Abstract: We investigate Nijenhuis deformations of $L_\infty$-algebras, a notion that unifies several Nijenhuis deformations, namely those of Lie algebras, Lie algebroids, Poisson structures and Courant structures. Additional examples, linked to Lie $n$-algebras and $n$-plectic manifolds, are included.

Introduction

$L_\infty$-algebras, introduced by Lada and Stasheff [15], who called them strongly homotopy Lie algebras, are collections of $n$-ary operations, assumed to satisfy some quadratic relations that reduce to the Jacobi identity, when only the binary operation is not trivial. These structures gained notoriety when Kontsevitch used $L_\infty$-morphisms to prove the existence of star-products on Poisson manifolds [10]. Voronov [23] derived an $L_\infty$-algebra from a Poisson element and an abelian subalgebra of a differential graded Lie algebra. For instance, an $L_\infty$-algebra encodes a Poisson structure in a neighborhood of a coisotropic submanifold, provided that a linear transversal is given, see [6] and [5]. This makes $L_\infty$-algebras a central tool for studying Poisson brackets, but there are more occurrences. Roytenberg and Weinstein [22] gave a description of the so-called Courant algebroids in terms of Lie 2-algebras. In the same vein, Rogers [19] encodes $n$-plectic manifolds by Lie $n$-algebras and Fréguier, Roger and Zambon [7] used this formalism to construct moment maps of those.

In this paper we develop a theory of Nijenhuis forms on $L_\infty$-algebras. Here, by Nijenhuis forms, we mean a generalization of the notion of Nijenhuis $(1, 1)$-tensors on manifolds, i.e., $(1, 1)$-tensors whose Nijenhuis torsion vanishes. On manifolds, Nijenhuis tensors are unary operations on the Lie algebra of vector fields. Since, when dealing with $L_\infty$-algebras, one has to replace Lie algebra brackets by collections of $n$-ary brackets for all integers $n \geq 1$, we also want to define Nijenhuis forms that are collections of $n$-ary operations for all integers $n \geq 1$. Our main idea is based on the fact that, given a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a linear endomorphism $N$ of $\mathfrak{g}$, $N$ is Nijenhuis if deforming twice by $N$
the original bracket yields the original bracket deformed by $N^2$. We translate this idea to $L_\infty$-algebras, where the brackets to be deformed are their $n$-ary brackets.

We present several examples of Nijenhuis forms on $L_\infty$-algebras. The first example is universal, in the sense that every $L_\infty$-structure admits it: the Euler map $S$, that multiplies an element by its degree. Nijenhuis operators on ordinary graded Lie algebras are among the most trivial examples. Poisson elements, and more generally, Maurer-Cartan elements of differential graded Lie algebras are also examples, which are not purely made of vector valued 1-forms, but which are the sum of a vector valued 1-form with a vector valued 0-form. Less trivial examples are given on Lie $n$-algebras. On those, we have Nijenhuis forms which are the sum of a family of vector valued $k$-forms. An interesting case is when the Lie $n$-algebra is associated to an $n$-plectic manifold [19]. The case of Lie 2-algebras is treated separately, and we have Nijenhuis forms which are the sum of a vector valued 1-form with a vector valued 2-form.

We discuss how Nijenhuis tensors on Courant algebroids [4, 12, 2, 3] fit in our definition of Nijenhuis forms on some $L_\infty$-algebras. In order to include Lie algebroids in our examples, we recall the concept of multiplicative $L_\infty$-algebras (related to $P_\infty$-algebras in [5]). In the last part of the paper, our examples come from well-known structures on Lie algebroids, defined by pairs of compatible tensors [14, 1, 3], such as $\Omega N$, Poisson-Nijenhuis [13] and $P\Omega$-structures.

Very recently, while we were about to finish this paper, a notion of Nijenhuis operator on Lie 2-algebras was introduced in [18], using a different perspective. That definition is a particular case of ours, as we explain in Remark 4.14.

The paper is organized in seven sections. In Section 1 we introduce a bracket of graded symmetric vector valued forms on a graded vector space that we call Richardson-Nijenhuis bracket, because it reduces to the usual Richardson-Nijenhuis bracket of vector valued forms on a (non-graded) vector space. With this graded bracket, we characterize $L_\infty$-structures as Poisson elements on the graded Lie algebra of graded symmetric vector valued forms. In Section 2 we present our main definition of Nijenhuis vector valued form with respect to an $L_\infty$-algebra, or more generally, with respect to a vector valued form of degree 1. Relaxing a bit the definition of Nijenhuis vector
valued form, yields the notions of weak Nijenhuis and co-boundary Nijen-
huis forms, which provide interesting examples to be discussed in the next
sections. Section 2 also contains the first examples of Nijenhuis forms on sym-
metric graded Lie algebras and symmetric differential graded Lie algebras:
the Euler map, Poisson and Maurer-Cartan elements. Section 3 is devoted
to Nijenhuis forms on Lie $n$-algebras. We construct examples of Nijenhuis
forms on general Lie $n$-algebras, in particular on those defined by $n$-plectic
manifolds. The case $n = 2$ is treated separately, in Section 4. There, we find
necessary and sufficient conditions to have a Nijenhuis form which is the sum
of vector valued 1-form with a vector valued 2-form. The importace of Lie 2-
algebras appears in Section 5, where we focus on Courant algebroids. Using a
construction established in [22], we associate a Lie 2-algebra to each Courant
algebroid and we relate $(1,1)$-tensors with vanishing Nijenhuis torsion on a
Courant algebroid, with Nijenhuis forms on the corresponding associated Lie
2-algebra. In Section 6, we study multiplicative $L_\infty$-algebras and its relation
with pre-Lie and Lie algebroids. We introduce the notions of extension by
derivation of $(1,1)$-tensors and of $k$-forms on a Lie algebroid, needed to con-
struct examples of Nijenhuis forms on Lie algebroids in the last section. In
Section 7, the last one, we obtain, out of $\Omega N$, Poisson-Nijenhuis and $P\Omega$
structures on a Lie algebroid, examples of weak Nijenhuis and co-boundary
Nijenhuis vector valued forms.

1. Richardson-Nijenhuis bracket and $L_\infty$-algebras

In this section we extend the usual Richardson-Nijenhuis bracket of vector
valued forms on vector spaces [9] to graded symmetric vector valued forms
on graded vector spaces. Then, we use it to characterize $L_\infty$-structures on
graded vector spaces. We start by fixing some notations on graded vector
spaces.

Let $E$ be a graded vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, that is, a vector
space of the form $\oplus_{i \in \mathbb{Z}} E_i$. For a given $i \in \mathbb{Z}$, the vector space $E_i$ is called
the component of degree $i$, elements of $E_i$ are called homogeneous elements
of degree $i$, and elements in the union $\cup_{i \in \mathbb{Z}} E_i$ are called the homogeneous
elements. We denote by $|X|$ the degree of a non-zero homogeneous element
$X$. Given a graded vector space $E = \oplus_{i \in \mathbb{Z}} E_i$ and an integer $p$, one may shift
to all the degrees by $p$ to get a new grading on the vector space $E$. We use the
notation $E[p]$ for the graded vector space $E$ after shifting the degrees by $p$,
that is, the graded vector space whose component of degree $i$ is $E_{i+p}$. 
We denote by \( S(E) \) the symmetric space of \( E \) which is, by definition, the quotient space of the tensor algebra \( \otimes E \) by the two-sided ideal \( I \subset \otimes E \) generated by elements of the type \( X \otimes Y - (-1)^{|X||Y|} Y \otimes X \), with \( X \) and \( Y \) arbitrary homogeneous elements in \( E \). For a given \( k \geq 0 \), \( S^k(E) \) is the image of \( \otimes^k E \) through the quotient map \( \otimes E \mapsto \otimes E / I = S(E) \) and one has the following decomposition

\[
S(E) = \bigoplus_{k \geq 0} S^k(E),
\]

where \( S^0(E) \) is simply the field \( \mathbb{K} \). Moreover, when all the components in the graded space \( E \) are of finite dimension, the dual of \( S^k(E) \) is isomorphic to \( S^k(E^*) \), for all \( k \geq 0 \). In this case, there is a one to one correspondence between

(i) graded symmetric \( k \)-linear maps on the graded vector space \( E \),
(ii) linear maps from the space \( S^k(E) \) to \( E \),
(iii) \( S^k(E^*) \otimes E \).

Elements of the space \( S^k(E^*) \otimes E \) are called \textit{symmetric vector valued \( k \)-forms}. Notice that \( S^0(E^*) \otimes E \), the space of vector valued zero-forms, is isomorphic to the space \( E \).

Having the decomposition \( S(E) = \bigoplus_{k \geq 0} S^k(E) \), every element in \( S(E) \) is the sum of finitely many elements in \( S^k(E) \), \( k \geq 0 \). We absolutely need to consider also infinite sums, which is often referred in the literature as taking the completion of \( S(E) \). By a \textit{formal sum}, we mean a sequence \( \phi : \mathbb{N} \cup \{0\} \to S(E) \) mapping an integer \( k \) to an element \( a_k \in S^k(E) \); we shall, by a slight abuse of notation, denote by \( \sum_{k=0}^\infty a_k \) such an element. We denote the set of all formal sums by \( \tilde{S}(E) \). The algebra structure on \( S(E) \) extends in an unique manner to \( \tilde{S}(E) \). For two formal sums \( a = \sum_{k=0}^\infty a_k \) and \( b = \sum_{k=0}^\infty b_k \) we define \( a + b \) to be \( \sum_{k=0}^\infty (a_k + b_k) \), while the product of \( a \) and \( b \) is the infinite sum \( \sum_{k=0}^\infty c_k \) with \( c_k = \sum_{i=0}^k a_i \cdot b_{k-i} \) (with \( \cdot \) being the product of \( S(E) \)).

When all the components in the graded space \( E \) are of finite dimension, there is a one to one correspondence between

(i) collections indexed by \( k \geq 0 \) of graded symmetric \( k \)-linear maps on the graded vector space \( E \),
(ii) collections indexed by \( k \geq 0 \) of linear maps from \( S^k(E) \) to \( E \),
(iii) \( \tilde{S}(E^*) \otimes E \).

Elements of the space \( \tilde{S}(E^*) \otimes E \) are called \textit{symmetric vector valued forms} and shall be written as infinite sums \( \sum K_i \) with \( K_i \in S^i(E^*) \otimes E \).
Let $E$ be a graded vector space, $E = \oplus_{i \in \mathbb{Z}} E_i$. The insertion operator of a symmetric vector valued $k$-form $K$ is an operator

$$\iota_K : S(E^*) \otimes E \rightarrow S(E^*) \otimes E$$

defined by

$$\iota_K L(X_1, \ldots, X_{k+l-1}) = \sum_{\sigma \in Sh(k,l-1)} \epsilon(\sigma) L(K(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), \ldots, X_{\sigma(k+l-1)}),$$

for all $L \in S^l(E^*) \otimes E$, $l \geq 0$ and $X_1, \ldots, X_{k+l-1} \in E$, where $Sh(i,j-1)$ stands for the set of $(i,j-1)$-unshuffles and $\epsilon(\sigma)$ is the Koszul sign which is defined as follows

$$x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} = \epsilon(\sigma) x_1 \otimes \cdots \otimes x_n,$$

for all $x_1, \cdots, x_n \in E$. If $L$ is an element in $S^0(E^*) \otimes E \simeq E$, then (1) should be understood as meaning that $\iota_K L = 0$, for all vector valued forms $K$ and

$$\iota_L K(X_1, \ldots, X_{k-1}) = K(L, X_1, \ldots, X_{k-1}),$$

for all vector valued $k$-form $K$.

Allowing $L$ and $K$ to be symmetric vector valued forms, that is, $L = \sum_{i \geq 0} L_i$ and $K = \sum_{i \geq 0} K_i$, with $L_i$ and $K_i$ vector valued $i$-forms, the previous definition of insertion operator extends in the obvious way. If $K$ is an element in $\tilde{S}^i(E^*)$, i.e. a linear form on $\tilde{S}^i(E)$, $i \geq 0$, one may define $\iota_K$ by a formula similar to (1). Moreover, $\iota_K : \tilde{S}(E^*) \rightarrow \tilde{S}(E^*)$, with $K \in \tilde{S}(E^*) \otimes E$, is the zero map if and only if $K = 0$.

Now, we define a bracket on the space $\tilde{S}(E^*) \otimes E$ as follows. Given a symmetric vector valued $k$-form $K \in \tilde{S}^k(E^*) \otimes E$ and a symmetric vector valued $l$-form $L \in \tilde{S}^l(E^*) \otimes E$, the Richardson-Nijenhuis bracket of $K$ and $L$ is the symmetric vector valued $(k + l - 1)$-form $[K, L]_{RN}$, given by

$$[K, L]_{RN} = \iota_K L - (-1)^{\bar{K} \bar{L}} \iota_L K,$$

where $\bar{K}$ is the degree of $K$ as a graded map, that is $K(X_1, \cdots, X_k) \in E_{1+\cdots+k+\bar{K}}$, for all $X_i \in E_i$. For an element $X \in E$, $\bar{X} = |X|$, that is, the degree of a vector valued 0-form, as a graded map, is just its degree as an element of $E$.

**Proposition 1.1.** The space $\tilde{S}(E^*) \otimes E$, equipped with the Richardson-Nijenhuis bracket, is a graded (skew-symmetric) Lie algebra.
If $K \in S^k(E^*) \otimes E$ is a vector valued $k$-form, an easy computation gives
\begin{equation}
K(X_1, \cdots, X_k) = [X_k, \cdots, [X_2, [X_1, K]_{RN}]_{RN} \cdots]_{RN},
\end{equation}
for all $X_1, \cdots, X_k \in E$.

In [16], the authors defined a multi-graded Richardson-Nijenhuis bracket, in a graded vector space, but their approach is different from ours.

Next, we recall the notion of $L_\infty$-algebra, following [8].

**Definition 1.2.** An $L_\infty$-algebra is a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E_i$ together with a family of symmetric vector valued forms $(l_i)_{i \geq 1}$ of degree 1, with $l_i : \otimes^i E \to E$ satisfying the following relation:
\begin{equation}
\sum_{i+j=n+1} \sum_{\sigma \in Sh(i,j-1)} \epsilon(\sigma) l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \cdots, X_{\sigma(n)}) = 0,
\end{equation}
for all $n \geq 1$ and all homogeneous $X_1, \cdots, X_n \in E$, where $\epsilon(\sigma)$ is the Koszul sign. The family of symmetric vector valued forms $(l_i)_{i \geq 1}$ is called an $L_\infty$-structure on the graded vector space $E$. Usually, we denote this $L_\infty$-structure by $\mu := \sum_{i \geq 1} l_i$ and we say, by an abuse of language, that $\mu$ has degree 1.

A slight generalization of an $L_\infty$-algebra is the so-called curved $L_\infty$-algebra. In this case, the family of symmetric vector valued forms is $(l_i)_{i \geq 0}$ that is, there is an extra symmetric vector valued 0-form $l_0 \in E_1$, called the curvature, such that $l_1(l_0) = 0$ and Equation (4) is replaced by
\begin{align*}
l_{n+1}(l_0, X_1, \cdots, X_n) &+ \sum_{i+j=n+1} \sum_{\sigma \in Sh(i,j-1)} \epsilon(\sigma) l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \cdots, X_{\sigma(n)}) = 0.
\end{align*}

There is an equivalent definition of $L_\infty$-algebra in terms of graded skew-symmetric vector valued forms $l'_i$ of degree $i - 2$. This was, in fact, the original definition introduced in [15]. The equivalence of both definitions is established by the so-called décalage isomorphism
\begin{equation}
l_i(X_1, \cdots, X_i) \mapsto (-1)^{(i-1)|X_1|+(i-2)|X_2|+\cdots+|X_{i-1}|} l'_i(X_1, \cdots, X_i),
\end{equation}
$X_1, \cdots, X_i \in E$. The family of graded skew-symmetric brackets $(l'_i)_{i \geq 1}$ defines an $L_\infty$-structure on the graded vector space $E$ if each $l'_i$ has degree $i - 2$
and
\[ \sum_{i+j=n+1} \sum_{\sigma \in \text{Sh}(i,j-1)} (-1)^{i(j-1)} \text{sign}(\sigma) l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \cdots, X_{\sigma(n)}) = 0, \]
for all \( n \geq 1 \) and all \( X_1, \cdots, X_n \in E \), with \( \text{sign}(\sigma) \) being the sign of the permutation \( \sigma \).

Next, we see that some well-known structures on (graded) vector spaces are examples of \( L_\infty \)-algebras.

We start with a symmetric graded Lie algebra, which is a graded vector space \( E = \bigoplus_{i \in \mathbb{Z}} E_i \) endowed with a binary graded symmetric bracket \( [.,.] = \mu \) of degree 1, satisfying the graded Jacobi identity i.e.
\[ [X, [Y, Z]] = (-1)^{|X|+1}[[[X, Y], Z] + (-1)^{|X|(|Y|+1)}[Y, [X, Z]]], \quad (5) \]
for all homogeneous elements \( X, Y, Z \in E \). Note that when the graded vector space is concentrated on degree \(-1\), that is, all the vector spaces \( E_i \) are zero, except \( E_{-1} \), then (5) is the usual Jacobi identity and we get a Lie algebra with symmetric bracket. We would like to remark that (5) can be written as
\[ \mu(\mu(X, Y), Z) + (-1)^{|Y||Z|}\mu(\mu(X, Z), Y) + (-1)^{|X|(|Y|+|Z|)}\mu(\mu(Y, Z), X) = 0, \quad (6) \]
for all homogeneous elements \( X, Y, Z \in E \). This means that a symmetric graded Lie algebra is simply an \( L_\infty \)-algebra such that all the multi-brackets are zero except the binary one. From this, we also conclude that a Lie algebra is an \( L_\infty \)-algebra on a graded vector space concentrated on degree \(-1\), for which all the brackets are zero except the binary bracket.

Another special case of an \( L_\infty \)-algebra is a symmetric differential graded Lie algebra. It is an \( L_\infty \)-structure on \( E = \bigoplus_{i \in \mathbb{Z}} E_i \), with all the brackets, except \( l_1 \) and \( l_2 \), being zero. In other words, a symmetric differential graded Lie algebra is a symmetric graded Lie algebra \( (\bigoplus_{i \in \mathbb{Z}} E_i, [.,.] = l_2) \) endowed with a differential \( d = l_1 \), that is, a linear map \( d : \bigoplus_{i \in \mathbb{Z}} E_i \to \bigoplus_{i \in \mathbb{Z}} E_i \) of degree 1 and squaring to zero, satisfying the compatibility condition
\[ d[X, Y] + [d(X), Y] + (-1)^{|X|}[X, d(Y)] = 0, \]
for all homogeneous elements \( X, Y \in E \). We shall denote a symmetric differential graded Lie algebra by \( (E, d, [.,.]) \) or by \( (E, l_1 + l_2) \).

We may also consider two particular cases of a curved \( L_\infty \)-algebra, that is to say, a curved symmetric graded Lie algebra and a curved symmetric
differential graded Lie algebra. More precisely, a \textit{curved} symmetric differential graded Lie algebra on a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E_i$ is a symmetric differential graded Lie algebra $(E, d, [., .])$ together with an element $\mathcal{C} \in E_1$ such that:

$$d(\mathcal{C}) = 0 \quad \text{and} \quad [\mathcal{C}, X] + d^2 X = 0, \quad \text{for all} \ X \in E.$$  

We shall denote the curved symmetric differential graded Lie algebra by $(E, \mathcal{C}, d, [., .])$ or by $(E, \mathcal{C} + l_1 + l_2)$. When $d = 0$, the curved symmetric differential graded Lie algebra is simply a curved symmetric graded Lie algebra.

The Richardson-Nijenhuis bracket on graded vector spaces, introduced previously, is intimately related to $L_\infty$-algebras. In the next theorem, that appears in an implicit form in [21], we use the Richardson-Nijenhuis bracket to characterize (curved) $L_\infty$-structures on a graded vector space.

**Theorem 1.3.** Let $E = \bigoplus_{i \in \mathbb{Z}} E_i$ be a graded vector space, $(l_i)_{i \geq 1} : \otimes^i E \to E$ be a family of symmetric vector valued forms on $E$ of degree 1 and $l_0 \in E_1$ be a symmetric vector valued 0-form. Set $\mu = \sum_{i \geq 1} l_i$ and $\mu' = \sum_{i \geq 0} l_i$. Then,

i) $\mu$ is an $L_\infty$-structure on $E$ if and only if $[\mu, \mu]_{RN} = 0$;

ii) $\mu'$ is a curved $L_\infty$-structure on $E$ if and only if $[\mu', \mu']_{RN} = 0$.

**Proof:** (i) It is a direct consequence of the following equalities that can be obtained from (1) and (2):

$$[\mu, \mu]_{RN} = \sum_{n \geq 1} \left( \sum_{i+j=n+1} [l_i, l_j]_{RN} \right) = 2 \sum_{n \geq 1} \left( \sum_{i+j=n+1} \iota l_i \iota l_j \right).$$

The proof of (ii) is easy.

Notice that for the case of symmetric graded Lie algebras, the statement of Theorem 1.3 appears in a natural way, since equation (6) is equivalent to

$$\frac{1}{2}(\iota_{\mu} \iota \mu + \iota \mu \iota \mu)(X, Y, Z) = \frac{1}{2}[\mu, \mu]_{RN}(X, Y, Z) = 0.$$

2. Nijenhuis forms on $L_\infty$ algebras: definition and first examples

In this section we define a Nijenhuis vector valued form with respect to a given vector valued form $\mu$ and deformation of $\mu$ by a Nijenhuis vector valued form. We show that deforming an $L_\infty$-structure by a Nijenhuis vector valued form, one gets an $L_\infty$-structure. Then, we present the first examples of Nijenhuis vector valued forms on some $L_\infty$-algebras.
Definition 2.1. Let $E$ be a graded vector space and $\mu$ be a symmetric vector valued form on $E$ of degree 1. A vector valued form $\mathcal{N}$ of degree zero is called

- **weak Nijenhuis** with respect to $\mu$ if
  \[
  [\mu, [\mathcal{N}, [\mathcal{N}, \mu]]_{RN}]_{RN} = 0,
  \]

- **co-boundary Nijenhuis** with respect to $\mu$ if there exists a vector valued form $\mathcal{K}$ of degree zero, such that
  \[
  [\mathcal{N}, [\mathcal{N}, \mu]]_{RN} = [\mathcal{K}, \mu]_{RN},
  \]

- **Nijenhuis** with respect to $\mu$ if there exists a vector valued form $\mathcal{K}$ of degree zero, such that
  \[
  [\mathcal{N}, [\mathcal{N}, \mu]]_{RN} = [\mathcal{K}, \mu]_{RN} \quad \text{and} \quad [\mathcal{N}, \mathcal{K}]_{RN} = 0.
  \]

Such a $\mathcal{K}$ is called a *square* of $\mathcal{N}$. If $\mathcal{N}$ contains an element of the underlying graded vector space, that is, $\mathcal{N}$ has a component which is a vector valued zero form, then $\mathcal{N}$ is called Nijenhuis (respectively, co-boundary Nijenhuis) vector valued form with *curvature*.

