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INFINITELY DIVISIBLE DISTRIBUTIONS IN INTEGER VALUED GARCH MODELS

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ABSTRACT: The aim of this paper is to propose an integer-valued model with conditional marginal distribution belonging to the general class of infinitely divisible discrete probability laws. With this proposal, we introduce a wide class of count series that includes, in particular, the Poisson $INGARCH$ model [**4**] and the negative Binomial and generalized Poisson models, introduced by Zhu in 2011 and 2012, respectively. The main probabilistic analysis of this model is here developed. Precisely, first and second order stationarity conditions are derived as well as the autocorrelation function. The existence of a strictly and ergodic solution is established in a subclass including the Poisson and generalized Poisson $INGARCH$ models.

KEYWORDS: integer-valued time series, $GARCH$ model, infinitely divisible discrete probability laws, compound-Poisson distributions.
AMS SUBJECT CLASSIFICATION (2000): 62G20, 62M10.

1. Introduction

For several years, the studies of time series models were dominated by real valued stochastic processes. However, many authors have underlined that such models do not give an adequate answer for integer-valued time series. For instance, when we deal with low dimension samples disregard the nature of the data leads, in general, to senseless results. Since this type of time series is quite common in various contexts and scientific fields, including medicine, economics, finance, epidemiology, tourism and queuing systems, over the past few years, different approaches to analyze and estimate this kind of data have been presented in literature.

Taking as reference the study associated with classical correlation structure of ARMA models, the general family of integer-valued ARMA models (or briefly, INARMA) have been introduced and developed with the scalar multiplication replaced by an integer valued operator with analogous properties, called thinning operation [**12**]. Using the same operator, different model families of integer values have been introduced as bilinear models ([**2**], [**3**]) or conditionally heteroskedastic ones ([**4**], [**13**], [**14**], [**15**], among others).

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The introduction of this last class of models seems very useful as, like it is observed in [**4**], to deal with these series of counts under hypotheses of homogeneous variance seems to unrealistic in many important situations. They present, in particular, a real sample in which the change of the series variability is evident, namely the time series of the number of cases of campylobacteriosis infections from January 1990 to the end of October 2000 in the north of the Province of Québec. To take into account these several kind of features, they propose an integer-valued process, analogous to the $GARCH$ model introduced by Bollerslev in 1986 but with Poisson deviates, denoted $INGARCH(p,q)$ model and defined as

$$\begin{cases} X_t|\underline{X}_{t-1} : \mathcal{P}(\lambda_t), \ \forall t \in \mathbb{Z}, \\ \lambda_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}, \end{cases}$$

with $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $i = 1, ..., p$, $j = 1, ..., q$, $\underline{X}_{t-1}$ the $\sigma-$field generated by $\{X_{t-i}, i \geq 1\}$ and where $\mathcal{P}(\lambda)$ is the Poisson distribution with parameter λ.

Replacing the distribution of deviates by other particular discrete ones, like negative Binomial or generalized Poisson, analogous integer-valued $GARCH$ models have been proposed and studied ([**14**], [**15**]).

With the aim of enlarging and unifying this class of $INGARCH$ models, we introduce in this paper an integer valued process with general infinitely divisible deviates. Thus, taking into account the equivalence between discrete infinitely divisible and compound-Poisson distributions ([**11**]) we define this conditional distribution using the general formulation of the characteristic function of a compound-Poisson law. With this new definition a wide set of probability distributions for deviates is considered that includes, in particular, those related to the models referred above. Precisely we may identify this set with the family of the probability distributions of a Poissonian random sum of independent variables with discrete distribution. For this general class of integer-valued processes, different kinds of stationarity are analyzed as well as the general property of ergodicity.

The reminder of the paper is organized as follows. In section 2 we define the model and present important particular cases. In section 3 we establish first and second order stationary conditions of the model. A necessary and sufficient condition for the existence of a strictly stationary and ergodic solution for a subclass of these models is also obtained. Some conclusion remarks and future developments are given in section 4.

2. Definition of the model

Let $X = (X_t, t \in \mathbb{Z})$ be a stochastic process with values in $\mathbb{N}_0$ and, for any $t \in \mathbb{Z}$, let $\underline{X}_{t-1}$ be the $\sigma-$field generated by $\{X_{t-i}, i \geq 1\}$.

Definition 2.1. *The process X is said to satisfy a Compound Poisson INteger-valued GARCH model with orders p and $q \in \mathbb{N}$, (CP-INGARCH(p,q)), if, $\forall t \in \mathbb{Z}$,*

$$\begin{cases} \Phi_{X_t|\underline{X}_{t-1}}(u) = e^{\lambda_t^*[\varphi_t(u)-1]}, \ u \in \mathbb{R} \\ E(X_t|\underline{X}_{t-1}) = \lambda_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j} \end{cases} \tag{1}$$

for some constants $\alpha_0 > 0$, $\alpha_i \geq 0$ $(i = 1, ..., p)$, $\beta_j \geq 0$ $(j = 1, ..., q)$, and where $(\varphi_t, t \in \mathbb{Z})$ is a family of characteristic functions on $\mathbb{R}$, $\underline{X}_{t-1}$-measurable and with first order derivative at zero, associated to a family of discrete laws with support $\mathbb{N}_0$ and where $\Phi_{X_t|\underline{X}_{t-1}}$ denotes the characteristic distribution function of $X_t|\underline{X}_{t-1}$.

The designation of compound Poisson integer-valued GARCH model follows from the formulation of the characteristic function of the conditional distribution of X_t. In fact, it's easy to prove that the characteristic function of a compound Poisson distribution is expressed as $\Phi(u) = e^{\lambda[\varphi(u)-1]}$, $u \in \mathbb{R}$, where φ is a characteristic function and λ a strictly positive real number ([**11**]).

In the previous definition, if $\beta_j = 0$, $j = 1, ..., q$, the $CP\text{-}INGARCH(p, q)$ model is simply denoted $CP\text{-}INARCH(p)$.

Observation 2.1. (1) *Consider $(\varphi_t, t \in \mathbb{Z})$ derivable at zero up to order 2. From the definition of the model and using the relationship between the characteristic function and the moments of a distribution we have*

$$\lambda_t = E(X_t|\underline{X}_{t-1}) = -i\Phi'_{X_t|\underline{X}_{t-1}}(0) = -i\lambda_t^*\varphi_t'(0) \Rightarrow \lambda_t^* = i \ \lambda_t/\varphi_t'(0),$$

$$V(X_t|\underline{X}_{t-1}) = -\lambda_t^*\varphi_t''(0) = -i \ \lambda_t[\varphi_t''(0)/\varphi_t'(0)],$$

where i denotes the imaginary unit. Note that the positivity of λ_t allows us to guarantee that $\varphi_t'(0)$ is nonzero.

(2) *As the conditional distribution of X_t is a discrete compound Poisson law, with support $\mathbb{N}_0$, then for any $t \in \mathbb{Z}$ and conditionally to $\underline{X}_{t-1}$, X_t can be identified in distribution as*

$$X_t \stackrel{d}{=} \sum_{i=1}^{N_t} X_{t,i}, \tag{2}$$

where $N_t \sim \mathcal{P}(\lambda_t^*)$ *and* $X_{t,1},\ldots, X_{t,N_t}$, *are discrete independent random variables, with support contained in* $\mathbb{N}_0$, *independent of* N_t *and having characteristic function* φ_t *with first derivative at zero. We note that the characteristic functions* φ_t *(respectively, the associated laws of probability) being* $\underline{X}_{t-1}$*-mensurable may be random functions (respectively, random measures).*

(3) *Let us consider the polynomials*

$$A(L) = \alpha_1 L + \ldots + \alpha_p L^p \text{ and } B(L) = 1 - \beta_1 L - \ldots - \beta_q L^q,$$

where L *is the backshift operator. To ensure the existence of the inverse of* $B(L)$ *we suppose that the roots of* $B(z) = 0$ *lie outside the unit circle which is equivalent to the hypothesis*

$$H_1 : \sum_{j=1}^{q} \beta_j < 1.$$

Thus, under this assumption, we can rewrite the conditional expectation of the model (1) *in the form*

$$B(L)\lambda_t = \alpha_0 + A(L)X_t$$

$$\Leftrightarrow \lambda_t = B^{-1}(L)[\alpha_0 + A(L)X_t] = \alpha_0 B^{-1}(1) + H(L)X_t$$

with $H(L) = B^{-1}(L)A(L) = \sum_{j=1}^{\infty} \psi_j L^j$, *where* ψ_j *are the coefficients of* z^j *in the Taylor expansion of the rational function* $A(z)/B(z)$ *in the neighbourhood of 0, i.e.,*

$$\lambda_t = \alpha_0 B^{-1}(1) + \sum_{j=1}^{\infty} \psi_j X_{t-j},$$

which expresses a $CP\text{-}INARCH(\infty)$ *representation of the model in study.*

Observation 2.2. *The model* (1) *include models already studied in the literature. Indeed,*

(1) [**4**] *introduced the model* $INGARCH(p, q)$ *mentioned in the introduction, which corresponds to the present model considering* φ_t *the characteristic function of the Dirac's law concentrated in* $\{1\}$, $\delta(1)$.

