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NIJENHUIS FORMS ON L_∞-ALGEBRAS

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ABSTRACT: We investigate Nijenhuis deformations of L_∞-algebras, a notion that unifies several Nijenhuis deformations, namely those of Lie algebras, Lie algebroids, Poisson structures and Courant structures. Additional examples, linked to Lie n-algebras and n-plectic manifolds, are included.

Introduction

L_∞-algebras, introduced by Lada and Stasheff [15], who called them strongly homotopy Lie algebras, are collections of n-ary operations, assumed to satisfy some quadratic relations that reduce to the Jacobi identity, when only the binary operation is not trivial. These structures gained notoriety when Kontsevitch used L_∞-morphisms to prove the existence of star-products on Poisson manifolds [10]. Voronov [23] derived an L_∞-algebra from a Poisson element and an abelian subalgebra of a differential graded Lie algebra. For instance, an L_∞-algebra encodes a Poisson structure in a neighborhood of a coisotropic submanifold, provided that a linear transversal is given, see [6] and [5]. This makes L_∞-algebras a central tool for studying Poisson brackets, but there are more occurences. Roytenberg and Weinstein [22] gave a description of the so-called Courant algebroids in terms of Lie 2-algebras. In the same vein, Rogers [19] encodes n-plectic manifolds by Lie n-algebras and Frégier, Roger and Zambon [7] used this formalism to construct moment maps of those.

In this paper we develop a theory of Nijenhuis forms on L_∞-algebras. Here, by Nijenhuis forms, we mean a generalization of the notion of Nijenhuis $(1,1)$-tensors on manifolds, i.e., $(1,1)$-tensors whose Nijenhuis torsion vanishes. On manifolds, Nijenhuis tensors are unary operations on the Lie algebra of vector fields. Since, when dealing with L_∞-algebras, one has to replace Lie algebra brackets by collections of n-ary brackets for all integers $n \geq 1$, we also want to define Nijenhuis forms that are collections of n-ary operations for all integers $n \geq 1$. Our main idea is based on the fact that, given a Lie algebra $(\mathfrak{g}, [.,.])$ and a linear endomorphism N of $\mathfrak{g}$, N is Nijenhuis if deforming twice by N

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the original bracket yields the original bracket deformed by N^2. We translate this idea to L_∞-algebras, where the brackets to be deformed are their n-ary brackets.

We present several examples of Nijenhuis forms on L_∞-algebras. The first example is universal, in the sense that every L_∞-structure admits it: the Euler map S, that multiplies an element by its degree. Nijenhuis operators on ordinary graded Lie algebras are among the most trivial examples. Poisson elements, and more generally, Maurer-Cartan elements of differential graded Lie algebras are also examples, which are not purely made of vector valued 1-forms, but which are the sum of a vector valued 1-form with a vector valued 0-form. Less trivial examples are given on Lie n-algebras. On those, we have Nijenhuis forms which are the sum of a family of vector valued k-forms. An interesting case is when the Lie n-algebra is associated to an n-plectic manifold [19]. The case of Lie 2-algebras is treated separately, and we have Nijenhuis forms which are the sum of a vector valued 1-form with a vector valued 2-form.

We discuss how Nijenhuis tensors on Courant algebroids [4, 12, 2, 3] fit in our defintion of Nijenhuis forms on some L_∞-algebras. In order to include Lie algebroids in our examples, we recall the concept of multiplicative L_∞-algebras (related to P_∞-algebras in [5]). In the last part of the paper, our examples come from well-known structures on Lie algebroids, defined by pairs of compatible tensors [14, 1, 3], such as ΩN-, Poisson-Nijenhuis [13] and $P\Omega$-structures.

Very recently, while we were about to finish this paper, a notion of Nijenhuis operator on Lie 2-algebras was introduced in [18], using a different perspective. That definition is a particular case of ours, as we explain in Remark 4.14.

The paper is organized in seven sections. In Section 1 we introduce a bracket of graded symmetric vector valued forms on a graded vector space that we call Richardson-Nijenhuis bracket, because it reduces to the usual Richardson-Nijenhuis bracket of vector valued forms on a (non-graded) vector space. With this graded bracket, we characterize L_∞-structures as Poisson elements on the graded Lie algebra of graded symmetric vector valued forms. In Section 2 we present our main definition of Nijenhuis vector valued form with respect to an L_∞-algebra, or more generally, with respect to a vector valued form of degree 1. Relaxing a bit the definition of Nijenhuis vector valued form, yields the notions of weak Nijenhuis and co-boundary Nijenhuis forms, which provide interesting examples to be discussed in the next sections. Section 2 also contains the first examples of Nijenhuis forms on symmetric graded Lie algebras and symmetric differential graded Lie algebras: the Euler map, Poisson and Maurer-Cartan elements. Section 3 is devoted to Nijenhuis forms on Lie n-algebras. We construct examples of Nijenhuis forms on general Lie n-algebras, in particular on those defined by n-plectic manifolds. The case $n = 2$ is treated separately, in Section 4. There, we find necessary and sufficient conditions to have a Nijenhuis form which is the sum of vector valued 1-form with a vector valued 2-form. The importance of Lie 2-algebras appears in Section 5, where we focus on Courant algebroids. Using a construction established in [22], we associate a Lie 2-algebra to each Courant algebroid and we relate $(1, 1)$-tensors with vanishing Nijenhuis torsion on a Courant algebroid, with Nijenhuis forms on the corresponding associated Lie 2-algebra. In Section 6, we study multiplicative L_∞-algebras and its relation with pre-Lie and Lie algebroids. We introduce the notions of extension by derivation of $(1, 1)$-tensors and of k-forms on a Lie algebroid, needed to construct examples of Nijenhuis forms on Lie algebroids in the last section. In Section 7, the last one, we obtain, out of ΩN-, Poisson-Nijenhuis and $P\Omega$-structures on a Lie algebroid, examples of weak Nijenhuis and co-boundary Nijenhuis vector valued forms.

1. Richardson-Nijenhuis bracket and L_∞-algebras

In this section we extend the usual Richardson-Nijenhuis bracket of vector valued forms on vector spaces [9] to graded symmetric vector valued forms on graded vector spaces. Then, we use it to characterize L_∞-structures on graded vector spaces. We start by fixing some notations on graded vector spaces.

Let E be a graded vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, that is, a vector space of the form $\oplus_{i\in\mathbb{Z}} E_i$. For a given $i \in \mathbb{Z}$, the vector space E_i is called the component of degree i, elements of E_i are called *homogeneous elements of degree* i, and elements in the union $\cup_{i\in\mathbb{Z}} E_i$ are called the *homogeneous elements*. We denote by $|X|$ the degree of a non-zero homogeneous element X. Given a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ and an integer p, one may shift all the degrees by p to get a new grading on the vector space E. We use the notation $E[p]$ for the graded vector space E after shifting the degrees by p, that is, the graded vector space whose component of degree i is E_{i+p}.

We denote by $S(E)$ the symmetric space of E which is, by definition, the quotient space of the tensor algebra $\otimes E$ by the two-sided ideal $I \subset \otimes E$ generated by elements of the type $X \otimes Y - (-1)^{|X||Y|} Y \otimes X$, with X and Y arbitrary homogeneous elements in E. For a given $k \geq 0$, $S^k(E)$ is the image of $\otimes^k E$ through the quotient map $\otimes E \mapsto \frac{\otimes E}{I} = S(E)$ and one has the following decomposition

$$S(E) = \oplus_{k \geq 0} S^k(E),$$

where $S^0(E)$ is simply the field $\mathbb{K}$. Moreover, when all the components in the graded space E are of finite dimension, the dual of $S^k(E)$ is isomorphic to $S^k(E^*)$, for all $k \geq 0$. In this case, there is a one to one correspondence between

(i) graded symmetric k-linear maps on the graded vector space E,
(ii) linear maps from the space $S^k(E)$ to E,
(iii) $S^k(E^*) \otimes E$.

Elements of the space $S^k(E^*) \otimes E$ are called *symmetric vector valued k-forms.* Notice that $S^0(E^*) \otimes E$, the space of vector valued zero-forms, is isomorphic to the space E.

Having the decomposition $S(E) = \oplus_{k \geq 0} S^k(E)$, every element in $S(E)$ is the sum of finitely many elements in $S^k(E)$, $k \geq 0$. We absolutely need to consider also infinite sums, which is often referred in the literature as taking the completion of $S(E)$. By a *formal sum*, we mean a sequence $\phi : \mathbb{N} \bigcup \{0\} \to S(E)$ mapping an integer k to an element $a_k \in S^k(E)$: we shall, by a slight abuse of notation, denote by $\sum_{k=0}^{\infty} a_k$ such an element. We denote the set of all formal sums by $\tilde{S}(E)$. The algebra structure on $S(E)$ extends in an unique manner to $\tilde{S}(E)$. For two formal sums $a = \sum_{k=0}^{\infty} a_k$ and $b = \sum_{k=0}^{\infty} b_k$ we define $a+b$ to be $\sum_{k=0}^{\infty}(a_k+b_k)$, while the product of a and b is the infinite sum $\sum_{k=0}^{\infty} c_k$ with $c_k = \sum_{i=0}^{k} a_i \cdot b_{k-i}$ (with $\cdot$ being the product of $S(E)$).

When all the components in the graded space E are of finite dimension, there is a one to one correspondence between

(i) collections indexed by $k \geq 0$ of graded symmetric k-linear maps on the graded vector space E,
(ii) collections indexed by $k \geq 0$ of linear maps from $S^k(E)$ to E,
(iii) $\tilde{S}(E^*) \otimes E$.

Elements of the space $\tilde{S}(E^*) \otimes E$ are called *symmetric vector valued forms* and shall be written as infinite sums $\sum K_i$ with $K_i \in S^i(E^*) \otimes E$.

Let E be a graded vector space, $E = \oplus_{i\in\mathbb{Z}} E_i$. The insertion operator of a symmetric vector valued k-form K is an operator

$$\iota_K : S(E^*) \otimes E \to S(E^*) \otimes E$$

defined by

$$\iota_K L(X_1, ..., X_{k+l-1}) = \sum_{\sigma\in Sh(k,l-1)} \epsilon(\sigma) L(K(X_{\sigma(1)}, ..., X_{\sigma(k)}), ..., X_{\sigma(k+l-1)}), \tag{1}$$

for all $L \in S^l(E^*) \otimes E$, $l \geq 0$ and $X_1, \cdots, X_{k+l-1} \in E$, where $Sh(i, j-1)$ stands for the set of $(i, j-1)$-unshuffles and $\epsilon(\sigma)$ is the Koszul sign which is defined as follows

$$X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} = \epsilon(\sigma) X_1 \otimes \cdots \otimes X_n,$$

for all $X_1, \cdots, X_n \in E$. If L is an element in $S^0(E^*) \otimes E \simeq E$, then (1) should be understood as meaning that $\iota_K L = 0$, for all vector valued forms K and

$$\iota_L K(X_1, ..., X_{k-1}) = K(L, X_1, ..., X_{k-1}),$$

for all vector valued k-form K.

Allowing L and K to be symmetric vector valued forms, that is, $L = \sum_{i\geq 0} L_i$ and $K = \sum_{i\geq 0} K_i$, with L_i and K_i vector valued i-forms, the previous definition of insertion operator extends in the obvious way.

If K is an element in $S^i(E^*)$, i.e. a linear form on $S^i(E)$, $i \geq 0$, one may define ι_K by a formula similar to (1). Moreover, $\iota_K : \tilde{S}(E^*) \to \tilde{S}(E^*)$, with $K \in \tilde{S}(E^*) \otimes E$, is the zero map if and only if $K = 0$.

Now, we define a bracket on the space $\tilde{S}(E^*) \otimes E$ as follows. Given a symmetric vector valued k-form $K \in S^k(E^*) \otimes E$ and a symmetric vector valued l-form $L \in S^l(E^*) \otimes E$, the *Richardson-Nijenhuis bracket* of K and L is the symmetric vector valued $(k + l - 1)$-form $[K, L]_{RN}$, given by

$$[K, L]_{RN} = \iota_K L - (-1)^{\bar{K}\bar{L}} \iota_L K, \tag{2}$$

where $\bar{K}$ is the degree of K as a graded map, that is $K(X_1, \cdots, X_k) \in E_{1+\cdots+k+\bar{K}}$, for all $X_i \in E_i$. For an element $X \in E$, $\bar{X} = |X|$, that is, the degree of a vector valued 0-form, as a graded map, is just its degree as an element of E.

Proposition 1.1. *The space $\tilde{S}(E^*)\otimes E$, equipped with the Richardson-Nijenhuis bracket, is a graded (skew-symmetric) Lie algebra.*

If $K \in S^k(E^*) \otimes E$ is a vector valued k-form, an easy computation gives

$$K(X_1, \cdots, X_k) = [X_k, \cdots, [X_2, [X_1, K]_{RN}]_{RN} \cdots]_{RN}, \tag{3}$$

for all $X_1, \cdots, X_k \in E$.

In [16], the authors defined a multi-graded Richardson-Nijenhuis bracket, in a graded vector space, but their approach is different from ours.

Next, we recall the notion of L_∞-algebra, following [8].

Definition 1.2. An L_∞-*algebra* is a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ together with a family of symmetric vector valued forms $(l_i)_{i\geq 1}$ of degree 1, with $l_i : \otimes^i E \to E$ satisfying the following relation:

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh(i,j-1)} \epsilon(\sigma) l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \cdots, X_{\sigma(n)}) = 0, \tag{4}$$

for all $n \geq 1$ and all homogeneous $X_1, \cdots, X_n \in E$, where $\epsilon(\sigma)$ is the Koszul sign. The family of symmetric vector valued forms $(l_i)_{i\geq 1}$ is called an L_∞-*structure* on the graded vector space E. Usually, we denote this L_∞-structure by $\mu := \sum_{i\geq 1} l_i$ and we say, by an abuse of language, that μ has degree 1.