It is obvious that the following implications hold:

$\mathcal{N}$ Nijenhuis $\Rightarrow$ $\mathcal{N}$ co-boundary Nijenhuis $\Rightarrow$ $\mathcal{N}$ weak Nijenhuis

Remark 2.2. It would be of course tempting to choose $\mathcal{K} = \iota_\mathcal{N}\mathcal{N}$ in Definition 2.1, having in mind what happens for manifolds, and the fact that $\iota_\mathcal{N}\mathcal{N} = \mathcal{N}^2$ for vector valued 1-forms. However, it is not what examples show to be a reasonable definition. Also, for $\mathcal{N}$ a vector valued 2-form we do not have, in general, $[\iota_\mathcal{N}\mathcal{N}, \mathcal{N}]_{RN} = 0$, which says $\iota_\mathcal{N}\mathcal{N}$ is not a good candidate for the square, except maybe for vector valued 1-forms.

Proposition 2.3. Let $(E, \mu)$ be a (curved) $L_\infty$-algebra and $\mathcal{N}$ be a symmetric vector valued form on $E$. Then $\mathcal{N}$ is weak Nijenhuis with respect to $\mu$ if and only if $[\mathcal{N}, \mu]_{RN}$ is a (curved) $L_\infty$-algebra.

Proof: First we observe that $[\mathcal{N}, \mu]_{RN}$ has degree 1 if and only if the degree of $\mathcal{N}$ is zero. Using the Jacobi identity, we get

\[
[\mu, [\mathcal{N}, [\mathcal{N}, \mu]]_{RN}]_{RN} = [[[\mu, \mathcal{N}]]_{RN}, [\mu, \mathcal{N}]]_{RN} + [\mathcal{N}, [[\mu, [\mathcal{N}, \mu]]_{RN}]]_{RN},
\]

which concludes the proof.
Given an $L_\infty$-structure $\mu$ and a symmetric vector valued form of degree zero $N$ on a graded vector space, we call $[N, \mu]_{RN}$ the deformation of $\mu$ by $N$ and denote the deformed structure by $\mu^N$. When $\mu$ is deformed $k$ times by $N$, the deformed structure is denoted by $\mu^{N,k}N$ or simply $\mu_k$ if there is no danger of confusion.

Weak Nijenhuis forms do not, in general, give hierarchies in any sense. However, Nijenhuis forms do.

**Theorem 2.4.** Let $\mathcal{N}$ be a Nijenhuis vector valued form with respect to a (curved) $L_\infty$-structure $\mu$ with square $\mathcal{K}$, on a graded vector space $E$. Then, for all integers $k \geq 1$, $\mu_k$ is a (curved) $L_\infty$-structure on $E$ and $N$ is Nijenhuis with square $\mathcal{K}$, with respect to $\mu_k$.

**Proof:** The case $k = 1$ follows from Proposition 2.3 together with the observation that if $\mathcal{N}$ is Nijenhuis, then it is also weak Nijenhuis with respect to $\mu$. Assume, by induction, that $\mathcal{N}$ is Nijenhuis with respect to $\mu_k$ with square $\mathcal{K}$. Then we have

$$[[\mathcal{N}, \mathcal{N}], \mu_k]_{RN} = [\mathcal{K}, \mu_k]_{RN},$$

that implies

$$[[\mathcal{N}, \mathcal{N}], [\mathcal{N}, \mu_k]_{RN}]_{RN} = [\mathcal{N}, [\mathcal{K}, \mu_k]_{RN}]_{RN}. \tag{7}$$

Applying the Jacobi identity on the right hand side of (7) and using the assumption that $\mathcal{N}$ and $\mathcal{K}$ commute with respect to the Richardson-Nijenhuis bracket, we get

$$[[\mathcal{N}, \mathcal{N}], \mu_{k+1}]_{RN} = [\mathcal{K}, \mu_{k+1}]_{RN}.$$

Thus, $\mathcal{N}$ is Nijenhuis with respect to $\mu_{k+1}$, with square $\mathcal{K}$. \hfill \blacksquare

Recall from [13] that a Nijenhuis operator on a graded Lie algebra $(E, \mu = [\cdot, \cdot])$ is a linear map $N : E \rightarrow E$ such that its Nijenhuis torsion with respect to $\mu$, defined by

$$T_\mu N(X, Y) := \mu(NX, NY) - N(\mu(NX, Y) + \mu(X, NY) - N(\mu(X, Y))), \tag{8}$$

for all $X, Y \in E$, is identically zero. For a binary bracket $\mu = [\cdot, \cdot]$, the deformed bracket by $N$ is denoted by $[\cdot, \cdot]_N$ and is given by $[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$. It has been shown in [13] that if $N$ is Nijenhuis on a Lie algebra $(E, [\cdot, \cdot])$, then $(E, [\cdot, \cdot]_N)$ is also a Lie algebra and $N$ is a morphism of Lie algebras. Also, it has been shown that $N$ is Nijenhuis if and only if deforming the original bracket of the Lie algebra twice by $N$ is
equivalent to deform it once by $N^2$, that is $([X, Y]_N) = [X, Y]_{N^2}$. This can be stated using the notion of Richardson-Nijenhuis bracket on the space of vector valued forms on a graded vector space $E$, as follows:

$$[N, [N, \mu]_{RN}] = [N^2, \mu]_{RN}.$$ 

So, we conclude that Nijenhuis operators in the usual and traditional sense are, of course, Nijenhuis in our sense also.

Next, we present the first examples of Nijenhuis vector valued forms on $L_\infty$-algebras. We start by introducing the Euler map $S$, the map that simply counts the degree of homogeneous elements in a graded vector space. More precisely, given a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E_i$, $S : E \to E$ is defined by $S(X) = -|X|X$, for all homogeneous elements $X \in E$ of degree $|X|$.

Notice that $S$, as a graded map, has degree zero, $\bar{S} = 0$. By a simple computation, using the definition of $S$, we get the following result.

**Lemma 2.5.** Let $E = \bigoplus_{i \in \mathbb{Z}} E_i$ be a graded vector space. Then,

$$[S, \alpha]_{RN} = \bar{\alpha} \alpha,$$

for every symmetric vector valued form $\alpha$ on $E$ of degree $\bar{\alpha}$.

**Proposition 2.6.** Let $\mu$ be a vector valued form of degree 1 on a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E_i$. The Euler map $S$ is a Nijenhuis vector valued form with respect to $\mu$ with square $\bar{S} = 0$.

**Proof:** Let $\mu = \sum_{i=1}^{\infty} l_i$. Applying Lemma 2.5 to each $l_i, 1 \leq i \leq \infty$, and taking the sum we get:

$$[S, \mu]_{RN} = \sum_{i=1}^{\infty} [S, l_i]_{RN} = \sum_{i=1}^{\infty} l_i = \mu.$$ 

Therefore

$$[S, [S, \mu]_{RN}] = [S, \mu]_{RN}.$$ 

Since $\bar{S} = 0$, Lemma 2.5 implies that $[S, S]_{RN} = 0$ and this completes the proof.

Of course, the result can be enlarged for every $\mu$-cocycle, that is, a vector valued form $\eta$ such that $[\mu, \eta]_{RN} = 0$.

**Proposition 2.7.** Let $\mu = \sum_{i \geq 1} l_i$ be a vector valued form of degree 1 on a graded vector space $E$. Then, for every element $\alpha$ of degree 0 in $\tilde{S}(E^*) \otimes E$ with $[\mu, \alpha]_{RN} = 0$, $S + \alpha$ is Nijenhuis with respect to $\mu$, with square $S$. 


Next, we give some examples of Nijenhuis forms on symmetric graded and symmetric differential graded Lie algebras. For that, we need to introduce the notions of Maurer Cartan and Poisson elements.

A **Maurer Cartan element** in a symmetric differential graded Lie algebra \((E, d, [, .])\) is an element \(e \in E_0\) such that
\[
d(e) - \frac{1}{2}[e, e] = 0.
\]

A **Maurer Cartan element** in a symmetric curved differential graded Lie algebra \((E, C, d, [, .])\) is an element \(e \in E_0\) such that
\[
(d(e) - C) - \frac{1}{2}[e, e] = 0.
\]

A **Poisson element** in a curved \(L_\infty\)-algebra \((E, \mu = \sum_{i \geq 0} l_i)\) is an element \(\pi \in E_0\), such that
\[
l_2(\pi, \pi) = 0.
\]

The next propositions provide examples of Nijenhuis vector valued forms on symmetric graded Lie algebras and symmetric differential graded Lie algebras.

**Proposition 2.8.** Let \(\mu = C + l_2\) be a curved symmetric graded Lie algebra structure on a graded vector space \(E = \bigoplus_{i \in \mathbb{Z}} E_i\) and \(\pi \in E_0\). Then, \(N = \pi + S\) is a Nijenhuis vector valued form (with curvature \(\pi\)) with respect to \(\mu\) and with square \(2\pi + S\) if, and only if, \(\pi\) is a Poisson element.

In this case, the deformed structure is the curved symmetric differential graded Lie algebra \((E, C + l_2(\pi, .) + l_2)\).

**Proof:** The proof of the equivalence is a direct consequence of the following equalities:
\[
[\pi + S, \mathcal{C} + l_2]_{RN} = l_2(\pi, .) + \mathcal{C} + l_2,
\]  
\[(\pi + S, [\pi + S, \mathcal{C} + l_2]_{RN})_{RN} = l_2(\pi, \pi) + \mathcal{C} + 2l_2(\pi, .) + l_2 = l_2(\pi, \pi) + [2\pi + S, \mathcal{C} + l_2]_{RN}
\]

and
\[
[\pi + S, 2\pi + S]_{RN} = 2[\pi, \pi]_{RN} + [\pi, S]_{RN} + 2[S, \pi]_{RN} + [S, S]_{RN} = 0,
\]
where we used \([\pi, S]_{RN} = [S, \pi]_{RN} = 0\). The last statement follows directly from (9) and Theorem 2.4.
Proposition 2.9. Let $\mu = \mathcal{C} + l_1 + l_2$ be a curved symmetric differential graded Lie algebra structure on a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E_i$ and $\pi \in E_0$. Then, $\mathcal{N} = \pi + S$ is a Nijenhuis vector valued form (with curvature $\pi$) with respect to $\mu$ and with square $2\pi + S$ if, and only if, $\pi$ is a Poisson element.

In this case, the deformed structure is the curved symmetric differential graded Lie algebra $(E, (\mathcal{C} + l_1(\pi)) + (l_1 + l_2(\pi, .)) + l_2)$.

Proof: The proof of the equivalence follows from:

$$[\pi + S, \mathcal{C} + l_1 + l_2]_{RN} = \mathcal{C} + l_1(\pi) + (l_2(\pi, .) + l_1) + l_2, \quad (10)$$

$$[\pi + S, [\pi + S, \mathcal{C} + l_1 + l_2]_{RN}]_{RN} = [\pi + S, \mathcal{C} + l_1 + l_2 + l_1(\pi) + l_2(\pi, .)]_{RN} = \mathcal{C} + l_1 + l_2 + 2l_1(\pi) + 2l_2(\pi, .) + l_2(\pi, \pi)$$

and

$$[\pi + S, 2\pi + S]_{RN} = 2[\pi, \pi]_{RN} + [\pi, S]_{RN} + 2[S, \pi]_{RN} + [S, S]_{RN} = 0.$$

The last statement follows directly from (10) and Theorem 2.4.

Notice that, in Proposition 2.9, if we start with a symmetric differential graded Lie algebra without curvature, that is, if $\mathcal{C} = 0$, then, the deformed structure is a curved symmetric differential graded Lie algebra with curvature $l_1(\pi)$.

Proposition 2.10. Let $\mu = \mathcal{C} + l_1 + l_2$ be a curved symmetric differential graded Lie algebra structure on a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E_i$ and $\pi \in E_0$. Then, $\mathcal{N} = \text{Id}_E + \pi$ is a Nijenhuis vector valued form (with curvature $\pi$) with respect to $\mu$ and with square $\text{Id}_E + \pi$ if, and only if, $\pi$ is a Maurer-Cartan element.

In this case, the deformed structure is the curved symmetric differential graded Lie algebra $(E, (l_1(\pi) - \mathcal{C}) + l_2(\pi, .) + l_2)$.

Proof: First notice that

$$[\pi + \text{Id}_E, \mathcal{C} + l_1 + l_2]_{RN} = (l_1(\pi) - \mathcal{C}) + l_2(\pi, .) + l_2 \quad (11)$$
and
\[
[\pi + Id_E, [\pi + Id_E, \mathcal{C} + l_1 + l_2]_{RN}]_{RN} = l_2(\pi, \pi) + l_2(\pi, \cdot) - l_1(\pi) + \mathcal{C} + l_2
\]
\[
= -\mathcal{C} - 2((l_1(\pi) - \mathcal{C}) - \frac{1}{2}l_2(\pi, \pi)) + l_1(\pi) + l_2(\pi, \cdot) + l_2
\]
\[
= -2((l_1(\pi) - \mathcal{C}) - \frac{1}{2}l_2(\pi, \pi)) + [\pi + Id_E, \mathcal{C} + l_1 + l_2]_{RN}.
\]
This, together with the fact that \([\pi + Id_E, \pi + Id_E]_{RN} = 0\), imply that \(Id_E + \pi\) is a Nijenhuis vector valued form with respect to \(\mu\) if, and only if, \(\pi\) is a Maurer-Cartan element of the curved symmetric differential graded Lie algebra \((E, \mu)\). The last statement follows from (11) and Theorem 2.4.

3. Nijenhuis forms on Lie \(n\)-algebras

Lie \(n\)-algebras are particular cases of \(L_\infty\)-algebras for which only \(n + 1\) brackets may be non-zero. We define Nijenhuis forms for this special case and we analyze, in particular, the Lie \(n\)-algebra defined by an \(n\)-plectic manifold.

A graded vector space \(E = \bigoplus_{i \in \mathbb{Z}} E_i\) is said to be concentrated in degrees \(p_1, \ldots, p_k\), with \(p_1, \ldots, p_k \in \mathbb{Z}\), if \(E_{p_1}, \ldots, E_{p_k}\) are the only non-zero components of \(E\).

**Definition 3.1.** A symmetric Lie \(n\)-algebra is a symmetric \(L_\infty\)-algebra whose underlying graded vector space is concentrated on degrees \(-n, \ldots, -1\).

**Remark 3.2.** Note that by degree reasons, the only non-zero symmetric vector valued forms (multi-brackets) are \(l_1, \ldots, l_{n+1}\).

**Proposition 3.3.** Let \((E = E_{-n} \oplus \cdots \oplus E_{-1}, \mu = l_1 + \cdots + l_{n+1})\) be a Lie \(n\)-algebra. Let \(k\) be an integer such that \(\frac{n+3}{2} \leq k \leq n + 1\) and \(N\) be any symmetric vector valued \(k\)-form of degree zero on \(E\). Then, \(\mathcal{N} = S + N\) is a Nijenhuis vector valued form with respect to \(\mu\), with square \(S + 2N\), and the deformed Lie \(n\)-algebra structure on \(E\) is of the form
\[
\mu^N = l_1 + \cdots + l_{k-1} + (l_k + [N, l_1]_{RN}) + \cdots + (l_{n+1} + [N, l_{n-k+2}]_{RN}).
\]

**Proof:** By Remark 3.2, any vector valued \((m+k-1)\)-form, with \(m \geq n-k+3\), is identically zero; hence
\[
[N, l_m]_{RN} = 0,
\]
for all \(m \geq n-k+3\). Also, any vector valued \((2k+m-2)\)-form, with \(m \geq 1\), is identically zero because, from the conditions \(\frac{n+3}{2} \leq k \leq n + 1\) and \(m \geq 1\),
we get $2k + m - 2 \geq n + 2$. Thus,

$$[N, [N, l_m]_{RN}]_{RN} = 0,$$

(13)

for all $m \geq 1$. From Equations (12) and (13), we get

$$[S + N, \mu]_{RN} = \mu + [N, l_1]_{RN} + \cdots + [N, l_{n-k+2}]_{RN}$$

(14)

and

$$[S + N, [S + N, \mu]_{RN}]_{RN} = \mu + 2[N, l_1]_{RN} + \cdots + 2[N, l_{n-k+2}]_{RN}.$$  

(15)

On the other hand, using Lemma 2.5, we have

$$[S + N, S + 2N]_{RN} = 0.$$  

(16)

Equations (15) and (16) show that $\mathcal{N} = S + N$ is a Nijenhuis vector valued form with respect to $\mu$, with square $S + 2N$, and Equation (14) shows that the deformed Lie $n$-algebra structure is

$$\mu^\mathcal{N} = l_1 + \cdots + l_{k-1} + (l_k + [N, l_1]_{RN}) + \cdots + (l_{n+1} + [N, l_{n-k+2}]_{RN}).$$

\[\square\]

Proposition 3.3 admits the following generalization.

**Proposition 3.4.** Let $(E = E_{-n} \oplus \cdots \oplus E_{-1}, \mu = l_1 + \cdots + l_{n+1})$ be a Lie $n$-algebra. Let $N_1, \ldots, N_l$ be a family of symmetric vector valued $k_1, \ldots, k_l$-forms, respectively, of degree zero on $E$, with $\frac{n+3}{2} \leq k_1 \leq \cdots \leq k_l \leq n + 1$. Then, $\mathcal{N} = S + \sum_{i=1}^l N_i$ is a Nijenhuis vector valued form with respect to $\mu$, with square $S + 2\sum_{i=1}^l N_i$. The deformed Lie $n$-algebra structure is

$$[S + \sum_{i=1}^l N_i, \mu]_{RN} = \mu + \left[\sum_{i=1}^l N_i, l_1\right]_{RN} + \cdots + \left[\sum_{i=1}^l N_i, l_{n-k+l+2}\right]_{RN}$$

$$+ \left[\sum_{i\neq l} N_i, l_{n-n+l+3}\right]_{RN} + \cdots + \left[\sum_{i\neq l} N_i, l_{n-k+3}\right]_{RN}$$

$$+ \left[\sum_{i\neq l, l-1} N_i, l_{n-k+3}\right]_{RN} + \cdots + \left[\sum_{i\neq l, l-2} N_i, l_{n-k+3}\right]_{RN}$$

$$+ \cdots + \left[N_1, l_{n-k+3}\right]_{RN} + \cdots + \left[N_1, l_{n-k+2}\right]_{RN}.$$  

**Proof:** Let $1 \leq i, j \leq l$. By Remark 3.2, any vector valued $(m + k_i - 1)$-form, with $m \geq n - k_i + 3$, is identically zero; hence,

$$[N_i, l_m]_{RN} = 0,$$

(17)
for all $m \geq n - k_i + 3$. Also, any vector valued $(k_i + k_j + m - 2)$-form, with $m \geq 1$ is identically zero, because out of the conditions $\frac{n+3}{2} \leq k_1 \leq \cdots \leq k_l \leq n + 1$ and $m \geq 1$ we get $k_i + k_j + m - 2 \geq n + 2$. Thus,

$$\left[ N_i, [N_j, l_m]_{RN} \right]_{RN} = 0, \quad (18)$$

for all $m \geq 1$. From Equations (17) and (18), we get

$$\left[ S + \sum_{i=1}^{l} N_i, \mu \right]_{RN} = \mu + \left[ \sum_{i=1}^{l} N_i, l_1 \right]_{RN} + \cdots + \left[ \sum_{i=1}^{l} N_i, l_{n-k_l+2} \right]_{RN}$$

$$+ \left[ \sum_{i\neq l} N_i, l_{n-k_l+3} \right]_{RN} + \cdots + \left[ \sum_{i\neq l} N_i, l_{n-k_{l-1}+2} \right]_{RN}$$

$$+ \cdots +$$

$$+ [N_1, l_{n-k_2+3}]_{RN} + \cdots + [N_1, l_{n-k_1+2}]_{RN}$$

and

$$\left[ S + \sum_{i=1}^{l} N_i, S + \sum_{i=1}^{l} N_i, \mu \right]_{RN} = \mu + 2 \left[ \sum_{i=1}^{l} N_i, \mu \right]_{RN} = \left[ S + 2 \sum_{i=1}^{l} N_i, \mu \right]_{RN}$$

It follows from the conditions $\frac{n+3}{2} \leq k_1 \leq \cdots \leq k_l \leq n + 1$ that, for $1 \leq i, j \leq l$, we have $k_i + k_j - 1 \geq n + 2$. Hence, $[N_i, N_j]_{RN} = 0$ for all $1 \leq i, j \leq l$, which implies that

$$\left[ S + \sum_{i=1}^{l} N_i, S + 2 \sum_{i=1}^{l} N_i \right]_{RN} = 0.$$

\textbf{Remark 3.5.} In Proposition 3.4 one may replace each vector valued $k_i$-form $N_i$ by a family of symmetric vector valued $k_i$-forms.

Next, we consider a particular class of Lie $n$-algebras, those associated to $n$-plectic manifolds. Let us recall some definitions from [19].

\textbf{Definition 3.6.} An $n$-plectic manifold is a manifold $M$ equipped with a non-degenerate and closed $(n + 1)$-form $\omega$. It is denoted by $(M, \omega)$.

An $(n - 1)$-form $\alpha$ on an $n$-plectic manifold $(M, \omega)$ is said to be a Hamiltonian form if there exists a smooth vector field $\chi_\alpha$ on $M$ such that $d\alpha = -\iota_{\chi_\alpha} \omega$. The vector field $\chi_\alpha$ is called the Hamiltonian vector field associated to $\alpha$. The
space of all Hamiltonian forms on an $n$-plectic manifold $(M, \omega)$ is denoted by $\Omega_{Ham}^{n-1}(M)$.