(2) *in* **[14]** *the NB-INGARCH*(p, q) *model was studied, in analogy with Ferland's model but where the* $X_t|\underline{X}_{t-1}$ *distribution is the negative Binomial law with parameters* (r, p_t) *with* $p_t = \frac{1}{1+\lambda_t}$ *and* $r \in \mathbb{N}$. *Considering, in the model* (1), φ_t *the characteristic function of a Logarithmic distribution with parameter* $1 - p_t$, *with* $p_t = \exp(-\lambda_t^*/r)$ *we recover, unless a scale factor, the previous model.*

(3) **[15]** *presents the GP-INGARCH*(p, q) *model, in analogy with previous ones, but taking as distribution of* $X_t|\underline{X}_{t-1}$ *the generalized Poisson law with parameters* (λ_t^*, κ) *where* $\lambda_t^* = (1 - \kappa)\lambda_t$ *and* $0 < \kappa < 1$. *This model results from the model* (1) *considering* φ_t *the characteristic function of the Borel's law with parameter* κ *(***[1]***,* **[12]***).*

These examples show that φ_t *can be a random function (case 2) or a deterministic one (cases 1 and 3). Moreover a wide class of processes is included in model* (1). *The following example shows how to obtain this kind of processes and also a particular situation where* (φ_t) *is a family of dependent on* t *determinist functions.*

(4) *Let us consider the process* $(X_t, t \in \mathbb{Z})$ *defined in* (2) *in which* $(X_{t,i}, t \in \mathbb{Z})$ *are independent random variables following the Binomial distribution with parameters* $r \in \mathbb{N}$ *and* $e^{-|t|}$, *that is,* $\varphi_t(u) = \left(e^{iu-|t|} + 1 - e^{-|t|}\right)^r$, $u \in \mathbb{R}$, $t \in \mathbb{Z}$. *To obtain model* (1) *it's enough to take* N_t *independent of* $X_{t,i}$ *and following* $\mathcal{P}(\lambda_t^*)$ *with* $\lambda_t^* = \frac{\lambda_t}{re^{-|t|}}$.

3. Stationarity properties

In time series modeling, to evaluate stability properties over time is important in statistical developments, in particular to reach good forecasts. The study of the stationarity of such models is thus a basic issue in their probabilistic analysis and will be the subject of this section.

3.1. First order stationarity. The following theorem gives a necessary and sufficient condition for first order stationarity of the general model introduced in (1).

Theorem 3.1. *Let* X *be a process satisfying the CP-INGARCH*(p, q) *model. This process is first order stationary if and only if* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

Proof. As X_t is a measurable positive function, is valid to apply operator expectation E and so we can write

$$\mu_t = E(X_t) = E(E(X_t|\mathcal{F}_{t-1})) = E(\lambda_t)$$

$$= \alpha_0 + \sum_{i=1}^{p} \alpha_i E(X_{t-i}) + \sum_{j=1}^{q} \beta_j E(\lambda_{t-j})$$

$$\Leftrightarrow \mu_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i \mu_{t-i} + \sum_{j=1}^{q} \beta_i \mu_{t-j}. \tag{3}$$

If there is a first order stationary solution then $E(X_t) = \mu$, $t \in \mathbb{Z}$, and hence

$$\left(1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j\right) \mu = \alpha_0.$$

As $\mu = E(X_t) = E(\lambda_t) > 0$ and $\alpha_0 > 0$, the parameters α_i and β_j necessarily verify $1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j > 0$.

Conversely, the non-homogeneous difference equation (3) has a stable solution, which is finite and independent of t, if and only if all the roots $z_1, ..., z_{\max(p,q)}$ of the equation

$$1 - \sum_{i=1}^{p} \alpha_i z^i - \sum_{j=1}^{q} \beta_j z^j = 0$$

lie outside the unit circle ([7]). As this property is equivalent to $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, the sufficient condition follows. ☐

Observation 3.1. *As a consequence of the previous theorem, if* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, *the processes* (X_t) *and* (λ_t) *are both first order stationary and we have*

$$E(X_t) = E(\lambda_t) = \mu = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}.$$

3.2. Second order stationarity. To develop the study on the second order stationarity of model (1) we assume that the family of characteristic functions $(\varphi_t, t \in \mathbb{Z})$ is derivable at zero up to order 2.

The general class of models considered and the complexity in the study of the second order stationarity in this class leads us to fix ourselves in the subclass of *CP-INGARCH*(p, q) models for which φ_t satisfies

$$H_2: \ -i\frac{\varphi_t''(0)}{\varphi_t'(0)} = v_0 + v_1 \lambda_t,$$

with $v_0 \geq 0$, $v_1 \geq 0$, not simultaneously zero. Although the restriction, it can be shown that a quite general subclass is considered, containing both random and deterministic characteristic functions since they have the general form $\varphi_t(u) = \exp \zeta(t) \int k(u)^{i(v_0+v_1\lambda_t)} du + \delta(t)$, with $k^{'}(0)/k(0) = 1$ and $u \in \mathbb{R}$. Let us note that the examples (1), (2) and (3) presented on the observation 2.2 belong to this subclass of models.

We begin by establishing a result which will be useful to obtain a sufficient condition of second order stationarity and gives us a representation of X in the state space. For simplicity, we consider $p = q$.

Proposition 3.1. *Let X be a first order stationary process following a CP-INGARCH(p,p) model such that H_2 is verified. The vector W_t, $t \in \mathbb{Z}$, of dimension $2p-1$ given by*

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_t X_{t-1}) \\ E(X_t X_{t-2}) \\ \cdots \\ E(X_t X_{t-(p-1)}) \\ E(\lambda_t \lambda_{t-1}) \\ E(\lambda_t \lambda_{t-2}) \\ \cdots \\ E(\lambda_t \lambda_{t-(p-1)}) \end{bmatrix}$$

satisfies an autoregressive equation of order p:

$$W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}, \tag{4}$$

where B_0 is a real vector of dimension $2p-1$ and B_k $(k = 1,...,p)$ are real squared matrices of order $2p-1$.

Proof. As previously remarked, $E(X_t^2)$, $E(X_t X_{t-k})$ and $E(\lambda_t \lambda_{t-k})$ are not necessarily finite, but as we have positive and measurable functions the involved integrals exist. We begin to calculate $E(X_t^2)$ for any $t \in \mathbb{Z}$. We have $E(X_t^2) = E[E(X_t^2 | \underline{X}_{t-1})]$ but

$$E(X_t^2 | \underline{X}_{t-1}) = V(X_t | \underline{X}_{t-1}) + \left[E(X_t | \underline{X}_{t-1})\right]^2 = -i \frac{\varphi_t^{''}(0)}{\varphi_t^{'}(0)} \lambda_t + \lambda_t^2 = v_0 \lambda_t + (1+v_1)\lambda_t^2$$

$$= v_0 \alpha_0 + (1+v_1)\alpha_0^2 + [v_0 + 2\alpha_0(1+v_1)] \left(\sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{p} \beta_j \lambda_{t-j} \right) + (1+v_1) \left[\sum_{i=1}^{p} \alpha_i^2 X_{t-i}^2 \right.$$

$$\left. + \sum_{\substack{i,j=1 \\ i \neq j}}^{p} \alpha_i \alpha_j X_{t-i} X_{t-j} + 2 \sum_{i=1}^{p} \sum_{j=1}^{p} \alpha_i \beta_j \lambda_{t-j} X_{t-i} + \sum_{i=1}^{p} \beta_i^2 \lambda_{t-i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{p} \beta_i \beta_j \lambda_{t-i} \lambda_{t-j} \right],$$

and so, using the first order stationary hypothesis, we conclude

$$E(X_t^2) = \widetilde{C} + (1+v_1)\left[\sum_{i=1}^{p}\alpha_i^2 E(X_{t-i}^2) + \sum_{\substack{i,j=1\\ i\neq j}}^{p}\alpha_i\alpha_j E(X_{t-i}X_{t-j})\right.$$

$$\left. +2\sum_{i,j=1}^{p}\alpha_i\beta_j E\left(X_{t-i}\lambda_{t-j}\right) + \sum_{i=1}^{p}\beta_i^2 E(\lambda_{t-i}^2) + \sum_{\substack{i,j=1\\ i\neq j}}^{p}\beta_i\beta_j E(\lambda_{t-i}\lambda_{t-j})\right]$$

$$= C + (1+v_1)\left[\sum_{i=1}^{p}\left(\alpha_i^2 + \frac{2\alpha_i\beta_i+\beta_i^2}{1+v_1}\right)E(X_{t-i}^2)\right.$$

$$\left. +2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\alpha_j(\alpha_i+\beta_i)E(X_{t-i}X_{t-j}) + 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\beta_j(\alpha_i+\beta_i)E(\lambda_{t-i}\lambda_{t-j})\right] \tag{5}$$

with $\widetilde{C} = v_0\mu + (1+v_1)\left[2\alpha_0\mu - \alpha_0^2\right]$ and $C = \widetilde{C} - v_0\mu\sum_{i=1}^{p}\left(2\alpha_i\beta_i + \beta_i^2\right)$ positive and independent of t constants, and where we took into account the following facts:

$$E(X_{t-i}\lambda_{t-j}) = \begin{cases} E(\lambda_{t-i}\lambda_{t-j}), & \text{if } j \geq i \\ E(X_{t-i}X_{t-j}), & \text{if } j < i \end{cases},\quad E(\lambda_t^2) = \frac{E(X_t^2) - v_0\mu}{1+v_1} \text{ and}$$

$$\mu - \alpha_0 = \mu\left(\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j\right).$$

Let us note that the relation between $E(X_t^2)$ and $E(\lambda_t^2)$ allow us to conclude that X is a second order process if and only if λ is a second order process.