A slight generalization of an L_∞-algebra is the so-called *curved* L_∞-algebra. In this case, the family of symmetric vector valued forms is $(l_i)_{i\geq 0}$ that is, there is an extra symmetric vector valued 0-form $l_0 \in E_1$, called the *curvature*, such that $l_1(l_0) = 0$ and Equation (4) is replaced by

$$l_{n+1}(l_0, X_1, \cdots, X_n) \\ + \sum_{i+j=n+1} \sum_{\sigma \in Sh(i,j-1)} \epsilon(\sigma) l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \cdots, X_{\sigma(n)}) = 0.$$

There is an equivalent definition of L_∞-algebra in terms of graded skew-symmetric vector valued forms l'_i of degree $i-2$. This was, in fact, the original definition introduced in [15]. The equivalence of both definitions is established by the so-called *décalage isomorphism*

$$l_i(X_1, \cdots, X_i) \mapsto (-1)^{(i-1)|X_1|+(i-2)|X_2|+\cdots+|X_{i-1}|} l'_i(X_1, \cdots, X_i),$$

$X_1, \cdots, X_i \in E$. The family of graded skew-symmetric brackets $(l'_i)_{i\geq 1}$ defines an L_∞-structure on the graded vector space E if each l'_i has degree $i-2$

and

$$\sum_{i+j=n+1}\sum_{\sigma\in Sh(i,j-1)}(-1)^{i(j-1)}\epsilon(\sigma)\,sign(\sigma)l_j(l_i(X_{\sigma(1)},\cdots,X_{\sigma(i)}),\cdots,X_{\sigma(n)})=0,$$

for all $n\geq 1$ and all $X_1,\cdots,X_n\in E$, with $sign(\sigma)$ being the sign of the permutation σ.

Next, we see that some well-known structures on (graded) vector spaces are examples of L_∞-algebras.

We start with a *symmetric graded Lie algebra*, which is a graded vector space $E=\oplus_{i\in\mathbb{Z}}E_i$ endowed with a binary graded symmetric bracket $[.,.]=\mu$ of degree 1, satisfying the graded Jacobi identity i.e.

$$[X,[Y,Z]]=(-1)^{|X|+1}[[X,Y],Z]+(-1)^{(|X|+1)(|Y|+1)}[Y,[X,Z]], \tag{5}$$

for all homogeneous elements $X,Y,Z\in E$. Note that when the graded vector space is concentrated on degree -1, that is, all the vector spaces E_i are zero, except E_{-1}, then (5) is the usual Jacobi identity and we get a Lie algebra with symmetric bracket. We would like to remark that (5) can be written as

$$\mu(\mu(X,Y),Z)+(-1)^{|Y||Z|}\mu(\mu(X,Z),Y)+(-1)^{|X|(|Y|+|Z|)}\mu(\mu(Y,Z),X)=0, \tag{6}$$

for all homogeneous elements $X,Y,Z\in E$. This means that a symmetric graded Lie algebra is simply an L_∞-algebra such that all the multi-brackets are zero except the binary one. From this, we also conclude that a *Lie algebra* is an L_∞-algebra on a graded vector space concentrated on degree -1, for which all the brackets are zero except the binary bracket.

Another special case of an L_∞-algebra is a *symmetric differential graded Lie algebra*. It is an L_∞-structure on $E=\oplus_{i\in\mathbb{Z}}E_i$, with all the brackets, except l_1 and l_2, being zero. In other words, a symmetric differential graded Lie algebra is a symmetric graded Lie algebra $(\oplus_{i\in\mathbb{Z}}E_i,[.,.]=l_2)$ endowed with a differential $\mathsf{d}=l_1$, that is, a linear map $\mathsf{d}:\oplus_{i\in\mathbb{Z}}E_i\to\oplus_{i\in\mathbb{Z}}E_i$ of degree 1 and squaring to zero, satisfying the compatibility condition

$$\mathsf{d}[X,Y]+[\mathsf{d}(X),Y]+(-1)^{|X|}[X,\mathsf{d}(Y)]=0,$$

for all homogeneous elements $X,Y\in E$. We shall denote a symmetric differential graded Lie algebra by $(E,\mathsf{d},[.,.])$ or by (E,l_1+l_2).

We may also consider two particular cases of a curved L_∞-algebra, that is to say, a curved symmetric graded Lie algebra and a curved symmetric differential graded Lie algebra. More precisely, a *curved* symmetric differential graded Lie algebra on a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ is a symmetric differential graded Lie algebra $(E, \mathsf{d}, [.,.])$ together with an element $\mathfrak{C} \in E_1$ such that:

$$\mathsf{d}(\mathfrak{C}) = 0 \quad \text{and} \quad [\mathfrak{C}, X] + \mathsf{d}^2 X = 0, \quad \text{for all } X \in E.$$

We shall denote the curved symmetric differential graded Lie algebra by $(E, \mathfrak{C}, \mathsf{d}, [.,.])$ or by $(E, \mathfrak{C} + l_1 + l_2)$. When $\mathsf{d} = 0$, the curved symmetric differential graded Lie algebra is simply a curved symmetric graded Lie algebra.

The Richardson-Nijenhuis bracket on graded vector spaces, introduced previously, is intimately related to L_∞-algebras. In the next theorem, that appears in an implicit form in [21], we use the Richardson-Nijenhuis bracket to characterize (curved) L_∞-structures on a graded vector space.

Theorem 1.3. *Let $E = \oplus_{i\in\mathbb{Z}} E_i$ be a graded vector space, $(l_i)_{i\geq 1} : \otimes^i E \to E$ be a family of symmetric vector valued forms on E of degree 1 and $l_0 \in E_1$ be a symmetric vector valued 0-form. Set $\mu = \sum_{i\geq 1} l_i$ and $\mu' = \sum_{i\geq 0} l_i$. Then,*

i) *μ is an L_∞-structure on E if and only if $[\mu, \mu]_{RN} = 0$;*

ii) *μ' is a curved L_∞-structure on E if and only if $[\mu', \mu']_{RN} = 0$.*

Proof: (i) It is a direct consequence of the following equalities that can be obtained from (1) and (2):

$$[\mu, \mu]_{RN} = \sum_{n\geq 1} \Big(\sum_{i+j=n+1} [l_i, l_j]_{RN} \Big) = 2 \sum_{n\geq 1} \Big(\sum_{i+j=n+1} \iota_{l_i} l_j \Big).$$

The proof of (ii) is easy. ■

Notice that for the case of symmetric graded Lie algebras, the statement of Theorem 1.3 appears in a natural way, since equation (6) is equivalent to

$$\frac{1}{2}(\iota_\mu \mu + \iota_\mu \mu)(X, Y, Z) = \frac{1}{2}[\mu, \mu]_{RN}(X, Y, Z) = 0.$$

2. Nijenhuis forms on L_∞ algebras: definition and first examples

In this section we define a Nijenhuis vector valued form with respect to a given vector valued form μ and deformation of μ by a Nijenhuis vector valued form. We show that deforming an L_∞-structure by a Nijenhuis vector valued form, one gets an L_∞-structure. Then, we present the first examples of Nijenhuis vector valued forms on some L_∞-algebras.

Definition 2.1. Let E be a graded vector space and μ be a symmetric vector valued form on E of degree 1. A vector valued form $\mathcal{N}$ of degree zero is called

- *weak Nijenhuis* with respect to μ if
$$\Big[\mu, \big[\mathcal{N}, [\mathcal{N}, \mu]_{RN}\big]_{RN}\Big]_{RN} = 0,$$
- *co-boundary Nijenhuis* with respect to μ if there exists a vector valued form $\mathcal{K}$ of degree zero, such that
$$\big[\mathcal{N}, [\mathcal{N}, \mu]_{RN}\big]_{RN} = [\mathcal{K}, \mu]_{RN},$$
- *Nijenhuis* with respect to μ if there exists a vector valued form $\mathcal{K}$ of degree zero, such that
$$\big[\mathcal{N}, [\mathcal{N}, \mu]_{RN}\big]_{RN} = [\mathcal{K}, \mu]_{RN} \quad \text{and} \quad [\mathcal{N}, \mathcal{K}]_{RN} = 0.$$
Such a $\mathcal{K}$ is called a *square* of $\mathcal{N}$. If $\mathcal{N}$ contains an element of the underlying graded vector space, that is, $\mathcal{N}$ has a component which is a vector valued zero form, then $\mathcal{N}$ is called Nijenhuis (respectively, co-boundary Nijenhuis) vector valued form with *curvature*.

It is obvious that the following implications hold:

$\mathcal{N}$ Nijenhuis $\Rightarrow$ $\mathcal{N}$ co-boundary Nijenhuis $\Rightarrow$ $\mathcal{N}$ weak Nijenhuis

Remark 2.2. It would be of course tempting to choose $\mathcal{K} = \iota_{\mathcal{N}}\mathcal{N}$ in Defintion 2.1, having in mind what happens for manifolds, and the fact that $\iota_{\mathcal{N}}\mathcal{N} = \mathcal{N}^2$ for vector valued 1-forms. However, it is not what examples show to be a reasonable definition. Also, for $\mathcal{N}$ a vector valued 2-form we do not have, in general, $[\iota_{\mathcal{N}}\mathcal{N}, \mathcal{N}]_{RN} = 0$, which says $\iota_{\mathcal{N}}\mathcal{N}$ is not a good candidate for the square, except maybe for vector valued 1-forms.

Proposition 2.3. *Let (E, μ) be a (curved) L_∞-algebra and $\mathcal{N}$ be a symmetric vector valued form on E. Then $\mathcal{N}$ is weak Nijenhuis with respect to μ if and only if $[\mathcal{N}, \mu]_{RN}$ is a (curved) L_∞-algebra.*

Proof: First we observe that $[\mathcal{N}, \mu]_{RN}$ has degree 1 if and only if the degree of $\mathcal{N}$ is zero. Using the Jacobi identity, we get
$$\begin{aligned}
[\mu, [\mathcal{N}, [\mathcal{N}, \mu]_{RN}]_{RN}]_{RN} &= [[\mu, \mathcal{N}]_{RN}, [\mu, \mathcal{N}]_{RN}]_{RN} + [\mathcal{N}, [\mu, [\mathcal{N}, \mu]_{RN}]_{RN}]_{RN} \\
&= [[\mu, \mathcal{N}]_{RN}, [\mu, \mathcal{N}]_{RN}]_{RN} \\
&= [[\mathcal{N}, \mu]_{RN}, [\mathcal{N}, \mu]_{RN}]_{RN},
\end{aligned}$$
which concludes the proof. ■

Given an L_∞-structure μ and a symmetric vector valued form of degree zero $\mathcal{N}$ on a graded vector space, we call $[\mathcal{N},\mu]_{_{RN}}$ the *deformation of* μ *by* $\mathcal{N}$ and denote the deformed structure by $\mu^{\mathcal{N}}$. When μ is deformed k times by $\mathcal{N}$, the deformed structure is denoted by $\mu^{\mathcal{N},\overset{k}{\dots},\mathcal{N}}$ or simply μ_k if there is no danger of confusion.

Weak Nijenhuis forms do not, in general, give hierarchies in any sense. However, Nijenhuis forms do.

Theorem 2.4. *Let $\mathcal{N}$ be a Nijenhuis vector valued form with respect to a (curved) L_∞-structure μ with square $\mathcal{K}$, on a graded vector space E. Then, for all integers $k \geq 1$, μ_k is a (curved) L_∞-structure on E and $\mathcal{N}$ is Nijenhuis with square $\mathcal{K}$, with respect to μ_k.*

Proof: The case $k=1$ follows from Proposition 2.3 together with the observation that if $\mathcal{N}$ is Nijenhuis, then it is also weak Nijenhuis with respect to μ. Assume, by induction, that $\mathcal{N}$ is Nijenhuis with respect to μ_k with square $\mathcal{K}$. Then we have

$$[\mathcal{N},[\mathcal{N},\mu_k]_{_{RN}}]_{_{RN}} = [\mathcal{K},\mu_k]_{_{RN}},$$

that implies

$$[\mathcal{N},[\mathcal{N},[\mathcal{N},\mu_k]_{_{RN}}]_{_{RN}}]_{_{RN}} = [\mathcal{N},[\mathcal{K},\mu_k]_{_{RN}}]_{_{RN}}. \tag{7}$$

Applying the Jacobi identity on the right hand side of (7) and using the assumption that $\mathcal{N}$ and $\mathcal{K}$ commute with respect to the Richardson-Nijenhuis bracket, we get

$$[\mathcal{N},[\mathcal{N},\mu_{k+1}]_{_{RN}}]_{_{RN}} = [\mathcal{K},\mu_{k+1}]_{_{RN}}.$$

Thus, $\mathcal{N}$ is Nijenhuis with respect to μ_{k+1}, with square $\mathcal{K}$. ■

Recall from [13] that a *Nijenhuis* operator on a graded Lie algebra $(E,\mu=[.,.])$ is a linear map $N: E\to E$ such that its Nijenhuis torsion with respect to μ, defined by

$$T_\mu N(X,Y) := \mu(NX,NY) - N(\mu(NX,Y)+\mu(X,NY)-N(\mu(X,Y))), \tag{8}$$

for all $X,\ Y\in E$, is identically zero. For a binary bracket $\mu=[.,.]$, the deformed bracket by N is denoted by $[.,.]_N$ and is given by $[X,Y]_N = [NX,Y]+[X,NY]-N[X,Y]$. It has been shown in [13] that if N is Nijenhuis on a Lie algebra $(E,[.,.])$, then $(E,[.,.]_N)$ is also a Lie algebra and N is a morphism of Lie algebras. Also, it has been shown that N is Nijenhuis if and only if deforming the original bracket of the Lie algebra twice by N is equivalent to deform it once by N^2, that is $([X,Y]_N)_N = [X,Y]_{N^2}$. This can be stated using the notion of Richardson-Nijenhuis bracket on the space of vector valued forms on a graded vector space E, as follows:

$$[N,[N,\mu]_{_{RN}}]_{_{RN}} = [N^2,\mu]_{_{RN}}.$$

So, we conclude that Nijenhuis operators in the usual and traditional sense are, of course, Nijenhuis in our sense also.

