+1$ entries of $\omega\mu$, and sorting them in ascending order. Next, the operators (29) act similarly on the resulting vector ignoring the entry m, then ignoring the entry $m-1$, and so on. Thus the weak composition indexing the new key polynomial $\kappa_{(0^{m-k},\alpha,0^{n-m})}$ (31) is such that $\alpha = (\alpha_1, \ldots, \alpha_k)$, where for each $i = k, \ldots, 1$, α_i is the maximum element of the last $\min\{i, n-m+1\}$ entries of $\omega\mu$ after deleting α_j, for $i < j \leq k$. ■

Therefore, for $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $1 \leq k \leq m \leq n$, and $n+1 \leq m+k$, (24) can be written explicitly as

$$\prod_{\substack{(i,j)\in\lambda\\ k\leq m}} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y) = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y). \qquad (32)$$

Then

$$\prod_{\substack{(i,j)\in\lambda\\ m\le k}}(1-x_iy_j)^{-1}=\prod_{\substack{(j,i)\in\overline{\lambda}\\ m\le k}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^m}\widehat{\kappa}_\mu(y)\pi_{\sigma(\overline{\lambda},SE)}\kappa_{\omega\mu}(x)$$
$$=\sum_{\mu\in\mathbb{N}^m}\widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x)=\sum_{\mu\in\mathbb{N}^m}\widehat{\kappa}_\mu(y)\kappa_{(0^{k-m},\alpha')}(x),\tag{33}$$

where α' is defined similarly as above, swapping k with m in Proposition 2.

In the identities (32) and (33), when respectively $m=n$ and $k=n$, one has for $\lambda=(n,n-1,\dots,n-k+1)$, with $1\le k\le n$,

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\substack{\mu\in\mathbb{N}^k\\ \nu=(\mu,0^{n-k})}}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y);$$

and for $\lambda=(m^{n-m+1},m-1,m-2,\dots,1)$, with $1\le m\le n$,

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\substack{\mu\in\mathbb{N}^m\\ \nu=(\mu,0^{n-m})}}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

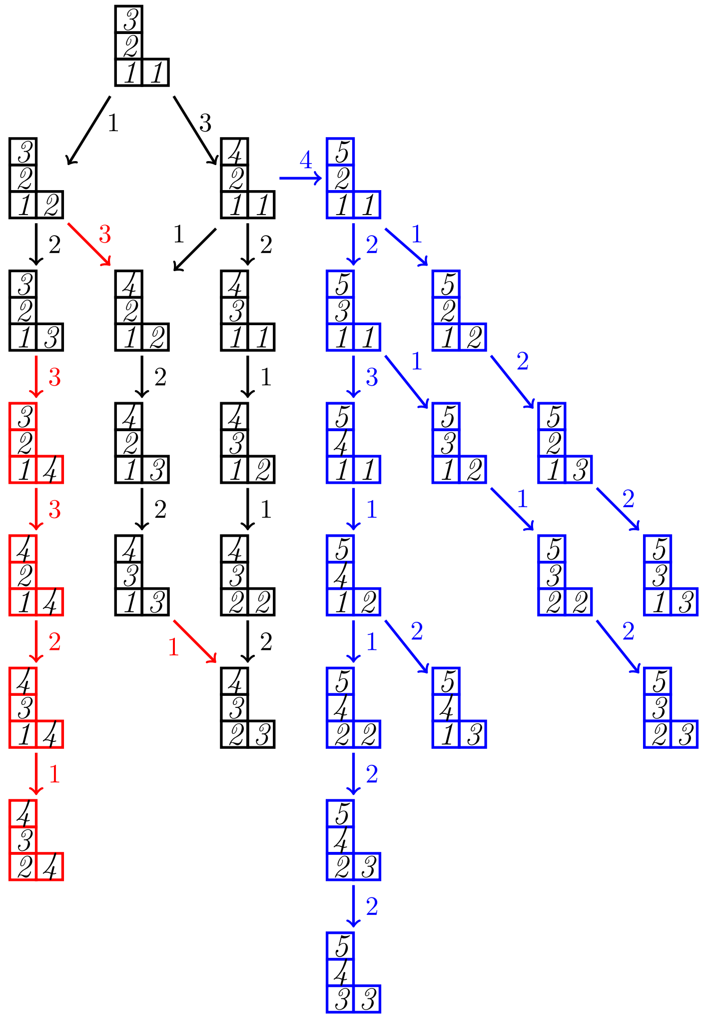
In particular, if $m=n=k$ ($\lambda=\overline{\lambda}$), we recover (18) from both previous identities. When $n+1=m+k$, from Proposition 2, identity (32) becomes

$$\prod_{\substack{(i,j)\in(m^k)\\ k\le m}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)s_{(\mu^+,0^{m-k})}(y)$$

$$=\sum_{\mu^+\in\mathbb{N}^k}\left(\sum_{\substack{\mu\in\mathbb{N}^k\\ \mu\le\omega\mu^+}}\widehat{\kappa}_\mu(x)\right)s_{(\mu^+,0^{m-k})}(y)=\sum_{\mu^+\in\mathbb{N}^k}s_{\mu^+}(x)s_{(\mu^+,0^{m-k})}(y),$$

and we recover identity (17) with $k\le m$. Similarly, (33) leads to (17) with $m\le k$.

Example 7. *Let $n=5$, $k=4$, $m=3$, $\mu=(1,1,2)$, and $\nu=(1,1,2,0,0)$. The black and blue tableaux constitute the vertices of the Demazure crystal $\mathfrak{B}_{\omega\nu}=\mathfrak{B}_{(0,0,2,1,1)}=\mathfrak{B}_{s_2s_1s_3s_2s_4s_3(2,1,1,0,0)}$. One has $\pi_2\pi_1\pi_3\pi_2\pi_3x^{(21100)}=\pi_2\pi_1\pi_2\pi_3x^{(21100)}=\kappa_{01210}(x)$. The black and the red tableaux are the vertices of the Demazure crystal $\mathfrak{B}_{(0,\omega\mu^+,0)}=\mathfrak{B}_{(0,1,1,2,0)}=\mathfrak{B}_{s_1s_2s_3s_2s_1\nu^+}$. The intersection $\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0,\omega\mu^+,0)}$ consists of the black tableaux which constitute the vertices of the Demazure crystal $\mathfrak{B}_{(0,\alpha,0)}=\mathfrak{B}_{(0,1,2,1,0)}=\mathfrak{B}_{s_2s_1s_2s_3(2,1,1,0,0)}$.*



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