On the other hand, when $k \geq 1$,

$$E(X_tX_{t-k}) = E\left[E(X_t|\mathcal{F}_{t-1})X_{t-k}\right] = E(\lambda_tX_{t-k}) = E\left(\left[\alpha_0 + \sum_{i=1}^{p}\alpha_iX_{t-i} + \sum_{j=1}^{p}\beta_j\lambda_{t-j}\right]X_{t-k}\right)$$

$$= \left[\alpha_0 - \frac{v_0\beta_k}{1+v_1}\right]\mu + \left[\alpha_k + \frac{\beta_k}{1+v_1}\right]E(X_{t-k}^2) + \sum_{i=k+1}^{p}\beta_iE(\lambda_{t-i}\lambda_{t-k})$$

$$+\sum_{i=1}^{k-1}(\alpha_i+\beta_i)E(X_{t-i}X_{t-k}) + \sum_{i=k+1}^{p}\alpha_iE(X_{t-i}X_{t-k}). \tag{6}$$

Similarly we obtain, for $k \geq 1$,

$$E(\lambda_t\lambda_{t-k}) = \left[\alpha_0 - \frac{v_0(\alpha_k+\beta_k)}{1+v_1}\right]\mu + \frac{\alpha_k+\beta_k}{1+v_1}E(X_{t-k}^2) + \sum_{i=k+1}^{p}\alpha_iE(X_{t-i}X_{t-k})$$

$$+\sum_{i=1}^{k-1}(\alpha_i+\beta_i)E(\lambda_{t-i}\lambda_{t-k})+\sum_{i=k+1}^{p}\beta_i E(\lambda_{t-i}\lambda_{t-k}). \tag{7}$$

Using the above expressions it is clear that $W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}$, with B_0 the vector and B_k $(k = 1, ..., p)$ the matrices presented in appendix A. □

For a $CP\text{-}INARCH(p)$ model, in which $\beta_i = 0$, $i = 1, ..., q$, the previous result assumes the form presented in the following corollary.

Corollary 3.1. *Let X be a first order stationary process following a CP-$INARCH(p)$ model such that H_2 is satisfied. The vector W_t, $t \in \mathbb{Z}$, of dimension p, given by*

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_tX_{t-1}) \\ \cdots \\ E(X_tX_{t-(p-1)}) \end{bmatrix}$$

follows an autoregressive equation of order p: $W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}$, with B_k $(k = 1, ..., p)$ squared matrices of order p with generic element $b_{ij}^{(k)}$ given by:

- *row $i = 1$:*
$$b_{1j}^{(k)} = \begin{cases} (1+v_1)\alpha_k^2, & \text{if } j = 1 \\ 2(1+v_1)\alpha_k\alpha_{j+k-1}, & \text{if } j = 2, ..., p \end{cases}$$
- *row $i \neq 1$:*
$$b_{ij}^{(k)} = \begin{cases} \alpha_{j+k-1}, & \text{if } i = k+1,\ j = 1, ..., p \\ \alpha_k, & \text{if } i = k+j,\ j = 2, ..., p \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha_i = 0$ for $i > p$, and B_0 the following vector of dimension p

$$B_0 = \begin{bmatrix} v_0\mu + \alpha_0(1+v_1)(2\mu - \alpha_0) \\ \alpha_0\mu \\ \cdots \\ \alpha_0\mu \end{bmatrix}.$$

Observation 3.2. *In proposition 3.1 we restrict ourselves to the case $p = q$ only to simplify the calculations. In fact, the same study can be made for the cases $p > q$, $p < q$ and the result is still valid, considering the vector W_t,*

$t \in \mathbb{Z}$, of dimension $p+q-1$,

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_t X_{t-1}) \\ \cdots \\ E(X_t X_{t-(p-1)}) \\ E(\lambda_t \lambda_{t-1}) \\ \cdots \\ E(\lambda_t \lambda_{t-(q-1)}) \end{bmatrix},$$

which satisfies an autoregressive equation of order $\max(p,q)$*:* $W_t = B_0 + \sum_{k=1}^{\max(p,q)} B_k W_{t-k}$*, where B_0 is a vector of dimension $p+q-1$ and B_k ($k = 1,...,\max(p,q)$) squared matrices of order $p+q-1$ whose components can be obtained from the generic elements of the matrices presented in appendix A.*

It is now possible to obtain a sufficient condition for weak stationarity of the process under study.

Theorem 3.2. *Let X be a first order stationary process following a CP-INGARCH(p,q) model such that H_2 is satisfied. This process is weakly stationary if*

$$P(L) = Id - \sum_{k=1}^{\max(p,q)} B_k L^k$$

is a polynomial matrix such that $\det P(z)$ *has all its roots outside the unit circle, where Id is the identity matrix of order $p+q-1$ and B_k ($k = 1,...,\max(p,q)$) are the squared matrices of the autorregressive equation* (4)*. Moreover,*

$$Cov(X_t, X_{t-i}) = e_{i+1}[P(1)]^{-1}B_0 - \mu^2, \ i = 0,...,p-1,$$
$$Cov(\lambda_t, \lambda_{t-i}) = e_{p+i}[P(1)]^{-1}B_0 - \mu^2, \ i = 1,...,q-1,$$

with e_i the order i row of the identity matrix.

Proof. Without loss of generality, let us consider $p \geq q$. As $\det P(z)$ has all roots outside the unit circle then $\det P(1) = \det\left(Id - \sum_{k=1}^{p} B_k\right) \neq 0$, that is, $P(1)$ is an invertible matrix. Thus, we obtain

$$W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k} \Leftrightarrow \left(Id - \sum_{k=1}^{p} B_k L^k\right) W_t = B_0$$

$$\Leftrightarrow \left(Id - \sum_{k=1}^{p} B_k L^k\right) W_t = \left(Id - \sum_{k=1}^{p} B_k\right) [P(1)]^{-1} B_0$$

$$\Leftrightarrow W_t - [P(1)]^{-1}B_0 = \sum_{k=1}^{p} B_k \left(W_{t-k} - [P(1)]^{-1}B_0\right).$$

The last equivalence shows that the sequence $(W_t - [P(1)]^{-1}B_0)_t$ satisfies an homogeneous linear recurrence equation under the stability condition. From Goldberg(1958) we conclude

$$\lim_t W_t = [P(1)]^{-1}B_0,$$

i.e., the solution of the equation is asymptotically independent of t. As W_t is asymptotically independent of t then, from the definition of W_t, follows the weak stationarity of (X_t) and (λ_t). Thus, we conclude the second order stationarity of the process. □

Theorem 3.3. *Consider a first order stationary CP-$INGARCH(1,1)$ model satisfying H_2. A necessary and sufficient condition for weak stationarity is*

$$(\alpha_1 + \beta_1)^2 + v_1\alpha_1^2 < 1.$$

Proof. From expression (5), we obtain, in this particular case, the non-homogeneous difference equation of first order

$$E(X_t^2) - [(\alpha_1 + \beta_1)^2 + v_1\alpha_1^2]E(X_{t-1}^2) = C,$$

where $C = v_0\mu + (1 + v_1)[2\alpha_0\mu - \alpha_0^2] - v_0\mu(\beta_1^2 + 2\alpha_1\beta_1) > 0$ and independent of t. If $(\alpha_1 + \beta_1)^2 + v_1\alpha_1^2 < 1$ then the above equation has an independent of t solution, that is, the process is second order stationary. On the other hand, if the process is second order stationary then $[1 - (\alpha_1 + \beta_1)^2 - v_1\alpha_1^2]E(X_t^2) = C \ \Rightarrow (\alpha_1 + \beta_1)^2 + v_1\alpha_1^2 < 1$. □

Let us now develop a necessary condition of weak stationarity for a CP-$INGARCH(p,p)$ model. To do that we will follow the idea present in [**5**] and [**14**] and, for simplicity, we restrict ourselves to the case $p = q$.

In that sense we consider $B = (b_{ij})$ the squared matrix of order $2p-2$ which terms are, to $i = 1, ..., p-1$, given by

$$b_{ij} = \begin{cases} \displaystyle\sum_{|k-i|=j} \alpha_k + \beta_{i-j}, & 1 \le j \le i-1 \\ \alpha_{2i} - 1, & j = i \\ \displaystyle\sum_{|k-i|=j} \alpha_k, & i+1 \le j \le p-1 \\ \beta_{i+j}, & p \le j \le 2p-i-1 \\ 0, & \text{otherwise} \end{cases}$$

$$b_{i+p-1,j}=\begin{cases}\alpha_{j+i}, & 1\leq j\leq p-i\\ \displaystyle\sum_{|k-i|=j-p+1}\beta_k+\alpha_{i-j+p-1}, & p\leq j\leq i+p-2\\ \beta_{2i}-1, & j=i+p-1\\ \displaystyle\sum_{|k-i|=j-p+1}\beta_k, & i+p\leq j\leq 2p-2\\ 0, & \text{otherwise}\end{cases}$$

where $\alpha_i=\beta_i=0$ for $i>p$. If B^{-1} exists, we denote its elements by d_{ij}. Consider also the vector $b=(b_{i0})$ with components

$$b_{i0}=\begin{cases}\alpha_i+\frac{\beta_i}{1+v_1}, & i=1,...,p-1\\ \frac{\alpha_{i-p+1}+\beta_{i-p+1}}{1+v_1}, & i=p,...,2p-2\end{cases}.$$

Theorem 3.4. *Let X be a process following a CP-INGARCH(p,p) model satisfying H_2 and such that $\alpha_0(1+v_1)>v_0$. If the process is second order stationary then all the roots of*

$$1-C_1z-...-C_pz^p=0$$

lie outside the unit circle, where for $v=1,...,p-1$,

$$C_v=(1+v_1)\left[\alpha_v^2+\frac{2\alpha_v\beta_v+\beta_v^2}{1+v_1}-2\sum_{(i,j)\in\{1,...,p\}^2:\ j-i=v}(\alpha_i+\beta_i)\sum_{u=1}^{2p-2}(\alpha_jd_{vu}+\beta_jd_{v+p-1,u})\,b_{u0}\right],$$

$$C_p=(1+v_1)\alpha_p^2+2\alpha_p\beta_p+\beta_p^2.$$

Proof. Let us start by recalling the existence of the CP-INARCH(∞) representation which results from the assumption of first order stationarity. From this representation and using the fact that X is a second order stationary process we conclude the second order stationarity of $\lambda=(\lambda_t,t\in\mathbb{Z})$. Let us denote $\gamma_k=E(X_tX_{t-k})$ and $\widetilde{\gamma}_k=E(\lambda_t\lambda_{t-k})$, with $k\in\mathbb{Z}$.