Next, we present the first examples of Nijenhuis vector valued forms on L_∞-algebras. We start by introducing the Euler map S, the map that simply counts the degree of homogeneous elements in a graded vector space. More precisely, given a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$, $S : E \to E$ is defined by $S(X) = -|X|X$, for all homogeneous elements $X \in E$ of degree $|X|$.

Notice that S, as a graded map, has degree zero, $\bar{S} = 0$. By a simple computation, using the definition of S, we get the following result.

Lemma 2.5. *Let $E = \oplus_{i\in\mathbb{Z}} E_i$ be a graded vector space. Then,*

$$[S,\alpha]_{_{RN}} = \bar{\alpha}\,\alpha,$$

for every symmetric vector valued form α on E of degree $\bar{\alpha}$.

Proposition 2.6. *Let μ be a vector valued form of degree 1 on a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$. The Euler map S is a Nijenhuis vector valued form with respect to μ with square S.*

Proof: Let $\mu = \sum_{i=1}^{\infty} l_i$. Applying Lemma 2.5 to each $l_i, 1 \leq i \leq \infty$, and taking the sum we get:

$$[S,\mu]_{_{RN}} = \sum_{i=1}^{\infty} [S,l_i]_{_{RN}} = \sum_{i=1}^{\infty} l_i = \mu.$$

Therefore

$$[S,[S,\mu]_{_{RN}}]_{_{RN}} = [S,\mu]_{_{RN}}.$$

Since $\bar{S} = 0$, Lemma 2.5 implies that $[S,S]_{_{RN}} = 0$ and this completes the proof. ∎

Of course, the result can be enlarged for every μ-cocycle, that is, a vector valued form η such that $[\mu,\eta]_{_{RN}} = 0$.

Proposition 2.7. *Let $\mu = \sum_{i\geq 1} l_i$ be a vector valued form of degree 1 on a graded vector space E. Then, for every element α of degree 0 in $\tilde{S}(E^*) \otimes E$ with $[\mu,\alpha]_{_{RN}} = 0$, $S + \alpha$ is Nijenhuis with respect to μ, with square S.*

Next, we give some examples of Nijenhuis forms on symmetric graded and symmetric differential graded Lie algebras. For that, we need to introduce the notions of Maurer Cartan and Poisson elements.

A *Maurer Cartan element* in a symmetric differential graded Lie algebra $(E, \mathsf{d}, [.,.])$ is an element $e \in E_0$ such that

$$\mathsf{d}(e) - \frac{1}{2}[e, e] = 0.$$

A *Maurer Cartan element* in a symmetric curved differential graded Lie algebra $(E,\ \mathfrak{C}, \mathsf{d}, [.,.])$ is an element $e \in E_0$ such that

$$(\mathsf{d}(e) - \mathfrak{C}) - \frac{1}{2}[e, e] = 0.$$

A *Poisson element* in a curved L_∞-algebra $(E, \mu = \sum_{i \geq 0} l_i)$ is an element $\pi \in E_0$, such that $l_2(\pi, \pi) = 0$.

The next propositions provide examples of Nijenhuis vector valued forms on symmetric graded Lie algebras and symmetric differential graded Lie algebras.

Proposition 2.8. *Let $\mu = \mathfrak{C} + l_2$ be a curved symmetric graded Lie algebra structure on a graded vector space $E = \oplus_{i \in \mathbb{Z}} E_i$ and $\pi \in E_0$. Then, $\mathcal{N} = \pi + S$ is a Nijenhuis vector valued form (with curvature π) with respect to μ and with square $2\pi + S$ if, and only if, π is a Poisson element.*

In this case, the deformed structure is the curved symmetric differential graded Lie algebra $(E, \mathfrak{C} + l_2(\pi, .) + l_2)$.

Proof: The proof of the equivalence is a direct consequence of the following equalities:

$$[\pi + S, \mathfrak{C} + l_2]_{_{RN}} = l_2(\pi, .) + \mathfrak{C} + l_2, \tag{9}$$

$$\begin{aligned} [\pi + S, [\pi + S, \mathfrak{C} + l_2]_{_{RN}}]_{_{RN}} &= l_2(\pi, \pi) + \mathfrak{C} + 2l_2(\pi, .) + l_2 \\ &= l_2(\pi, \pi) + [2\pi + S, \mathfrak{C} + l_2]_{_{RN}} \end{aligned}$$

and

$$[\pi + S, 2\pi + S]_{_{RN}} = 2[\pi, \pi]_{_{RN}} + [\pi, S]_{_{RN}} + 2[S, \pi]_{_{RN}} + [S, S]_{_{RN}} = 0,$$

where we used $[\pi, S]_{_{RN}} = [S, \pi]_{_{RN}} = 0$. The last statement follows directly from (9) and Theorem 2.4. ■

Proposition 2.9. *Let $\mu = \mathfrak{C} + l_1 + l_2$ be a curved symmetric differential graded Lie algebra structure on a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ and $\pi \in E_0$. Then, $\mathcal{N} = \pi + S$ is a Nijenhuis vector valued form (with curvature π) with respect to μ and with square $2\pi + S$ if, and only if, π is a Poisson element.*

In this case, the deformed structure is the curved symmetric differential graded Lie algebra $(E, (\mathfrak{C} + l_1(\pi)) + (l_1 + l_2(\pi, .)) + l_2)$.

Proof: The proof of the equivalence follows from:

$$[\pi + S, \mathfrak{C} + l_1 + l_2]_{_{RN}} = \mathfrak{C} + l_1(\pi) + (l_2(\pi, .) + l_1) + l_2, \tag{10}$$

$$\begin{aligned}
[\pi + S, [\pi + S, \mathfrak{C} + l_1 + l_2]_{_{RN}}]_{_{RN}} &= [\pi + S, \mathfrak{C} + l_1 + l_2 + l_1(\pi) + l_2(\pi, .)]_{_{RN}} \\
&= \mathfrak{C} + l_1 + l_2 + 2l_1(\pi) + 2l_2(\pi, .) + l_2(\pi, \pi) \\
&= [2\pi + S, \mathfrak{C} + l_1 + l_2]_{_{RN}} + l_2(\pi, \pi)
\end{aligned}$$

and

$$[\pi + S, 2\pi + S]_{_{RN}} = 2[\pi, \pi]_{_{RN}} + [\pi, S]_{_{RN}} + 2[S, \pi]_{_{RN}} + [S, S]_{_{RN}} = 0.$$

The last statement follows directly from (10) and Theorem 2.4. ■

Notice that, in Proposition 2.9, if we start with a symmetric differential graded Lie algebra without curvature, that is, if $\mathfrak{C} = 0$, then, the deformed structure is a curved symmetric differential graded Lie algebra with curvature $l_1(\pi)$.

Proposition 2.10. *Let $\mu = \mathfrak{C} + l_1 + l_2$ be a curved symmetric differential graded Lie algebra structure on a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ and $\pi \in E_0$. Then, $\mathcal{N} = Id_E + \pi$ is a Nijenhuis vector valued form (with curvature π) with respect to μ and with square $Id_E + \pi$ if, and only if, π is a Maurer-Cartan element.*

In this case, the deformed structure is the curved symmetric differential graded Lie algebra $(E, (l_1(\pi) - \mathfrak{C}) + l_2(\pi, .) + l_2)$.

Proof: First notice that

$$[\pi + Id_E, \mathfrak{C} + l_1 + l_2]_{_{RN}} = (l_1(\pi) - \mathfrak{C}) + l_2(\pi, .) + l_2 \tag{11}$$

and

$$\begin{aligned}[\pi + Id_E, [\pi + Id_E, \mathfrak{C} + l_1 + l_2]_{RN}]_{RN} &= l_2(\pi,\pi) + l_2(\pi,.) - l_1(\pi) + \mathfrak{C} + l_2 \\ &= -\mathfrak{C} - 2((l_1(\pi) - \mathfrak{C}) - \frac{1}{2}l_2(\pi,\pi)) + l_1(\pi) + l_2(\pi,.) + l_2 \\ &= -2((l_1(\pi) - \mathfrak{C}) - \frac{1}{2}l_2(\pi,\pi)) + [\pi + Id_E, \mathfrak{C} + l_1 + l_2]_{RN}.\end{aligned}$$

This, together with the fact that $[\pi+Id_E, \pi+Id_E]_{RN} = 0$, imply that $Id_E+\pi$ is a Nijenhuis vector valued form with respect to μ if, and only if, π is a Maurer-Cartan element of the curved symmetric differential graded Lie algebra (E, μ). The last statement follows from (11) and Theorem 2.4. ∎

3. Nijenhuis forms on Lie n-algebras

Lie n-algebras are particular cases of L_∞-algebras for which only $n+1$ brackets may be non-zero. We define Nijenhuis forms for this special case and we analyze, in particular, the Lie n-algebra defined by an n-plectic manifold.

A graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ is said to be *concentrated in degrees* $p_1, \cdots p_k$, with $p_1, \cdots, p_k \in \mathbb{Z}$, if $E_{p_1}, \cdots, E_{p_k}$ are the only non-zero components of E.

Definition 3.1. A symmetric Lie n-algebra is a symmetric L_∞-algebra whose underlying graded vector space is concentrated on degrees $-n, \cdots, -1$.

Remark 3.2. Note that by degree reasons, the only non-zero symmetric vector valued forms (multi-brackets) are $l_1, \cdots, l_{n+1}$.

Proposition 3.3. *Let* $(E = E_{-n} \oplus \cdots \oplus E_{-1}, \mu = l_1 + \cdots + l_{n+1})$ *be a Lie* n*-algebra. Let* k *be an integer such that* $\frac{n+3}{2} \leq k \leq n+1$ *and* N *be any symmetric vector valued* k*-form of degree zero on* E*. Then,* $\mathcal{N} = S + N$ *is a Nijenhuis vector valued form with respect to* μ*, with square* $S + 2N$*, and the deformed Lie* n*-algebra structure on* E *is of the form*

$$\mu^{\mathcal{N}} = l_1 + \cdots + l_{k-1} + (l_k + [N, l_1]_{RN}) + \cdots + (l_{n+1} + [N, l_{n-k+2}]_{RN}).$$

Proof: By Remark 3.2, any vector valued $(m+k-1)$-form, with $m \geq n-k+3$, is identically zero; hence

$$[N, l_m]_{RN} = 0, \tag{12}$$

for all $m \geq n-k+3$. Also, any vector valued $(2k+m-2)$-form, with $m \geq 1$, is identically zero because, from the conditions $\frac{n+3}{2} \leq k \leq n+1$ and $m \geq 1$, we get $2k+m-2 \geq n+2$. Thus,

$$[N,[N,l_m]_{RN}]_{RN}=0, \tag{13}$$

for all $m \geq 1$. From Equations (12) and (13), we get

$$[S+N,\mu]_{RN}=\mu+[N,l_1]_{RN}+\cdots+[N,l_{n-k+2}]_{RN} \tag{14}$$

and

$$[S+N,[S+N,\mu]_{RN}]_{RN}=\mu+2[N,l_1]_{RN}+\cdots+2[N,l_{n-k+2}]_{RN}. \tag{15}$$

On the other hand, using Lemma 2.5, we have

$$[S+N,S+2N]_{RN}=0. \tag{16}$$

Equations (15) and (16) show that $\mathcal{N}=S+N$ is a Nijenhuis vector valued form with respect to μ, with square $S+2N$, and Equation (14) shows that the deformed Lie n-algebra structure is

$$\mu^{\mathcal{N}}=l_1+\cdots+l_{k-1}+(l_k+[N,l_1]_{RN})+\cdots+(l_{n+1}+[N,l_{n-k+2}]_{RN}).$$

■

Proposition 3.3 admits the following generalization.

Proposition 3.4. *Let $(E=E_{-n}\oplus\cdots\oplus E_{-1},\mu=l_1+\cdots+l_{n+1})$ be a Lie n-algebra. Let $N_1,\cdots,N_l$ be a family of symmetric vector valued $k_1,\cdots,k_l$-forms, respectively, of degree zero on E, with $\frac{n+3}{2}\leq k_1\leq\cdots\leq k_l\leq n+1$. Then, $\mathcal{N}=S+\sum_{i=1}^l N_i$ is a Nijenhuis vector valued form with respect to μ, with square $S+2\sum_{i=1}^l N_i$. The deformed Lie n-algebra structure is*

$$\begin{aligned}\Big[S+\textstyle\sum_{i=1}^l N_i,\mu\Big]_{RN} &= \mu+\Big[\textstyle\sum_{i=1}^l N_i,l_1\Big]_{RN}+\cdots+\Big[\textstyle\sum_{i=1}^l N_i,l_{n-k_l+2}\Big]_{RN}\\ &+\Big[\textstyle\sum_{i\neq l} N_i,l_{n-k_l+3}\Big]_{RN}+\cdots+\Big[\textstyle\sum_{i\neq l} N_i,l_{n-k_{l-1}+2}\Big]_{RN}\\ &+\Big[\textstyle\sum_{i\neq l,l-1} N_i,l_{n-k_l+3}\Big]_{RN}+\cdots+\Big[\textstyle\sum_{i\neq l,l-1} N_i,l_{n-k_{l-1}+2}\Big]_{RN}\\ &+\cdots+\\ &+[N_1,l_{n-k_2+3}]_{RN}+\cdots+[N_1,l_{n-k_1+2}]_{RN}.\end{aligned}$$

Proof: Let $1\leq i,j\leq l$. By Remark 3.2, any vector valued $(m+k_i-1)$-form, with $m\geq n-k_i+3$, is identically zero; hence,

$$[N_i,l_m]_{RN}=0, \tag{17}$$

for all $m \geq n - k_i + 3$. Also, any vector valued $(k_i + k_j + m - 2)$-form, with $m \geq 1$ is identically zero, because out of the conditions $\frac{n+3}{2} \leq k_1 \leq \cdots \leq k_l \leq n+1$ and $m \geq 1$ we get $k_i + k_j + m - 2 \geq n + 2$. Thus,

$$\left[N_i, [N_j, l_m]_{RN}\right]_{RN} = 0, \tag{18}$$

for all $m \geq 1$. From Equations (17) and (18), we get

$$\begin{aligned}
\left[S + \textstyle\sum_{i=1}^{l} N_i, \mu\right]_{RN} &= \mu + \left[\textstyle\sum_{i=1}^{l} N_i, l_1\right]_{RN} + \cdots + \left[\textstyle\sum_{i=1}^{l} N_i, l_{n-k_l+2}\right]_{RN} \\
&+ \quad \left[\textstyle\sum_{i\neq l} N_i, l_{n-k_l+3}\right]_{RN} + \cdots + \left[\textstyle\sum_{i\neq l} N_i, l_{n-k_{l-1}+2}\right]_{RN} \\
&+ \quad \left[\textstyle\sum_{i\neq l,l-1} N_i, l_{n-k_l+3}\right]_{RN} + \cdots + \left[\textstyle\sum_{i\neq l,l-1} N_i, l_{n-k_{l-1}+2}\right]_{RN} \\
&+ \quad \cdots + \\
&+ \quad [N_1, l_{n-k_2+3}]_{RN} + \cdots + [N_1, l_{n-k_1+2}]_{RN}
\end{aligned}$$

and

$$\left[S + \sum_{i=1}^{l} N_i, \left[S + \sum_{i=1}^{l} N_i, \mu\right]_{RN}\right]_{RN} = \mu + 2\left[\sum_{i=1}^{l} N_i, \mu\right]_{RN} = \left[S + 2\sum_{i=1}^{l} N_i, \mu\right]_{RN}$$

It follows from the conditions $\frac{n+3}{2} \leq k_1 \leq \cdots \leq k_l \leq n+1$ that, for $1 \leq i, j \leq l$, we have $k_i + k_j - 1 \geq n + 2$. Hence, $[N_i, N_j]_{RN} = 0$ for all $1 \leq i, j \leq l$, which implies that

$$\left[S + \sum_{i=1}^{l} N_i, S + 2\sum_{i=1}^{l} N_i\right]_{RN} = 0.$$

■

Remark 3.5. In Proposition 3.4 one may replace each vector valued k_i-form N_i by a family of symmetric vector valued k_i-forms.