Given two Hamiltonian forms $\alpha, \beta$ on an $n$-plectic manifold $(M, \omega)$, with Hamiltonian vector fields $\chi_\alpha$ and $\chi_\beta$, respectively, one may define a bracket $\{., .\}$ by setting

$$\{\alpha, \beta\} := \iota_{\chi_\alpha} \iota_{\chi_\beta} \omega.$$ 

It turns out that $\{\alpha, \beta\}$ is a Hamiltonian form with associated Hamiltonian vector field $[\chi_\alpha, \chi_\beta]$, see [19].

Following [19], we may associate to an $n$-plectic manifold $(M, \omega)$ a symmetric Lie $n$-algebra.

**Theorem 3.7.** Let $(M, \omega)$ be an $n$-plectic manifold. Set

$$E_i = \begin{cases} \Omega_{Ham}^{n-1}(M), & \text{if } i = -1, \\ \Omega^{n+i}(M), & \text{if } -n \leq i \leq -2 \end{cases}$$

and $E = \bigoplus_{i=-n}^{-1} E_i$. Let the collection $l_k : E \times \ldots \times E \to E$, $k \geq 1$, of symmetric multi-linear maps be defined as

$$l_1(\alpha) = \begin{cases} (-1)^{|\alpha|} d\alpha, & \text{if } \alpha \not\in E_{-1}, \\ 0, & \text{if } \alpha \in E_{-1}, \end{cases}$$

$$l_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} 0, & \text{if } \alpha_i \not\in E_{-1} \text{ for some } 0 \leq i \leq k, \\ (-1)^{\frac{k+1}{2}} \iota_{\chi_{\alpha_1}} \ldots \iota_{\chi_{\alpha_k}} \omega, & \text{if } \alpha_i \in E_{-1} \text{ for all } 0 \leq i \leq k \text{ and } k \text{ is even}, \\ (-1)^{\frac{k-1}{2}} \iota_{\chi_{\alpha_1}} \ldots \iota_{\chi_{\alpha_k}} \omega, & \text{if } \alpha_i \in E_{-1} \text{ for all } 0 \leq i \leq k \text{ and } k \text{ is odd}, \end{cases}$$

for $k \geq 2$, where $\chi_{\alpha_i}$ is the Hamiltonian vector field associated to $\alpha_i$. Then, $(E, (l_k)_{k \geq 1})$ is a symmetric Lie $n$-algebra.

**Proof:** In [19], an $L_\infty$-algebra is defined to be a graded vector space $L$ equipped with a collection $l_k : L^\otimes k \to L$ of skew-symmetric maps, with $\bar{l}_k = k-2$, satisfying a relation so called graded Jacobi identity. However, by translations of degrees in the graded vector space as $L_i \to L_{-i}$, it is equivalent to say an $L_\infty$-algebra is a graded vector space $L$ equipped with a collection $\{l_k : L^\otimes k \to L\}$ of skew-symmetric maps, with $\bar{l}_k = 2-k$, satisfying a certain graded Jacobi identity. Now, it is enough to shift, by 1, the degrees of the graded vector space in Theorem 3.14. in [19] and use the décalage isomorphism to get the desired result. 

\[\square\]
In the next proposition we give an example of a Nijenhuis vector valued form, with respect to the $L_\infty$-algebra (Lie $n$-algebra) structure associated to a given $n$-plectic manifold, which is the sum of a symmetric vector valued 1-form with a symmetric vector valued $i$-form, with $i = 2, \ldots, n$.

**Proposition 3.8.** Let $(M, \omega)$ be an $n$-plectic manifold with the associated symmetric Lie $n$-algebra structure $\mu = l_1 + \cdots + l_{n+1}$. For any $n$-form $\eta$ on the manifold $M$, and any $i = 2, \ldots, n$, define $\tilde{\eta}_i$ to be the symmetric vector valued $i$-form of degree zero given by

$$\tilde{\eta}_i(\beta_1, \ldots, \beta_i) = \begin{cases} \iota_{\chi_{\beta_1}} \cdots \iota_{\chi_{\beta_i}} \eta, & \text{if } \beta_i \in E_{-1}, \\ 0, & \text{otherwise} \end{cases} \quad (19)$$

where $\chi_{\beta_1}, \ldots, \chi_{\beta_n}$ are the Hamiltonian vector fields of $\beta_1, \ldots, \beta_n$, respectively. Then, $S + \tilde{\eta}_i$ is a Nijenhuis vector valued form with respect to $\mu$, with square $S + 2\tilde{\eta}_i$. The deformed structure is

$$[S + \tilde{\eta}_i, \mu]_{RN} = \mu + [\tilde{\eta}_i, l_1]_{RN} + [\tilde{\eta}_i, l_2]_{RN}.$$

The proof of Proposition 3.8 is based on the following lemma.

**Lemma 3.9.** For all $2 \leq i \leq n$, and all homogeneous elements $\alpha_1, \ldots, \alpha_i \in E$, we have:

1. $\tilde{\eta}_i(l_1(\alpha_1), \alpha_2, \ldots, \alpha_i) = 0$,
2. $[\tilde{\eta}_i, l_m]_{RN} = \begin{cases} 0, & m \geq 3 \\ -\iota_2 \tilde{\eta}_i, & m = 2 \\ d \circ \tilde{\eta}_i, & m = 1 \end{cases}$
3. $[\tilde{\eta}_i, [\tilde{\eta}_i, l_m]_{RN}]_{RN} = 0, \ m \geq 1$.

**Proof:** We start by noticing that from its definition, $\tilde{\eta}_i$ vanishes on $\bigoplus_{i=2}^{n} E_i$ and $\text{Im} \tilde{\eta}_i \subset E_{-i}$, $i \geq 2$. So, to prove item (1), the only case we have to investigate is when $\alpha_1 \in E_{-2}$ and $l_1(\alpha_1), \alpha_2, \ldots, \alpha_i$ are all Hamiltonian forms. Let $\chi_{l_1(\alpha_1)}$ be the Hamiltonian vector field associated to $l_1(\alpha_1)$. Then, we have

$$\iota_{\chi_{l_1(\alpha_1)}} \omega = -d(l_1(\alpha_1)) = -d^2 \alpha_1 = 0,$$

thus $\chi_{l_1(\alpha_1)} = 0$, by the non-degeneracy of $\omega$. This proves item (1).
Let us now compute $[\tilde{\eta}_i, l_m]_{RN}$. When $m \geq 3$, from the definitions of $l_m$ and $\tilde{\eta}_i$, we get
\[
l_m(\tilde{\eta}_i(\alpha_1, \cdots, \alpha_i), \cdots, \alpha_{m+i-1}) = 0
\]
and
\[
\tilde{\eta}_i(l_m(\alpha_1, \cdots, \alpha_m), \cdots, \alpha_{m+i-1}) = 0,
\]
for all $\alpha_1, \cdots, \alpha_i, \cdots, \alpha_{m+i-1} \in E$, $i \geq 2$, so that $[\tilde{\eta}_i, l_m]_{RN} = 0$. Since $\tilde{\eta}_i$ takes value in $E_{-i}$, we have $\iota_{\tilde{\eta}_i}l_2 = 0$, hence $[\tilde{\eta}_i, l_2]_{RN} = -\iota_{\tilde{\eta}_i}l_2\tilde{\eta}_i$. From item (1) and definition of $\tilde{\eta}_i$ we get $[\tilde{\eta}_i, l_1]_{RN} = d \circ \tilde{\eta}_i$.

Last, we prove item (3). For $m \geq 3$, $[\tilde{\eta}_i, [\tilde{\eta}_i, l_m]_{RN}]_{RN} = 0$ is a direct consequence of item (2). The case $m = 2$ follows from the fact that $\tilde{\eta}_i$ does not take value in $E_{-1}$, so $l_2(\tilde{\eta}_i(\alpha_1, \cdots, \alpha_i), \alpha_{i+1}) = 0$, for all $\alpha_1, \cdots, \alpha_{i+1} \in E$. Hence, using item (2) we get
\[
\iota_{\tilde{\eta}_i}[\tilde{\eta}_i, l_2]_{RN} = 0 \quad \text{and} \quad \iota_{[\tilde{\eta}_i, l_2]_{RN}}{}\tilde{\eta}_i = 0,
\]
which gives $[\tilde{\eta}_i, [\tilde{\eta}_i, l_2]_{RN}]_{RN} = 0$. Similar arguments as those used above prove that $[\tilde{\eta}_i, [\tilde{\eta}_i, l_1]_{RN}]_{RN} = 0$.

**Proof:** (of Proposition 3.8) From Lemma 3.9 we have
\[
[S + \tilde{\eta}_i, \mu]_{RN} = \mu + [\tilde{\eta}_i, l_1]_{RN} + [\tilde{\eta}_i, l_2]_{RN}
\]
and applying $[S + \tilde{\eta}_i, \cdot]_{RN}$ to both sides of Equation (20), we get
\[
[S + \tilde{\eta}_i, [S + \tilde{\eta}_i, \mu]_{RN}]_{RN} = \mu + 2[\tilde{\eta}_i, l_1]_{RN} + 2[\tilde{\eta}_i, l_2]_{RN} = [S + 2\tilde{\eta}_i, \mu]_{RN}.
\]
Now, the equation
\[
[S + \tilde{\eta}_i, S + \tilde{\eta}_i]_{RN} = 0,
\]
holds, for all $i \geq 2$, as a consequence of $\iota_{\tilde{\eta}_i}\tilde{\eta}_i = 0$.

From Proposition 3.8 we immediately get the following result.

**Theorem 3.10.** Let $\eta$ be an arbitrary $n$-form on an $n$-plectic manifold $(M, \omega)$. Let $(E = E_{-n} \oplus \cdots \oplus E_{-1}, \mu = l_1 + \cdots + l_{n+1})$ be the Lie $n$-algebra associated to $(M, \omega)$. For each $2 \leq i \leq n$, define the maps $\tilde{\eta}_i$ as in (19). Then, $\mathcal{N} := S + \sum_{i=2}^{n} \tilde{\eta}_i$ is a Nijenhuis vector valued form with respect to the Lie $n$-algebra structure $\mu$, with square $S + 2 \sum_{i=2}^{n} \tilde{\eta}_i$. Moreover, the deformed structure is of the form $[\mathcal{N}, \mu]_{RN} = \sum_{i=1}^{n+1} l_i^N$, with $l_i^N$ being the component in
the vector valued form \([\mathcal{N}, \mu]_{\mathbb{R}^n}\) which is a vector valued i-form, and is given by:

\[
l_i^N = \begin{cases} 
  l_1, & \text{for } i = 1, \\
  l_i + d \circ \tilde{\eta}_i - \iota_{\tilde{\eta}_i} \eta_{i-1}, & \text{for } i \geq 2.
\end{cases}
\]

A special case of the previous theorem is considered in the next proposition.

**Proposition 3.11.** Let \((M, \omega)\) be an n-plectic manifold and \(\alpha\) a Hamiltonian form on \((M, \omega)\). For each \(2 \leq i \leq n\), define the maps \(\tilde{\alpha}_i\) as

\[
\tilde{\alpha}_i(\beta_1, \ldots, \beta_i) = \begin{cases} 
  \iota_{\chi_{\beta_1}} \cdots \iota_{\chi_{\beta_i}} \omega, & \text{if } \beta_k \in E_{-1} \text{ for all } 1 \leq k \leq i, \\
  0, & \text{otherwise}
\end{cases}
\]

where \(\chi_\alpha, \chi_{\beta_1}, \ldots, \chi_{\beta_i}\) are the Hamiltonian vector fields associated to the Hamiltonian forms \(\alpha, \beta_1, \ldots, \beta_i\), respectively. Then, \(S + \sum_{i=2}^n \tilde{\alpha}_i\) is a Nijenhuis vector valued form with respect to the Lie n-algebra structure \(\mu = l_1 + \cdots + l_{n+1}\), associated to the n-plectic manifold \((M, \omega)\).

Theorem 3.10 can be easily generalized if, instead of taking one n-form on the manifold \(M\), we take a family of n-forms on \(M\).

**Theorem 3.12.** Let \((\eta^j)_{j \geq 1}\) be a family of n-forms on an n-plectic manifold \((M, \omega)\). Let \((E = E_{-n} \oplus \cdots \oplus E_{-1}, \mu = l_1 + \cdots + l_{n+1})\) be the Lie n-algebra associated to \((M, \omega)\). For each \(2 \leq i \leq n\), define the vector valued i-forms \((\tilde{\eta}^j)_i\) as

\[
(\tilde{\eta}^j)_i(\beta_1, \ldots, \beta_i) = \begin{cases} 
  \iota_{\chi_{\beta_1}} \cdots \iota_{\chi_{\beta_i}} \eta^j, & \text{if } \beta_k \in E_{-1} \text{ for all } 1 \leq k \leq i, \\
  0, & \text{otherwise}
\end{cases}
\]

where \(\chi_{\beta_1}, \ldots, \chi_{\beta_i}\) are the Hamiltonian vector fields associated to the Hamiltonian forms \(\beta_1, \ldots, \beta_i\), respectively. Then, \(\mathcal{N} := S + \sum_{j=1}^n \sum_{i=2}^n \tilde{\eta}^j)_i\) is a Nijenhuis vector valued form with respect to the Lie n-algebra structure \(\mu\).

4. The case of Lie 2-algebras

In this section we treat the case of Lie 2-algebras. We show how to construct Nijenhuis forms with respect to Lie 2-algebras, which are the sum of a vector valued 1-form with a vector valued 2-form.

We start by recalling that a Lie 2-algebra is a pair \((E, \mu)\), where \(E\) is a graded vector space with degrees concentrated in \(-2\) and \(-1\), that is \(E = E_{-2} \oplus E_{-1}\), and \(\mu = l_1 + l_2 + l_3\) with \(l_1, l_2\) and \(l_3\) being symmetric vector...
valued 1-form, 2-form and 3-form, respectively, all of them of degree 1. By
degree reasons, the brackets $l_1$ and $l_3$ are not identically zero in the following
cases:

$$l_1 : E_{-2} \to E_{-1}, \quad l_3 : E_{-1} \times E_{-1} \times E_{-1} \to E_{-2},$$

while the binary bracket $l_2$ has two parts

$$l_2|_{E_{-1} \times E_{-2}} : E_{-1} \times E_{-2} \to E_{-2}, \quad l_2|_{E_{-1} \times E_{-1}} : E_{-1} \times E_{-1} \to E_{-1}.$$ 

The equation $[\mu, \mu]_{RN} = 0$ gives the following relations (by degree reasons,
all the missing cases are identically zero):

1. $[l_1, l_2]_{RN}(f, g) = 0,$
2. $[l_1, l_2]_{RN}(X, f) = 0,$
3. $(2[l_1, l_3]_{RN} + [l_2, l_2]_{RN})(X, Y, f) = 0,$
4. $(2[l_1, l_3]_{RN} + [l_2, l_2]_{RN})(X, Y, Z) = 0,$
5. $[l_2, l_3]_{RN}(X, Y, Z, W) = 0,$

with $X, Y, Z, W \in E_{-1}$ and $f, g \in E_{-2}.$

Let us set

$$l_1 = \partial, \quad l_3 = \omega$$

and, for all $X, Y \in E_{-1}$ and $f \in E_{-2},$

$$l_2|_{E_{-1} \times E_{-1}}(X, Y) = [X, Y]_2 \quad \text{and} \quad l_2|_{E_{-1} \times E_{-2}}(X, f) = \chi(X)f,$$

with $\chi : E_{-1} \to \text{End}(E_{-2}).$ Then, we have:

**Lemma 4.1.** A vector valued form $\mu = l_1 + l_2 + l_3,$ with associated quadruple
$(\partial, \chi, [, ,]_2, \omega)$ given by (26) and (27), is a Lie 2-algebra structure on $E = E_{-2} \oplus E_{-1}$ if and only if

1. $\chi(\partial f)g = -\chi(\partial g)f,$
2. $[X, \partial f]_2 = \partial(\chi(X)f),$ 
3. $\chi([X, Y]_2)f + \chi(Y)\chi(X)f - \chi(X)\chi(Y)f + \omega(X, Y, \partial f) = 0,$
4. $[[X, Y]_2, Z]_2 + \text{c.p.} = \partial(\omega(X, Y, Z)),$
5. $\chi(W)\omega(X, Y, Z) - \chi(Z)\omega(X, Y, W) + \chi(Y)\omega(X, Z, W)$
6. $-\chi(X)\omega(Y, Z, W) = -\omega([X, Y]_2, Z, W) + \omega([X, Z]_2, Y, W) - \omega([X, W]_2, Y, Z) - \omega([Y, Z]_2, X, W) + \omega([Y, W]_2, X, Z) - \omega([Z, W]_2, X, Y),$

for all $X, Y, Z, W \in E_{-1}$ and $f \in E_{-2}.$
Proof: We have the following equivalences, by applying the definition of Richardson-Nijenhuis bracket: (21) ⇔ (28), (22) ⇔ (29), (23) ⇔ (30), (24) ⇔ (31) and (25) ⇔ (32).

The quadruple (∂, χ, [., .], ω) of Lemma 4.1 is the quadruple associated to the Lie 2-algebra structure µ = l₁ + l₂ + l₃.

There is an associated Chevalley-Eilenberg differential to each Lie 2-algebra. Before giving its definition, we need the next lemma.

Lemma 4.2. Let (E = E₂ ⊕ E₁, µ = l₁ + l₂ + l₃) be a Lie 2-algebra with corresponding quadruple (∂, χ, [., .], ω) and η ∈ S^k(E*) ⊗ E be a vector valued k-form of degree k − 2. Then,

\[ [\eta, l₂]_{RN}(X₀, \ldots, Xₖ) = \sum_{i=0}^{k} (-1)^i \chi(X_i) \eta(X₀, \ldots, \hat{X}_i, \ldots, Xₖ) \]
\[ + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \eta([X_i, X_j]₂, X₀, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, Xₖ), \]

(33)

for all X₀, . . . , Xₖ ∈ E₁, where \( \hat{X}_i \) means the absence of \( X_i \).

Proof: By degree reasons, η has to be of the form \( \eta : E₁ \times \ldots \times E₁ → E₂ \). Using the Richardson-Nijenhuis bracket definition one gets Equation (33).

Definition 4.3. Let \( E = E₂ ⊕ E₁ \) be a graded vector space concentrated on degrees −2 and −1, \( S_k(E) \subset S^k(E*) ⊗ E \) be the subspace of all symmetric vector valued k-forms of degree k − 2 and \( S^*_E := \oplus_{k≥1} S_k(E) \). Let \( \chi : E₁ → End(E₂) \) be a representation of vector spaces and \( [., .] : E₁ × E₁ → E₁ \) a graded symmetric bilinear map. Then, the Chevalley-Eilenberg differential \( d^{CE} \) is the map

\[ d^{CE} : S^*_E → S^*_E \]

such that, if \( \eta ∈ S_k(E) \), then \( d^{CE}η ∈ S_{k+1}(E) \) is defined by

\[ d^{CE}η(X₀, \ldots, Xₖ) = \sum_{i=0}^{k} (-1)^i \chi(X_i) \eta(X₀, \ldots, \hat{X}_i, \ldots, Xₖ) \]
\[ + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \eta([X_i, X_j], X₀, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, Xₖ), \]

for all \( X₀, . . . , Xₖ ∈ E₁ \), where \( \hat{X}_i \) means for the absence of \( X_i \).
In general, the operator $d^{CE}$ does not square to zero. However, according to Lemma 4.2 it can be written as

$$d^{CE} = [\cdot, l_2]_{RN},$$

and we get, from the graded Jacobi identity of the Richardson-Nijenhuis bracket, that $d^{CE}$ squares to zero if and only if $[l_2, l_2]_{RN} = 0$.

Next, we explain how a crossed module of Lie algebras can be seen as a Lie 2-algebra. Let us first recall the definition of a crossed module of Lie algebras $\mathcal{C}$:

**Definition 4.4.** A crossed module of Lie algebras $(g, [\cdot, \cdot]^g)$ and $(h, [\cdot, \cdot]^h)$ is a homomorphism $\partial : g \to h$ together with an action by derivation of $h$ on $g$, that is, a linear map $\chi : h \to \text{Hom}(g, g)$ such that

$$\partial(\chi(h)g) = [h, \partial(g)]^h,$$

for all $g \in g$, $h \in h$ \hspace{1cm} (34)

and

$$\chi(\partial(g_1))g_2 = [g_1, g_2]^g,$$

for all $g_1, g_2 \in g$. \hspace{1cm} (35)

Such a crossed module will be denoted by $(g, h, \partial, \chi)$.

From a Lie 2-algebra with vanishing vector valued 3-form, we may get a crossed module of Lie algebras.

**Proposition 4.5.** Let $(E = E_{-2} \oplus E_{-1}, \mu = l_1 + l_2 + l_3)$ be a Lie 2-algebra, with corresponding quadruple $(\partial, \chi, [\cdot, \cdot]_2, \omega)$ given by (26) and (27). If $\omega = 0$, then $(E_{-2}, E_{-1}, \partial, \chi)$ is a crossed module of Lie algebras.

Proposition 3.3 provides the construction of Nijenhuis forms on Lie $n$-algebras. However, for the case $n = 2$, that proposition does not give the possibility of having a Nijenhuis vector valued 2-form. We intend to give an example of Nijenhuis vector valued form with respect to a Lie 2-algebra structure $\mu$ on a graded vector space $E_{-2} \oplus E_{-1}$ which is not purely a 1-form, i.e. not just a collection of maps from $E_i$ to $E_i$, $i = 1, 2$. As we have mentioned before, elements of degree zero in $\tilde{S}(E^*) \otimes E$ are necessarily of the form $N + \alpha$ with $N : E \to E$ a linear endomorphism preserving the degree and $\alpha : E \times E \to E$ a symmetric vector valued 2-form of degree zero.

**Theorem 4.6.** Let $\mu = l_1 + l_2 + l_3$ be a Lie 2-algebra structure on a graded vector space $E = E_{-2} \oplus E_{-1}$ and $\alpha$ a symmetric vector valued 2-form of degree
zero. Then, \( S + \alpha \) is a Nijenhuis vector valued form with respect to \( \mu \), with square of \( S + 2\alpha \), if and only if
\[
\alpha(l_1(\alpha(X,Y)), Z) + c.p. = 0,
\]
for all \( X, Y, Z \in E_{-1} \).