In what follows we use the expressions obtained for $E(X_t^2)$, $E(X_tX_{t-k})$ and $E(\lambda_t\lambda_{t-k})$ in proposition 3.1. From (5) we have,

$$\gamma_0=C+(1+v_1)\left[\sum_{i=1}^{p}\left(\alpha_i^2+\frac{2\alpha_i\beta_i+\beta_i^2}{1+v_1}\right)\gamma_0+2\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)(\alpha_j\gamma_v+\beta_j\widetilde{\gamma}_v)\right],\quad (8)$$

with $C=v_0\mu\left[1-\sum_{i=1}^{p}(2\alpha_i\beta_i+\beta_i^2)\right]+(1+v_1)\left[2\alpha_0\mu-\alpha_0^2\right]>0$ independent of t. From (6) it follows that for $k=1,...,p-1$,

$$\gamma_k=\left(\alpha_0-\frac{v_0\beta_k}{1+v_1}\right)\mu+\left(\alpha_k+\frac{\beta_k}{1+v_1}\right)\gamma_0+\sum_{i=k+1}^{p}\beta_i\widetilde{\gamma}_{i-k}+\sum_{i=1}^{k-1}(\alpha_i+\beta_i)\gamma_{k-i}+\sum_{i=k+1}^{p}\alpha_i\gamma_{i-k}$$

$$\Leftrightarrow \gamma_k - \sum_{|i-k|=1} \alpha_i \gamma_1 - ... - \sum_{|i-k|=k} \alpha_i \gamma_k - ... - \sum_{|i-k|=p-1} \alpha_i \gamma_{p-1} - \sum_{k-i=1} \beta_i \gamma_1 - ... - \sum_{k-i=k-1} \beta_i \gamma_{k-1}$$

$$- \sum_{i-k=1} \beta_i \widetilde{\gamma}_1 - ... - \sum_{i-k=p-k} \beta_i \widetilde{\gamma}_{p-k} = \left(\alpha_0 - \frac{v_0 \beta_k}{1+v_1}\right) \mu + \left(\alpha_k + \frac{\beta_k}{1+v_1}\right) \gamma_0,$$

or equivalently,

$$\sum_{u=1}^{p-1} b_{ku} \gamma_u + \sum_{u=1}^{p-k} b_{k,u+p-1} \widetilde{\gamma}_u = - \left[\left(\alpha_0 - \frac{v_0 \beta_k}{1+v_1}\right) \mu + \left(\alpha_k + \frac{\beta_k}{1+v_1}\right) \gamma_0 \right] \quad (9)$$

with

$$b_{ku} = \begin{cases} \sum_{|i-k|=u} \alpha_i + \beta_{k-u}, & 1 \leq u \leq k-1 \\ \alpha_{2k} - 1, & u = k \\ \sum_{|i-k|=u} \alpha_i, & k+1 \leq u \leq p-1 \end{cases}$$

and $b_{k,u+p-1} = \beta_{u+k}$, $u = 1, ..., p-k$, and where we consider $\alpha_i = \beta_i = 0$, $i > p$. Similarly we get from (7), for $k = 1, ..., p-1$,

$$\widetilde{\gamma}_k = \left(\alpha_0 - \frac{v_0(\alpha_k + \beta_k)}{1+v_1}\right) \mu + \frac{\alpha_k + \beta_k}{1+v_1} \gamma_0 + \sum_{i=k+1}^{p} \alpha_i \gamma_{i-k} + \sum_{i=1}^{k-1} (\alpha_i + \beta_i) \widetilde{\gamma}_{k-i} + \sum_{i=k+1}^{p} \beta_i \widetilde{\gamma}_{i-k}$$

$$\Leftrightarrow (1 - \beta_{2k}) \widetilde{\gamma}_k - \sum_{|i-k|=1} \beta_i \widetilde{\gamma}_1 - ... - \sum_{|i-k|=p-1} \beta_i \widetilde{\gamma}_{p-1} - \sum_{k-i=1} \alpha_i \widetilde{\gamma}_1 - ... - \sum_{k-i=k-1} \alpha_i \widetilde{\gamma}_{k-1}$$

$$- \sum_{i-k=1} \alpha_i \gamma_1 - ... - \sum_{i-k=p-k} \alpha_i \gamma_{p-k} = \left(\alpha_0 - \frac{v_0(\alpha_k + \beta_k)}{1+v_1}\right) \mu + \frac{\alpha_k + \beta_k}{1+v_1} \gamma_0,$$

or equivalently,

$$\sum_{u=1}^{p-k} b_{k+p-1,u} \gamma_u + \sum_{u=1}^{p-1} b_{k+p-1,u+p-1} \widetilde{\gamma}_u = - \left[\left(\alpha_0 - \frac{v_0(\alpha_k + \beta_k)}{1+v_1}\right) \mu + \frac{\alpha_k + \beta_k}{1+v_1} \gamma_0 \right] \quad (10)$$

with

$$b_{k+p-1,u+p-1} = \begin{cases} \sum_{|i-k|=u} \beta_i + \alpha_{k-u}, & 1 \leq u \leq k-1 \\ \beta_{2k} - 1, & u = k \\ \sum_{|i-k|=u} \beta_i, & k+1 \leq u \leq p-1 \end{cases}$$

and $b_{k+p-1,u} = \alpha_{u+k}$, $u = 1, ..., p-k$. Let $i, j = 1, ..., 2p-2$, $B = (b_{ij})$ and $B^{-1} = (d_{ij})$ its inverse which existence is a consequence of the first order stationarity (appendix B). Thus, from expressions (9) and (10) and using the invertibility of B we obtain

$$\widehat{\gamma}=\begin{bmatrix}\gamma_1\\ \vdots\\ \gamma_{p-1}\\ \widetilde{\gamma}_1\\ \vdots\\ \widetilde{\gamma}_{p-1}\end{bmatrix}=-B^{-1}\left(\begin{bmatrix}\alpha_0\mu\\ \vdots\\ \alpha_0\mu\\ \alpha_0\mu\\ \vdots\\ \alpha_0\mu\end{bmatrix}+v_0\mu\begin{bmatrix}\alpha_1-b_{10}\\ \vdots\\ \alpha_{p-1}-b_{p-1,0}\\ -b_{p,0}\\ \vdots\\ -b_{2p-2,0}\end{bmatrix}+\gamma_0 b\right),$$

where b is the vector previously introduced. So, for $l=1,...,2p-2$,

$$\widehat{\gamma}_l=-\alpha_0\mu\sum_{u=1}^{2p-2}d_{lu}+v_0\mu\left[\sum_{u=1}^{p-1}(b_{u0}-\alpha_u)d_{lu}+\sum_{u=p}^{2p-2}b_{u0}d_{lu}\right]-\sum_{u=1}^{2p-2}d_{lu}b_{u0}\gamma_0.$$

Taking the last part of (8) and, using the previous expression, we get

$$2\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)\left(\alpha_j\gamma_v+\beta_j\widetilde{\gamma}_v\right)$$

$$=\widehat{C}-2\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)\left[\alpha_j\sum_{u=1}^{2p-2}d_{vu}b_{u0}+\beta_j\sum_{u=1}^{2p-2}d_{v+p-1,u}b_{u0}\right]\gamma_0,$$

where $\widehat{C}$ is a positive constant independent of t, as proved in appendix B.
Then replacing this expression in (8), we finally get

$$\gamma_0=C_0+(1+v_1)\left[\sum_{i=1}^{p}\left(\alpha_i^2+\frac{2\alpha_i\beta_i+\beta_i^2}{1+a_1}\right)\gamma_0\right.$$

$$\left.-2\sum_{v=1}^{p-1}\sum_{j-i=v}(\alpha_i+\beta_i)\left(\alpha_j\sum_{u=1}^{2p-2}d_{vu}b_{u0}+\beta_j\sum_{u=1}^{2p-2}d_{v+p-1,u}b_{u0}\right)\gamma_0\right]$$

$$=C_0+(1+v_1)\left[\left(\alpha_p^2+\frac{2\alpha_p\beta_p+\beta_p^2}{1+v_1}\right)\gamma_0+\sum_{v=1}^{p-1}\left\{\left(\alpha_v^2+\frac{2\alpha_v\beta_v+\beta_v^2}{1+v_1}\right)\right.\right.$$

$$\left.\left.-2\sum_{j-i=v}(\alpha_i+\beta_i)\sum_{u=1}^{2p-2}\left(\alpha_j d_{vu}+\beta_j d_{v+p-1,u}\right)b_{u0}\right\}\gamma_0\right],$$

or,

$$\gamma_0 = C_0 + \sum_{v=1}^{p} C_v \gamma_0 \Leftrightarrow \left(1 - \sum_{v=1}^{p} C_v\right) \gamma_0 = C_0,$$

where $C_0 = C + (1 + v_1)\widehat{C} > 0$ and C_v are the coefficients defined in the statement of the theorem. Hence, the previous equality implies $1-\sum_{v=1}^{p} C_v > 0$, that is, the roots of equation $1 - C_1 z - ... - C_p z^p = 0$ lie outside the unit circle. $\square$

Observation 3.3. *The previous theorem is also valid in the case $p \neq q$. In fact, it is sufficient to consider B the squared matrix of order $p + q - 2$ with components deducible from the case $p = q$, considering the adequate coefficients equal to zero, and the equation $1 - C_1 z - ... - C_r z^r = 0$, with $r = \max(p, q)$ and $C_1, ..., C_r$ analogous to that presented when $p = q$.*

Let us point out that when X follows a CP-$INARCH(p)$ model we easily obtain $\widehat{C} = -2\alpha_0\mu \sum_{v=1}^{p-1} \sum_{j-i=v} \alpha_i\alpha_j \sum_{u=1}^{p-1} d_{vu} > 0$ in appendix B. Therefore, in this case, we do not need to ensure that $\alpha_0(1+v_1) > v_0$ and the last theorem assumes the following form.