Next, we consider a particular class of Lie n-algebras, those associated to n-plectic manifolds. Let us recall some definitions from [19].

Definition 3.6. An *n-plectic manifold* is a manifold M equipped with a non-degenerate and closed $(n+1)$-form ω. It is denoted by (M, ω).

An $(n-1)$-form α on an n-plectic manifold (M, ω) is said to be a *Hamiltonian form* if there exists a smooth vector field χ_α on M such that $\mathsf{d}\alpha = -\iota_{\chi_\alpha}\omega$. The vector field χ_α is called the *Hamiltonian vector field* associated to α. The space of all Hamiltonian forms on an n-plectic manifold (M,ω) is denoted by $\Omega^{n-1}_{Ham}(M)$.

Given two Hamiltonian forms α,β on an n-plectic manifold (M,ω), with Hamiltonian vector fields χ_α and χ_β, respectively, one may define a bracket $\{.,.\}$ by setting

$$\{\alpha,\beta\} := \iota_{\chi_\alpha}\iota_{\chi_\beta}\omega.$$

It turns out that $\{\alpha,\beta\}$ is a Hamiltonian form with associated Hamiltonian vector field $[\chi_\alpha,\chi_\beta]$, see [19].

Following [19], we may associate to an n-plectic manifold (M,ω) a symmetric Lie n-algebra.

Theorem 3.7. *Let (M,ω) be an n-plectic manifold. Set*

$$E_i = \begin{cases} \Omega^{n-1}_{Ham}(M), & \text{if} \quad i=-1,\\ \Omega^{n+i}(M), & \text{if} \ -n\leq i\leq -2 \end{cases}$$

and $E=\oplus_{i=-n}^{-1}E_i$. Let the collection $l_k : E\times \overset{k}{\ldots}\times E\to E$, $k\geq 1$, of symmetric multi-linear maps be defined as

$$l_1(\alpha) = \begin{cases} (-1)^{|\alpha|}\mathsf{d}\alpha, & \text{if} \quad \alpha\notin E_{-1},\\ 0, & \text{if} \quad \alpha\in E_{-1}, \end{cases}$$

$$l_k(\alpha_1,\cdots,\alpha_k) =$$
$$= \begin{cases} 0, & \text{if } \alpha_i\notin E_{-1} \text{ for some } \ 0\leq i\leq k,\\ (-1)^{\frac{k}{2}+1}\iota_{\chi_{\alpha_1}}\cdots\iota_{\chi_{\alpha_k}}\omega, & \text{if } \alpha_i\in E_{-1} \text{ for all } \ 0\leq i\leq k \ \text{ and } \ k \text{ is even},\\ (-1)^{\frac{k-1}{2}}\iota_{\chi_{\alpha_1}}\cdots\iota_{\chi_{\alpha_k}}\omega, & \text{if } \alpha_i\in E_{-1} \text{ for all } \ 0\leq i\leq k \ \text{ and } \ k \text{ is odd}, \end{cases}$$

for $k\geq 2$, where χ_{α_i} is the Hamiltonian vector field associated to α_i. Then, $(E,(l_k)_{k\geq 1})$ is a symmetric Lie n-algebra.

Proof: In [19], an L_∞-algebra is defined to be a graded vector space L equipped with a collection $l_k : L^{\otimes^k}\to L$ of skew-symmetric maps, with $\bar{l_k} = k-2$, satisfying a relation so called graded Jacobi identity. However, by translations of degrees in the graded vector space as $L_i\to L_{-i}$, it is equivalent to say an L_∞-algebra is a graded vector space L equipped with a collection $\{l_k : L^{\otimes^k}\to L\}$ of skew-symmetric maps, with $\bar{l_k} = 2-k$, satisfying a certain graded Jacobi identity. Now, it is enough to shift, by 1, the degrees of the graded vector space in Theorem 3.14. in [19] and use the décalage isomorphism to get the desired result. ■

In the next proposition we give an example of a Nijenhuis vector valued form, with respect to the L_∞-algebra (Lie n-algebra) structure associated to a given n-plectic manifold, which is the sum of a symmetric vector valued 1-form with a symmetric vector valued i-form, with $i = 2, \cdots, n$.

Proposition 3.8. *Let (M, ω) be an n-plectic manifold with the associated symmetric Lie n-algebra structure $\mu = l_1 + \cdots + l_{n+1}$. For any n-form η on the manifold M, and any $i = 2, \ldots, n$, define $\widetilde{\eta}_i$ to be the symmetric vector valued i-form of degree zero given by*

$$\widetilde{\eta}_i(\beta_1, \cdots, \beta_i) = \begin{cases} \iota_{\chi_{\beta_1}} \cdots \iota_{\chi_{\beta_i}} \eta, & \text{if } \beta_i \in E_{-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

where $\chi_{\beta_1}, \cdots, \chi_{\beta_n}$ are the Hamiltonian vector fields of $\beta_1, \cdots, \beta_n$, respectively. Then, $S + \widetilde{\eta}_i$ is a Nijenhuis vector valued form with respect to μ, with square $S + 2\widetilde{\eta}_i$. The deformed structure is

$$[S + \widetilde{\eta}_i, \mu]_{RN} = \mu + [\widetilde{\eta}_i, l_1]_{RN} + [\widetilde{\eta}_i, l_2]_{RN}.$$

The proof of Proposition 3.8 is based on the following lemma.

Lemma 3.9. *For all $2 \leq i \leq n$, and all homogeneous elements $\alpha_1, \cdots, \alpha_i \in E$, we have:*

(1) $\widetilde{\eta}_i(l_1(\alpha_1), \alpha_2, \cdots, \alpha_i) = 0$,

(2) $[\widetilde{\eta}_i, l_m]_{RN} = \begin{cases} 0, & m \geq 3 \\ -\iota_{l_2}\widetilde{\eta}_i, & m = 2 \\ \mathsf{d} \circ \widetilde{\eta}_i, & m = 1 \end{cases}$

(3) $[\widetilde{\eta}_i, [\widetilde{\eta}_i, l_m]_{RN}]_{RN} = 0, \ m \geq 1$.

Proof: We start by noticing that from its definition, $\widetilde{\eta}_i$ vanishes on $\oplus_{i=-n}^{-2} E_i$ and $\operatorname{Im} \widetilde{\eta}_i \subset E_{-i}$, $i \geq 2$. So, to prove item (1), the only case we have to investigate is when $\alpha_1 \in E_{-2}$ and $l_1(\alpha_1), \alpha_2, \cdots, \alpha_i$ are all Hamiltonian forms. Let $\chi_{l_1(\alpha_1)}$ be the Hamiltonian vector field associated to $l_1(\alpha_1)$. Then, we have

$$\iota_{\chi_{l_1(\alpha_1)}} \omega = -\mathsf{d}(l_1(\alpha_1)) = -\mathsf{d}^2 \alpha_1 = 0,$$

thus $\chi_{l_1(\alpha_1)} = 0$, by the non-degeneracy of ω. This proves item (1).

Let us now compute $[\widetilde{\eta}_i, l_m]_{RN}$. When $m \geq 3$, from the definitions of l_m and $\widetilde{\eta}_i$, we get

$$l_m(\widetilde{\eta}_i(\alpha_1, \cdots, \alpha_i), \cdots, \alpha_{m+i-1}) = 0$$

and

$$\widetilde{\eta}_i(l_m(\alpha_1, \cdots, \alpha_m), \cdots, \alpha_{m+i-1}) = 0,$$

for all $\alpha_1, \cdots, \alpha_{i+m-1} \in E$, $i \geq 2$, so that $[\widetilde{\eta}_i, l_m]_{RN} = 0$. Since $\widetilde{\eta}_i$ takes value in E_{-i}, we have $\iota_{\widetilde{\eta}_i} l_2 = 0$, hence $[\widetilde{\eta}_i, l_2]_{RN} = -\iota_{l_2}\widetilde{\eta}_i$. From item (1) and definition of $\widetilde{\eta}_i$ we get $[\widetilde{\eta}_i, l_1]_{RN} = \mathsf{d} \circ \widetilde{\eta}_i$.

Last, we prove item (3). For $m \geq 3$, $[\widetilde{\eta}_i, [\widetilde{\eta}_i, l_m]_{RN}]_{RN} = 0$ is a direct consequence of item (2). The case $m = 2$ follows from the fact that $\widetilde{\eta}_i$ does not take value in E_{-1}, so $l_2(\widetilde{\eta}_i(\alpha_1, \cdots, \alpha_i), \alpha_{i+1}) = 0$, for all $\alpha_1, \cdots, \alpha_{i+1} \in E$. Hence, using item (2) we get

$$\iota_{\widetilde{\eta}_i}[\widetilde{\eta}_i, l_2]_{RN} = 0 \quad \text{and} \quad \iota_{[\widetilde{\eta}_i, l_2]_{RN}}\widetilde{\eta}_i = 0,$$

which gives $[\widetilde{\eta}_i, [\widetilde{\eta}_i, l_2]_{RN}]_{RN} = 0$. Similar arguments as those used above prove that $[\widetilde{\eta}_i, [\widetilde{\eta}_i, l_1]_{RN}]_{RN} = 0$. ■

Proof: (of Proposition 3.8) From Lemma 3.9 we have

$$[S + \widetilde{\eta}_i, \mu]_{RN} = \mu + [\widetilde{\eta}_i, l_1]_{RN} + [\widetilde{\eta}_i, l_2]_{RN} \tag{20}$$

and applying $[S + \widetilde{\eta}_i, .]_{RN}$ to both sides of Equation (20), we get

$$\begin{aligned} [S + \widetilde{\eta}_i, [S + \widetilde{\eta}_i, \mu]_{RN}]_{RN} &= \mu + 2[\widetilde{\eta}_i, l_1]_{RN} + 2[\widetilde{\eta}_i, l_2]_{RN} \\ &= [S + 2\widetilde{\eta}_i, \mu]_{RN}. \end{aligned}$$

Now, the equation

$$[S + \widetilde{\eta}_i, S + \widetilde{\eta}_i]_{RN} = 0,$$

holds, for all $i \geq 2$, as a consequence of $\iota_{\widetilde{\eta}_i}\widetilde{\eta}_i = 0$. ■

From Proposition 3.8 we immediately get the following result.

Theorem 3.10. *Let η be an arbitrary n-form on an n-plectic manifold (M, ω). Let $(E = E_{-n} \oplus \cdots \oplus E_{-1}, \mu = l_1 + \cdots + l_{n+1})$ be the Lie n-algebra associated to (M, ω). For each $2 \leq i \leq n$, define the maps $\widetilde{\eta}_i$ as in (19). Then, $\mathcal{N} := S + \sum_{i=2}^n \widetilde{\eta}_i$ is a Nijenhuis vector valued form with respect to the Lie n-algebra structure μ, with square $S + 2\sum_{i=2}^n \widetilde{\eta}_i$. Moreover, the deformed structure is of the form $[\mathcal{N}, \mu]_{RN} = \sum_{i=1}^{n+1} l_i^{\mathcal{N}}$, with $l_i^{\mathcal{N}}$ being the component in* *the vector valued form* $[\mathcal{N},\mu]_{RN}$ *which is a vector valued* i*-form, and is given by:*

$$l_i^{\mathcal{N}} = \begin{cases} l_1, & \textit{for } i=1,\\ l_i + \mathsf{d}\circ\widetilde{\eta_i} - \iota_{l_2}\widetilde{\eta_{i-1}}, & \textit{for } i\geq 2. \end{cases}$$

A special case of the previous theorem is considered in the next proposition.