**Proof:** By degree reasons, the only case where the vector valued 3-form \( [\alpha, [\alpha, l_1]_{RN}]_{RN} \) is not identically zero is when it is evaluated on elements of \( E_{-1} \). In this case, we get
\[
[\alpha, [\alpha, l_1]_{RN}]_{RN}(X, Y, Z) = [\alpha, l_1]_{RN}(\alpha(X,Y), Z) + c.p.
- \alpha([\alpha, l_1]_{RN}(X,Y), Z) + c.p.
= -2\alpha(l_1(\alpha(X,Y)), Z) + c.p.,
\]
for all \( X, Y, Z \in E_{-1} \). Again by degree reasons, \( [\alpha, [\alpha, l_2]_{RN}]_{RN} \) and \( [\alpha, l_3]_{RN} \) are identically zero. So, we have
\[
[S + \alpha, [S + \alpha, l_1 + l_2 + l_3]_{RN}]_{RN} =
\]
\[
= [S + \alpha, l_1 + l_2 + l_3]_{RN} + [\alpha, l_1]_{RN} + [\alpha, l_2]_{RN}
\]
\[
= l_1 + l_2 + l_3 + 2[\alpha, l_1]_{RN} + 2[\alpha, l_2]_{RN} + [\alpha, [\alpha, l_1]_{RN}]_{RN}
\]
\[
= [S + 2\alpha, l_1 + l_2 + l_3]_{RN} + [\alpha, [\alpha, l_1]_{RN}]_{RN}.
\]
(37)

On the other hand, Lemma 2.5 and Equation (2) imply that
\[
[S + \alpha, S + 2\alpha]_{RN} = 0.
\]
(38)

Equations (36), (37) and (38) show that \( S + \alpha \) is a Nijenhuis vector valued form with respect to \( \mu \), with square \( S + 2\alpha \), if and only if \( \alpha(l_1(\alpha(X,Y)), Z) + c.p. = 0 \), for all \( X, Y, Z \in E_{-1} \).

**Corollary 4.7.** Let \( \mu = l_1 + l_2 + l_3 \) be a Lie 2-algebra structure on a graded vector space \( E = E_{-2} \oplus E_{-1} \), with \( l_1 = 0 \). Then, for every vector valued 2-form \( \alpha \) of degree zero, \( S + \alpha \) is a Nijenhuis vector valued form with respect to \( \mu \), with square \( S + 2\alpha \).

Combining Theorems 4.6 and 2.4 we get the following proposition.

**Proposition 4.8.** Let \( \mu = l_1 + l_2 + l_3 \) be a Lie 2-algebra structure on a graded vector space \( E = E_{-2} \oplus E_{-1} \). Let \( \alpha \) be a vector valued 2-form of degree zero such that \( \alpha(l_1(\alpha(X,Y)), Z) + c.p. = 0 \), for all \( X, Y, Z \in E_{-1} \). Let \( \mu_k \) stand for the vector valued form defined by \( \mu_k = [S + \alpha, [S + \alpha, \cdots, [S + \cdots]_{RN} = 0. \)

(38)
\[ \alpha, \mu \]_{RN} \cdot \cdot \cdot [\mu]_{RN}, \text{ with } k \text{ copies of } S + \alpha. \] Then, \( S + \alpha \) is a Nijenhuis vector valued form with respect to all the terms of the hierarchy of successive deformations \( \mu_k \), with square \( S + 2\alpha \).

If \( \mu = l_1 + l_2 + l_3 \) is a Lie 2-algebra on \( E = E_{-2} \oplus E_{-1} \) with \( l_1 = 0 \), then \( [., .]_2, \) given by (27), is a Lie bracket on \( E_{-1} \). Also, the condition \( [l_2, l_3]_{RN} = 0 \) means that \( l_3 \) is a Chevalley-Eilenberg-closed 3-form of this Lie algebra \( E_{-1} \) valued in \( E_{-2} \). This kind of Lie 2-algebras are usually called \textit{string Lie algebras}. A Lie 2-algebra \( (E_{-2} \oplus E_{-1}, l_1 + l_2 + l_3) \) with \( l_2 = l_3 = 0 \) and \( l_1 \) invertible, is called a \textit{trivial} Lie 2-algebra. The next example is an application of Theorem 4.6 to a trivial Lie 2-algebra.

**Example 4.9.** Let \( g \) be a vector space and \( [.,.]_g \) be a skew-symmetric bilinear map on \( g \). Let \( E_{-1} := \{ -1 \} \times g \), \( E_{-2} := \{ -2 \} \times g \) and let \( \partial : E_{-2} \to E_{-1} \) be given by \( (-2, x) \mapsto (-1, x) \). Define \( \alpha : E_{-1} \times E_{-1} \to E_{-2} \) to be vector valued 2-form on the graded vector space \( E = E_{-2} \oplus E_{-1} \) as \( ((-1, x), (-1, y)) \mapsto (-2, [x, y]_g) \). Then, as a direct consequence of Theorem 4.6, we have that \( S + \alpha \) is Nijenhuis with respect to \( \partial \) if and only if \( [.,.]_g \) is a Lie bracket.

Let us now look at the deformed Lie 2-algebra structure.

**Proposition 4.10.** Let \( \mu = l_1 + l_2 + l_3 \) be a Lie 2-algebra structure on a graded vector space \( E = E_{-2} \oplus E_{-1} \), with associated quadruple \((\partial, [.,.]_2, \chi, \omega)\). Let \( \alpha \) be a symmetric vector valued 2-form of degree zero on \( E \) and set \( N = S + \alpha \). The deformed structure \( \mu^N \) is associated to the quadruple \((\partial', [.,.]_2', \chi', \omega')\):

\[
\partial' f = \partial f, \\
[X, Y]_2' = [X, Y]_2 + \partial(\alpha(X, Y)), \\
\chi'(X)f = \chi(X)f - \alpha(\partial f, X), \\
\omega'(X, Y, Z) = \omega(X, Y, Z) + d^{CE} \alpha(X, Y, Z),
\]

(39)

for all \( X, Y, Z \in E_{-1} \) and \( f \in E_{-2} \).

**Proof:** The statement follows from the following easy relations:

\[
[S + \alpha, \mu]_{RN} = l_1 + (l_2 + [\alpha, l_1]_{RN}) + (l_3 + [\alpha, l_2]_{RN}); \\
[\alpha, l_1]_{RN}(X, Y) = l_1(\alpha(X, Y)), \quad \text{for all } X, Y \in E_{-1}; \\
[\alpha, l_1]_{RN}(X, f) = -\alpha(l_1(f), X), \quad \text{for all } X \in E_{-1}, f \in E_{-2}; \\
[\alpha, l_2]_{RN} = d^{CE} \alpha.
\]
Notice that, in the case of Proposition 4.10, the vector valued form $S - \alpha$ has the inverse effect of $S + \alpha$, that is, $[S - \alpha, [S + \alpha, \mu]]_{RN} = \mu$.

As we have seen previously, string Lie algebras on $E_{-2} \oplus E_{-1}$ are in one to one correspondence with Lie algebra structures on $\mathfrak{g} := E_{-1}$ together with a representation of the Lie algebra $\mathfrak{g}$ on the vector space $V := E_{-2}$ and a Chevalley-Eilenberg 3-cocycle $\omega$ for this representation. Hence, we denote string Lie algebras as triples $(\mathfrak{g}, V, \omega)$. According to Proposition 4.8, the deformation of a string Lie algebra $(\mathfrak{g}, V, \omega)$ by $S + \alpha$, just amounts to change the 3-cocycle $\omega$ into $\omega + d^{CE} \alpha$. So that, for string Lie algebras, adding up a coboundary, i.e., changing $(\mathfrak{g}, V, \omega)$ into $(\mathfrak{g}, V, \omega + d^{CE} \alpha)$ can be seen as a Nijenhuis transformation by $S + \alpha$.

A Lie 2-subalgebra of a Lie 2-algebra $(E = E_{-2} \oplus E_{-1}, \mu = l_1 + l_2 + l_3)$ is a Lie 2-algebra $(E' = E'_2 \oplus E'_{-1}, \mu' = l'_1 + l'_2 + l'_3)$ with $E'_{-2} \subset E_{-2}$ and $E'_{-1} \subset E_{-1}$ vector subspaces,

\[ l'_1 = l_1|_{E'}, \quad l'_2 = l_2|_{E' \times E'} \quad \text{and} \quad l'_3 = l_3|_{E' \times E' \times E'}. \]

Let us now investigate Lie 2-algebras structures for which $\chi = 0$. There may be quite a few such Lie 2-algebras but we are going to show that, after a Nijenhuis transformation of the form $S + \alpha$, such Lie 2-algebras will be decomposed as a direct sum of a string Lie algebra with a trivial Lie 2-algebra.

**Proposition 4.11.** Given a Lie 2-algebra structure $l_1 + l_2 + l_3$ on a graded vector space $E = E_{-2} \oplus E_{-1}$ and corresponding quadruple $(\partial, [\cdot, \cdot], 2, \chi, \omega)$, with $\chi = 0$, there exists a Nijenhuis form $S + \alpha$, with $\alpha$ a vector valued 2-form of degree zero, such that the deformed bracket $[S + \alpha, l_1 + l_2 + l_3]$ is the direct sum of a string Lie 2-algebra with a trivial $L_\infty$-algebra.

**Proof:** We set $E'_{-1} := \text{Im}(\partial)$, $E'_{-2} := \text{Ker}(\partial)$ and we choose two subspaces $E'_{-2} \subset E_{-2}$ and $E'_{-1} \subset E_{-1}$ such that the following are direct sums: $E'_{-2} \oplus E'_{-2} = E_{-2}$ and $E'_{-1} \oplus E'_{-1} = E_{-1}$. Since $\chi = 0$, by (29), the bracket $[\cdot, \cdot]_2$ vanishes on $E_{-1} \times E'_{-1}$; so that, there exists a unique skew-symmetric bilinear map $\alpha : E_{-1} \times E_{-1} \to E'_{-2}$ such that

\[ \partial \alpha(X, Y) = -\text{pr}_{E'_{-1}}([X, Y]_2), \quad \text{for all} \quad X, Y \in E_{-1}, \]

where $\text{pr}_{E'_{-1}}$ stands for the projection on $E'_{-1}$ with respect to $E'_{-1}$. Note that $\alpha(X, Y) = 0$ if $X$ or $Y$ belong to $E'_{-1}$, therefore we have $\alpha(\partial \alpha(X, Y), Z) = 0.$
for all \( X, Y, Z \in E_{-1} \). Hence, by Theorem 4.6, \( S + \alpha \) is Nijenhuis form with square \( S + 2\alpha \). We claim that, for the deformed bracket \( l'_1 + l'_2 + l'_3 := [S + \alpha, l_1 + l_2 + l_3]_{RN} \), \( (E_{-1}^s \oplus E_{-2}^s, l'_1 + l'_2 + l'_3) \) and \( (E_{-1}^l \oplus E_{-2}^l, l'^{l}_{1} + l'^{l}_{2} + l'^{l}_{3}) \) are Lie 2-subalgebras of \( (E = E_{-2} \oplus E_{-1}, \mu = l_1 + l_2 + l_3) \), where \( l'_i \) and \( l'^{l}_{i} \) stand for the restrictions of \( l'_i \) to \( E_{-1}^s \oplus E_{-2}^s \) and \( E_{-1}^l \oplus E_{-2}^l \), respectively.

We also claim that \( (E_{-1}^s \oplus E_{-2}^s, l'_1 + l'_2 + l'_3) \) is a string Lie 2-algebra while \( (E_{-1}^l \oplus E_{-2}^l, l'^{l}_{1} + l'^{l}_{2} + l'^{l}_{3}) \) is a trivial Lie 2-algebra, and that their direct sum is isomorphic to \( (E_{-2} \oplus E_{-1}, l'_1 + l'_2 + l'_3) \).

Let \( (\partial', [\cdot, \cdot]', \chi', \omega') \) stand for the corresponding quadruple associated to the deformed structure \( l'_1 + l'_2 + l'_3 \). From \([l_1, l_2]_{RN} = 0\) we get \( l_2(l_1(f), X) = 0\), for all \( f \in E_{-2} \). This means that \( l_2 \) vanishes on \( E_{-1}^l \). Also, since \( \alpha(X, Y) = 0 \) if \( X \) or \( Y \) belongs to \( E_{-1}^l \), by Equations (39), we have that \( \chi' = 0 \) and \([\cdot, \cdot]' = 0\) and hence \( l'_2 \) vanishes on \( E_{-1}^l \). From \([l_1, l_3]_{RN} = 0\), we get that \( \omega(X, Y, Z) \) vanishes for all \( X \in E_{-1}^l \), so by Equations (39) the restriction of \( l'_3 \) to \( E_{-1}^l \) vanishes. Since the restriction of \( l'_1 \) to \( E_{-1}^l \) is a bijection onto its image, the restriction of \( l'_1 + l'_2 + l'_3 \) to \( E_{-1}^l \oplus E_{-2}^l \) is a Lie 2-subalgebra and it is a trivial Lie 2-algebra.

Next we prove that \( (E_{-2}^s \oplus E_{-1}^s, l'_1 + l'_2 + l'_3) \) is a Lie 2-subalgebra with \( l'_2(E_{-2}^s) = 0 \) and hence is a string Lie algebra. Let \( X, Y \in E_{-1}^s \). Then, by Equations (39) we have

\[
l'_2(X, Y) = [X, Y]_2 + \partial \alpha(X, Y) = [X, Y]_2 - pr_{E_{-1}^l}([X, Y]_2).
\]

This implies that

\[
l'_2(X, Y) \in E_{-1}^s. \tag{40}
\]

Let \( X, Y, Z \in E_{-1}^s \). Then, we have \( l'_1(X) = l'_1(Y) = l'_1(Z) = 0 \). Hence, from

\[
(2[l'_1, l'_3]_{RN} + [l'_2, l'_2]_{RN})(X, Y, Z) = 0,
\]

we get

\[
l'_1(l'_3(X, Y, Z)) = l'_2(l'_2(X, Y), Z). \tag{41}
\]

Using Relation (40), the right hand side of Equation (41) belongs to \( E_{-1}^s \), while according to the definition of \( E_{-1}^l \), the left hand side of Equation (41) belongs to \( E_{-1}^l \) and since \( E_{-1} = E_{-1}^l \oplus E_{-1}^s \) is a direct sum, both sides of Equation (41) should be zero. This implies that

\[
l'_3(X, Y, Z) \in E_{-2}^s. \tag{42}
\]

Relation (40) and Equation (42) show that \( (E_{-2}^s \oplus E_{-1}^s, l'_1 + l'_2 + l'_3) \) is a Lie 2-subalgebra. Also, by definition of \( E_{-2}^s \), we have \( l'_1(E_{-2}^s) = 0 \). This completes the proof.
Next, it is interesting to see that Lie algebras themselves can be seen as Nijenhuis forms. We start by noticing that any vector valued 2-form of degree zero on a graded vector space $E_{-2} \oplus E_{-1}$ is of the form

$$\alpha(X, Y) = \begin{cases} -\alpha(Y, X), & \text{if } X, Y \in E_{-1}, \\ 0, & \text{otherwise}. \end{cases}$$

(43)

This, together with the fact that $\alpha$ always takes value in $E_{-2}$, imply that

$$\alpha(\alpha(X, Y), Z) + c.p. = 0,$$

(44)

for all $X, Y, Z \in E_{-1}$. Equations (43) and (44) mean that any symmetric vector valued 2-form $\alpha$ on an arbitrary graded vector space $E_{-2} \oplus E_{-1}$ is a Lie algebra (not a graded Lie algebra). In the next proposition, we show that there is also a way to get a Lie bracket on a graded vector space $E = E_{-2} \oplus E_{-1}$ from a Nijenhuis form with respect to a Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$ on the vector space $E$.

**Proposition 4.12.** Let $(E = E_{-2} \oplus E_{-1}, \mu = l_1 + l_2 + l_3)$ be a Lie 2-algebra, with corresponding quadruple $(\partial, [., .], \chi, \omega)$. Let $\alpha$ be a vector valued 2-form of degree zero and define a bilinear map $\tilde{\alpha}$ by setting

$$\tilde{\alpha}(X, Y) = \begin{cases} \alpha(X, Y), & \text{for } X, Y \in E_{-1}, \\ \alpha(\partial X, Y), & \text{for } X \in E_{-2}, Y \in E_{-1}, \\ \alpha(X, \partial Y), & \text{for } X \in E_{-1}, Y \in E_{-2}, \\ \alpha(\partial X, \partial Y), & \text{for } X, Y \in E_{-2}. \end{cases}$$

Then, $S + \alpha$ is Nijenhuis vector valued 2-form with respect to $\mu$, with square $S + 2\alpha$, if and only if $(E, \tilde{\alpha})$ is a Lie algebra.

*Proof:* By definition, $\tilde{\alpha}$ is a skew-symmetric bilinear map on the vector space $E$ and we have

$$\tilde{\alpha}(\tilde{\alpha}(X, Y), Z) + c.p. = \alpha(\partial \alpha(X, Y), Z) + c.p.,$$

$$\tilde{\alpha}(\tilde{\alpha}(f, Y), Z) + c.p. = \alpha(\partial \alpha(f, Y), Z) + c.p.,$$

(45)

for all $X, Y, Z \in E_{-1}$ and $f \in E_{-2}$. Hence, Theorem 4.6 together with (45) imply that $\tilde{\alpha}$ is a Lie bracket on the vector space $E$ if, and only if, $S + \alpha$ is a Nijenhuis form with respect to $\mu$, with square $S + 2\alpha$.

Last, we give a result involving weak Nijenhuis forms on a Lie 2-algebra.
Proposition 4.13. Let $\partial : E_{-2} \to E_{-1}$ be a Lie 2-algebra structure on a graded vector space $E = E_{-2} \oplus E_{-1}$, that is, a Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$ on $E$, with $l_1 = \partial$ and $l_2 = l_3 = 0$. Let $\alpha$ be a symmetric vector valued 2-form of degree zero on the graded vector space $E$. If $S + \alpha$ is a weak Nijenhuis vector valued form with respect to $\partial$, then $E_{-1}$ is a Lie algebra with a representation on $E_{-2}$.

Proof: According to Proposition 2.3, $S + \alpha$ is a weak Nijenhuis vector valued form with respect to $\partial$ if and only if $[[S + \alpha, \partial]_{RN}, [S + \alpha, \partial]_{RN}]_{RN} = 0$ or to $[[\alpha, \partial]_{RN}, [\alpha, \partial]_{RN}]_{RN} = 0$. Therefore, $S + \alpha$ is a weak Nijenhuis vector valued form with respect to $\partial$ if and only if

$$\partial \alpha(\partial \alpha(X, Y), Z) + \text{c.p.}(X, Y, Z) = 0 \quad (46)$$

and

$$\alpha(\partial \alpha(X, Y), \partial f) + \text{c.p.}(X, Y, \partial f) = 0, \quad (47)$$

for all $X, Y, Z \in E_{-1}$ and $f \in E_{-2}$. Equation (46) means that $[X, Y] := \partial \alpha(X, Y)$ defines a Lie bracket on $E_{-1}$ since clearly it is skew-symmetric. If we define a map $\cdot : E_{-1} \times E_{-2} \to E_{-2}$ by setting $X \cdot f := \alpha(X, \partial f)$, then (47) can be written as

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f),$$

which means that $\cdot$ is a representation of $E_{-1}$ on $E_{-2}$.

Remark 4.14. A notion of Nijenhuis operator on a Lie 2-algebra independently appeared in [18], while the present paper was about to be completed. This notion is a particular case of ours, by the following reasons. First, in [18], a Nijenhuis operator is necessarily a vector valued 1-form. Second, if $\mathcal{N} = (N_0, N_1)$ is a Nijenhuis operator in the sense of Definition 3.2. in [18], with respect to a Lie 2-algebra $l_1 + l_2 + l_3$, then

$$[\mathcal{N}, [\mathcal{N}, l_i]_{RN}]_{RN} = [\mathcal{N}^2, l_i]_{RN}$$

holds for $i = 1, 2$ and 3, which means that $\mathcal{N}$ is a Nijenhuis vector valued form, in our sense, with square $\mathcal{N}^2$. 
5. Nijenhuis forms on Courant algebroids

We recall that one can associate a Lie 2-algebra to a Courant algebroid [22]. We use this construction to see how $(1, 1)$-tensors on a Courant algebroid, with vanishing Nijenhuis torsion, are related with Nijenhuis forms with respect to the associated Lie 2-algebra.

**Definition 5.1.** A **Courant algebroid** is a vector bundle $E \rightarrow M$ together with a non-degenerate inner product $\langle ., . \rangle$, a morphism of vector bundles $\rho : E \rightarrow TM$ and a bilinear operator $\circ : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$, such that the following axioms hold:

(i) $(\Gamma(E), \circ)$ is a Leibniz algebra, i.e., $X \circ (Y \circ Z) = (X \circ Y) \circ Z + Y \circ (X \circ Z)$,

(ii) $\rho(X)(Y, Z) = \langle X \circ Y, Z \rangle + \langle Y, X \circ Z \rangle$,

(iii) $\rho(X)(Y, Z) = \langle X, Y \circ Z \rangle + \langle X, Z \circ Y \rangle$,

for all $X, Y, Z \in \Gamma(E)$.

When item (i) in Definition 5.1 does not hold, the quadruple $(E, \circ, \rho, \langle ., . \rangle)$ is called a **pre-Courant algebroid** [2].

The next proposition is stated in [11], for Courant algebroids. Since the proof does not use the fact of $\circ$ being a Leibniz bracket, the result also holds for pre-Courant algebroids.

**Proposition 5.2.** For every pre-Courant algebroid $(E, \circ, \rho, \langle ., . \rangle)$ we have

$$X \circ (fY) = f(X \circ Y) + (\rho(X)f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$.

**Corollary 5.3.** Let $(E, \circ, \rho, \langle ., . \rangle)$ and $(E, \circ', \rho', \langle ., . \rangle)$ be two pre-Courant algebroids. If $\circ = \circ'$, then $\rho = \rho'$.

**Proof:** Assume that $(E, \circ, \rho, \langle ., . \rangle)$ and $(E, \circ', \rho', \langle ., . \rangle)$ are both pre-Courant algebroids. By Proposition 5.2 we have

$$(\rho(X)f)Y = (\rho'(X)f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, which implies that $\rho = \rho'$.