Corollary 3.2. *Let X be a process following a CP-$INARCH(p)$ model satisfying H_2. If the process is second order stationary, then the roots of the equation $1 - C_1 z - ... - C_p z^p = 0$ are outside the unit circle, where for $u, l = 1, ..., p - 1$,*

$$C_u = (1 + v_1) \left[\alpha_u^2 - \sum_{v=1}^{p-1} \sum_{|i-j|=v} \alpha_i\alpha_j d_{vu} b_{u0} \right], \; C_p = (1 + v_1)\, \alpha_p^2,$$

$$b_{l0} = \alpha_l, \; b_{ll} = \sum_{|i-l|=l} \alpha_i - 1 \text{ and for } u \neq l, \; b_{lu} = \sum_{|i-l|=u} \alpha_i,$$

with $B = (b_{ij})$ and $B^{-1} = (d_{ij})$ squared matrices of order $p - 1$.

In the following we present some examples to illustrate the conditions of second order stationarity displayed.

Example 3.1. *Let us consider a CP-$INGARCH(2, 2)$ model verifying H_2 and such that $\sum_{i=1}^{2} (\alpha_i + \beta_i) < 1$. To examine the sufficient condition of second order stationarity we consider the polynomial matrix $P(z) = Id -$*

$B_1 z - B_2 z^2$, *with* B_1 *and* B_2 *the squared matrices of order* 3 *given by*

$$B_1 = \begin{bmatrix} (\alpha_1+\beta_1)^2+v_1\alpha_1^2 & 2(1+v_1)\alpha_2(\alpha_1+\beta_1) & 2(1+v_1)\beta_2(\alpha_1+\beta_1) \\ \alpha_1+\frac{\beta_1}{1+v_1} & \alpha_2 & \beta_2 \\ \frac{\alpha_1+\beta_1}{1+v_1} & \alpha_2 & \beta_2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} (\alpha_2+\beta_2)^2+v_1\alpha_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the determinant of this polynomial is

$$\begin{aligned} \det(P(z)) &= 1 - \left[(\alpha_1+\beta_1)^2+\alpha_2+\beta_2+v_1\alpha_1^2\right] z \\ &- \left[(\alpha_1+\beta_1)^2(\alpha_2+\beta_2)+(\alpha_2+\beta_2)^2+v_1(\alpha_2^2-\alpha_1^2\beta_2+\alpha_1^2\alpha_2+2\alpha_1\alpha_2\beta_1)\right] z^2 \\ &- \left[-(\alpha_2+\beta_2)^3-v_1\alpha_2^2(\alpha_2+\beta_2)\right] z^3. \end{aligned}$$

So

$$\det(P(1)) > 0 \Leftrightarrow (\alpha_1+\beta_1)^2(1+\alpha_2+\beta_2)+(\alpha_2+\beta_2)\left[1+\alpha_2+\beta_2-(\alpha_2+\beta_2)^2\right]$$
$$+v_1\left(\alpha_1^2[1+\alpha_2-\beta_2]+\alpha_2^2[1-\alpha_2-\beta_2]+2\alpha_1\alpha_2\beta_1\right) < 1, \tag{11}$$

that is, whenever the coefficients of the model satisfy this inequality, the process X *is second order stationary. In order to obtain the necessary condition, we study the roots of the equation* $1 - C_1 z - C_2 z^2 = 0$ *with*

$$\begin{aligned} C_1 &= (\alpha_1+\beta_1)^2+v_1\alpha_1^2-2(1+v_1)\left[(\alpha_1+\beta_1)\sum_{u=1}^{2}(\alpha_2 d_{1u}+\beta_2 d_{2u})b_{u0}\right] \\ &= (\alpha_1+\beta_1)^2+v_1\alpha_1^2+2(1+v_1)(\alpha_1+\beta_1)\frac{\alpha_2 b_{10}+\beta_2 b_{20}}{1-\alpha_2-\beta_2} \\ &= \left[\frac{1+\alpha_2+\beta_2}{1-\alpha_2-\beta_2}\right](\alpha_1+\beta_1)^2+v_1\left[\frac{\alpha_1(1+\alpha_2-\beta_2)+2\alpha_2\beta_1}{1-\alpha_2-\beta_2}\right]\alpha_1, \end{aligned}$$

$$C_2 = (\alpha_2+\beta_2)^2+v_1\alpha_2^2,$$

since the matrices B *and* B^{-1} *and the vector* b *are given by*

$$b = \begin{bmatrix} \alpha_1+\frac{\beta_1}{1+v_1} \\ \frac{\alpha_1+\beta_1}{1+v_1} \end{bmatrix},\ B = \begin{bmatrix} \alpha_2-1 & \beta_2 \\ \alpha_2 & \beta_2-1 \end{bmatrix},\ B^{-1} = \begin{bmatrix} \frac{\beta_2-1}{1-\alpha_2-\beta_2} & \frac{-\beta_2}{1-\alpha_2-\beta_2} \\ \frac{-\alpha_2}{1-\alpha_2-\beta_2} & \frac{\alpha_2-1}{1-\alpha_2-\beta_2} \end{bmatrix}.$$

Hence, the roots of equation are outside the unit circle if and only if $C_1 + C_2 < 1$, *which coincides with the sufficient condition* (11) *and so is also a necessary and sufficient condition of weak stationarity.*

We can also deduce from last result the necessary and sufficient condition for second order stationarity of a CP-$INARCH(2)$ model. Indeed, if $\beta_1 = \beta_2 = 0$ we obtain

$$(1+v_1)\left[\alpha_1^2(1+\alpha_2)+\alpha_2^2(1-\alpha_2)\right]+\alpha_2<1.$$

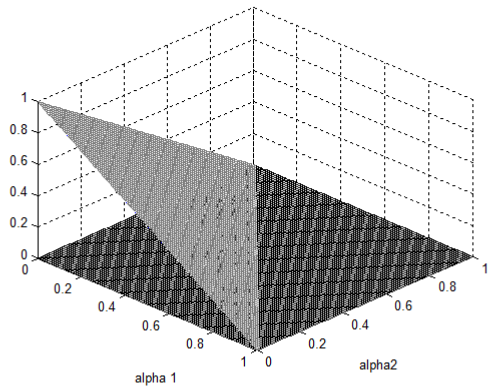

FIGURE 1. Frontier of the first order stationary region of a CP-$INARCH(2)$ model.

Figures 1, 2 and 3 illustrate, for $v_1 = 0$, the first and second stationarity regions of the CP-$INARCH(2)$ model. The region plotted in Figure 3 is obviously the intersection of the first order stationarity region with the set of points (α_1, α_2) verifying last condition.

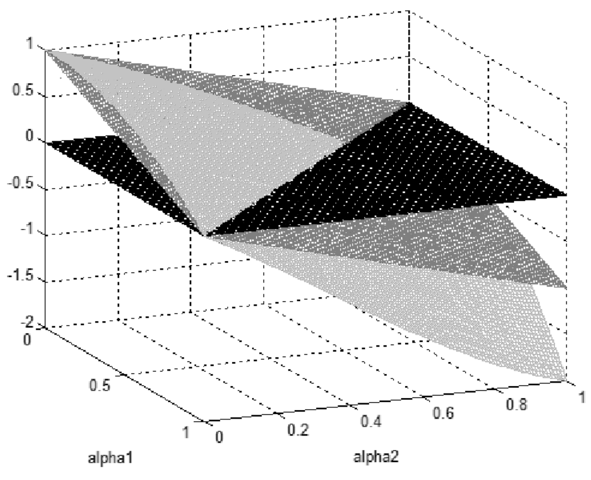

FIGURE 2. Frontiers of first and second order stationary regions of a CP-$INARCH(2)$ model.

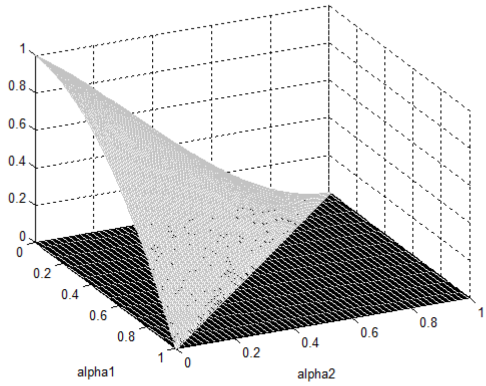

FIGURE 3. Frontier of the weak stationarity region of a CP-$INARCH(2)$ model.