Proposition 3.11. *Let* (M,ω) *be an* n*-plectic manifold and* α *a Hamiltonian form on* (M,ω)*. For each* $2\leq i\leq n$*, define the maps* $\widetilde{\alpha}_i$ *as*

$$\widetilde{\alpha}_i(\beta_1,\cdots,\beta_i) = \begin{cases} \iota_{\chi_\alpha}\iota_{\chi_{\beta_1}}\cdots\iota_{\chi_{\beta_i}}\omega, & \textit{if } \ \beta_k\in E_{-1} \textit{ for all } 1\leq k\leq i,\\ 0, & \textit{otherwise} \end{cases}$$

where $\chi_\alpha,\chi_{\beta_1},\cdots,\chi_{\beta_i}$ *are the Hamiltonian vector fields associated to the Hamiltonian forms* $\alpha,\beta_1,\cdots,\beta_i$*, respectively. Then,* $S+\sum_{i=2}^n\widetilde{\alpha}_i$ *is a Nijenhuis vector valued form with respect to the Lie* n*-algebra structure* $\mu = l_1+\cdots+l_{n+1}$*, associated to the* n*-plectic manifold* (M,ω)*.*

Theorem 3.10 can be easily generalized if, instead of taking one n-form on the manifold M, we take a family of n-forms on M.

Theorem 3.12. *Let* $(\eta^j)_{j\geq 1}$ *be a family of* n*-forms on an* n*-plectic manifold* (M,ω)*. Let* $(E=E_{-n}\oplus\cdots\oplus E_{-1}, \mu = l_1+\cdots+l_{n+1})$ *be the Lie* n*-algebra associated to* (M,ω)*. For each* $2\leq i\leq n$*, define the vector valued* i*-forms* $\widetilde{(\eta^j)}_i$ *as*

$$\widetilde{(\eta^j)}_i(\beta_1,\cdots,\beta_i) = \begin{cases} \iota_{\chi_{\beta_1}}\cdots\iota_{\chi_{\beta_i}}\eta^j, & \textit{if } \ \beta_k\in E_{-1} \textit{ for all } 1\leq k\leq i,\\ 0, & \textit{otherwise} \end{cases}$$

where $\chi_{\beta_1},\cdots,\chi_{\beta_i}$ *are the Hamiltonian vector fields associated to the Hamiltonian forms* $\beta_1,\cdots,\beta_i$*, respectively. Then,* $\mathcal{N} := S+\sum_{j\geq 1}\sum_{i=2}^n\widetilde{(\eta^j)}_i$ *is a Nijenhuis vector valued form with respect to the Lie* n*-algebra structure* μ*.*

4. The case of Lie 2-algebras

In this section we treat the case of Lie 2-algebras. We show how to construct Nijenhuis forms with respect to Lie 2-algebras, which are the sum of a vector valued 1-form with a vector valued 2-form.

We start by recalling that a Lie 2-algebra is a pair (E,μ), where E is a graded vector space with degrees concentrated in -2 and -1, that is $E = E_{-2}\oplus E_{-1}$, and $\mu = l_1+l_2+l_3$ with l_1, l_2 and l_3 being symmetric vector valued 1-form, 2-form and 3-form, respectively, all of them of degree 1. By degree reasons, the brackets l_1 and l_3 are not identically zero in the following cases:

$$l_1 : E_{-2} \to E_{-1}, \quad l_3 : E_{-1} \times E_{-1} \times E_{-1} \to E_{-2},$$

while the binary bracket l_2 has two parts

$$l_2|_{E_{-1}\times E_{-2}} : E_{-1} \times E_{-2} \to E_{-2}, \quad l_2|_{E_{-1}\times E_{-1}} : E_{-1} \times E_{-1} \to E_{-1}.$$

The equation $[\mu,\mu]_{_{RN}} = 0$ gives the following relations (by degree reasons, all the missing cases are identically zero):

$$[l_1, l_2]_{_{RN}}(f, g) = 0, \tag{21}$$

$$[l_1, l_2]_{_{RN}}(X, f) = 0, \tag{22}$$

$$\left(2[l_1, l_3]_{_{RN}} + [l_2, l_2]_{_{RN}}\right)(X, Y, f) = 0, \tag{23}$$

$$(2[l_1, l_3]_{_{RN}} + [l_2, l_2]_{_{RN}})(X, Y, Z) = 0, \tag{24}$$

$$[l_2, l_3]_{_{RN}}(X, Y, Z, W) = 0, \tag{25}$$

with $X, Y, Z, W \in E_{-1}$ and $f, g \in E_{-2}$.

Let us set

$$l_1 = \partial, \quad l_3 = \omega \tag{26}$$

and, for all $X, Y \in E_{-1}$ and $f \in E_{-2}$,

$$l_2|_{E_{-1}\times E_{-1}}(X, Y) = [X, Y]_2 \quad \text{and} \quad l_2|_{E_{-1}\times E_{-2}}(X, f) = \chi(X)f, \tag{27}$$

with $\chi : E_{-1} \to End(E_{-2})$. Then, we have:

Lemma 4.1. *A vector valued form $\mu = l_1 + l_2 + l_3$, with associated quadruple $(\partial, \chi, [.,.]_2, \omega)$ given by (26) and (27), is a Lie 2-algebra structure on $E = E_{-2} \oplus E_{-1}$ if and only if*

$$\chi(\partial f)g = -\chi(\partial g)f, \tag{28}$$

$$[X, \partial f]_2 = \partial(\chi(X)f), \tag{29}$$

$$\chi([X, Y]_2)f + \chi(Y)\chi(X)f - \chi(X)\chi(Y)f + \omega(X, Y, \partial f) = 0, \tag{30}$$

$$[[X, Y]_2, Z]_2 + c.p. = \partial(\omega(X, Y, Z)), \tag{31}$$

$$\begin{aligned}\chi(W)\omega(X, Y, Z) - \chi(Z)\omega(X, Y, W) + \chi(Y)\omega(X, Z, W)\\ -\chi(X)\omega(Y, Z, W) =\\ -\omega([X, Y]_2, Z, W) + \omega([X, Z]_2, Y, W) - \omega([X, W]_2, Y, Z)\\ -\omega([Y, Z]_2, X, W) + \omega([Y, W]_2, X, Z) - \omega([Z, W]_2, X, Y),\end{aligned} \tag{32}$$

for all $X, Y, Z, W \in E_{-1}$ and $f \in E_{-2}$.

Proof: We have the following equivalences, by applying the definition of Richardson-Nijenhuis bracket: (21) $\Leftrightarrow$ (28), (22) $\Leftrightarrow$ (29), (23) $\Leftrightarrow$ (30), (24) $\Leftrightarrow$ (31) and (25) $\Leftrightarrow$ (32). ■

The quadruple $(\partial, \chi, [.,.]_2, \omega)$ of Lemma 4.1 is the *quadruple associated to the Lie 2-algebra structure* $\mu = l_1 + l_2 + l_3$.

There is an associated Chevalley-Eilenberg differential to each Lie 2-algebra. Before giving its definition, we need the next lemma.

Lemma 4.2. *Let* $(E = E_{-2} \oplus E_{-1}, \mu = l_1 + l_2 + l_3)$ *be a Lie 2-algebra with corresponding quadruple* $(\partial, \chi, [.,.]_2, \omega)$ *and* $\eta \in S^k(E^*) \otimes E$ *be a vector valued* k*-form of degree* $k-2$. *Then,*

$$\begin{aligned}[][\eta, l_2]_{RN}(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i \chi(X_i)\eta(X_0, \cdots, \widehat{X_i}, \cdots, X_k) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \eta([X_i, X_j]_2, X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k),\end{aligned} \tag{33}$$

for all $X_0, \dots, X_k \in E_{-1}$, *where* $\widehat{X_i}$ *means the absence of* X_i.

Proof: By degree reasons, η has to be of the form $\eta : E_{-1} \times \overset{k}{\dots} \times E_{-1} \to E_{-2}$. Using the Richardson-Nijenhuis bracket definition one gets Equation (33). ■

Definition 4.3. Let $E = E_{-2} \oplus E_{-1}$ be a graded vector space concentrated on degrees -2 and -1, $S_k(E) \subset S^k(E^*) \otimes E$ be the subspace of all symmetric vector valued k-forms of degree $k-2$ and $S^\bullet(E) := \oplus_{k \ge 1} S_k(E)$. Let $\chi : E_{-1} \to End(E_{-2})$ be a representation of vector spaces and $[.,.] : E_{-1} \times E_{-1} \to E_{-1}$ a graded symmetric bilinear map. Then, the *Chevalley-Eilenberg differential* d^{CE} is the map

$$\mathsf{d}^{CE} : S^\bullet(E) \to S^\bullet(E)$$

such that, if $\eta \in S_k(E)$, then $\mathsf{d}^{CE}\eta \in S_{k+1}(E)$ is defined by

$$\begin{aligned}\mathsf{d}^{CE}\eta(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i \chi(X_i)\eta(X_0, \cdots, \widehat{X_i}, \cdots, X_k) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \eta([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k),\end{aligned}$$

for all $X_0, \dots, X_k \in E_{-1}$, where $\widehat{X_i}$ means for the absence of X_i.

In general, the operator d^{CE} does not square to zero. However, according to Lemma 4.2 it can be written as

$$\mathsf{d}^{CE} = [.,l_2]_{_{RN}},$$

and we get, from the graded Jacobi identity of the Richardson-Nijenhuis bracket, that d^{CE} squares to zero if and only if $[l_2,l_2]_{_{RN}} = 0$.

Next, we explain how a crossed module of Lie algebras can be seen as a Lie 2-algebra. Let us first recall the definition of a crossed module of Lie algebras [24]:

Definition 4.4. A *crossed module* of Lie algebras $(\mathfrak{g},\ [.,.]^{\mathfrak{g}})$ and $(\mathfrak{h},\ [.,.]^{\mathfrak{h}})$ is a homomorphism $\partial:\mathfrak{g}\to\mathfrak{h}$ together with an action by derivation of $\mathfrak{h}$ on $\mathfrak{g}$, that is, a linear map $\chi:\mathfrak{h}\to Hom(\mathfrak{g},\mathfrak{g})$ such that

$$\partial(\chi(h)g) = [h,\partial(g)]^{\mathfrak{h}},\quad \text{for all } \ g\in\mathfrak{g},\ h\in\mathfrak{h} \tag{34}$$

and

$$\chi(\partial(g_1))g_2 = [g_1,g_2]^{\mathfrak{g}},\quad \text{for all } \ g_1,g_2\in\mathfrak{g}. \tag{35}$$

Such a crossed module will be denoted by $(\mathfrak{g},\ \mathfrak{h},\ \partial,\ \chi)$.

From a Lie 2-algebra with vanishing vector valued 3-form, we may get a crossed module of Lie algebras.

Proposition 4.5. *Let $(E = E_{-2}\oplus E_{-1},\ \mu = l_1+l_2+l_3)$ be a Lie 2-algebra, with corresponding quadruple $(\partial,\chi,[.,.]_2,\omega)$ given by (26) and (27). If $\omega = 0$, then $(E_{-2},E_{-1},\partial,\chi)$ is a crossed module of Lie algebras.*

Proposition 3.3 provides the construction of Nijenhuis forms on Lie n-algebras. However, for the case $n=2$, that proposition does not give the possibility of having a Nijenhuis vector valued 2-form. We intend to give an example of Nijenhuis vector valued form with respect to a Lie 2-algebra structure μ on a graded vector space $E_{-2}\oplus E_{-1}$ which is not purely a 1-form, i.e. not just a collection of maps from E_i to E_i, $i=1,2$. As we have mentioned before, elements of degree zero in $\tilde{S}(E^*)\otimes E$ are necessarily of the form $N+\alpha$ with $N:E\to E$ a linear endomorphism preserving the degree and $\alpha: E\times E\to E$ a symmetric vector valued 2-form of degree zero.

Theorem 4.6. *Let $\mu = l_1+l_2+l_3$ be a Lie 2-algebra structure on a graded vector space $E = E_{-2}\oplus E_{-1}$ and α a symmetric vector valued 2-form of degree* *zero. Then, $S+\alpha$ is a Nijenhuis vector valued form with respect to μ, with square of $S+2\alpha$, if and only if*

$$\alpha(l_1(\alpha(X,Y)),Z)+c.p.=0,$$

for all $X,Y,Z\in E_{-1}$.

Proof: By degree reasons, the only case where the vector valued 3-form $[\alpha,[\alpha,l_1]_{RN}]_{RN}$ is not identically zero is when it is evaluated on elements of E_{-1}. In this case, we get

$$\begin{aligned}[\alpha,[\alpha,l_1]_{RN}]_{RN}(X,Y,Z) &= [\alpha,l_1]_{RN}(\alpha(X,Y),Z)+c.p.\\ &\quad -\alpha([\alpha,l_1]_{RN}(X,Y),Z)+c.p.\\ &= -2\alpha(l_1(\alpha(X,Y)),Z)+c.p.,\end{aligned}\tag{36}$$

for all $X,Y,Z\in E_{-1}$. Again by degree reasons, $[\alpha,[\alpha,l_2]_{RN}]_{RN}$ and $[\alpha,l_3]_{RN}$ are identically zero. So, we have

$$\begin{aligned}&[S+\alpha,[S+\alpha,l_1+l_2+l_3]_{RN}]_{RN}=\\ &= [S+\alpha,l_1+l_2+l_3+[\alpha,l_1]_{RN}+[\alpha,l_2]_{RN}]_{RN}\\ &= l_1+l_2+l_3+2[\alpha,l_1]_{RN}+2[\alpha,l_2]_{RN}+[\alpha,[\alpha,l_1]_{RN}]_{RN}\\ &= [S+2\alpha,l_1+l_2+l_3]_{RN}+[\alpha,[\alpha,l_1]_{RN}]_{RN}.\end{aligned}\tag{37}$$

On the other hand, Lemma 2.5 and Equation (2) imply that

$$[S+\alpha,S+2\alpha]_{RN}=0.\tag{38}$$

Equations (36), (37) and (38) show that $S+\alpha$ is a Nijenhuis vector valued form with respect to μ, with square $S+2\alpha$, if and only if $\alpha(l_1(\alpha(X,Y)),Z)+c.p.=0$, for all $X,Y,Z\in E_{-1}$. ■

Corollary 4.7. *Let $\mu=l_1+l_2+l_3$ be a Lie 2-algebra structure on a graded vector space $E=E_{-2}\oplus E_{-1}$, with $l_1=0$. Then, for every vector valued 2-form α of degree zero, $S+\alpha$ is a Nijenhuis vector valued form with respect to μ, with square $S+2\alpha$.*

Combining Theorems 4.6 and 2.4 we get the following proposition.