We intend to define Nijenhuis deformation of Courant structures. Let $(E, \circ, \rho, \langle ., . \rangle)$ be a Courant algebroid. For a given endomorphism $N : E \rightarrow E$, the deformed bracket by $N$ is a bilinear operation $\circ^N$, defined as:

$$X \circ^N Y := NX \circ Y + X \circ NY - N(X \circ Y),$$
for all $X, Y \in \Gamma(E)$. The deformation of $\rho$ by $N$ is the map $\rho^N$ given by $\rho^N(X) = \rho(NX)$, $X \in \Gamma(E)$. The Nijenhuis torsion of $N$, with respect to the bracket $\circ$, is defined as:

$$T_\circ N(X, Y) := NX \circ NY - N(X \circ^N Y),$$

for all $X, Y \in \Gamma(E)$. A direct computation shows that

$$T_\circ N = \frac{1}{2}(\circ^N N - \circ^{N^2}).$$

All maps $N : \Gamma(E) \to \Gamma(E)$ that will be considered here are $C^\infty(M)$-linear, that is to say they are $(1, 1)$-tensors, that is, smooth sections of endomorphisms of $E$. We denote an endomorphism (vector bundle morphism) of $E$ and the induced map on $\Gamma(E)$ by the same letter.

According to [4], for every vector bundle $E \to M$, if $(\Gamma(E), \circ)$ is a Leibniz algebra and $N : E \to E$ is any endomorphism whose Nijenhuis torsion vanishes, then the pair $(\Gamma(E), \circ^N)$ is a Leibniz algebra. However, given a Courant algebroid $(E, \circ, \rho, \langle . , . \rangle)$ and a $(1, 1)$-tensor $N$, $(E, \circ^N, \rho^N, \langle . , . \rangle)$ may fail to be a pre-Courant algebroid, even if the Nijenhuis torsion of $N$ vanishes. Indeed, from [4] we have the following:

**Theorem 5.4.** If $N$ is an endomorphism on a pre-Courant algebroid $(E, \circ, \rho, \langle . , . \rangle)$, then the quadruple $(E, \circ^N, \rho^N, \langle . , . \rangle)$ is a pre-Courant algebroid if and only if

$$X \circ (N + N^*)Y = (N + N^*)(X \circ Y) \text{ and } (N + N^*)(Y \circ Y) = ((N + N^*)Y) \circ Y$$

for all $X, Y \in \Gamma(E)$, where $N^*$ stands for the transpose of $N$, with respect to $\langle . , . \rangle$.

**Remark 5.5.** In fact, Theorem 5.4 is slightly different from Theorem 4 in [4], because there, the authors start from a Courant algebroid. But the same proof is still valid for the case of pre-Courant.

A Casimir function or simply a Casimir on a Courant algebroid $(E, \circ, \rho, \langle . , . \rangle)$ is a function $f \in C^\infty(M)$ such that $\rho(X)f = 0$, for all $X \in \Gamma(E)$. It is easy to check that $f$ is a Casimir if and only if $\mathcal{D}f = 0$, where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is given by

$$\langle \mathcal{D}f, X \rangle = \rho(X)f.$$ (48)

Also, if $f$ is a Casimir, then

$$(fX) \circ Y = f(X \circ Y) = X \circ (fY)$$ (49)
holds for all sections \( X, Y \in \Gamma(E) \).

The next lemma is a slight generalization of a result in [4].

**Lemma 5.6.** Given a pre-Courant algebroid \((E, \circ, \rho, \langle ., . \rangle)\) and a map \( N : \Gamma(E) \to \Gamma(E) \), if \( N + N^* = \lambda Id_{\Gamma(E)} \), for some Casimir function \( \lambda \in C^\infty(M) \), then \((E, \circ^N, \rho^N, \langle ., . \rangle)\) is a pre-Courant algebroid.

**Proof:** This lemma is a direct consequence of Theorem 5.4 together with (49).

**Theorem 5.7.** Let \((E, \circ, \rho, \langle ., . \rangle)\) be a Courant algebroid and \( N \) a \( (1,1) \)-tensor on \( E \) whose Nijenhuis torsion vanishes and such that \( N + N^* = \lambda Id_{\Gamma(E)} \), with \( \lambda \) being a Casimir function. Then, \((E, \circ^N, \rho^N, \langle ., . \rangle)\) is a Courant algebroid.

**Proof:** Note that \((E, \circ)\) is a Leibniz algebra, so that \((E, \circ^N)\) is also a Leibniz algebra since the Nijenhuis torsion of \( N \) vanishes. This, together with Lemma 5.6, prove the theorem.

**Remark 5.8.** For a (pre-)Courant algebroid \((E, \circ, \rho, \langle ., . \rangle)\), and a \( (1,1) \)-tensor \( N \) on \( E \) with \( N + N^* = \lambda Id_{\Gamma(E)} \) and \( \lambda \) a Casimir function, we have

\[
\rho^N(X)f = \rho(NX)f = \langle NX, Df \rangle = \langle X, N^*Df \rangle = \langle X, (\lambda Id_{\Gamma(E)})Df \rangle,
\]

for all \( X \in \Gamma(E), f \in C^\infty(M) \). This means that the operator \( D^N : C^\infty(M) \to \Gamma(E) \) associated with the (pre-)Courant algebroid \((E, \circ^N, \rho^N, \langle ., . \rangle)\), is given by

\[
D^N = (\lambda Id_{\Gamma(E)}) \circ D.
\]

If we consider the skew-symmetrization of \( \circ \), we obtain the bracket \([., .] \) used in the original definition of Courant algebroid [17]:

\[
[X, Y] = \frac{1}{2}(X \circ Y - Y \circ X),
\]

with \( X, Y \in \Gamma(E) \). The deformation of \([., .] \) by a \( (1,1) \)-tensor \( N \) on \( E \) is the bracket \([., .]_N \) on \( \Gamma(E) \), given by

\[
[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y] = \frac{1}{2}(X \circ^N Y - Y \circ^N X).
\]

\(^*\)In [4], \( \lambda \) is a real number.
The next lemma is an axiom included in the original definition of Courant algebroid [20].

**Lemma 5.9.** Let \((E, \circ, \rho, \langle ., . \rangle)\) be a Courant algebroid and \(D\) its associated operator, given by (48). Then,

\[
[X, fY] = f[X, Y] + (\rho(X)f)Y - \frac{1}{2}\langle X, Y \rangle Df,
\]

for all \(X, Y \in \Gamma(E)\) and \(f \in C^\infty(M)\), where \([., .]\) is the bracket given by (51).

**Remark 5.10.** From the proof of Proposition 2.6.5 in [20], we realize that Lemma 5.9 also holds in the case of a pre-Courant algebroid.

In [22], it was proved that to each Courant algebroid corresponds a Lie 2-algebra. The result in [22] is established using the graded skew-symmetric version of a Lie 2-algebra and the definition of Courant algebroid with skew-symmetric bracket. With our conventions it goes as follows.

Let \((E, \circ, \rho, \langle ., . \rangle)\) be a Courant algebroid over \(M\), with associated operator \(D\), given by (48). Consider the graded vector space \(V = C^\infty(M) \oplus \Gamma(E)\), where the elements of \(C^\infty(M)\) have degree \(-2\) and the elements of \(\Gamma(E)\) have degree \(-1\), and the following symmetric vector valued forms \(l_1, l_2, l_3\) on \(V\), defined by:

\[
\begin{align*}
\quad l_1f &= Df \\
\quad l_2(X, Y) &= \frac{1}{2}(X \circ Y - Y \circ X) \\
\quad l_2(X, f) &= \frac{1}{2}\langle X, Df \rangle, \\
\quad l_3(X, Y, Z) &= \frac{1}{12}(X \circ Y - Y \circ X, Z) + \text{c.p.},
\end{align*}
\]

for all \(X, Y, Z \in \Gamma(E)\) and \(f \in C^\infty(M)\), with \(l_1, l_2, l_3\) being identically zero in all the other cases. Notice that \(l_2|_{\Gamma(E) \times \Gamma(E)}\) coincides with the bracket \([., .]\) given by (51).

**Proposition 5.11.** If \((E, \circ, \rho, \langle ., . \rangle)\) is Courant (respectively, pre-Courant) algebroid, then the pair \((V, l_1 + l_2 + l_3)\), constructed in above, is a symmetric Lie (respectively, pre-Lie\(^\dagger\) ) 2-algebra.

We call this symmetric Lie 2-algebra the symmetric Lie 2-algebra associated to the Courant algebroid \((E, \circ, \rho, \langle ., . \rangle)\).

\(^\dagger\)A pre-Lie 2-algebra is a pair \((E = E_{-2} \oplus E_{-1}, l_1 + l_2 + l_3)\), where \(E\) is a graded vector space concentrated in degrees \(-2\) and \(-1\), and \(l_1, l_2, l_3\) are symmetric graded vector valued 1-form, 2-form and 3-form, respectively, of degree 1.
Starting with a $(1, 1)$-tensor on a Courant algebroid with vanishing Nijenhuis torsion we construct, in the next proposition, a Nijenhuis form for the Lie 2-algebra associated to that Courant structure. First, we need the following lemma.

**Lemma 5.12.** Let \((E, \circ, \rho, (\cdot, \cdot))\) be a pre-Courant algebroid with the associated symmetric pre-Lie 2-algebra structure \(\mu = l_1 + l_2 + l_3\), on the graded vector space \(V = C^\infty(M) \oplus \Gamma(E)\). Let \(N\) be a $(1, 1)$-tensor on \(E\) such that

\[
N + N^* = \lambda \text{Id}_{\Gamma(E)},
\]

with \(\lambda\) a Casimir function. Then, the pre-Lie 2-algebra structure associated to the pre-Courant algebroid \((E, \circ, \rho^N, (\cdot, \cdot))\) is \([N, l_1 + l_2 + l_3]_{RN}\), with \(N\) defined as follows:

\[
N|_{\Gamma(E)} = N \quad \text{and} \quad N|_{C^\infty(M)} = \lambda \text{Id}_{C^\infty(M)}. \tag{53}
\]

**Proof:** Let us denote the pre-Lie 2-algebra associated to the pre-Courant algebroid \((E, \circ, \rho^N, (\cdot, \cdot))\) by \(l_1^N + l_2^N + l_3^N\). Using (50) and (52) and taking into account the fact that \(D\) is a derivation, we have, for all \(f \in C^\infty(M)\) and for all \(X, Y, Z \in \Gamma(E)\),

\[
l_1^N f = D^N f = \lambda Df - N Df = l_1(Nf) - N l_1(f) = [N, l_1]_{RN}(f), \tag{54}
\]

\[
l_2^N(X, Y) = \frac{1}{2}(X \circ^N Y - Y \circ^N X) = l_2(NX, Y) + l_2(X, NY) - N l_2(X, Y) = [N, l_2]_{RN}(X, Y), \tag{55}
\]

\[
l_2^N(X, f) = \frac{1}{2}(X, D^N f) = \frac{1}{2}(X, (-N + \lambda \text{Id}_{\Gamma(E)}) D f) = \frac{1}{2}(X, N^* D f) = \frac{1}{2}(NX, D f) = l_2(NX, f) + \lambda l_2(X, f) - \lambda l_2(X, f) = l_2(NX, f) + l_2(X, N f) - N l_2(X, f) = [N, l_2]_{RN}(X, f) \tag{56}
\]
and
\[
l_3^N(X, Y, Z) = \frac{1}{12}(X \circ^N Y - Y \circ^N X, Z) + c.p.(X, Y, Z)
\]
\[
= \frac{1}{6}(l_2^N(X, Y), Z) + c.p.(X, Y, Z)
\]
\[
= \frac{1}{6}(\langle l_2(NX, Y) + l_2(X, NY) + (N^*-\lambda \text{Id}_{\Gamma(E)})l_2(X, Y), Z\rangle + c.p.(X, Y, Z)
\]
\[
= \frac{1}{6}(\langle l_2(NX, Y), Z\rangle + \langle l_2(X, NY), Z\rangle + \langle l_2(X, Y), NZ \rangle - \lambda \langle l_2(X, Y), Z\rangle
\]
\[
+ c.p.(X, Y, Z)
\]
\[
= \frac{1}{6}(\langle l_2(NX, Y), Z\rangle + c.p.(NX, Y, Z) + \langle l_2(X, NY), Z\rangle + c.p.(X, NY, Z)
\]
\[
+ \langle l_2(X, Y), NZ \rangle + c.p.(X, Y, NZ) - \lambda \langle l_2(X, Y), Z\rangle + c.p.(X, Y, Z))
\]
\[
= l_3(NX, Y, Z) + l_3(X, NY, Z) + l_3(X, Y, NZ) - \lambda l_3(X, Y, Z)
\]
\[
= [N, l_3]_{RN}(X, Y, Z).
\]  

Equations (54), (55), (56) and (57) complete the proof.  

For the case of a Courant algebroid, we have the following result.

**Corollary 5.13.** Let \((E, \circ, \rho, \langle \cdot, \cdot \rangle)\) be a Courant algebroid with the associated symmetric Lie-2 algebra structure \(\mu = l_1 + l_2 + l_3\), on the graded vector space \(V = C^\infty(M) \oplus \Gamma(E)\). Let \(N\) be a \((1, 1)\)-tensor on \(E\) such that

\[
\begin{align*}
N + N^* & = \lambda \text{Id}_{\Gamma(E)}, \\
(\Gamma(E), \circ^N) & \text{ is a Leibniz algebra},
\end{align*}
\]

with \(\lambda\) a Casimir function. Then, the Lie 2-algebra structure associated to the Courant algebroid \((E, \circ^N, \rho^N, \langle \cdot, \cdot \rangle)\) is \([N, l_1 + l_2 + l_3]\), with \(N\) given by (53).

**Proposition 5.14.** Let \((E, \circ, \rho, \langle \cdot, \cdot \rangle)\) be a Courant algebroid with the associated symmetric Lie 2-algebra structure \(\mu = l_1 + l_2 + l_3\), on the graded vector space \(V = C^\infty(M) \oplus \Gamma(E)\). Let \(N\) be a \((1, 1)\)-tensor on \(E\) whose Nijenhuis torsion with respect to the bracket \(\circ\) vanishes and satisfies the following conditions

\[
\begin{align*}
N + N^* & = \lambda \text{Id}_{\Gamma(E)}, \\
N^2 + (N^2)^* & = \gamma \text{Id}_{\Gamma(E)},
\end{align*}
\]
with $\lambda$ and $\gamma$ Casimir functions. Define $\mathcal{N}$ and $\mathcal{K}$ as

$$\mathcal{N}|_{\Gamma(E)} = N \text{ and } \mathcal{N}|_{C^\infty(M)} = \lambda \text{Id}_{C^\infty(M)},$$

$$\mathcal{K}|_{\Gamma(E)} = N^2 = \lambda N + \frac{\gamma - \lambda^2}{2} \text{Id}|_{\Gamma(E)} \text{ and } \mathcal{K}|_{C^\infty(M)} = \gamma \text{Id}_{C^\infty(M)}.$$ 

Then, $\mathcal{N}$ is a Nijenhuis vector valued 1-form with respect to $\mu$, with square $\mathcal{K}$.

**Proof:** Since the Nijenhuis torsion of $N$ vanishes, $(E, o^N)$ and $(E, o^{N^2})$ are Leibniz algebras [4], [2]. Applying Corollary 5.13 for the Courant algebroid $(E, o, \rho, \langle ., . \rangle)$, the $(1, 1)$-tensor $N$ and the vector valued 1-form $\mathcal{N}$, twice, we get

$$l_{1}^{N,N} + l_{2}^{N,N} + l_{3}^{N,N} = [\mathcal{N}, [\mathcal{N}, l_{1} + l_{2} + l_{3}]]_{RN},$$

(58)

where $l_{1}^{N,N} + l_{2}^{N,N} + l_{3}^{N,N}$ stands for the Lie 2-algebra structure associated to the Courant algebroid $(E, o^{N,N}, \rho^{N,N}, \langle ., . \rangle)$. Applying again Corollary 5.13 for the Courant algebroid $(E, o, \rho, \langle ., . \rangle)$, the $(1, 1)$-tensor $N^2$ and the vector valued 1-form $\mathcal{K}$, we get

$$l_{1}^{N^2} + l_{2}^{N^2} + l_{3}^{N^2} = [\mathcal{K}, l_{1} + l_{2} + l_{3}]_{RN},$$

(59)

where $l_{1}^{N^2} + l_{2}^{N^2} + l_{3}^{N^2}$ stands for the Lie 2-algebra structure associated to the Courant algebroid $(E, o^{N^2}, \rho^{N^2}, \langle ., . \rangle)$. On the other hand, since the Nijenhuis torsion of $N$ vanishes, the Courant algebroids $(E, o^{N,N}, \rho^{N,N}, \langle ., . \rangle)$ and $(E, o^{N^2}, \rho^{N^2}, \langle ., . \rangle)$ coincide. Therefore, (58) and (59) imply that

$$[\mathcal{N}, [\mathcal{N}, l_{1} + l_{2} + l_{3}]]_{RN} = [\mathcal{K}, l_{1} + l_{2} + l_{3}]_{RN}.$$ 

Finally, an easy computation shows that $[\mathcal{N}, \mathcal{K}]_{RN}$ vanishes both on functions and on sections of $E$.

Since the Lie 2-algebra structure entirely encodes the Courant algebroid structure, there was a hope that we could, given a Courant structure, find a Nijenhuis deformation by a Nijenhuis tensor which is the sum of a vector valued 1-form and a vector valued 2-form of the corresponding Lie 2-algebra structure, and prove, eventually, that the Lie 2-algebra structure obtained by this procedure comes from a Courant structure. But this fails, at least when the anchor is not identically zero, as it is shown in the next proposition. First, notice that every $C^\infty(M)$-linear vector valued form of degree 0 on $E_{-2} \oplus E_{-1}$, where $E_{-2} := C^\infty(M)$ and $E_{-1} := \Gamma(E)$, is the sum of a 2-form $\alpha$, a $(1, 1)$-tensor $N$ and an endomorphism of $C^\infty(M)$ of the form $f \mapsto \lambda f$ for
some smooth function $\lambda$. Hence, we denote a $C^\infty(M)$-linear vector valued form of degree zero on $E_{-2} \oplus E_{-1}$ as a sum, $\lambda + N + \alpha$.

**Theorem 5.15.** Let $(\circ, \rho, \langle ., . \rangle)$ be a Courant structure on a vector bundle $E \rightarrow M$ with the associated Lie 2-algebra structure $l_1 + l_2 + l_3$ on the graded vector space $V = E_{-2} \oplus E_{-1}$, where $E_{-2} := C^\infty(M)$ and $E_{-1} := \Gamma(E)$. Let $N = \lambda + N + \alpha$ be a $C^\infty(M)$-linear vector valued form of degree zero on $V$. Assume also that $\rho$ is not equal to zero on a dense subset of the base manifold. If $[N, l_1 + l_2 + l_3]_{RN}$ is the Lie 2-algebra associated to a Courant structure with the same scalar product $\langle ., . \rangle$, then

1. $\lambda$ is a Casimir,
2. $\alpha = 0$,
3. $N + N^* = \lambda \text{Id}_{\Gamma(E)}$.

In this case, the Courant structure that $[N, l_1 + l_2 + l_3]_{RN}$ is associated to, is $(\circ^N, \rho^N, \langle ., . \rangle)$.

**Proof:** Set $\mu = l_1 + l_2 + l_3$ and denote the $i$-form component of $[N, \mu]_{RN}$ by $[N, \mu]_{RN}^i$, $i = 1, 2$. Then, for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, we have

$$[N, \mu]_{RN}^1(f) = ([\lambda, l_1]_{RN} + [N, l_1]_{RN})(f) = l_1(\lambda f) - NL_1(f) = \lambda l_1(f) + fl_1(\lambda) - NL_1(f).$$

The first equation in (52) implies that, if $[N, \mu]_{RN}$ is a Lie 2-algebra associated to a Courant algebroid, then $[N, \mu]_{RN}^1$ has to be a derivation, and this happens if and only if $l_1(\lambda) = 0$. So, we get that $\lambda$ is a Casimir and

$$[N, \mu]_{RN}^1(f) = (\lambda \text{Id}_{\Gamma(E)} - N)l_1(f).$$

On the other hand,

$$[N, \mu]_{RN}^2(X, f) = ([\lambda, l_2]_{RN} + [N, l_2]_{RN} + [\alpha, l_1]_{RN})(X, f) = l_2(X, \lambda f) - NL_2(X, f) + l_2(NX, f) - \alpha(X, l_1(f)) = \frac{1}{2}\lambda \langle X, l_1(f) \rangle - \frac{1}{2}\lambda \langle X, l_1(\lambda) \rangle + \frac{1}{2}\langle NX, l_1(f) \rangle - \frac{1}{2}\langle NX, l_1(f) \rangle - \alpha(X, l_1(f)) = \frac{1}{2}\langle NX, l_1(f) \rangle - \alpha(X, l_1(f)).$$

and the same computations for $(f, X)$ instead of $(X, f)$ gives

$$[N, \mu]_{RN}^2(f, X) = \frac{1}{2}\langle NX, l_1(f) \rangle - \alpha(l_1(f), X).$$
Since $[\mathcal{N}, \mu]_{\mathcal{R}N}^2(X, f) = [\mathcal{N}, \mu]_{\mathcal{R}N}^2(f, X)$, from (61) and (62) we get $\alpha(X, l_1(f)) = 0$, for all $X \in \Gamma(E)$ and $f \in C^\infty(M)$; so,

$$[\mathcal{N}, \mu]_{\mathcal{R}N}^2(X, f) = \frac{1}{2} \langle NX, l_1(f) \rangle.$$  \hspace{1cm} (63)

For any $X, Y \in \Gamma(E)$, we have

$$[\mathcal{N}, \mu]_{\mathcal{R}N}^2(X, Y) = ([\lambda, l_2]_{\mathcal{R}N} + [N, l_2]_{\mathcal{R}N} + [\alpha, l_1]_{\mathcal{R}N})(X, Y) = l_2(NX, Y) + l_2(X, NY) - Nl_2(X, Y) + l_1 \alpha(X, Y).$$  \hspace{1cm} (64)

According to Lemma 5.9, if $[\mathcal{N}, \mu]_{\mathcal{R}N}$ is a Lie 2-algebra associated to a Courant structure, then we must have:

$$[\mathcal{N}, \mu]_{\mathcal{R}N}^2(X, fY) = f[N, \mu]_{\mathcal{R}N}^2(X, Y) + 2[N, \mu]_{\mathcal{R}N}^2(X, f)Y - \frac{1}{2} \langle X, Y \rangle [N, \mu]_{\mathcal{R}N}^1(f).$$  \hspace{1cm} (65)

Using (60), (63) and (64), we get

$$[\mathcal{N}, \mu]_{\mathcal{R}N}^2(X, fY) = l_2(NX, fY) + l_2(X, NfY) - Nl_2(X, fY) + l_1 \alpha(X, fY)$$

$$= fl_2(NX, Y) + 2l_2(NX, fY) - \frac{1}{2} \langle NX, Y \rangle l_1(f)$$

$$+ fl_2(X, NY) + 2l_2(X, f)NY - \frac{1}{2} \langle X, NY \rangle l_1(f)$$

$$- fNl_2(X, NY) - 2l_2(X, f)NY + \frac{1}{2} \langle X, Y \rangle Nl_1(f)$$

$$+ fl_1 \alpha(X, Y) + \alpha(X, Y)l_1(f)$$

$$= f(l_2(NX, Y) + l_2(X, NY) - Nl_2(X, Y) + l_1 \alpha(X, Y)) + 2l_2(NX, f)Y$$

$$- \frac{1}{2} \langle X, (N + N^*)Y \rangle l_1(f) + \frac{1}{2} \langle X, Y \rangle Nl_1(f) + \alpha(X, Y)l_1(f)$$  \hspace{1cm} (66)

and

$$f[N, \mu]_{\mathcal{R}N}^2(X, Y) + 2[N, \mu]_{\mathcal{R}N}^2(X, f)Y - \frac{1}{2} \langle X, Y \rangle [N, \mu]_{\mathcal{R}N}^1(f)$$  \hspace{1cm} (67)

$$= f(l_2(NX, Y) + l_2(X, NY) - Nl_2(X, NY) + l_1 \alpha(X, Y)) + 2l_2(NX, f)Y$$

$$- \frac{1}{2} \langle X, Y \rangle (\lambda \operatorname{Id}_{\Gamma(E)} - N)l_1(f).$$

Now, Equations (65), (66) and (67) show that

$$\frac{1}{2} \langle X, (N + N^* - \lambda \operatorname{Id}_{\Gamma(E)})Y \rangle l_1(f) = \alpha(X, Y)l_1(f),$$
for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$. Since $\alpha$ is skew-symmetric, $\langle \cdot , (N+N^*-\lambda \text{Id}) \cdot \rangle$ is symmetric on $\Gamma(E) \times \Gamma(E)$ and the anchor is not zero everywhere, which implies that $l_1(f)$ is not always zero, we have $\alpha = 0$ and $N + N^* - \lambda \text{Id}_{\Gamma(E)} = 0$.