Example 3.2. *Let us now consider a CP-$INGARCH(p,p)$ model with $\alpha_1 = \ldots = \alpha_{p-1} = \beta_1 = \ldots = \beta_{p-1} = 0$ verifying H_2 and such that $\alpha_p + \beta_p < 1$.*

To analyze the sufficient condition for weak stationarity of X we consider the polynomial matrix resulting from the Theorem 3.2

$$P(z) = Id - B_1 z - \ldots - B_p z^p$$

$$= \begin{bmatrix} 1-[(\alpha_p+\beta_p)^2+v_1\alpha_p^2]z^p & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-\alpha_p z^{p-1} & -\alpha_p z^{p-2} & \cdots & -\alpha_p z & -\beta_p z^{p-1} & -\beta_p z^{p-2} & \cdots & -\beta_p z^2 & -\beta_p z \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_p z^{p-1} & -\alpha_p z^{p-2} & \cdots & -\alpha_p z & 1-\beta_p z^{p-1} & -\beta_p z^{p-2} & \cdots & -\beta_p z^2 & -\beta_p z \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

with $B_k (k = 1, \ldots, p)$ squared matrices of order $2p-1$. In what follows denote by $P_{ij}(z)$ the submatrix of $P(z)$ obtained by deleting the row i and the column j. Applying Laplace theorem to the first row of the matrix $P(z)$ we have

$$\det P(z) = [1 - ((\alpha_p+\beta_p)^2 + v_1\alpha_p^2)z^p] \det P_{11}(z)$$

with

$$\det P_{11}(z) = \begin{vmatrix} 1-\alpha_p z^{p-1} & -\alpha_p z^{p-2} & \cdots & -\alpha_p z & -\beta_p z^{p-1} & -\beta_p z^{p-2} & \cdots & -\beta_p z \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -\alpha_p z^{p-1} & -\alpha_p z^{p-2} & \cdots & -\alpha_p z & 1-\beta_p z^{p-1} & -\beta_p z^{p-2} & \cdots & -\beta_p z \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= (1-\alpha_p z^{p-1}) \det Q_{11}(z) + \alpha_p z^{p-2} \det Q_{12}(z) - ... + (-1)^{p+1}\alpha_p z \det Q_{1,p-1}(z)$$

$$+(-1)^{p+2}\beta_p z^{p-1} \det Q_{1p}(z) + (-1)^{p+3}\beta_p z^{p-2} \det Q_{1,p+1}(z) + ... + \beta_p z \det Q_{1,2p-2}(z),$$

again using the Laplace theorem in the first row of the matrix $P_{11}(z)$ *and taking* $Q_{ij}(z)$ *the submatrix of* $P_{11}(z)$ *obtained by deleting the row* i *and the column* j. *Let us note that* $\det Q_{12}(z) = ... = \det Q_{1,p-1}(z) = \det Q_{1,p+1}(z) = ... = \det Q_{1,2p-2}(z) = 0$ *because all the matrices have a null row.*

Now, applying Laplace theorem on row $p-1$ *of the matrices* $Q_{11}(z)$, $Q_{1p}(z)$ *we obtain*

$$\det Q_{11}(z) = \begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -\alpha_p z^{p-2} & \cdots & -\alpha_p z & 1-\beta_p z^{p-1} & -\beta_p z^{p-2} & \cdots & -\beta_p z \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = (-1)^{2p-2}(1-\beta_p z^{p-1}),$$

$$\det Q_{1p}(z) = \begin{vmatrix} 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ -\alpha_p z^{p-1} & -\alpha_p z^{p-2} & \cdots & -\alpha_p z & -\beta_p z^{p-2} & \cdots & -\beta_p z \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix} = (-1)^{p+1}\alpha_p z^{p-1},$$

since when we delete that line, a column of zeros appears except when we consider the term in the position $(p-1, p-1)$ *and* $(1,1)$, *respectively. This allow us to conclude*

$$\det P_{11}(z) = (1-\alpha_p z^{p-1})(1-\beta_p z^{p-1}) + (-1)^{2p+3}\alpha_p\beta_p z^{2p-2} = 1-(\alpha_p+\beta_p)z^{p-1}$$

and finally

$$\det P(z) = [1-((\alpha_p+\beta_p)^2+v_1\alpha_p^2)z^p][1-(\alpha_p+\beta_p)z^{p-1}]$$
$$= 1-(\alpha_p+\beta_p)z^{p-1}-[(\alpha_p+\beta_p)^2+v_1\alpha_p^2][1-(\alpha_p+\beta_p)z^{p-1}]z^p.$$

We deduce that the sufficient condition for second order stationarity of the considered model, and taking into account that $\alpha_p+\beta_p<1$, is given by

$$\det(P(1))>0 \Leftrightarrow (\alpha_p+\beta_p)^2+v_1\alpha_p^2<1.$$

Finally this condition is also the necessary condition obtained in Theorem 3.4 *as it reduces to ensure that the roots of $1-C_pz^p=0$, with $C_p=(\alpha_p+\beta_p)^2+v_1\alpha_p^2$, lie outside the unit circle, that is, $C_p<1$. Let us note that in this case it is not necessary to ensure that $\alpha_0(1+v_1)>v_0$ because in the proof of theorem* 3.4 *we have $\widehat{C}=0$ and so the constant C_0 is always positive.*

3.3. Strict stationarity. In this section we study the existence of strictly stationary solutions for the class of models previously introduced. The study undertaken allows us to establish the existence of strictly stationary and ergodic processes in a subclass of CP-$INGARCH(p,q)$ models for which the characteristic function is deterministic. In this subclass, we begin by building a first order stationary process solution of the model that, under certain conditions, will be strictly stationary and ergodic.

3.3.1. *Construction of a process solution when φ_t is deterministic.* Let us consider model (1) associated to a given family of characteristic functions $(\varphi_t, t\in\mathbb{Z})$ such that the hypothesis H_1 is satisfied. We assume $H_3:\varphi_t$ is deterministic.

Let $(U_t, t\in\mathbb{Z})$ be a sequence of independent real random variables distributed according to a discrete compound-Poisson law with characteristic function

$$\Phi_{U_t}(u) = exp\left\{\frac{\alpha_0}{B(1)}\frac{i}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}.$$

For each $t\in\mathbb{Z}$ and $k\in\mathbb{N}$, let $\mathcal{Z}_{t,k}=\{Z_{t,k,j}\}_{j\in\mathbb{N}}$ be a sequence of independent discrete compound-Poisson random variables with characteristic function

$$\Phi_{Z_{t,k,j}}(u) = exp\left\{\psi_k\frac{i}{\varphi_{t+k}'(0)}\left[\varphi_{t+k}(u)-1\right]\right\},$$

where $(\psi_j, j\in\mathbb{N})$ is the sequence of coefficients associated to the CP-$INARCH(\infty)$ representation of the model. We note that $E(U_t)=\alpha_0B^{-1}(1)=$

ψ_0, $E(Z_{t,k,j}) = \psi_k$ and that $Z_{t,k,j}$ are identically distributed for each $(t,k) \in \mathbb{Z} \times \mathbb{N}$. We also assume that all the variables U_s, $Z_{t,k,j}$, $s,t \in \mathbb{Z}, k,j \in \mathbb{N}$, are mutually independent. Based on these random variables, we define the sequence $X_t^{(n)}$ as follows:

$$X_t^{(n)} = \begin{cases} 0, & n < 0 \\ U_t, & n = 0 \\ U_t + \sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^{(n-k)}} Z_{t-k,k,j}, & n > 0 \end{cases}, \tag{12}$$

where we consider the convention $\sum_{j=1}^{0} Z_{t-k,k,j} = 0$. Let us recall the definition of *thinning operation*: considering a non-negative integer-valued random variable W and $\phi \geq 0$, the thinning operation is defined by

$$\phi \circ W = \begin{cases} \sum_{j=1}^{W} V_j, & \text{if } W > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\{V_j\}$, called counting series, is a sequence of i.i.d. non-negative integer-valued random variables, independent of W and such that $E(V_j) = \phi$. An important property of this operation is that $E\left(\phi \circ W\right) = \phi E\left(W\right)$ ([**6**]). Using this definition, $X_t^{(n)}$, $n > 0$, is rewritten in the form

$$X_t^{(n)} = U_t + \sum_{k=1}^{n} \psi_k^{(t-k)} \circ X_{t-k}^{(n-k)}, \tag{13}$$

where the notation $(\psi_k^{(\tau)} \circ)$ means that the sequence of random variables of mean ψ_k involved in the thinning operation corresponds to time τ. Similarly to [**4**] this representation shows that

$$X_t^{(n)} = f\left(U_{t-n}, ..., U_t\right), \ n \geq 0.$$

In what follows we present some properties of the sequence $X_t^{(n)}$, which will be of interest in the study of its behavior.

Property 3.1. *If* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ *then* $\{(X_t^{(n)}, t \in \mathbb{Z}), \ n \in \mathbb{Z}\}$ *is a sequence of first order stationary processes such that, as* $n \to \infty$*,*

$$\mu_n = E\left(X_t^{(n)}\right) \longrightarrow \mu.$$

Proof. We start by noting that $E(X_t^{(n)})$ does not depend on t, $\forall n \in \mathbb{Z}$. The result is trivial for $n < 0$. For $n = 0$ we obtain $E(X_t^{(0)}) = E(U_t) = \psi_0$, which is also independent of t. Let us consider now, as induction hypothesis, that for an arbitrarily fixed value of t and for $n > 0$, $E(X_t^{(n)})$ is independent of t. Therefore,

$$E\left(X_t^{(n+1)}\right) = E\left(U_t + \sum_{k=1}^{n+1} \psi_k^{(t-k)} \circ X_{t-k}^{(n+1-k)}\right) = \psi_0 + \sum_{k=1}^{n+1} \psi_k E\left(X_{t-k}^{(n+1-k)}\right)$$

$$= g\left(E\left(X_{t-n-1}^{(0)}\right), ..., E\left(X_{t-1}^{(n)}\right)\right),$$

that is an independent function of t. So

$$\mu_n = E\left(X_t^{(n)}\right) = \begin{cases} 0, & n < 0 \\ \psi_0, & n = 0 \\ \psi_0 + \sum_{k=1}^{n} \psi_k \mu_{n-k}, & n > 0 \end{cases},$$

which for $n > 0$ is equivalent to

$$\mu_n = \sum_{k=1}^{\infty} \psi_k \mu_{n-k} + \psi_0 = B^{-1}(L)\left[A(L)\mu_n + \alpha_0\right] \Longleftrightarrow B(L)\mu_n = A(L)\mu_n + \alpha_0$$

$$\Longleftrightarrow K(L)\mu_n = \alpha_0,$$

taking $K(L) = B(L) - A(L)$.