Proposition 4.8. *Let $\mu=l_1+l_2+l_3$ be a Lie 2-algebra structure on a graded vector space $E=E_{-2}\oplus E_{-1}$. Let α be a vector valued 2-form of degree zero such that $\alpha(l_1(\alpha(X,Y)),Z)+c.p.=0$, for all $X,Y,Z\in E_{-1}$. Let μ_k stand for the vector valued form defined by $\mu_k=[S+\alpha,[S+\alpha,\cdots,[S+$*

$\alpha, \mu]_{RN} \cdots]_{RN}]_{RN}$, with k copies of $S+\alpha$. Then, $S+\alpha$ is a Nijenhuis vector valued form with respect to all the terms of the hierarchy of successive deformations μ_k, with square $S+2\alpha$.

If $\mu = l_1+l_2+l_3$ is a Lie 2-algebra on $E = E_{-2}\oplus E_{-1}$ with $l_1 = 0$, then $[.,.]_2$, given by (27), is a Lie bracket on E_{-1}. Also, the condition $[l_2, l_3]_{RN} = 0$ means that l_3 is a Chevalley-Eilenberg-closed 3-form of this Lie algebra E_{-1} valued in E_{-2}. This kind of Lie 2-algebras are usually called *string Lie algebras*. A Lie 2-algebra $(E_{-2}\oplus E_{-1}, l_1+l_2+l_3)$ with $l_2 = l_3 = 0$ and l_1 invertible, is called a *trivial* Lie 2-algebra. The next example is an application of Theorem 4.6 to a trivial Lie 2-algebra.

Example 4.9. Let $\mathfrak{g}$ be a vector space and $[.,.]_{\mathfrak{g}}$ be a skew-symmetric bilinear map on $\mathfrak{g}$. Let $E_{-1} := \{-1\}\times\mathfrak{g}$, $E_{-2} := \{-2\}\times\mathfrak{g}$ and let $\partial : E_{-2}\to E_{-1}$ be given by $(-2, x)\mapsto(-1, x)$. Define $\alpha : E_{-1}\times E_{-1}\to E_{-2}$ to be vector valued 2-form on the graded vector space $E = E_{-2}\oplus E_{-1}$ as $((-1, x),(-1, y))\mapsto(-2,[x, y]_{\mathfrak{g}})$. Then, as a direct consequence of Theorem 4.6, we have that $S+\alpha$ is Nijenhuis with respect to ∂ if and only if $[.,.]_{\mathfrak{g}}$ is a Lie bracket.

Let us now look at the deformed Lie 2-algebra structure.

Proposition 4.10. *Let $\mu = l_1+l_2+l_3$ be a Lie 2-algebra structure on a graded vector space $E = E_{-2}\oplus E_{-1}$, with associated quadruple $(\partial, [.,.]_2, \chi, \omega)$. Let α be a symmetric vector valued 2-form of degree zero on E and set $\mathcal{N} = S+\alpha$. The deformed structure $\mu^{\mathcal{N}}$ is associated to the quadruple $(\partial', [.,.]'_2, \chi', \omega')$:*

$$\begin{array}{rcl} \partial' f & = & \partial f, \\ [X,Y]'_2 & = & [X,Y]_2 + \partial(\alpha(X,Y)), \\ \chi'(X)f & = & \chi(X)f - \alpha(\partial f, X), \\ \omega'(X,Y,Z) & = & \omega(X,Y,Z) + \mathsf{d}^{CE}\alpha(X,Y,Z), \end{array} \tag{39}$$

for all $X, Y, Z\in E_{-1}$ and $f\in E_{-2}$.

Proof: The statement follows from the following easy relations:

$$\begin{array}{l} [S+\alpha,\mu]_{RN} = l_1 + (l_2 + [\alpha, l_1]_{RN}) + (l_3 + [\alpha, l_2]_{RN}); \\ [\alpha, l_1]_{RN}(X,Y) = l_1(\alpha(X,Y)), \ \text{ for all } X, Y\in E_{-1}; \\ [\alpha, l_1]_{RN}(X,f) = -\alpha(l_1(f), X), \ \text{ for all } X\in E_{-1}, f\in E_{-2}; \\ [\alpha, l_2]_{RN} = \mathsf{d}^{CE}\alpha. \end{array}$$

■

Notice that, in the case of Proposition 4.10, the vector valued form $S-\alpha$ has the inverse effect of $S+\alpha$, that is, $[S-\alpha,[S+\alpha,\mu]_{RN}]_{RN}=\mu$.

As we have seen previously, string Lie algebras on $E_{-2}\oplus E_{-1}$ are in one to one correspondence with Lie algebra structures on $\mathfrak{g}:=E_{-1}$ together with a representation of the Lie algebra $\mathfrak{g}$ on the vector space $V:=E_{-2}$ and a Chevalley-Eilenberg 3-cocycle ω for this representation. Hence, we denote string Lie algebras as triples $(\mathfrak{g},V,\omega)$. According to Proposition 4.8, the deformation of a string Lie algebra $(\mathfrak{g},V,\omega)$ by $S+\alpha$, just amounts to change the 3-cocycle ω into $\omega+\mathsf{d}^{CE}\alpha$. So that, for string Lie algebras, adding up a coboundary, i.e., changing $(\mathfrak{g},V,\omega)$ into $(\mathfrak{g},V,\omega+\mathsf{d}^{CE}\alpha)$ can be seen as a Nijenhuis transformation by $S+\alpha$.

A *Lie 2-subalgebra* of a Lie 2-algebra $(E=E_{-2}\oplus E_{-1},\mu=l_1+l_2+l_3)$ is a Lie 2-algebra $(E'=E'_{-2}\oplus E'_{-1},\mu'=l'_1+l'_2+l'_3)$ with $E'_{-2}\subset E_{-2}$ and $E'_{-1}\subset E_{-1}$ vector subspaces,

$$l'_1=l_1|_{E'},\ l'_2=l_2|_{E'\times E'}\ \text{ and }\ l'_3=l_3|_{E'\times E'\times E'}.$$

Let us now investigate Lie 2-algebras structures for which $\chi=0$. There may be quite a few such Lie 2-algebras but we are going to show that, after a Nijenhuis transformation of the form $S+\alpha$, such Lie 2-algebras will be decomposed as a direct sum of a string Lie algebra with a trivial Lie 2-algebra.

Proposition 4.11. *Given a Lie 2-algebra structure $l_1+l_2+l_3$ on a graded vector space $E=E_{-2}\oplus E_{-1}$ and corresponding quadruple $(\partial,[.,.]_2,\chi,\omega)$, with $\chi=0$, there exists a Nijenhuis form $S+\alpha$, with α a vector valued 2-form of degree zero, such that the deformed bracket $[S+\alpha,l_1+l_2+l_3]$ is the direct sum of a string Lie 2-algebra with a trivial L_∞-algebra.*

Proof: We set $E^t_{-1}:=\mathrm{Im}(\partial)$, $E^s_{-2}:=\mathrm{Ker}(\partial)$ and we choose two subspaces $E^t_{-2}\subset E_{-2}$ and $E^s_{-1}\subset E_{-1}$ such that the following are direct sums: $E^t_{-2}\oplus E^s_{-2}=E_{-2}$ and $E^t_{-1}\oplus E^s_{-1}=E_{-1}$. Since $\chi=0$, by (29), the bracket $[.,.]_2$ vanishes on $E_{-1}\times E^t_{-1}$; so that, there exists a unique skew-symmetric bilinear map $\alpha:E_{-1}\times E_{-1}\to E^t_{-2}$ such that

$$\partial\alpha(X,Y)=-pr_{E^t_{-1}}([X,Y]_2),\ \text{for all}\ X,Y\in E_{-1},$$

where $pr_{E^t_{-1}}$ stands for the projection on E^t_{-1} with respect to E^s_{-1}. Note that $\alpha(X,Y)=0$ if X or Y belong to E^t_{-1}, therefore we have $\alpha(\partial\alpha(X,Y),Z)=0$, for all $X, Y, Z \in E_{-1}$. Hence, by Theorem 4.6, $S + \alpha$ is Nijenhuis form with square $S + 2\alpha$. We claim that, for the deformed bracket $l'_1 + l'_2 + l'_3 := [S + \alpha, l_1 + l_2 + l_3]_{RN}$, $(E^s_{-1} \oplus E^s_{-2}, l'^s_1 + l'^s_2 + l'^s_3)$ and $(E^t_{-1} \oplus E^t_{-2}, l'^t_1 + l'^t_2 + l'^t_3)$ are Lie 2-subalgebras of $(E = E_{-2} \oplus E_{-1}, \mu = l_1 + l_2 + l_3)$, where l'^s_i and l'^t_i stand for the restrictions of l'_i to $E^s_{-1} \oplus E^s_{-2}$ and $E^t_{-1} \oplus E^t_{-2}$, respectively. We also claim that $(E^s_{-1} \oplus E^s_{-2}, l'^s_1 + l'^s_2 + l'^s_3)$ is a string Lie 2-algebra while $(E^t_{-1} \oplus E^t_{-2}, l'^t_1 + l'^t_2 + l'^t_3)$ is a trivial Lie 2-algebra, and that their direct sum is isomorphic to $(E_{-2} \oplus E_{-1}, l'_1 + l'_2 + l'_3)$.

Let $(\partial', [.,.]', \chi', \omega')$ stand for the corresponding quadruple associated to the deformed structure $l'_1 + l'_2 + l'_3$. From $[l_1, l_2]_{RN} = 0$ we get $l_2(l_1(f), X) = 0$, for all $f \in E_{-2}$. This means that l_2 vanishes on E^t_{-1}. Also, since $\alpha(X, Y) = 0$ if X or Y belongs to E^t_{-1}, by Equations (39), we have that $\chi' = 0$ and $[.,.]'_2 = 0$ and hence l'_2 vanishes on E^t_{-1}. From $[l_1, l_3]_{RN} = 0$, we get that $\omega(X, Y, Z)$ vanishes for all $X \in E^t_{-1}$, so by Equations (39) the restriction of l'_3 to E^t_{-1} vanishes. Since the restriction of l_1 to E^t_{-2} is a bijection onto its image, the restriction of $l'_1 + l'_2 + l'_3$ to $E^t_{-1} \oplus E^t_{-2}$ is a Lie 2-subalgebra and it is a trivial Lie 2-algebra.

Next we prove that $(E^s_{-2} \oplus E^s_{-1}, l'_1 + l'_2 + l'_3)$ is a Lie 2-subalgebra with $l'_1(E^s_{-2}) = 0$ and hence is a string Lie algebra. Let $X, Y \in E^s_{-1}$. Then, by Equations (39) we have

$$l'_2(X, Y) = [X, Y]_2 + \partial\alpha(X, Y) = [X, Y]_2 - pr_{E^t_{-1}}([X, Y]_2).$$

This implies that

$$l'_2(X, Y) \in E^s_{-1}. \tag{40}$$

Let $X, Y, Z \in E^s_{-1}$. Then, we have $l'_1(X) = l'_1(Y) = l'_1(Z) = 0$. Hence, from $(2[l'_1, l'_3]_{RN} + [l'_2, l'_2]_{RN})(X, Y, Z) = 0$, we get

$$l'_1(l'_3(X, Y, Z)) = l'_2(l'_2(X, Y), Z). \tag{41}$$

Using Relation (40), the right hand side of Equation (41) belongs to E^s_{-1}, while according to the definition of E^t_{-1}, the left hand side of Equation (41) belongs to E^t_{-1} and since $E_{-1} = E^t_{-1} \oplus E^s_{-1}$ is a direct sum, both sides of Equation (41) should be zero. This implies that

$$l'_3(X, Y, Z) \in E^s_{-2}. \tag{42}$$

Relation (40) and Equation (42) show that $(E^s_{-2} \oplus E^s_{-1}, l'_1 + l'_2 + l'_3)$ is a Lie 2-subalgebra. Also, by definition of E^s_{-2}, we have $l'_1(E^s_{-2}) = 0$. This completes the proof. ∎

Next, it is interesting to see that Lie algebras themselves can be seen as Nijenhuis forms. We start by noticing that any vector valued 2-form of degree zero on a graded vector space $E_{-2} \oplus E_{-1}$ is of the form

$$\alpha(X,Y) = \begin{cases} -\alpha(Y,X), & \text{if } X,Y \in E_{-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{43}$$

This, together with the fact that α always takes value in E_{-2}, imply that

$$\alpha(\alpha(X,Y),Z) + c.p. = 0, \tag{44}$$

for all $X,Y,Z \in E_{-1}$. Equations (43) and (44) mean that any symmetric vector valued 2-form α on an arbitrary graded vector space $E_{-2} \oplus E_{-1}$ is a Lie algebra (not a graded Lie algebra). In the next proposition, we show that there is also a way to get a Lie bracket on a graded vector space $E = E_{-2} \oplus E_{-1}$ from a Nijenhuis form with respect to a Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$ on the vector space E.

Proposition 4.12. *Let $(E = E_{-2} \oplus E_{-1},\ \mu = l_1 + l_2 + l_3)$ be a Lie 2-algebra, with corresponding quadruple $(\partial, [.,.]_2, \chi, \omega)$. Let α be a vector valued 2-form of degree zero and define a bilinear map $\tilde{\alpha}$ by setting*

$$\tilde{\alpha}(X,Y) = \begin{cases} \alpha(X,Y), & \text{for } X,Y \in E_{-1}, \\ \alpha(\partial X, Y), & \text{for } X \in E_{-2}, Y \in E_{-1}, \\ \alpha(X, \partial Y), & \text{for } X \in E_{-1}, Y \in E_{-2}, \\ \alpha(\partial X, \partial Y), & \text{for } X,Y \in E_{-2}. \end{cases}$$

Then, $S + \alpha$ is Nijenhuis vector valued 2-form with respect to μ, with square $S + 2\alpha$, if and only if $(E, \tilde{\alpha})$ is a Lie algebra.