\textbf{Corollary 5.16.} Let $(E, \circ, \rho, \langle \cdot , \cdot \rangle)$ be a Courant algebroid with anchor $\rho$ being different from zero on a dense subset of $E$ and let $\mu$ be the associated Lie 2-algebra structure on the graded vector space $C^\infty(M) \oplus \Gamma(E)$. Then, there is a one to one correspondence between:

(i) quadruples $(N, K, \lambda, \gamma)$ with $N, K$ being $(1,1)$-tensors on $E$ and $\lambda, \gamma$ being Casimir functions satisfying the following conditions:

\[
\begin{align*}
\circ^{N,N} &= \circ^K, \\
NK - KN &= 0, \\
N + N^* &= \lambda \text{Id}_{\Gamma(E)}, \\
K + K^* &= \gamma \text{Id}_{\Gamma(E)}, \\
\langle \Gamma(E), \circ^N \rangle \text{ and } \langle \Gamma(E), \circ^K \rangle &\text{ are Leibniz algebras.}
\end{align*}
\]

(ii) Nijenhuis vector valued forms $N$ with respect to $\mu$, with square $K$, such that the deformed brackets $[N, \mu]_{RN}$ and $[K, \mu]_{RN}$ are Lie 2-algebras associated to Courant structures with the same scalar product.

\textbf{Proof:} Given a quadruple $(N, K, \lambda, \gamma)$ satisfying conditions in item (i), we define vector valued 1-forms $N$ and $K$ on the graded vector space $C^\infty(M) \oplus \Gamma(E)$ as $N(f) = \lambda f$, $K(f) = \gamma f$, $N(X) = NX$ and $K(X) = KX$, for all $X \in \Gamma(E)$ and $f \in C^\infty(M)$. We prove that $N$ is a Nijenhuis vector valued form with respect to $\mu$, with square $K$. First, notice that using Corollary 5.3, the assumption $\circ^{N,N} = \circ^K$ implies that $(E, \circ^{N,N}, \rho^{N,N}, \langle \cdot , \cdot \rangle)$ and $(E, \circ^K, \rho^K, \langle \cdot , \cdot \rangle)$ are the same pre-Courant algebroid, hence, they have the same associated pre-Lie 2-algebras. On the other hand, using Lemma 5.12, the pre-Lie 2-algebra associated to the pre-Courant algebroid $(E, \circ^{N,N}, \rho^{N,N}, \langle \cdot , \cdot \rangle)$ is $[N, N, \mu]_{RN}$ and the pre-Lie 2-algebra associated to the pre-Courant algebroid $(E, \circ^K, \rho^K, \langle \cdot , \cdot \rangle)$ is $[K, \mu]_{RN}$. Hence,

\[
[N, [N, \mu]_{RN}]_{RN} = [K, \mu]_{RN}, \quad (68)
\]

Also, using the assumption $NK - KN = 0$, we get

\[
[N, K]_{RN} = 0. \quad (69)
\]
Equations (68) and (69) show that $\mathcal{N}$ is a Nijenhuis vector valued 1-form with respect to $\mu$, with square $K$. By Corollary 5.13, $[\mathcal{N}, \mu]_{RN}$ is a Lie 2-algebra associated to the Courant algebroid $(E, \circ^N, \rho, \langle.,.\rangle)$ and $[\mathcal{K}, \mu]_{RN}$ is a Lie 2-algebra associated to the Courant algebroid $(E, \circ^K, \rho, \langle.,.\rangle)$.

Conversely, assume that $\mathcal{N}$ is a Nijenhuis vector valued form with respect to $\mu$, with square $K$, such that $[\mathcal{N}, \mu]_{RN}$ and $[\mathcal{K}, \mu]_{RN}$ are Lie 2-algebras associated to Courant algebroids. Then, by Theorem 5.15, $\mathcal{N}$ is of the form $\lambda + N$ with $N + N^* = \lambda \text{Id}_{\Gamma(E)}$ and $\mathcal{K}$ is of the form $\gamma + K$, with $K + K^* = \gamma \text{Id}_{\Gamma(E)}$. Moreover, the Courant algebroid which is associated to the Lie 2-algebra $[\mathcal{N}, \mu]_{RN}$ (respectively, $[\mathcal{K}, \mu]_{RN}$) is $(E, \circ^N, \rho, \langle.,.\rangle)$ (respectively, $(E, \circ^K, \rho, \langle.,.\rangle)$). From this, we get that $(\Gamma(E), \circ^N)$ and $(\Gamma(E), \circ^K)$ are Leibniz algebras. Since $\mathcal{N}$ is a Nijenhuis vector valued form with respect to $\mu$, with square $\mathcal{K}$, we have

$$[\mathcal{N}, [\mathcal{N}, \mu]_{RN}]_{RN} = [\mathcal{K}, \mu]_{RN}$$

(70)

and

$$[\mathcal{N}, \mathcal{K}]_{RN} = 0.$$  

(71)

Applying both sides of Equation (70) on a pair of sections $X, Y \in \Gamma(E)$ we get $X \circ^N Y = X \circ^K Y$, which implies $\circ^N = \circ^K$. Lastly, Equation (71) implies $KN - NK = 0$.

Using Lemma 5.9 and Remark 5.10, and also taking into account the fact that the operator $\mathcal{D}$ associated to a pre-Courant algebroid $(E, \circ, \rho, \langle.,.\rangle)$, given by (48), is a derivation, we may restate Theorem 5.15.

**Theorem 5.17.** Let $(\circ, \rho, \langle.,.\rangle)$ be a Courant structure on a vector bundle $E \rightarrow M$, with the associated symmetric Lie 2-algebra structure $l_1 + l_2 + l_3$ on the graded vector space $V = E_{-2} \oplus E_{-1}$, where $E_{-2} := C^\infty(M)$ and $E_{-1} := \Gamma(E)$. Let $\mathcal{N} = \lambda + N + \alpha$ be a $C^\infty(M)$-linear vector valued form of degree zero on $V$. Assume also that $\rho$ is not equal to zero on a dense subset of the base manifold. If $[\mathcal{N}, l_1 + l_2 + l_3]_{RN} = l_1' + l_2' + l_3'$, where the vector valued forms $l_1', l_2', l_3'$ are obtained from a pre-Courant algebroid, with the same scalar product, by the construction given in (52), then

1. $\lambda$ is a Casimir,
2. $\alpha = 0$,
3. $N + N^* = \lambda \text{Id}_{\Gamma(E)}$.

In this case, the Courant structure that $[\mathcal{N}, l_1 + l_2 + l_3]_{RN}$ is associated to, is $(\circ^N, \rho^N, \langle.,.\rangle)$. 
And this leads to the next result:

**Corollary 5.18.** Let \((E, \circ, \rho, \langle.,.\rangle)\) be a Courant algebroid with anchor \(\rho\) being different from zero on a dense subset of \(E\), with the associated Lie 2-algebra structure \(\mu = l_1 + l_2 + l_3\) on the graded vector space \(\mathcal{C}^\infty(M) \oplus \Gamma(E)\). Then, there is a one to one correspondence between:

(i) quadruples \((N, K, \lambda, \gamma)\) with \(N, K\) being \((1,1)\)-tensors and \(\lambda, \gamma\) being Casimir functions satisfying the following conditions:

\[
\begin{align*}
&\circ^{N,N} = \circ^K, \\
&NK - KN = 0, \\
&N + N^* = \lambda \text{Id}_{\Gamma(E)}, \\
&K + K^* = \gamma \text{Id}_{\Gamma(E)}.
\end{align*}
\]

(ii) Nijenhuis vector valued forms \(N\) with respect to \(\mu\), with square \(K\), such that the deformed bracket is of the form \([N, \mu]_{RN} = l'_1 + l'_2 + l'_3\) and \(l'_1, l'_2, l'_3\) are constructed by the procedure in (52) obtained from a pre-Courant algebroid, with the same scalar product.

**Proof:** Let \(N\) be a Nijenhuis vector valued form with respect to the Lie 2-algebra structure \(\mu = l_1 + l_2 + l_3\), with square \(K\), and assume that \([N, \mu]_{RN}\) is obtained from a pre-Courant algebroid. Let \(N|_{\mathcal{C}^\infty(M)} = N\), \(N|_{\mathcal{C}^\infty(M)} = \lambda \text{Id}_{\Gamma(E)}\), \(K|_{\Gamma(E)} = K\) and \(K|_{\mathcal{C}^\infty(M)} = \gamma \text{Id}_{\mathcal{C}^\infty(M)}\). By Theorem 5.17, \(N + N^* = \lambda \text{Id}_{\Gamma(E)}\) and \((E, \circ^N, \rho^N, \langle.,.\rangle)\) is a pre-Courant algebroid (it is, in fact, the pre-Courant algebroid which \([N, \mu]_{RN}\) is obtained from). Hence, by Lemma 5.6, \((E, \circ^{N,N}, \rho^{N,N}, \langle.,.\rangle)\) is a pre-Courant algebroid. Now, Lemma 5.12 implies that \([K, \mu]_{RN} = [N, [N, \mu]_{RN}]_{RN}\) is obtained from the pre-Courant algebroid \((E, \circ^{N,N}, \rho^{N,N}, \langle.,.\rangle)\), by the construction given in (52). Therefore, by Theorem 5.17, \(K + K^* = \gamma \text{Id}_{\Gamma(E)}\). The assumption \([N, K]_{RN} = 0\) implies that \(NK - KN = 0\), while \([N, [N, \mu]_{RN}]_{RN} = [K, \mu]_{RN}\) implies that \(\circ^{N,N} = \circ^K\).

Conversely, assume that we are given a quadruple \((N, K, \lambda, \gamma)\) satisfying the properties in (72). By Lemma 5.6, \((E, \circ^N, \rho^N, \langle.,.\rangle)\) is a pre-Courant and by Lemma 5.12, the pre-Lie 2-algebra structure associated to the pre-Courant algebroid \((E, \circ^N, \rho^N, \langle.,.\rangle)\) is \([N, \mu]_{RN}\). Similar arguments prove that the pre-Lie 2-algebra structure associated to the pre-Courant algebroid \((E, \circ^{N,N}, \rho^{N,N}, \langle.,.\rangle)\) is \([N, [N, \mu]_{RN}]_{RN}\) and the pre-Lie 2-algebra structure
associated to the pre-Courant algebroid \((E, \circ^K, \rho^K, \langle, \rangle)\) is \([\mathcal{K}, \mu]_{\mathbb{R}^N}\). Now, the assumption \(\circ^{N,N} = \circ^K\) and Lemma 5.3 imply that \((E, \circ^{N,N}, \rho^{N,N}, \langle, \rangle)\) and \((E, \circ^K, \rho^K, \langle, \rangle)\) are the same pre-Courant algebroid; therefore, we have \([\mathcal{N}, [\mathcal{N}, \mu]]_{\mathbb{R}^N} = [\mathcal{K}, \mu]_{\mathbb{R}^N}\). It follows from the assumption \(NK - KN = 0\) that \([\mathcal{N}, \mathcal{K}] = 0\). Hence, \(\mathcal{N}\) is a Nijenhuis vector valued form with respect to the Lie 2-algebra structure \(\mu\), with square \(\mathcal{K}\).

6. Multiplicative \(L_\infty\)-structures

Adapting the notion of \(P_\infty\)-structure on a graded vector space [5] to the symmetric graded case, we define, in this section, multiplicative \(L_\infty\)-structures. We classify all multiplicative \(L_\infty\)-structures on \(\Gamma(\wedge A)[2]\), for \(A \to M\) an arbitrary vector bundle over a manifold \(M\). When \(A \to M\) is equipped with a Lie algebroid structure, given a \((1, 1)\)-tensor \(N\) on \(A\), we define the extension of \(N\) by derivation, which is a symmetric vector valued 1-form on \(\Gamma(\wedge A)[2]\), of degree zero. For a \(k\)-form on the Lie algebroid, we also define its extension by derivation, yielding a symmetric vector valued form \(k\)-form of degree \(k - 2\). These multi-derivations will be used in the next section to construct examples of Nijenhuis forms.

There is an important graded Lie subalgebra of \((\tilde{S}(E^*) \otimes E, [\cdot, \cdot]_{\mathbb{R}^N})\), when \(E\) itself is equipped with a graded commutative associative algebra structure on \(E[2]\), denoted by \(\wedge\), that is, a bilinear operation such that for all \(X \in E_i, Y \in E_j, Z \in E_k\)

- \(X \wedge Y \in E_{i+j+2}\),
- \((X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)\),
- \(X \wedge Y = (-1)^{|X||Y|}Y \wedge X\),

where \(|X| = i + 2\) and \(|Y| = j + 2\).

**Definition 6.1.** Let \(E\) be a graded vector space equipped with an associative graded commutative algebra structure, that is a graded symmetric bilinear map \(\wedge\) of degree zero which is associative. An element \(D \in S^d(E^*) \otimes E\) is called a multi-derivation vector valued \(d\)-form, if

\[
D(X_1, \ldots, X_{i-1}, Y \wedge Z, X_{i+1}, \ldots, X_d) = (-1)^{|Z|(|X_{i+1}| + \cdots |X_d|)}D(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_d) \wedge Z \\
+ (-1)^{|Y|(|X_1| + \cdots |X_{i-1}| + d)}Y \wedge D(X_1, \ldots, X_{i-1}, Z, X_{i+1}, \ldots, X_d),
\]  

\[(73)\]
for all $X_1, \cdots, X_d, Y, Z \in E$, where $\bar{D}$ is the degree of $D$ as a graded map.

Remark 6.2. The graded commutativity of the product $\wedge$ implies that Equation (73) is equivalent to

$$D(X_1, \cdots, X_{d-1}, Y \wedge Z) = D(X_1, \cdots, X_{d-1}, Y) \wedge Z + (-1)^{|Y||Z|} D(X_1, \cdots, X_{d-1}, Z) \wedge Y.$$ 

We denote the space of all multi-derivation vector valued forms by $\text{MultiDer}(E)$. Elements of $S_1(E^*) \otimes E$ are simply called derivations. By definition, $E \subset \text{MultiDer}(E)$ and we have the following:

**Proposition 6.3.** $\text{MultiDer}(E)$ is a graded Lie subalgebra of $(\tilde{S}(E^*) \otimes E, [, ,]_{RN})$.

We shall use the following lemmas in the proof of Proposition 6.3.

**Lemma 6.4.** Let $D_1$ and $D_2$ be two derivations. Then, $[D_1, D_2]_{RN}$ is also a derivation.

**Proof:** We have

$$[D_1, D_2]_{RN} = D_2 \circ D_1 - (-1)^{D_1 D_2} D_1 \circ D_2 = -(-1)^{D_1 D_2} [D_1, D_2],$$

where $[,]$ is the graded commutator on the space of derivations of the graded associative commutative algebra $(E, \wedge)$. Since $[D_1, D_2]$ is a derivation, so is $[D_1, D_2]_{RN}$. □

**Lemma 6.5.** If $D \in S^d(E^*) \otimes E$ is a multi-derivation vector valued $d$-form, then for all $X \in E$, $[X, D]_{RN}$ is a multi-derivation vector valued $(d - 1)$-form.

**Proof:** It is a direct consequence of

$$[X, D]_{RN}(X_1, \cdots, X_{d-2}, Y \wedge Z) = D(X, X_1, \cdots, X_{d-2}, Y \wedge Z),$$

which holds for all elements $Y, Z, X_1, \cdots, X_{d-2} \in E$. □

**Proof:** (of Proposition 6.3) Let $D, D'$ be two multi-derivation vector valued $d$-form and $d'$-form, respectively. We show that $[D, D']_{RN}$ is a multi-derivation vector valued $(d + d' - 1)$-form, using induction on the number $n = d + d' - 1$. Lemmas 6.4 and 6.5 prove the case $n = 1$. Assume, by induction, that $[D, D']_{RN}$ is a multi-derivation vector valued $(d + d' - 1)$-form and let $D_1$ and
Let \( D_1 \) and \( D_2 \) be two multi-derivation vector valued \( d_1 \)- and \( d_2 \)-forms respectively, such that \( d_1 + d_2 - 1 = n + 1 \). From (3) we have
\[
[D_1, D_2]_{\mathcal{R}N}(X_1, \cdots, X_{d_1+d_2-2}, Y \wedge Z) = [Y \wedge Z, [X_{d_1+d_2-2}, \cdots, [X_1, [D_1, D_2]_{\mathcal{R}N} \cdots]_{\mathcal{R}N}]_{\mathcal{R}N},
\]
or, using the graded Jacobi identity of \([.,.]_{\mathcal{R}N}\),
\[
[D_1, D_2]_{\mathcal{R}N}(X_1, \cdots, X_{d_1+d_2-2}, Y \wedge Z) = [Y \wedge Z, [X_{d_1+d_2-2}, \cdots, [[X_1, D_1]_{\mathcal{R}N}, D_2]_{\mathcal{R}N} \cdots]_{\mathcal{R}N}]_{\mathcal{R}N}
\]
\[+ (-1)^{D_1 X_1} [Y \wedge Z, [X_{d_1+d_2-2}, \cdots, [D_1, [X_1, D_2]_{\mathcal{R}N} \cdots]_{\mathcal{R}N}]_{\mathcal{R}N},
\]
for all \( X_1, \cdots, X_{d_1+d_2-2}, Y, Z \in E \). By Lemma 6.5, \([X_1, D_1]_{\mathcal{R}N}\) and \([X_1, D_2]_{\mathcal{R}N}\) are multi-derivation vector valued \((d_1-1)\)- and \((d_2-1)\)-forms respectively, and hence using the assumption of induction, \([[[X_1, D_1]_{\mathcal{R}N}, D_2]_{\mathcal{R}N} \cdots [D_1, [X_1, D_2]_{\mathcal{R}N}]_{\mathcal{R}N}\) are multi-derivation vector valued \(n\)-forms. Therefore,
\[
[D_1, D_2]_{\mathcal{R}N}(X_1, \cdots, X_{d_1+d_2-2}, Y \wedge Z) = [[[X_1, D_1]_{\mathcal{R}N}, D_2]_{\mathcal{R}N}(X_2, \cdots, X_{d_1+d_2-2}, Y \wedge Z)
\]
\[+ (-1)^{D_1 X_1} [D_1, [X_1, D_2]_{\mathcal{R}N}]_{\mathcal{R}N}(X_2, \cdots, X_{d_1+d_2-2}, Y \wedge Z)
\]
\[= [[[X_1, D_1]_{\mathcal{R}N}, D_2]_{\mathcal{R}N}(X_2, \cdots, X_{d_1+d_2-2}, Y) \wedge Z
\]
\[+ (-1)^{Y} [Y]_{\mathcal{R}N}(X_2, \cdots, X_{d_1+d_2-2}, Z) \wedge Y
\]
\[+ (-1)^{D_1 X_1} [D_1, [X_1, D_2]_{\mathcal{R}N}]_{\mathcal{R}N}(X_2, \cdots, X_{d_1+d_2-2}, Y) \wedge Z
\]
\[+ (-1)^{D_1 X_1} [D_1, [X_1, D_2]_{\mathcal{R}N}]_{\mathcal{R}N}(X_2, \cdots, X_{d_1+d_2-2}, Z) \wedge Y
\]
\[= [D_1, D_2]_{\mathcal{R}N}(X_1, \cdots, X_{d_1+d_2-2}, Y) \wedge Z
\]
\[+ (-1)^{Y} [Y]_{\mathcal{R}N}(X_1, \cdots, X_{d_1+d_2-2}, Z) \wedge Y,
\]
which completes the induction and also the proof (see Remark 6.2).

Let us now define multiplicative \(L_\infty\)-algebra.

**Definition 6.6.** An \(L_\infty\)-structure \( \mu = \sum_{i=1}^{\infty} l_i \) on a graded vector space \( E \) equipped with a graded commutative product \( \wedge : E_i \times E_j \to E_{i+j} \) is called **multiplicative** if all the multi-linear brackets \( l_i \) are multi-derivations.

Next, we discuss the relation between multiplicative \(L_\infty\)-structures and Lie algebroids.