Thus, the sequence $\{\mu_n\}$ satisfies a finite difference equation of degree $\max(p, q)$ with constant coefficients. The characteristic polynomial $K(z)$ of this equation has all its roots outside of the unit circle since $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, and so, $\{(X_t^{(n)}, t \in \mathbb{Z}),\ n \in \mathbb{Z}\}$ is a sequence of first order stationary processes. From this stationarity, we deduce

$$\lim_{n\to\infty} \mu_n = \frac{\psi_0}{1 - \sum_{k=1}^{\infty} \psi_k} = \frac{\alpha_0 B^{-1}(1)}{1 - H(1)} = \frac{\alpha_0}{K(1)} = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j} = \mu.$$

$\square$

Property 3.2. *If $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ and φ_t derivable at zero up to order 2, then the sequence $\{(X_t^{(n)}, t \in \mathbb{Z}),\ n \in \mathbb{Z}\}$ converges almost surely, in L^1 and L^2 to a process $(X_t^*, t \in \mathbb{Z})$.*

Proof. Let's begin by showing that $\{(X_t^{(n)}, t \in \mathbb{Z}), n \in \mathbb{Z}\}$ is a non-decreasing sequence. Indeed, when $n = 0$ and for a fixed value of t, we have

$$X_t^{(1)} - X_t^{(0)} = U_t + \sum_{j=1}^{X_{t-1}^{(0)}} Z_{t-1,1,j} - U_t = \sum_{j=1}^{U_{t-1}} Z_{t-1,1,j} \geq 0,$$

because this is a random sum of non-negative integer random variables. Supposing that for any fixed t and for $n > 0$ we have $X_t^{(n)} - X_t^{(n-1)} \geq 0$ we obtain

$$X_t^{(n+1)} - X_t^{(n)} = \sum_{k=1}^{n} \sum_{j=X_{t-k}^{(n-k)}+1}^{X_{t-k}^{(n+1-k)}} Z_{t-k,k,j} + \sum_{j=1}^{U_{t-n-1}} Z_{t-n-1,n+1,j},$$

which is obviously a non-negative process.

Using the monotony of the sequence and the hypothesis on the model coefficients we prove that $\{(X_t^{(n)}, t \in \mathbb{Z}), n \in \mathbb{Z}\}$ converges almost surely to a process, $(X_t^*, t \in \mathbb{Z})$, that is almost surely finite, using Borel-Cantelli theorem like in Proposition 2 of [**4**].

Applying Beppo Lévi's theorem we conclude that the first moment of X_t^* is finite since by the property 3.1

$$\mu = \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} E\left(X_t^{(n)}\right) = E\left(X_t^*\right)$$

and consequently the convergence of $\{(X_t^{(n)}, t \in \mathbb{Z}), n \in \mathbb{Z}\}$ in L^1 is deduced. For its convergence in L^2, we proceed as [**4**] in Propositions 4 and 5, noting that

$$V\left(Z_{t-k,k,j}\right) = \Phi''_{Z_{t-k,k,j}}(0) - \psi_k^2 = i\frac{\varphi''_{t+k}(0)}{\varphi'_{t+k}(0)}\psi_k = R_t(\psi_k) < \infty.$$

□

3.3.2. *Stationarity and Ergodicity.* Taking into account the results of the previous section, we obtain the next theorem.

Theorem 3.5. *A CP-INGARCH(p,q) model, satisfying hypothesis H_3, admits a solution. This solution is first order stationary if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.*

Proof. We start by proving that the limit (X_t^*) of the sequence $(X_t^{(n)})$ is a solution of the model (1) showing that, for any $u \in \mathbb{R}$,

$$\Phi_{X_t^*|\underline{X}_{t-1}^*}(u) = e^{\lambda_t^*[\varphi_t(u)-1]},$$

with $E(X_t^*|\underline{X}_{t-1}^*) = \lambda_t = -i\ \varphi_t'(0)\lambda_t^*$.

Similarly to [**4**], section 2.6, we state that for a fixed t, the sequence $Y_t^{(n)} = r_t^{(n)} - X_t^{(n)}$ with

$$r_t^{(n)} = U_t + \sum_{k=1}^{n} \sum_{j=1}^{X_{t-k}^*} Z_{t-k,k,j},$$

converges in mean to zero, when $n \to \infty$. Then $Y_t^{(n)}$ and $X_t^* - X_t^{(n)}$ converge in probability to zero. Moreover,

$$X_t^* - r_t^{(n)} = (X_t^* - X_t^{(n)}) + (X_t^{(n)} - r_t^{(n)}) = (X_t^* - X_t^{(n)}) - Y_t^{(n)}.$$

This equality allows us to conclude that the sequence $r_t^{(n)}$ converges in probability to X_t^* and then $r_t^{(n)}|\underline{X}_{t-1}^*$ converges in law to $X_t^*|\underline{X}_{t-1}^*$ and so, by Paul Lévy theorem,

$$\lim_{n\to+\infty} \Phi_n = \Phi_{X_t^*|\underline{X}_{t-1}^*}(u), \ \forall u \in \mathbb{R},$$

where Φ_n denotes the characteristic function of $r_t^{(n)}|\underline{X}_{t-1}^*$.

Let us obtain Φ_n. Conditionally to $\underline{X}_{t-1}^*$, we have

$$\Phi_{\sum_{j=1}^{X_{t-k}^*} Z_{t-k,k,j}}(u) = \prod_{j=1}^{X_{t-k}^*} \Phi_{Z_{t-k,k,j}}(u) = \exp\left\{\sum_{j=1}^{X_{t-k}^*} \psi_k \frac{i}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}$$
$$= \exp\left\{\psi_k X_{t-k}^* \frac{i}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}.$$

From the independence of the variables involved in the definition of $r_t^{(n)}$, we obtain

$$\Phi_n(u) = \exp\left(\frac{\alpha_0}{B(1)}\frac{i}{\varphi_t'(0)}\left[\varphi_t(u)-1\right] + \sum_{k=1}^{n}\psi_k X_{t-k}^* \frac{i}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right)$$
$$= \exp\left\{\left(\frac{\alpha_0}{B(1)} + \sum_{k=1}^{n}\psi_k X_{t-k}^*\right)\frac{i}{\varphi_t'(0)}\left[\varphi_t(u)-1\right]\right\}.$$

Considering

$$\lambda_t^{(n)} = \frac{\alpha_0}{B(1)} + \sum_{k=1}^{n}\psi_k X_{t-k}^*,$$

we deduce that $\lambda_t^{(n)} \to \lambda_t$ and thus $\frac{i}{\varphi_t'(0)}\lambda_t^{(n)} \to \lambda_t^*$, when $n \to \infty$. So,

$$\lim_{n\to+\infty} \Phi_n(u) = \exp\left\{\lambda_t^*\left[\varphi_t(u)-1\right]\right\} = \Phi_{X_t^*|\underline{X}_{t-1}^*}(u), \ \forall u \in \mathbb{R},$$

which shows that the almost sure limit of $X_t^{(n)}$, X_t^*, is a solution of the model (1). The first order stationarity of the solution is a consequence of Theorem 3.1. □

Now, let us consider φ_t deterministic and independent of t, i.e., $\varphi_t \equiv \varphi$, $\forall t \in \mathbb{Z}$. In this subclass, it is possible to establish the strict stationarity of (X_t^*), as well as its ergodicity.

Theorem 3.6. *Let $\varphi_t \equiv \varphi,\ t \in \mathbb{Z}$, in the model $CP\text{-}INGARCH(p,q)$ defined by* (1)*.*

- **(a):** $\{(X_t^{(n)}, t \in \mathbb{Z}), n \in \mathbb{Z}\}$ *is a sequence of strictly stationary and ergodic processes;*
- **(b):** *There exists a strictly stationary and ergodic process that satisfies the model* (1)*, if and only if* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$*. Moreover, its first two moments are finite.*

Proof.

- **(a):** The proof of strict stationarity follows the procedure presented in Proposition 3 of [**4**], since in this particular case the sequences $(U_t, t \in \mathbb{Z})$ and $(\mathcal{Z}_{t,k}, t \in \mathbb{Z}, k \in \mathbb{N})$ defined in section 3.3.1 are of i.i.d. random variables. Moreover, $(X_t^{(n)})$ is a sequence of ergodic processes, because it is a measurable function of the sequence of i.i.d. random variables $\{(U_t, \mathcal{Z}_{t,j}), t \in \mathbb{Z}, j \in \mathbb{N}\}$;
- **(b):** In theorem 3.5 we proved that $(X_t^*, t \in \mathbb{Z})$ is a solution of (1). So, it is enough to prove that when $\varphi_t \equiv \varphi$, the almost sure limit is strictly stationary and ergodic. From (a), $(X_t^{(n)})$ is a sequence of strictly stationary processes. Otherwise, $(X_t^{(n)})$ converge almost surely to (X_t^*) if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$. So, considering without loss of generality, the indexes $\{1, ..., k\}$, we have for any $h \in \mathbb{Z}$,

$$(X_1^{(n)}, ..., X_k^{(n)}) \longrightarrow_{n \Rightarrow +\infty} (X_1^*, ..., X_k^*),$$

$$(X_{1+h}^{(n)}, ..., X_{k+h}^{(n)}) \longrightarrow_{n \Rightarrow +\infty} (X_{1+h}^*, ..., X_{k+h}^*),$$

almost surely and, in consequence, in law. Taking into account the strict stationarity of $(X_t^{(n)})$ and the limit unicity, it's easy to conclude that (X_t^*) is a strictly stationary process.