Proof: By definition, $\tilde{\alpha}$ is a skew-symmetric bilinear map on the vector space E and we have

$$\begin{aligned} \tilde{\alpha}(\tilde{\alpha}(X,Y),Z) + c.p. &= \alpha(\partial\alpha(X,Y),Z) + c.p., \\ \tilde{\alpha}(\tilde{\alpha}(f,Y),Z) + c.p. &= \alpha(\partial\alpha(\partial f,Y),Z) + c.p., \end{aligned} \tag{45}$$

for all $X,Y,Z \in E_{-1}$ and $f \in E_{-2}$. Hence, Theorem 4.6 together with (45) imply that $\tilde{\alpha}$ is a Lie bracket on the vector space E if, and only if, $S + \alpha$ is a Nijenhuis form with respect to μ, with square $S + 2\alpha$. ■

Last, we give a result involving weak Nijenhuis forms on a Lie 2-algebra.

Proposition 4.13. *Let $\partial : E_{-2} \to E_{-1}$ be a Lie 2-algebra structure on a graded vector space $E = E_{-2} \oplus E_{-1}$, that is, a Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$ on E, with $l_1 = \partial$ and $l_2 = l_3 = 0$. Let α be a symmetric vector valued 2-form of degree zero on the graded vector space E. If $S + \alpha$ is a weak Nijenhuis vector valued form with respect to ∂, then E_{-1} is a Lie algebra with a representation on E_{-2}.*

Proof: According to Proposition 2.3, $S+\alpha$ is a weak Nijenhuis vector valued form with respect to ∂ if and only if $[S+\alpha, \partial]_{RN}$ is an L_∞-structure on the graded vector space E which, in turn, is equivalent to

$$[[S+\alpha,\partial]_{RN}, [S+\alpha,\partial]_{RN}]_{RN} = 0$$

or to

$$[[\alpha,\partial]_{RN}, [\alpha,\partial]_{RN}]_{RN} = 0.$$

Therefore, $S+\alpha$ is a weak Nijenhuis vector valued form with respect to ∂ if and only if

$$\partial\alpha(\partial\alpha(X,Y),Z) + c.p.(X,Y,Z) = 0 \tag{46}$$

and

$$\alpha(\partial\alpha(X,Y),\partial f) + c.p.(X,Y,\partial f) = 0, \tag{47}$$

for all $X, Y, Z \in E_{-1}$ and $f \in E_{-2}$. Equation (46) means that $[X,Y] := \partial\alpha(X,Y)$ defines a Lie bracket on E_{-1} since clearly it is skew-symmetric. If we define a map $\cdot : E_{-1} \times E_{-2} \to E_{-2}$ by setting $X \cdot f := \alpha(X, \partial f)$, then (47) can be written as

$$[X,Y]\cdot f = X\cdot(Y\cdot f) - Y\cdot(X\cdot f),$$

which means that $\cdot$ is a representation of E_{-1} on E_{-2}. ■

Remark 4.14. A notion of Nijenhuis operator on a Lie 2-algebra independently appeared in [18], while the present paper was about to be completed. This notion is a particular case of ours, by the following reasons. First, in [18], a Nijenhuis operator is necessarily a vector valued 1-form. Second, if $\mathcal{N} = (N_0, N_1)$ is a Nijenhuis operator in the sense of Definition 3.2. in [18], with respect to a Lie 2-algebra $l_1 + l_2 + l_3$, then

$$[\mathcal{N}, [\mathcal{N}, l_i]_{RN}]_{RN} = [\mathcal{N}^2, l_i]_{RN}$$

holds for $i = 1, 2$ and 3, which means that $\mathcal{N}$ is a Nijenhuis vector valued form, in our sense, with square $\mathcal{N}^2$.

5. Nijenhuis forms on Courant algebroids

We recall that one can associate a Lie 2-algebra to a Courant algebroid [22]. We use this construction to see how $(1,1)$-tensors on a Courant algebroid, with vanishing Nijenhuis torsion, are related with Nijenhuis forms with respect to the associated Lie 2-algebra.

Definition 5.1. A *Courant algebroid* is a vector bundle $E \to M$ together with a non-degenerate inner product $\langle .,. \rangle$, a morphism of vector bundles $\rho : E \to TM$ and a bilinear operator $\circ : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, such that the following axioms hold:

(i) $(\Gamma(E), \circ)$ is a Leibniz algebra, i.e., $X \circ (Y \circ Z) = (X \circ Y) \circ Z + Y \circ (X \circ Z)$,
(ii) $\rho(X)\langle Y, Z\rangle = \langle X \circ Y, Z\rangle + \langle Y, X \circ Z\rangle$,
(iii) $\rho(X)\langle Y, Z\rangle = \langle X, Y \circ Z\rangle + \langle X, Z \circ Y\rangle$,

for all $X, Y, Z \in \Gamma(E)$.

When item (i) in Definition 5.1 does not hold, the quadruple $(E, \circ, \rho, \langle .,. \rangle)$ is called a *pre-Courant algebroid* [2].

The next proposition is stated in [11], for Courant algebroids. Since the proof does not use the fact of $\circ$ being a Leibniz bracket, the result also holds for pre-Courant algebroids.

Proposition 5.2. *For every pre-Courant algebroid $(E, \circ, \rho, \langle .,. \rangle)$ we have*

$$X \circ (fY) = f(X \circ Y) + (\rho(X)f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$.

Corollary 5.3. *Let $(E, \circ, \rho, \langle .,. \rangle)$ and $(E, \circ', \rho', \langle .,. \rangle)$ be two pre-Courant algebroids. If $\circ = \circ'$, then $\rho = \rho'$.*

Proof: Assume that $(E, \circ, \rho, \langle .,. \rangle)$ and $(E, \circ, \rho', \langle .,. \rangle)$ are both pre-Courant algebroids. By Proposition 5.2 we have

$$(\rho(X)f)Y = (\rho'(X)f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, which implies that $\rho = \rho'$. ■

We intend to define Nijenhuis deformation of Courant structures. Let $(E, \circ, \rho, \langle .,. \rangle)$ be a Courant algebroid. For a given endomorphism $N : E \to E$, the deformed bracket by N is a bilinear operation $\circ^N$, defined as:

$$X \circ^N Y := NX \circ Y + X \circ NY - N(X \circ Y),$$

for all $X, Y \in \Gamma(E)$. The deformation of ρ by N is the map ρ^N given by $\rho^N(X) = \rho(NX)$, $X \in \Gamma(E)$. The Nijenhuis torsion of N, with respect to the bracket $\circ$, is defined as:

$$T_\circ N(X,Y) := NX \circ NY - N(X \circ^N Y),$$

for all $X, Y \in \Gamma(E)$. A direct computation shows that

$$T_\circ N = \frac{1}{2}(\circ^{N,N} - \circ^{N^2}).$$

All maps $N : \Gamma(E) \to \Gamma(E)$ that will be considered here are $\mathcal{C}^\infty(M)$-linear, that is to say they are $(1,1)$-tensors, that is, smooth sections of endomorphisms of E. We denote an endomorphism (vector bundle morphism) of E and the induced map on $\Gamma(E)$ by the same letter.

According to [4], for every vector bundle $E \to M$, if $(\Gamma(E), \circ)$ is a Leibniz algebra and $N : E \to E$ is any endomorphism whose Nijenhuis torsion vanishes, then the pair $(\Gamma(E), \circ^N)$ is a Leibniz algebra. However, given a Courant algebroid $(E, \circ, \rho, \langle .,. \rangle)$ and a $(1,1)$-tensor N, $(E, \circ^N, \rho^N, \langle .,. \rangle)$ may fail to be a pre-Courant algebroid, even if the Nijenhuis torsion of N vanishes. Indeed, from [4] we have the following:

Theorem 5.4. *If N is an endomorphism on a pre-Courant algebroid $(E, \circ, \rho, \langle .,. \rangle)$, then the quadruple $(E, \circ^N, \rho^N, \langle .,. \rangle)$ is a pre-Courant algebroid if and only if*

$$X \circ (N + N^*)Y = (N + N^*)(X \circ Y) \text{ and } (N + N^*)(Y \circ Y) = ((N + N^*)Y) \circ Y$$

for all $X, Y \in \Gamma(E)$, where N^* stands for the transpose of N, with respect to $\langle .,. \rangle$.

Remark 5.5. In fact, Theorem 5.4 is slightly different from Theorem 4 in [4], because there, the authors start from a Courant algebroid. But the same proof is still valid for the case of pre-Courant.

A *Casimir function* or simply a Casimir on a Courant algebroid $(E, \circ, \rho, \langle .,. \rangle)$ is a function $f \in \mathcal{C}^\infty(M)$ such that $\rho(X)f = 0$, for all $X \in \Gamma(E)$. It is easy to check that f is a Casimir if and only if $\mathcal{D}f = 0$, where $\mathcal{D} : \mathcal{C}^\infty(M) \to \Gamma(E)$ is given by

$$\langle \mathcal{D}f, X \rangle = \rho(X)f. \tag{48}$$

Also, if f is a Casimir, then

$$(fX) \circ Y = f(X \circ Y) = X \circ (fY) \tag{49}$$

holds for all sections $X, Y \in \Gamma(E)$.

The next lemma is a slight generalization of a result in [4].[*]

Lemma 5.6. *Given a pre-Courant algebroid $(E, \circ, \rho, \langle ., . \rangle)$ and a map $N : \Gamma(E) \to \Gamma(E)$, if $N + N^* = \lambda Id_{\Gamma(E)}$, for some Casimir function $\lambda \in \mathcal{C}^\infty(M)$, then $(E, \circ^N, \rho^N, \langle ., . \rangle)$ is a pre-Courant algebroid.*

Proof: This lemma is a direct consequence of Theorem 5.4 together with (49). ■

Theorem 5.7. *Let $(E, \circ, \rho, \langle ., . \rangle)$ be a Courant algebroid and N a $(1,1)$-tensor on E whose Nijenhuis torsion vanishes and such that*

$$N + N^* = \lambda Id_{\Gamma(E)},$$

with λ being a Casimir function. Then, $(E, \circ^N, \rho^N, \langle ., . \rangle)$ is a Courant algebroid.

Proof: Note that $(E, \circ)$ is a Leibniz algebra, so that $(E, \circ^N)$ is also a Leibniz algebra since the Nijenhuis torsion of N vanishes. This, together with Lemma 5.6, prove the theorem. ■

Remark 5.8. For a (pre-)Courant algebroid $(E, \circ, \rho, \langle ., . \rangle)$, and a $(1,1)$-tensor N on E with $N + N^* = \lambda Id_{\Gamma(E)}$ and λ a Casimir function, we have

$$\rho^N(X)f = \rho(NX)f = \langle NX, \mathcal{D}f \rangle = \langle X, N^*\mathcal{D}f \rangle = \langle X, (-N + \lambda Id_{\Gamma(E)})\mathcal{D}f \rangle,$$

for all $X \in \Gamma(E)$, $f \in \mathcal{C}^\infty(M)$. This means that the operator $\mathcal{D}^N : \mathcal{C}^\infty(M) \to \Gamma(E)$ associated with the (pre-)Courant algebroid $(E, \circ^N, \rho^N, \langle ., . \rangle)$, is given by

$$\mathcal{D}^N = (-N + \lambda Id_{\Gamma(E)}) \circ \mathcal{D}. \tag{50}$$

If we consider the skew-symmetrization of $\circ$, we obtain the bracket $[., .]$ used in the original definition of Courant algebroid [17]:

$$[X, Y] = \frac{1}{2}(X \circ Y - Y \circ X), \tag{51}$$

with $X, Y \in \Gamma(E)$. The deformation of $[., .]$ by a $(1,1)$-tensor N on E is the bracket $[., .]_N$ on $\Gamma(E)$, given by

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y] = \frac{1}{2}(X \circ^N Y - Y \circ^N X).$$

[*]In [4], λ is a real number.

The next lemma is an axiom included in the original definition of Courant algebroid [20].

Lemma 5.9. *Let $(E, \circ, \rho, \langle ., . \rangle)$ be a Courant algebroid and $\mathcal{D}$ its associated operator, given by (48). Then,*

$$[X, fY] = f[X, Y] + (\rho(X)f)Y - \frac{1}{2}\langle X, Y\rangle \mathcal{D}f,$$

for all $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, where $[., .]$ is the bracket given by (51).

Remark 5.10. From the proof of Proposition 2.6.5 in [20], we realize that Lemma 5.9 also holds in the case of a pre-Courant algebroid.

In [22], it was proved that to each Courant algebroid corresponds a Lie 2-algebra. The result in [22] is established using the graded skew-symmetric version of a Lie 2-algebra and the definition of Courant algebroid with skew-symmetric bracket. With our conventions it goes as follows.

Let $(E, \circ, \rho, \langle ., . \rangle)$ be a Courant algebroid over M, with associated operator $\mathcal{D}$, given by (48). Consider the graded vector space $V = \mathcal{C}^\infty(M) \oplus \Gamma(E)$, where the elements of $\mathcal{C}^\infty(M)$ have degree -2 and the elements of $\Gamma(E)$ have degree -1, and the following symmetric vector valued forms l_1, l_2 and l_3 on V, defined by:

$$\begin{array}{rcl} l_1 f &=& \mathcal{D}f \\ l_2(X, Y) &=& \frac{1}{2}(X \circ Y - Y \circ X) \\ l_2(X, f) &=& \frac{1}{2}\langle X, \mathcal{D}f\rangle, \\ l_3(X, Y, Z) &=& \frac{1}{12}\langle X \circ Y - Y \circ X, Z\rangle + c.p., \end{array} \tag{52}$$

for all $X, Y, Z \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, with l_1, l_2 and l_3 being identically zero in all the other cases. Notice that $l_2|_{\Gamma(E)\times\Gamma(E)}$ coincides with the bracket $[., .]$ given by (51).

Proposition 5.11. *If $(E, \circ, \rho, \langle ., . \rangle)$ is Courant (respectively, pre-Courant) algebroid, then the pair $(V, l_1 + l_2 + l_3)$, constructed in above, is a symmetric Lie (respectively, pre-Lie*[†]*) 2-algebra.*

We call this symmetric Lie 2-algebra the symmetric Lie 2-algebra *associated* to the Courant algebroid $(E, \circ, \rho, \langle ., . \rangle)$.