A **pre-Lie algebroid** structure on a vector bundle \( A \to M \) over a manifold \( M \) is a pair \((\rho, [.,.]\)) with \( \rho : A \to TM \) a vector bundle morphism over
the identity of $M$, called anchor map, and $[,]$ a skew-symmetric bilinear endomorphism of $\Gamma(A)$ subject to the so-called Leibniz identity:

$$[X, fY] = f[X, Y] + (\rho(X)f)Y,$$

for all $X, Y \in \Gamma(A)$ and all $f \in C^\infty(M)$. When, moreover, $[,]$ is a Lie algebra bracket, the pair $([,] , \rho)$ is called a Lie algebroid structure on $A \to M$. We denote by $[,]_{SN}$ the Schouten-Nijenhuis bracket on the the space of multivectors of the (pre-)Lie algebroid $A$ and by $d^A$ the (pre-)differential of $A$. We recall that

$$[X, f]_{SN} = \rho(X)f, [P, Q]_{SN} = -(-1)^{pq}[Q, P]_{SN},$$

$$[P, Q \wedge R]_{SN} = [P, Q]_{SN} \wedge R + (-1)^{qr}[P, R]_{SN} \wedge Q,$$

for all $X \in \Gamma(A)$, $P \in \Gamma(\wedge^{p+1}A)$, $Q \in \Gamma(\wedge^{q+1}A)$, $R \in \Gamma(\wedge^{r+1}A)$ and $f \in C^\infty(M)$ and that

$$d^A\omega(X_0, \ldots, X_k) := \sum_{i=0}^k (-1)^i\rho(X_i)\omega(\widehat{X}_i) + \sum_{0 \leq i < j \leq k} (-1)^{i+j}\omega([X_i, X_j], \widehat{X}_{i,j}),$$

for all $X_0, \ldots, X_k \in \Gamma(A), \omega \in \Gamma(\wedge^kA^*)$, where $\widehat{X}_i$ and $\widehat{X}_{i,j}$ stand for $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k$ and $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k$ respectively. Notice that in the above expression, we have implicitly identified elements of $\Gamma(\wedge^kA^*)$ with skew-symmetric $k$-linear maps from $\Gamma(A) \times \ldots \times \Gamma(A)$ to $C^\infty(M)$.

Let $([,] , \rho)$ be a pre-Lie algebroid structure on a vector bundle $A \to M$. Set $E_i := \Gamma(\wedge^{i+1}A)$ and $E = \bigoplus_{i \geq -1} E_i$, with $E_{-1} = \Gamma(\wedge^0A) = C^\infty(M)$. The Schouten-Nijenhuis bracket is a graded skew-symmetric bracket of degree zero on $E = \bigoplus_{i \geq -1} E_i$ and it is known that a pre-Lie algebroid structure $(\rho, [,])$ is a Lie algebroid structure on the vector bundle $A \to M$, if and only if $[,]_{SN}$ is a graded Lie algebra bracket on $E = \Gamma(\wedge A)[1]$. It is also well known that the pre-differential $d^A$ is a derivation of $\Gamma(\wedge A^*)$ and that $d^A$ squares to zero if and only if $(A, [,] , \rho)$ is Lie algebroid.

The discussion above leads to the conclusion that there are two ways to see Lie algebroids as $L_\infty$-structures: the first one will make it an $L_\infty$-structure on $\Gamma(\wedge A)$, and the second one will make it an $L_\infty$-structure on $\Gamma(\wedge A^*)[13]$. More precisely:
Proposition 6.7. Let \( A \to M \) be a vector bundle and \( A^* \to M \) its dual. There is a one to one correspondence between:

(i) pre-Lie algebroid structures \((\rho, [\cdot, \cdot])\) on \( A \to M \),
(ii) binary multi-derivations of \( \Gamma(\wedge A)[2] \) of degree 1,
(iii) unary multi-derivations of \( \Gamma(\wedge A^*)[2] \) of degree 1.

The one to one correspondence above restricts to a one to one correspondence between:

(i') Lie algebroid structures \((\rho, [\cdot, \cdot])\) on \( A \to M \),
(ii') multiplicative \( L_\infty \)-structures on \( \Gamma(\wedge A)[2] \) given by a binary bracket,
(iii') multiplicative \( L_\infty \)-structures on \( \Gamma(\wedge A^*)[2] \) given by a unary bracket.

Given a \((1, 1)\)-tensor \( N \) on a Lie algebroid \((A, [\cdot, \cdot], \rho)\), we define a linear map \( \overline{N} \) on the graded vector space \( \Gamma(\wedge A)[2] \), by setting

\[
\overline{N}(f) := 0,
\]

for all \( f \in C^\infty(M) \), and

\[
\overline{N}(P) := \sum_{i=1}^{p} P_1 \wedge \cdots \wedge P_{i-1} \wedge N(P_i) \wedge P_{i+1} \wedge \cdots \wedge P_p,
\]

for all monomial multi-sections \( P = P_1 \wedge \cdots \wedge P_p \in \Gamma(\wedge A)[2] \). The map \( \overline{N} \) is called the extension of \( N \) by derivation on the graded vector space \( \Gamma(\wedge A)[2] \). It is a multi-derivation on the graded vector space \( \Gamma(\wedge A)[2] \), hence a symmetric vector valued 1-form on \( \Gamma(\wedge A)[2] \), and has degree zero.

For a \( k \)-form on a Lie algebroid, we also consider its extension by derivation. More precisely, if \( \kappa \in \Gamma(\wedge^k A^*) \), the extension of \( \kappa \) by derivation is a \( k \)-linear map, denoted by \( \overline{\kappa} \), given by

\[
\overline{\kappa}(P_1, \cdots, P_k) := \sum_{i_1, \cdots, i_k=1}^{p_1, \cdots, p_k} (-1)^{\bullet} \kappa(P_{1,i_1}, \cdots, P_{k,i_k}) \overline{P_{1,i_1}} \wedge \cdots \wedge \overline{P_{k,i_k}},
\]

for all homogeneous multi-sections \( P_i = P_{i,1} \wedge \cdots \wedge P_{i,p_i} \in \Gamma(\wedge^{p_i} A) \), with \( i = 1, \cdots, k \), where \( 1 \leq i_j \leq p_j \) for all \( 1 \leq j \leq k \),

\[
\overline{P_{j,i_j}} = P_{j,1} \wedge \cdots \wedge P_{j,i_j-1} \wedge P_{j,i_j+1} \wedge \cdots \wedge P_{j,p_j} \in \Gamma(\wedge^{p_j-1} A)
\]

and

\[
\bullet = 2p_1 + 3p_2 + \cdots + (k+1)p_k + i_1 + \cdots + i_k + 1.
\]
It follows from its definition that \( \kappa \) is a multi-derivation on the graded vector space \( \Gamma(\wedge A)[2] \) and that it is a symmetric vector valued \( k \)-form of degree \( k - 2 \) on \( \Gamma(\wedge A)[2] \).

**Lemma 6.8.** Let \( (A, [\cdot, \cdot], \rho) \) be a Lie algebroid, \( \alpha \in \Gamma(\wedge^k A^*) \) be a \( k \)-form and \( \beta \in \Gamma(\wedge^l A^*) \) be an \( l \)-form. Then,

\[
\left[ \alpha, \beta \right]_{RN} = 0.
\]

**Proof:** The fact that \( \alpha \) (respectively \( \beta \)) is a vector valued \( k \)-form (respectively \( l \)-form) of degree \( k - 2 \) (respectively \( l - 2 \)), imply that \( \left[ \alpha, \beta \right]_{RN} \) is a vector valued \( (k+l-1) \)-form of degree \( k+l-4 \) on the graded vector space \( \Gamma(\wedge A) = \bigoplus_{i \geq 0} \Gamma(\wedge^i A) \). Therefore, for all \( l, k \geq 0 \) the restriction of \( \left[ \alpha, \beta \right]_{RN} \) to the space of sections is zero and hence we have \( \left[ \alpha, \beta \right]_{RN} = 0 \), because \( \left[ \alpha, \beta \right]_{RN} \) is a multi-derivation and it is uniquely determined on the space of sections.

According to Proposition 6.7, for a given Lie algebroid \( (A, [\cdot, \cdot], \rho) \), the bracket \( l^{[\cdot, \cdot]}_2 \) given by

\[
l^{[\cdot, \cdot]}_2(P, Q) = (-1)^{p-1}[P, Q]_{SN}, \quad P \in \Gamma(\wedge^p A), Q \in \Gamma(\wedge^q A), \quad (75)
\]

defines a multiplicative graded Lie algebra structure on \( \Gamma(\wedge A)[2] \). When we deform the bracket \( [\cdot, \cdot] \) by \( N \) as

\[
\left[ X, Y \right]_N = \left[ NX, Y \right] + \left[ X, NY \right] - N \left[ X, Y \right],
\]

for all \( X, Y \in \Gamma(A) \), of course we may consider \( l^{[\cdot, \cdot]}_2 \) using Equation (75) and we may take the Schouten-Nijenhuis bracket \( [\cdot, \cdot]^N_{SN} \) corresponding to the deformed bracket \( [\cdot, \cdot]_N \). Note that the bracket \( l^{[\cdot, \cdot]}_2 \) is not necessarily a multiplicative graded Lie algebra structure. On the other hand, since \( l^{[\cdot, \cdot]}_2 \) is a symmetric vector valued 2-form of degree 1 and \( N \) is a (symmetric) vector valued 1-form of degree zero, we can consider the deformation of \( l^{[\cdot, \cdot]}_2 \) by \( N \).

The following lemma shows the relation between \( \left[ N, l^{[\cdot, \cdot]}_2 \right]_{RN} \) and \( l^{[\cdot, \cdot]}_2 \).

**Lemma 6.9.** Let \( N \) be a \((1,1)\)-tensor on a Lie algebroid \((A, [\cdot, \cdot], \rho)\). Then, we have

\[
\left[ N, l^{[\cdot, \cdot]}_2 \right]_{RN} = l^{[\cdot, \cdot]}_2 N.
\]
Proof: The proof follows directly from the fact that the Schouten-Nijenhuis bracket on $\Gamma(\wedge A)$ associated to the bracket $[,]_N$ is given by

$$[P, Q]_N^SN = [NP, Q]_SN + [P, NQ]_SN - N[P, Q]_SN,$$

for all $P, Q \in \Gamma(\wedge A)$, see [13].

We will need the following lemma for our next purpose.

Lemma 6.10. Let $(A, [,], \rho)$ be a Lie algebroid, with differential $dA$ and associated multiplicative graded Lie algebra structure $l_2^{[\cdot]}$ on $\Gamma(\wedge A)[2]$. Then,

$$\left[\alpha, l_2^{[\cdot]}\right]_{RN} = dA\alpha,$$

for all $\alpha \in \Gamma(\wedge^n A^*)$.

Proof: We shall prove the statement for $n = 2$. A similar proof can be done for any $n \geq 1$. First note that $\left[\alpha, l_2^{[\cdot]}\right]_{RN}$ is a vector valued 3-form of degree 1 on the graded vector space $\Gamma(\wedge A)[2]$. This implies that the restriction of $\left[\alpha, l_2^{[\cdot]}\right]_{RN}$ to $\Gamma(A)$ is of the form:

$$\left[\alpha, l_2^{[\cdot]}\right]_{RN} |_{\Gamma(A) \times \Gamma(A) \times \Gamma(A)} : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \to C^\infty(M)$$

and, by degree reasons, any other restriction of $\left[\alpha, l_2^{[\cdot]}\right]_{RN}$ is zero. On the other hand, by Proposition 6.3, $\left[\alpha, l_2^{[\cdot]}\right]_{RN}$ is a multi-derivation, so that its restriction to the sections $\Gamma(A)$ is a $C^\infty(M)$-linear map. Therefore, $\left[\alpha, l_2^{[\cdot]}\right]_{RN} \in \Gamma(\wedge^3 A^*)$. Next, we show that

$$\left[\alpha, l_2^{[\cdot]}\right]_{RN} |_{\Gamma(A) \times \Gamma(A) \times \Gamma(A)} = dA\alpha$$

and this together with the fact that $\left[\alpha, l_2^{[\cdot]}\right]_{RN}$ is a multi-derivation will imply that $\left[\alpha, l_2^{[\cdot]}\right]_{RN} = dA\alpha$, by the uniqueness of extension by derivation of $dA\alpha$ to the graded vector space $\Gamma(\wedge A)[2]$. A direct computation shows that

$$\left[\alpha, l_2^{[\cdot]}\right]_{RN} (X, Y, Z) = [\alpha(X, Y), Z]_SN - \alpha([X, Y]_SN, Z) + c.p.,$$
for all \( X, Y, Z \in \Gamma(A) \). Hence, Equation (74) together with the definition of \( d^A \) imply that

\[
\left[ \alpha, l^2_{[\cdot,\cdot]} \right]_{RN}(X, Y, Z) = \rho(Z)\alpha(X, Y) - \alpha([X, Y], Z) + c.p. = d\alpha^A(X, Y, Z).
\]

This completes the proof.

7. Nijenhuis forms on multiplicative \( L_\infty \)-structures associated to Lie algebroids

In this section we consider several structures defined by tensors and pairs of tensors on a Lie algebroid and, by using their extensions by derivations, we construct Nijenhuis forms (weak Nijenhuis and co-boundary Nijenhuis, in some cases) with respect to the graded Lie algebra associated to the Lie algebroid structure.

Let \((A, [\cdot,\cdot], \rho)\) be a Lie algebroid and \( N : A \to A \) an endomorphism. Then, as in the case of Lie algebras, the Nijenhuis torsion of \( N \) with respect to the Lie bracket \([\cdot,\cdot]\), denoted by \( T_{[\cdot,\cdot]}N \), is defined by Equation (8) and again a direct computation shows that

\[
T_{[\cdot,\cdot]}N(X, Y) = \frac{1}{2} \left([X, Y]_{N,N} - [X, Y]_{N^2}\right),
\]

for all \( X, Y \in \Gamma(A) \). A \((1,1)\)-tensor \( N \) on a Lie algebroid \((A, [\cdot,\cdot], \rho)\) is said to be Nijenhuis if the Nijenhuis torsion of \( N \), with respect to the Lie algebroid bracket \([\cdot,\cdot]\), vanishes. As a consequence of Lemma 6.9, we have the following proposition:

**Proposition 7.1.** For every Nijenhuis tensor \( N \) on a Lie algebroid \((A, [\cdot,\cdot], \rho)\), the extension \( \overline{N} \) of \( N \) by derivation is a Nijenhuis vector valued 1-form with respect to the multiplicative graded Lie algebra structure \( l^2_{[\cdot,\cdot]} \) on the graded vector space \( \Gamma(\wedge A)[2] \), with square \( (N^2) \).

**Proof:** Applying Lemma 6.9 twice, for the tensor \( N \) and the bracket \( l^2_{[\cdot,\cdot]} \), we get

\[
\left[ \overline{N}, \left[ \overline{N}, l^2_{[\cdot,\cdot]} \right]_{RN} \right]_{RN} = l^2_{[\cdot,\cdot]}N,N. \quad \text{The same lemma gives} \quad \left[ N^2, l^2_{[\cdot,\cdot]} \right]_{RN} = l^2_{[\cdot,\cdot]}N^2.
\]

Since \( N \) is a Nijenhuis \((1,1)\)-tensor on \( A \), we have \( l^2_{[\cdot,\cdot]}N,N = l^2_{[\cdot,\cdot]}N^2 \), which implies that

\[
\left[ \overline{N}, \left[ \overline{N}, l^2_{[\cdot,\cdot]} \right]_{RN} \right]_{RN} = \left[ N^2, l^2_{[\cdot,\cdot]} \right]_{RN}. \quad \text{Also,} \quad (N^2) \quad \text{and} \quad \overline{N} \quad \text{commute}
\]

with respect to the Richardson-Nijenhuis bracket. ■
In the next proposition we obtain a Nijenhuis vector valued form which is the sum of a vector valued 1-form with a vector valued 2-form.

**Proposition 7.2.** Let \((A, [[., .], \rho])\) be a Lie algebroid, with differential \(d^A\) and associated multiplicative graded Lie algebra structure \(l_2^{[\cdot, \cdot]}\) on \(\Gamma(\wedge A)[2]\). Then, for every section \(\alpha \in \Gamma(\wedge^2 A^*)\), \(S + \alpha\) is a Nijenhuis vector valued form with respect to \(l_2^{[\cdot, \cdot]}\), with square \(S + 2\alpha\). The deformed structure is \(l_2^{[\cdot, \cdot]} + d^A \alpha\).

**Proof:** As a direct consequence of Lemma 6.10, we have

\[
\left[ S + \alpha, l_2^{[\cdot, \cdot]} \right]_{RN} = \left[ S + \alpha, l_2^{[\cdot, \cdot]} \right]_{RN} + d^A \alpha.
\]

A simple computation gives

\[
\left[ S + \alpha, \left[ S + \alpha, l_2^{[\cdot, \cdot]} \right]_{RN} \right]_{RN} = \left[ S + 2\alpha, l_2^{[\cdot, \cdot]} \right]_{RN}
\]

and the fact that \([S + \alpha, S + 2\alpha]_{RN} = 0\) completes the proof.

Our next purpose is to use well-known structures on a Lie algebroid defined by pairs of compatible tensors, such as \(\Omega N\)-, Poisson-Nijenhuis and \(P\Omega\)-structures [14, 1, 3], to construct Nijenhuis forms on the multiplicative graded Lie algebra associated to the Lie algebroid. We start by recalling what an \(\Omega N\)-structure is.

**Definition 7.3.** [1, 14] Let \((A, [[., .], \rho])\) be a Lie algebroid, with differential \(d^A\), \(N\) be a \((1, 1)\)-tensor on \(A\) and \(\alpha \in \Gamma(\wedge^2 A^*)\) a 2-form. Let \(\alpha_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)\) be a bilinear map, defined as

\[
\alpha_N(X, Y) = \alpha(NX, Y). \tag{76}
\]

Then, the pair \((\alpha, N)\) is an \(\Omega N\)-structure on the Lie algebroid \(A\) if \(\alpha(NX, Y) = \alpha(X, NY)\) for all \(X, Y \in \Gamma(A)\) (which amounts to \(\alpha_N\) being skew-symmetric and therefore a 2-form on \(A\)), and \(\alpha\) and \(\alpha_N\) are \(d^A\)-closed.

**Lemma 7.4.** Let \((A, [[., .], \rho])\) be a Lie algebroid, with differential \(d^A\) and with the associated multiplicative graded Lie algebra structure \(l_2^{[\cdot, \cdot]}\) on the graded vector space \(\Gamma(\wedge A)[2]\). Let \(N\) be a \((1, 1)\)-tensor on the Lie algebroid and \(\alpha \in \Gamma(\wedge^2 A^*)\) be a 2-form such that \(\alpha_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)\) given by \(\eqref{76}\) is skew-symmetric and therefore a 2-form on \(A\). Then,

\[
i) \quad [N, \alpha]_{RN} = 2\alpha_N,
\]

\[
ii) \quad [N + \alpha, l_2^{[\cdot, \cdot]}]_{RN} = l_2^{[\cdot, \cdot]} + d^A \alpha
\]
If \( \mathcal{N} \) is Nijenhuis, then
\[
\left[ \mathcal{N} + \alpha, \left[ \mathcal{N} + \alpha, l_2^{[\cdot, \cdot]} \right] \right]_{\mathcal{R}_N} = \left[ \mathcal{N}^2 + l_2^{[\cdot, \cdot]}, \alpha \right]_{\mathcal{R}_N} - 2 \alpha \mathcal{N} + 2 \left[ \mathcal{N}, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N}.
\]

**Proof:**

i) First notice that for all \( X, Y \in \Gamma(A) \) we have
\[
\left[ \mathcal{N}, \alpha \right]_{\mathcal{R}_N}(X, Y) = \alpha(\mathcal{N}X, Y) - \alpha(\mathcal{N}Y, X) = 2\alpha_N(X, Y).
\]

Since \( \mathcal{N} \) and \( \alpha \) are both derivations, by Lemma 6.4, \( \left[ \mathcal{N}, \alpha \right]_{\mathcal{R}_N} \) is a derivation and hence it is the unique extension of \( 2\alpha_N \) by derivation.

ii) It is a direct consequence of Lemma 6.9 together with Lemma 6.10.

iii) Using item (ii) and Lemma 6.9, we have
\[
\left[ \mathcal{N} + \alpha, \left[ \mathcal{N} + \alpha, l_2^{[\cdot, \cdot]} \right] \right]_{\mathcal{R}_N} = \left[ \mathcal{N} + \alpha, l_2^{[\cdot, \cdot]} + \mathcal{d}^A \alpha \right]_{\mathcal{R}_N}
\]
\[
= l_2^{[\cdot, \cdot]} \mathcal{N} + \left[ \mathcal{N}, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N} + \left[ \alpha, l_2^{[\cdot, \cdot]} \right]_{\mathcal{R}_N} + \left[ \alpha, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N}.
\]

But, using Lemma 6.9 and the graded Jacobi identity we have
\[
\left[ \alpha, l_2^{[\cdot, \cdot]} \right]_{\mathcal{R}_N} = \left[ \alpha, \left[ \mathcal{N}, l_2^{[\cdot, \cdot]} \right] \right]_{\mathcal{R}_N}
\]
\[
= \left[ [\alpha, \mathcal{N}]_{\mathcal{R}_N}, l_2^{[\cdot, \cdot]} \right]_{\mathcal{R}_N} + \left[ \mathcal{N}, \left[ \alpha, l_2^{[\cdot, \cdot]} \right] \right]_{\mathcal{R}_N}
\]
\[
= \left[ -2\alpha_N, l_2^{[\cdot, \cdot]} \right]_{\mathcal{R}_N} + \left[ \mathcal{N}, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N}
\]

and, by Lemma 6.8, \( \left[ \alpha, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N} = 0 \). Hence, since \( \mathcal{N} \) is Nijenhuis, we get
\[
\left[ \mathcal{N} + \alpha, \left[ \mathcal{N} + \alpha, l_2^{[\cdot, \cdot]} \right] \right]_{\mathcal{R}_N} = \left[ \mathcal{N}^2 - 2\alpha_N, l_2^{[\cdot, \cdot]} \right]_{\mathcal{R}_N} + 2 \left[ \mathcal{N}, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N}
\]
\[
= \left[ \mathcal{N}^2, l_2^{[\cdot, \cdot]} \right]_{\mathcal{R}_N} - 2 \mathcal{d}^A \alpha_N + 2 \left[ \mathcal{N}, \mathcal{d}^A \alpha \right]_{\mathcal{R}_N}.
\]

The next proposition is now immediate.