In what concerns the ergodicity it follows from the fact that the limit of a measurable function is still measurable.

Regarding the necessary condition, we observe that if (X_t^*) is a strictly stationary solution of model (1) it is also first order stationary as, by property 3.2, it is a process of L^1. So, by Theorem 3.1, $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

□

Observation 3.4. *Under the conditions of the previous theorem it follows that* $\{X_t^*\}_{t\in\mathbb{Z}}$ *is also a weakly stationary solution of the model because it is a strictly stationary second order process.*

4. Conclusion

In this paper a general class of $INGARCH$ models was introduced, including as particular cases some recent contributions on the modeling of integer-valued time series ([**4**], [**14**], [**15**]). This generality is achieved considering that the distribution of X_t given its past belongs to the family of infinitely divisible discrete laws and defining the model by means of the corresponding characteristic function.

Conditions for first and second order stationarity are given and the existence of a strict stationary and ergodic solution is established in a subclass which includes, in particular, the Poisson and generalized Poisson $INGARCH$ models. Moreover, we are strongly convicted that sufficient condition of second order stationarity is also a necessary one as it happens in the cases developed in examples 3.1 and 3.2. Other particular cases were studied, namely the corresponding to the $CP\text{-}INGARCH(3)$ model for which that result is still easily established.

The probabilistic study here developed will be very useful in future statistical studies as, in particular, those related to the model estimation. Other probabilistic studies may be considered in future as, for instance, those of moments greater than 2 which will be essential in the evaluation of other features of the model like leptokurtic or Taylor properties ([**8**]). Despite of all this future work in order to implement these general models in practice, we should stress that the studies here developed, unifying and enlarging several approaches recently considered in the literature, presents a significant contribution to the modeling of integer valued time series.

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Appendix A. Autorregressive equation of W_t

From (5), (6) and (7) it follows that the vector W_t verifies the autorregressive equation $W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}$, of order p, where denoting by $v = 1 + v_1$

$$B_0 = \begin{bmatrix} C \\ \mu\left(\alpha_0 - \frac{v_0\beta_1}{v}\right) \\ \cdots \\ \mu\left(\alpha_0 - \frac{v_0\beta_{p-1}}{v}\right) \\ \mu\left(\alpha_0 - \frac{v_0(\alpha_1+\beta_1)}{v}\right) \\ \cdots \\ \mu\left(\alpha_0 - \frac{v_0(\alpha_{p-1}+\beta_{p-1})}{v}\right) \end{bmatrix},$$

and B_k $(k = 1, ..., p)$ are the squared matrices having generic element $b_{ij}^{(k)}$ given by:

- row $i = 1$:
$$b_{1j}^{(k)} = \begin{cases} v\alpha_k^2 + 2\alpha_k\beta_k + \beta_k^2, & j = 1 \\ 2v(\alpha_k + \beta_k)\alpha_{j+k-1}, & j = 2, ..., p \\ 2v(\alpha_k + \beta_k)\beta_{j+k-p}, & j = p+1, ..., 2p-1 \end{cases}$$
- row $i = k+1$, $(k \neq p)$:
$$b_{k+1,j}^{(k)} = \begin{cases} \alpha_k + \frac{\beta_k}{v}, & j = 1 \\ \alpha_{j+k-1}, & j = 2, ..., p \\ \beta_{j+k-p}, & j = p+1, ..., 2p-1 \end{cases}$$
- row $i = k+p$:
$$b_{k+p,j}^{(k)} = \begin{cases} \frac{\alpha_k+\beta_k}{v}, & j = 1 \\ \alpha_{j+k-1}, & j = 2, ..., p \\ \beta_{j+k-p}, & j = p+1, ..., 2p-1 \end{cases}$$
- row $i = k+j$:
$$b_{k+j,j}^{(k)} = \begin{cases} \alpha_k + \beta_k, & j = 2, ..., p-k, p+1, ..., 2p-1-k \\ 0 & j = p-k+1, ..., p \end{cases}$$

and for any other case $b_{ij}^{(k)} = 0$, where we consider $\alpha_i = \beta_i = 0$, for $i > p$. The general form of these matrices are presented in next page.

$$B_1 = \begin{bmatrix} v\alpha_1^2+2\alpha_1\beta_1+\beta_1^2 & 2v\alpha_2(\alpha_1+\beta_1) & 2v\alpha_3(\alpha_1+\beta_1) & \cdots & 2v\alpha_{p-1}(\alpha_1+\beta_1) & 2v\alpha_p(\alpha_1+\beta_1) & 2v\beta_2(\alpha_1+\beta_1) & 2v\beta_3(\alpha_1+\beta_1) & \cdots & 2v\beta_{p-1}(\alpha_1+\beta_1) & 2v\beta_p(\alpha_1+\beta_1) \\ \alpha_1+\frac{\beta_1}{v} & \alpha_2 & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p & \beta_2 & \beta_3 & \cdots & \beta_{p-1} & \beta_p \\ 0 & \alpha_1+\beta_1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_1+\beta_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1+\beta_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{\alpha_1+\beta_1}{v} & \alpha_2 & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p & \beta_2 & \beta_3 & \cdots & \beta_{p-1} & \beta_p \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_1+\beta_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha_1+\beta_1 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} v\alpha_2^2+2\alpha_2\beta_2+\beta_2^2 & 2v\alpha_3(\alpha_2+\beta_2) & 2v\alpha_4(\alpha_2+\beta_2) & \cdots & 2v\alpha_{p-1}(\alpha_2+\beta_2) & 2v\alpha_p(\alpha_2+\beta_2) & 0 & 2v\beta_3(\alpha_2+\beta_2) & \cdots & 2v\beta_{p-1}(\alpha_2+\beta_2) & 2v\beta_p(\alpha_2+\beta_2) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_2+\frac{\beta_2}{v} & \alpha_3 & \alpha_4 & \cdots & \alpha_{p-1} & \alpha_p & 0 & \beta_3 & \cdots & \beta_{p-1} & \beta_p & 0 \\ 0 & \alpha_2+\beta_2 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_2+\beta_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\alpha_2+\beta_2}{v} & \alpha_3 & \alpha_4 & \cdots & \alpha_{p-1} & \alpha_p & 0 & \beta_3 & \cdots & \beta_{p-1} & \beta_p & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha_2+\beta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha_2+\beta_2 & 0 & 0 \end{bmatrix}$$

$\cdots$

$$B_p = \begin{bmatrix} v\alpha_p^2+2\alpha_p\beta_p+\beta_p^2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

Appendix B. Invertibility of B and positivity of $\widehat{C}$

By definition, a matrix $B = (b_{ij}) \in \mathbb{R}^{(2p-2)\times(2p-2)}$ is strictly diagonally dominant by rows when

$$|b_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{2p-2} |b_{ij}|,\ \ i = 1, ..., 2p-2.$$

As the process X, being second order stationary, is also first order stationary, we have from Theorem 3.1

$$\sum_{l=1}^{p} (\alpha_l + \beta_l) < 1 \Leftrightarrow (\alpha_{2i} + \beta_{2i}) + \sum_{|l-i|\neq i} (\alpha_l + \beta_l) < 1$$

$$\Rightarrow \begin{cases} |\alpha_{2i} - 1| > \beta_{2i} + \sum_{|l-i|\neq i} (\alpha_l + \beta_l) - (\alpha_i + \beta_i), & \text{if } i = 1, ..., p-1 \\ |\beta_{2i} - 1| > \alpha_{2i} + \sum_{|l-i|\neq i} (\alpha_l + \beta_l) - (\alpha_i + \beta_i), & \text{if } i = p, ..., 2p-2 \end{cases},$$

that is, B is strictly diagonally dominant by rows. From Levy-Desplanques theorem ([**9**], pp. 352, 392) we know that a strictly diagonally dominant by rows matrix admits inverse. In addition, as B is strictly diagonally dominant by rows, the same happens to $-B$. As $-b_{lu} \leq 0$, for $u \neq l$, and $-b_{ll} > 0$ we conclude that $-B$ is a M-matrix ([**10**], p. 30), that means, $(-B)^{-1} \geq 0$ $\Rightarrow B^{-1} \leq 0 \Rightarrow d_{ij} \leq 0$. This allow us to conclude that the constant $\widehat{C}$ given by

$$\widehat{C} = -2\alpha_0\mu \sum_{v=1}^{p-1} \sum_{j-i=v} (\alpha_i + \beta_i) \left[\alpha_j \sum_{u=1}^{2p-2} d_{vu} + \beta_j \sum_{u=1}^{2p-2} d_{v+p-1,u}\right]$$

$$+2\ \frac{v_0\mu}{1+v_1} \sum_{v=1}^{p-1} \sum_{j-i=v} (\alpha_i + \beta_i) \left[\alpha_j \sum_{u=1}^{p-1} \beta_u d_{vu} + \alpha_j \sum_{u=p}^{2p-2} (\alpha_{u-p+1} + \beta_{u-p+1}) d_{vu}\right.$$

$$\left. + \beta_j \sum_{u=1}^{p-1} \beta_u d_{v+p-1,u} + \beta_j \sum_{u=p}^{2p-2} (\alpha_{u-p+1} + \beta_{u-p+1}) d_{v+p-1,u}\right]$$

is positive, under the assumption $\alpha_0(1 + v_1) > v_0$.

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