**Proposition 7.5.** Let \((A, [\cdot, \cdot], \rho)\) be a Lie algebroid, with differential \( \mathcal{d}^A \) and with associated multiplicative graded Lie algebra structure \( l_2^{[\cdot, \cdot]} \) on the graded vector space \( \Gamma(\Lambda A)[2] \). If \((\alpha, \mathcal{N})\) is an \( \Omega_N \)-structure on the Lie algebroid \( A \), then \( \mathcal{N} + \alpha \) is a Nijenhuis vector valued form, with respect to \( l_2^{[\cdot, \cdot]} \), with square \( \mathcal{N}^2 + \alpha_N \).
Proof: Let \((\alpha, N)\) be an \(\Omega N\)-structure on the Lie algebroid \(A\). Then, \(d^A\alpha_N = 0\) and, by Lemma 6.10, we have \(\left[\overline{\alpha_N}, \overline{l_2^{[\cdot, \cdot]}}\right]_{RN} = 0\). It follows from item (iii) in Lemma 7.4, that \(\left[\alpha_N, \overline{N + \alpha} \right]_{RN} = 0\). Since \(\left[\alpha_N, \overline{N + \alpha} \right]_{RN} = \left[\alpha_N, \overline{N^2 + \alpha} \right]_{RN} = 2(\overline{\alpha_N})_N - 2\overline{\alpha_{N^2}} = 0\), the proof is complete.

We are now going to see how to include Poisson-Nijenhuis structures among our examples of Nijenhuis structures on \(L_\infty\)-algebras. Let us first fix and recall some notations and notions.

Let \((A, \mu = [\cdot, \cdot], \rho)\) be a Lie algebroid, \(\pi \in \Gamma(\wedge^2A)\) a bivector and \(N : \pi \rightarrow \pi\) a vector bundle morphism. We denote by \(N^*\) the morphism \(\pi \rightarrow \pi\) given by \(\langle N^*\alpha, X \rangle = \langle \alpha, NX \rangle\), for all \(X, Y \in \Gamma(A)\). We consider the morphism induced by \(\pi, \pi^# : A^* \rightarrow A\), given by \(\langle \beta, \pi^#\alpha \rangle = \pi(\alpha, \beta)\), and we denote by \(\pi_N\) the bivector defined by

\[
\pi_N(\alpha, \beta) = \langle \beta, N\pi^#\alpha \rangle = \langle N^*\beta, \pi^#\alpha \rangle,
\]

for all \(\alpha, \beta \in \Gamma(A^*)\). A bracket \(\{\cdot, \cdot\}_\pi^\mu\) can be defined on \(\Gamma(A^*)\), the space of 1-forms on the Lie algebroid \((A, \mu = [\cdot, \cdot], \rho)\), as follows:

\[
\{\alpha, \beta\}_\pi^\mu = \mathcal{L}^A_{\pi^#(\alpha)}\beta - \mathcal{L}^A_{\pi^#(\beta)}\alpha - d^A(\pi(\alpha, \beta)),
\]

for all \(\alpha, \beta \in \Gamma(A^*)\). It is well known that if \(\pi\) is a Poisson bivector on the Lie algebroid \((A, \mu = [\cdot, \cdot], \rho)\), that is \([\pi, \pi]_{SN} = 0\), then \((\Gamma(A^*), \{\cdot, \cdot\}_\pi)\) is a Lie algebra and if this is the case, then \(\pi^#\) is a Lie algebra morphism form the Lie algebra \((\Gamma(A^*), \{\cdot, \cdot\}^\mu)\) to the Lie algebra \((\Gamma(A), \mu)\).

For every Poisson structure \(\pi\) on a Lie algebroid \(A\), the triple \((\Gamma(\wedge A)[1], [\cdot, \cdot]_{SN}, \pi, \{\cdot, \cdot\}_{SN})\) is a skew-symmetric differential graded Lie algebra, so that the pair \((l_1^{[\cdot, \cdot]}, l_2^{[\cdot, \cdot]})\) given by

\[
l_1^{[\cdot, \cdot]}(P) = [\pi, P]_{SN} \quad \text{and} \quad l_2^{[\cdot, \cdot]}(P, Q) := (-1)^{(p-1)}[P, Q]_{SN},
\]

where \(P \in \Gamma(\wedge^pA)\) and \(Q \in \Gamma(\wedge^qA)\), is an \(L_\infty\)-structure on the graded vector space \(\Gamma(\wedge A)[2]\), which is clearly multiplicative. This \(L_\infty\)-structure is called
the $L_\infty$-structure associated to the Poisson structure $\pi$ and the Lie algebroid $A$.

Now, we recall the notion of Poisson-Nijenhuis structure on a Lie algebroid.

**Definition 7.6.** [13] Let $(A, \mu = [\cdot, \cdot], \rho)$ be a Lie algebroid, $\pi \in \Gamma(\wedge^2 A)$ a bivector and $N$ a $(1, 1)$-tensor on $A$. Then, the pair $(\pi, N)$ is a Poisson-Nijenhuis structure on the Lie algebroid $(A, \mu = [\cdot, \cdot], \rho)$ if

i) $N$ is a Nijenhuis $(1, 1)$-tensor with respect to the Lie bracket $\mu$,

ii) $\pi$ is a Poisson bivector,

iii) $N \circ \pi^\# = \pi^\# \circ N^*$,

iv) $(\{\alpha, \beta\}_\pi^\mu)_{N^*} = \{\alpha, \beta\}_\pi^\mu N^*$,

for all $\alpha, \beta \in \Gamma(A^*)$, where $(\{\cdot, \cdot\}_\pi^\mu)_{N^*}$ is the deformation of the Lie bracket $\{\cdot, \cdot\}_\pi$ by $N^*$ and $(\cdot, \cdot)_\pi^\mu$ is the bracket determined by the pair $(\pi, \mu^N = [\cdot, \cdot]_N)$ according to formula (77).

Notice that $\pi^\# N = N \pi^\# = \pi^# N$ and hence,

$$N(\pi) = 2 \pi_N. \quad (78)$$

Recall from [13] that if $(\pi, N)$ is a Poisson-Nijenhuis structure on a Lie algebroid $(A, \mu = [\cdot, \cdot], \rho)$, then $(A, \mu^N = [\cdot, \cdot]_N, \rho \circ N)$ and $(A^*, \{\cdot, \cdot\}_\pi^\mu, \rho \circ \pi^\#)$ are Lie algebroids. Also,

$$((\cdot, \cdot)_\pi^\mu)_{N^*}, \rho \circ \pi^\# \circ N^* \), (\cdot, \cdot)^\mu_\pi N \circ N \circ \pi^\#), \text{ and } (\cdot, \cdot)^\mu_\pi N, \rho \circ \pi^\#$$

define the same Lie algebroid structure on $A^*$. Moreover, identifying the graded vector spaces $\Gamma(\wedge A^{**})$ and $\Gamma(\wedge A)$, the differential $d_{A^*}^{A^*}$ coincide with the linear map $[\pi, \cdot]_{SN}$. Hence, we have

$$d_{\{\cdot, \cdot\}_{\pi^\mu}}^{A^*} = d_{\{\cdot, \cdot\}_{\pi_N}}^{A^*},$$

which is equivalent to

$$(\pi, \cdot)_{SN}^N = (\pi_N, \cdot)_{SN}, \quad (79)$$

where $[\cdot, \cdot]_{SN}^N$ is the Schouten-Nijenhuis bracket with respect to the Lie bracket $[\cdot, \cdot]_N$.

**Lemma 7.7.** Let $(\pi, N)$ be a Poisson-Nijenhuis structure on a Lie algebroid $(A, [\cdot, \cdot], \rho)$. Then,

$$\left[N, l_{[\cdot, \cdot]}^\pi\right]_{\pi N} \ = \ [\pi, N(P)]_{SN} - N[\pi, P]_{SN} = [-\pi_N, P]_{SN},$$
for all $P \in \Gamma(\wedge A)$.

**Proof:** The first equality follows directly from the definition of $l_1^{[,\cdot],\pi}$. For the second equality, observe that for all $P \in \Gamma(\wedge A)$ we have

$$[\pi, P]_N^SN = [N(\pi), P]_SN + [\pi, N(P)]_SN - N[\pi, P]_SN,$$

where $[\cdot, \cdot]_SN$ stands for the Schouten-Nijenhuis bracket with respect to the Lie bracket $[\cdot, \cdot]_N$. Hence, using (78) and (79), we have

$$[\pi, N(P)]_SN - N[\pi, P]_SN = [\pi, P]_SN^N - [N(\pi), P]_SN = [\pi, P]_SN^N - 2[\pi_N, P]_SN$$

$$= \left([\pi, P]_SN^N - [\pi_N, P]_SN\right) - [\pi_N, P]_SN$$

$$= -[\pi_N, P]_SN.$$

\[\blacksquare\]

**Proposition 7.8.** Let $(\pi, N)$ be a Poisson-Nijenhuis structure on a Lie algebroid $(A, [\cdot, \cdot], \rho)$. Then, the derivation $N$ is a weak Nijenhuis tensor for the $L_\infty$-structure associated to the Poisson structure $\pi$ and the Lie algebroid $(A, [\cdot, \cdot], \rho)$.

In this case, the deformed structure $[N, l_1^{[,\cdot],\pi} + l_2^{[,\cdot]}]_{RN}$ is the $L_\infty$-structure associated to the Poisson structure $-\pi_N$ and the Lie algebroid $(A, [\cdot, \cdot], \rho \circ N)$.

**Proof:** Lemmas 7.7 and 6.9 imply that

$$\left[N, l_1^{[,\cdot],\pi} + l_2^{[,\cdot]}\right]_{RN} = -l_1^{[,\cdot],\pi_N} + l_2^{[,\cdot]}_N.$$

Hence,

$$\left[N, \left[N, l_1^{[,\cdot],\pi} + l_2^{[,\cdot]}\right]\right]_{RN} = \left[l_1^{[,\cdot],\pi_N,N} + l_2^{[,\cdot],N,N}\right] = l_1^{[,\cdot],\pi_{N^2}} + l_2^{[,\cdot],N^2}$$

$$= \left[N^2, l_1^{[,\cdot],\pi} + l_2^{[,\cdot]}\right]_{RN} - 2 \left[N^2, l_1^{[,\cdot],\pi}\right]_{RN}.$$

Denoting $\mu = l_1^{[,\cdot],\pi} + l_2^{[,\cdot]}$ and using the fact that $\pi_{N^2}$ is a Poisson bivector on the Lie algebroid $(A, [\cdot, \cdot], \rho)$ and hence $(\Gamma(\wedge A)[2], l_1^{[,\cdot],\pi_{N^2}} + l_2^{[,\cdot]})$ is a
symmetric differential graded Lie algebra, we have
\[
\left[ \mu, [N, [N, \mu]]_{RN} \right]_{RN} = \left[ \mu, [N^2, \mu]_{RN} \right]_{RN} - 2\left[ \mu, [N^2, l_1^{[\ldots]}, \pi]_{RN} \right]_{RN} = -2\left[ \mu, [N^2, l_1^{[\ldots], \pi}]_{RN} \right]_{RN} = 2\left[ \mu, [N^2, l_1^{[\ldots], \pi}]_{RN} \right]_{RN} = 2\left[ \mu, l_1^{[\ldots], \pi, N^2} \right]_{RN}.
\]

and
\[
\left[ [l_1^{[\ldots], \pi}, l_1^{[\ldots], \pi, N^2}]_{RN} (P) \right] = l_1^{[\ldots], \pi, N^2} (l_1^{[\ldots], \pi} (P)) + l_1^{[\ldots], \pi} (l_1^{[\ldots], \pi, N^2} (P)) = \left[ \pi_{N^2}, [\pi, P]_{SN} \right]_{SN} + \left[ \pi, [\pi_{N^2}, P]_{SN} \right]_{SN} = \left[ \pi, \pi_{N^2} \right]_{SN} , P \right]_{SN} = 0.
\]
Hence, we have
\[
\left[ \mathcal{N} + \pi, \left[ \mathcal{N} + \pi, l_1^{(\ldots, \pi)} + l_2^{(\ldots, \pi)} \right]_{\mathcal{RN}} \right]_{\mathcal{RN}} = \left[ \mathcal{N} + \pi, -l_1^{(\ldots, \pi)} \pi + l_2^{(\ldots, \pi)} - l_1^{(\ldots, \pi)} \right]_{\mathcal{RN}} = l_1^{(\ldots, \pi)} N.\label{eq:83}
\]
But \( l_1^{(\ldots, \pi)}(\pi) = [\pi, \pi]_{\mathcal{SN}} = 0 \), \( l_1^{(\ldots, \pi)} N(\pi) = [\pi N, \pi]_{\mathcal{SN}} = 0 \) and \( l_2^{(\ldots, \pi)} N(P) + l_2^{(\ldots, \pi)} N(P) = [\pi N, P]_{\mathcal{SN}} - [\pi, P]_{\mathcal{SN}} = 0 \), for all \( P \in \Gamma(\wedge A) \), where \([\ldots, \pi]_{\mathcal{SN}}\) is the Schouten-Nijenhuis bracket associated to the Lie bracket \([\ldots, \pi]_{\mathcal{SN}}\). Hence, (83) can be written as
\[
\left[ \mathcal{N} + \pi, \left[ \mathcal{N} + \pi, l_1^{(\ldots, \pi)} + l_2^{(\ldots, \pi)} \right]_{\mathcal{RN}} \right]_{\mathcal{RN}} = l_1^{(\ldots, \pi)} N,\label{eq:83}
\]
Similar computations as in (80), (81) and (82) show that \( [\mu, [\mathcal{N}, [\mathcal{N}, \mu]]_{\mathcal{RN}}]_{\mathcal{RN}} = 0 \), which means that \( \mathcal{N} \) is weak Nijenhuis vector valued form with respect to the symmetric differential graded Lie algebra structure \( \mu = l_1^{(\ldots, \pi)} + l_2^{(\ldots, \pi)} \) on the graded vector space \( \Gamma(\wedge A)[2] \).

The next proposition establishes a relation between Poisson-Nijenhuis structures and co-boundary Nijenhuis tensors on a Lie algebroid.

**Proposition 7.10.** Let \((A, [\ldots, \ldots], \rho)\) be a Lie algebroid, \( \pi \in \Gamma(\wedge^2 A) \) a bivector and \( N \) a \((1,1)\)-tensor on \( A \) such that
\[
N \circ \pi^# = \pi^# \circ N^*.
\]
Then, \( \mathcal{N} + \pi \) is a co-boundary Nijenhuis vector valued form with curvature, with respect to the multiplicative graded Lie algebra structure \( l_2^{(\ldots, \pi)} \) on the graded vector space \( \Gamma(\wedge A)[2] \), with square \( \mathcal{N}^2 \), if and only if \((\pi, N)\) is a Poisson-Nijenhuis structure on the Lie algebroid \((A, [\ldots, \ldots], \rho)\). The deformed structure \([\mathcal{N}, l_2^{(\ldots, \pi)}]_{\mathcal{RN}}\) is the \( L_\infty \)-structure (indeed a differential graded Lie algebra structure) associated to the Poisson structure \( \pi \) on the Lie algebroid \((A, [\ldots, \ldots], \rho \circ N)\).

**Proof:** Assume that \((\pi, N)\) is a Poisson-Nijenhuis structure on the Lie algebroid \((A, [\ldots, \ldots], \rho)\). Then,
\[
\left[ \mathcal{N} + \pi, l_2^{(\ldots, \pi)} \right]_{\mathcal{RN}} = l_2^{(\ldots, \pi)} N.\label{eq:83}
\]
and, by (79), we get
\[
\left[ N + \pi, \left[ N + \pi, l_2^{[\cdots]} \right]_{RN} \right]_{RN} = l_2^{[\cdots]N,N} + l_2^{[\cdots]N,N} - l_1^{[\cdots]N,N}
\]
\[
= l_2^{[\cdots]N,N} = \left[ N^2, l_2^{[\cdots]} \right]_{RN},
\]
which means that \( \overline{N} + \pi \) is a co-boundary Nijenhuis with respect to the multiplicative graded Lie algebra structure \( l_2^{[\cdots]} \) on the graded vector space \( \Gamma(\wedge A)[2] \), with square \( N^2 \).

Conversely, assume that \( \overline{N} + \pi \) be a co-boundary Nijenhuis with respect to the multiplicative graded Lie algebra structure \( l_2^{[\cdots]} \) on the graded vector space \( \Gamma(\wedge A)[2] \), with square \( N^2 \), that is,
\[
\left[ \overline{N} + \pi, \left[ \overline{N} + \pi, l_2^{[\cdots]} \right]_{RN} \right]_{RN} = \left[ N^2, l_2^{[\cdots]} \right]_{RN}. \quad (84)
\]
Decomposing by homogeneous components, we get
\[
\left[ \overline{N} + \pi, \left[ \overline{N} + \pi, l_2^{[\cdots]} \right]_{RN} \right]_{RN} = l_2^{[\cdots]N,N} + \left( \left[ \overline{N}, l_2^{[\cdots]}(\pi, .) \right]_{RN} + l_2^{[\cdots]N}(\pi, .) \right) + l_2^{[\cdots]}(\pi, \pi). \quad (85)
\]
From (84) and (85), we get
\[
\left[ N^2, l_2^{[\cdots]} \right]_{RN} = l_2^{[\cdots]N,N}, \quad (86)
\]
\[
\left( \left[ N, l_2^{[\cdots]}(\pi, .) \right]_{RN} + l_2^{[\cdots]N}(\pi, .) \right) = 0 \quad (87)
\]
and
\[
l_2^{[\cdots]}(\pi, \pi) = 0. \quad (88)
\]
Equation (86) is equivalent to \( l_2^{[\cdots]N,N} = l_2^{[\cdots]N^2} \), or to \([\cdots]_{N,N} = [\cdots]_{N^2} \), which means that \( N \) is a Nijenhuis tensor on \( A \). Equation (88) means that \( \pi \) is Poisson, while Equation (87) gives
\[
\left( \left[ N, l_2^{[\cdots]}(\pi, .) \right]_{RN} + l_2^{[\cdots]N}(\pi, .) \right)(P) = 0,
\]
or
\[
\left[ N, l_2^{[\cdots]}(\pi, .) \right]_{RN}(P) = [\pi, P]_{SN}^N, \quad (89)
\]
for all \( P \in \Gamma(\wedge A) \). The definition of \([.,.]^{N}_{SN}\) gives
\[
\pi, P^{N}_{SN} = [N(\pi), P]_{SN} + [\pi, N(P)]_{SN} - N[\pi, P]_{SN} = 2[\pi N, P]_{SN} + [N, l^{[\pi, .]}_{1}]_{RN}(P),
\]
where in the second equality we used \( N(\pi) = 2\pi N \) and the definition of the Richardson-Nijenhuis bracket. From (89) and (90), we get
\[
[\pi, P]_{SN} = [\pi N, P]_{SN}.
\]
and this completes the proof that \((\pi, N)\) is a Poisson-Nijenhuis structure on the Lie algebroid \((A, [., .], \rho)\).

Last, we shall say a few words about the so-called \(P\Omega\)-structures [1, 14]. Recall that a \(P\Omega\)-structure on a Lie algebroid \((A, \rho, [., .])\) is a pair \((\pi, \omega)\) where \(\pi \in \Gamma(\wedge^2 A)\) is a Poisson element and \(\omega \in \Gamma(\wedge^2 A^*)\) is a 2-form, with \(d^A\alpha = 0\). The 2-form \(\omega \in \Gamma(\wedge^2 A^*)\) determines a morphism \(\omega^\flat : A \to A^*\), given by \(\langle Y, \omega^\flat(X) \rangle = \omega(X, Y)\). Defining a \((1, 1)\) tensor \(N := \pi^\# \circ \omega^\flat\), it is known that \((\pi, N)\) is a Poisson-Nijenhuis structure while \((\omega, N)\) is an \(\Omega N\)-structure.

**Proposition 7.11.** Let \((\pi, \omega)\) be a \(P\Omega\)-structure on a Lie algebroid \((A, [., .], \rho)\). Then, \(N = \omega + \pi\) is a co-boundary Nijenhuis form, with curvature, with respect to the multiplicative graded Lie algebra structure \(l^{[., .]}_{2}\) on the graded vector space \(\Gamma(\wedge A)[2]\), with square \(N\), where \(N = \pi^\# \circ \omega^\flat\). The deformed structure is \(-l^{[\pi, .]}_{1}\).

**Proof:** Observe that
\[
l^{[\pi, .]}_{1}(P) = [\pi, P]_{SN} = -l^{[\pi, .]}_{2}(P) = -[\pi, l^{[\pi, .]}_{2}]_{RN}(P)
\]
for all \( P \in \Gamma(\wedge^2 A) \). This means that
\[
l^{[\pi, .]}_{1} = -[\pi, l^{[\pi, .]}_{2}]_{RN}.
\]
Hence,
\[
[N, l^{[\pi, .]}_{2}]_{RN} = -l^{[\pi, .]}_{1} + d^A\omega = -l^{[\pi, .]}_{1},
\]
where in the second equality we used the definition of the Richardson-Nijenhuis bracket. From (89) and (90), we get
\[
[\pi, P]_{SN} = [\pi N, P]_{SN}.
\]
and this completes the proof that \((\pi, N)\) is a Poisson-Nijenhuis structure on the Lie algebroid \((A, [., .], \rho)\).
which proves the last claim (and proves that \( N \) is weak Nijenhuis vector valued form with respect to \( l_2^{[\cdots]} \), since \( l_1^{[\cdots],\pi} \) is an \( L_\infty \)-structure on \( \Gamma(\wedge A)[2] \). Equations (92) and (91) imply that

\[
\left[ N, \left[ N, l_2^{[\cdots]} \right] \right]_{RN} = - \left[ N, l_1^{[\cdots],\pi} \right]_{RN} = - \left[ \omega, l_1^{[\cdots],\pi} \right]_{RN} - \left[ \pi, \pi \right]_{SN} = \left[ \omega, l_2^{[\cdots]} \right]_{RN}.
\]

This shows that \( N \) is a co-boundary Nijenhuis vector valued form with respect to the graded Lie algebra structure \( l_2^{[\cdots]} \), on the graded vector space \( \Gamma(\wedge A)[2] \), with square \( [\omega, \pi]_{RN} \). A direct computation shows that \( [\pi, \omega]_{RN} = -N \) and completes the proof.

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