[†]A pre-Lie 2-algebra is a pair $(E = E_{-2} \oplus E_{-1}, l_1 + l_2 + l_3)$, where E is a graded vector space concentrated in degrees -2 and -1, and l_1, l_2 and l_3 are symmetric graded vector valued 1-form, 2-form and 3-form, respectively, of degree 1.

Starting with a $(1,1)$-tensor on a Courant algebroid with vanishing Nijenhuis torsion we construct, in the next proposition, a Nijenhuis form for the Lie 2-algebra associated to that Courant structure. First, we need the following lemma.

Lemma 5.12. *Let $(E, \circ, \rho, \langle ., . \rangle)$ be a pre-Courant algebroid with the associated symmetric pre-Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$, on the graded vector space $V = \mathcal{C}^\infty(M) \oplus \Gamma(E)$. Let N be a $(1,1)$-tensor on E such that*

$$N + N^* = \lambda \, \mathrm{Id}_{\Gamma(E)},$$

with λ a Casimir function. Then, the pre-Lie 2-algebra structure associated to the pre-Courant algebroid $(E, \circ^N, \rho^N, \langle ., . \rangle)$ is $[\mathcal{N}, l_1 + l_2 + l_3]_{RN}$, with $\mathcal{N}$ defined as follows:

$$\mathcal{N}|_{\Gamma(E)} = N \quad \textit{and} \quad \mathcal{N}|_{\mathcal{C}^\infty(M)} = \lambda \, \mathrm{Id}_{\mathcal{C}^\infty(M)}. \tag{53}$$

Proof: Let us denote the pre-Lie 2-algebra associated to the pre-Courant algebroid $(E, \circ^N, \rho^N, \langle ., . \rangle)$ by $l_1^N + l_2^N + l_3^N$. Using (50) and (52) and taking into account the fact that $\mathcal{D}$ is a derivation, we have, for all $f \in \mathcal{C}^\infty(M)$ and for all $X, Y, Z \in \Gamma(E)$,

$$l_1^N f = \mathcal{D}^N f = \lambda \mathcal{D} f - N \mathcal{D} f = l_1(\mathcal{N} f) - \mathcal{N} l_1(f) = [\mathcal{N}, l_1]_{RN}(f), \tag{54}$$

$$\begin{aligned} l_2^N(X,Y) &= \tfrac{1}{2}(X \circ^N Y - Y \circ^N X) = l_2(NX, Y) + l_2(X, NY) - N l_2(X,Y) \\ &= [\mathcal{N}, l_2]_{RN}(X,Y), \end{aligned} \tag{55}$$

$$\begin{aligned} l_2^N(X,f) &= \tfrac{1}{2}\langle X, \mathcal{D}^N f \rangle = \tfrac{1}{2}\langle X, (-N + \lambda \, \mathrm{Id}_{\Gamma(E)}) \mathcal{D} f \rangle = \tfrac{1}{2}\langle X, N^* \mathcal{D} f \rangle \\ &= \tfrac{1}{2}\langle NX, \mathcal{D} f \rangle = l_2(NX, f) + \lambda l_2(X, f) - \lambda l_2(X, f) \\ &= l_2(\mathcal{N} X, f) + l_2(X, \mathcal{N} f) - \mathcal{N} l_2(X, f) \\ &= [\mathcal{N}, l_2]_{RN}(X, f) \end{aligned} \tag{56}$$

and

$$
\begin{aligned}
l_3^N(X,Y,Z) &= \frac{1}{12}\langle X \circ^N Y - Y \circ^N X, Z\rangle + c.p.(X,Y,Z)\\
&= \frac{1}{6}\langle l_2^N(X,Y), Z\rangle + c.p.(X,Y,Z)\\
&= \frac{1}{6}(\langle l_2(NX,Y)+l_2(X,NY) + (N^*-\lambda Id_{\Gamma(E)})l_2(X,Y), Z\rangle)+c.p.(X,Y,Z)\\
&= \frac{1}{6}(\langle l_2(NX,Y),Z\rangle + \langle l_2(X,NY),Z\rangle + \langle l_2(X,Y),NZ\rangle - \lambda\langle l_2(X,Y),Z\rangle)\\
&\quad +c.p.(X,Y,Z)\\
&= \frac{1}{6}(\langle l_2(NX,Y),Z\rangle + c.p.(NX,Y,Z) + \langle l_2(X,NY),Z\rangle + c.p.(X,NY,Z)\\
&\quad +\langle l_2(X,Y),NZ\rangle + c.p.(X,Y,NZ) - \lambda\langle l_2(X,Y),Z\rangle + c.p.(X,Y,Z))\\
&= l_3(\mathcal{N}X,Y,Z) + l_3(X,\mathcal{N}Y,Z) + l_3(X,Y,\mathcal{N}Z) - \mathcal{N}l_3(X,Y,Z)\\
&= [\mathcal{N}, l_3]_{RN}(X,Y,Z). \qquad (57)
\end{aligned}
$$

Equations (54), (55), (56) and (57) complete the proof. ■

For the case of a Courant algebroid, we have the following result.

Corollary 5.13. *Let $(E, \circ, \rho, \langle ., .\rangle)$ be a Courant algebroid with the associated symmetric Lie-2 algebra structure $\mu = l_1 + l_2 + l_3$, on the graded vector space $V = \mathcal{C}^\infty(M) \oplus \Gamma(E)$. Let N be a $(1,1)$-tensor on E such that*

$$
\begin{cases}
N + N^* = \lambda\, \mathrm{Id}_{\Gamma(E)},\\
(\Gamma(E), \circ^N) \text{ is a Leibniz algebra,}
\end{cases}
$$

with λ a Casimir function. Then, the Lie 2-algebra structure associated to the Courant algebroid $(E, \circ^N, \rho^N, \langle ., .\rangle)$ is $[\mathcal{N}, l_1 + l_2 + l_3]$, with $\mathcal{N}$ given by (53).

Proposition 5.14. *Let $(E, \circ, \rho, \langle ., .\rangle)$ be a Courant algebroid with the associated symmetric Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$, on the graded vector space $V = \mathcal{C}^\infty(M) \oplus \Gamma(E)$. Let N be a $(1,1)$-tensor on E whose Nijenhuis torsion with respect to the bracket $\circ$ vanishes and satisfies the following conditions*

$$
\begin{cases}
N + N^* = \lambda\, \mathrm{Id}_{\Gamma(E)}\\
N^2 + (N^2)^* = \gamma\, \mathrm{Id}_{\Gamma(E)},
\end{cases}
$$

with λ and γ Casimir functions. Define $\mathcal{N}$ and $\mathcal{K}$ as

$$\mathcal{N}|_{\Gamma(E)} = N \;\; \textit{and} \;\; \mathcal{N}|_{\mathcal{C}^\infty(M)} = \lambda \, \mathrm{Id}_{\mathcal{C}^\infty(M)},$$

$$\mathcal{K}|_{\Gamma(E)} = N^2 = \lambda N + \frac{\gamma - \lambda^2}{2} \mathrm{Id}_{\Gamma(E)} \;\; \textit{and} \;\; \mathcal{K}|_{\mathcal{C}^\infty(M)} = \gamma \, \mathrm{Id}_{\mathcal{C}^\infty(M)}.$$

Then, $\mathcal{N}$ is a Nijenhuis vector valued 1-form with respect to μ, with square $\mathcal{K}$.

Proof: Since the Nijenhuis torsion of N vanishes, $(E, \circ^N)$ and $(E, \circ^{N^2})$ are Leibniz algebras [4], [2]. Applying Corollary 5.13 for the Courant algebroid $(E, \circ, \rho, \langle ., . \rangle)$, the $(1,1)$-tensor N and the vector valued 1-form $\mathcal{N}$, twice, we get

$$l_1^{N,N} + l_2^{N,N} + l_3^{N,N} = [\mathcal{N}, [\mathcal{N}, l_1 + l_2 + l_3]_{_{RN}}]_{_{RN}}, \tag{58}$$

where $l_1^{N,N} + l_2^{N,N} + l_3^{N,N}$ stands for the Lie 2-algebra structure associated to the Courant algebroid $(E, \circ^{N,N}, \rho^{N,N}, \langle ., . \rangle)$. Applying again Corollary 5.13 for the Courant algebroid $(E, \circ, \rho, \langle ., . \rangle)$, the $(1,1)$-tensor N^2 and the vector valued 1-form $\mathcal{K}$, we get

$$l_1^{N^2} + l_2^{N^2} + l_3^{N^2} = [\mathcal{K}, l_1 + l_2 + l_3]_{_{RN}}, \tag{59}$$

where $l_1^{N^2} + l_2^{N^2} + l_3^{N^2}$ stands for the Lie 2-algebra structure associated to the Courant algebroid $(E, \circ^{N^2}, \rho^{N^2}, \langle ., . \rangle)$. On the other hand, since the Nijenhuis torsion of N vanishes, the Courant algebroids $(E, \circ^{N,N}, \rho^{N,N}, \langle ., . \rangle)$ and $(E, \circ^{N^2}, \rho^{N^2}, \langle ., . \rangle)$ coincide. Therefore, (58) and (59) imply that

$$[\mathcal{N}, [\mathcal{N}, l_1 + l_2 + l_3]_{_{RN}}]_{_{RN}} = [\mathcal{K}, l_1 + l_2 + l_3]_{_{RN}}.$$

Finally, an easy computation shows that $[\mathcal{N}, \mathcal{K}]_{_{RN}}$ vanishes both on functions and on sections of E. ■

Since the Lie 2-algebra structure entirely encodes the Courant algebroid structure, there was a hope that we could, given a Courant structure, find a Nijenhuis deformation by a Nijenhuis tensor which is the sum of a vector valued 1-form and a vector valued 2-form of the corresponding Lie 2-algebra structure, and prove, eventually, that the Lie 2-algebra structure obtained by this procedure comes from a Courant structure. But this fails, at least when the anchor is not identically zero, as it is shown in the next proposition. First, notice that every $C^\infty(M)$-linear vector valued form of degree 0 on $E_{-2} \oplus E_{-1}$, where $E_{-2} := \mathcal{C}^\infty(M)$ and $E_{-1} := \Gamma(E)$, is the sum of a 2-form α, a $(1,1)$-tensor N and an endomorphism of $C^\infty(M)$ of the form $f \mapsto \lambda f$ for some smooth function λ. Hence, we denote a $C^\infty(M)$-linear vector valued form of degree zero on $E_{-2} \oplus E_{-1}$ as a sum, $\lambda + N + \alpha$.

Theorem 5.15. *Let $(\circ, \rho, \langle ., . \rangle)$ be a Courant structure on a vector bundle $E \to M$ with the associated Lie 2-algebra structure $l_1 + l_2 + l_3$ on the graded vector space $V = E_{-2} \oplus E_{-1}$, where $E_{-2} := \mathcal{C}^\infty(M)$ and $E_{-1} := \Gamma(E)$. Let $\mathcal{N} = \lambda + N + \alpha$ be a $C^\infty(M)$-linear vector valued form of degree zero on V. Assume also that ρ is not equal to zero on a dense subset of the base manifold. If $[\mathcal{N}, l_1 + l_2 + l_3]_{RN}$ is the Lie 2-algebra associated to a Courant structure with the same scalar product $\langle ., . \rangle$, then*

(1) λ is a Casimir,
(2) $\alpha = 0$,
(3) $N + N^* = \lambda Id_{\Gamma(E)}$.

In this case, the Courant structure that $[\mathcal{N}, l_1 + l_2 + l_3]_{RN}$ is associated to, is $(\circ^N, \rho^N, \langle ., . \rangle)$.

Proof: Set $\mu = l_1 + l_2 + l_3$ and denote the i-form component of $[\mathcal{N}, \mu]_{RN}$ by $[\mathcal{N}, \mu]^i_{RN}$, $i = 1, 2$. Then, for all $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, we have

$$\begin{aligned} [\mathcal{N}, \mu]^1_{RN}(f) &= ([\lambda, l_1]_{RN} + [N, l_1]_{RN})(f) \\ &= l_1(\lambda f) - N l_1(f) \\ &= \lambda l_1(f) + f l_1(\lambda) - N l_1(f). \end{aligned}$$

The first equation in (52) implies that, if $[\mathcal{N}, \mu]_{RN}$ is a Lie 2-algebra associated to a Courant algebroid, then $[\mathcal{N}, \mu]^1_{RN}$ has to be a derivation, and this happens if and only if $l_1(\lambda) = 0$. So, we get that λ is a Casimir and

$$[\mathcal{N}, \mu]^1_{RN}(f) = (\lambda Id_{\Gamma(E)} - N) l_1(f). \tag{60}$$

On the other hand,

$$\begin{aligned} [\mathcal{N}, \mu]^2_{RN}(X, f) &= ([\lambda, l_2]_{RN} + [N, l_2]_{RN} + [\alpha, l_1]_{RN})(X, f) \\ &= l_2(X, \lambda f) - \lambda l_2(X, f) + l_2(NX, f) - \alpha(X, l_1(f)) \\ &= \tfrac{1}{2}\lambda \langle X, l_1(f) \rangle - \tfrac{1}{2}\lambda \langle X, l_1(f) \rangle + \tfrac{1}{2}\langle NX, l_1(f) \rangle - \alpha(X, l_1(f)) \\ &= \tfrac{1}{2}\langle NX, l_1(f) \rangle - \alpha(X, l_1(f)), \end{aligned} \tag{61}$$

and the same computations for (f, X) instead of (X, f) gives

$$[\mathcal{N}, \mu]^2_{RN}(f, X) = \frac{1}{2}\langle NX, l_1(f) \rangle - \alpha(l_1(f), X). \tag{62}$$

Since $[\mathcal{N},\mu]^2_{RN}(X,f) = [\mathcal{N},\mu]^2_{RN}(f,X)$, from (61) and (62) we get $\alpha(X,l_1(f)) = 0$, for all $X \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$; so,

$$[\mathcal{N},\mu]^2_{RN}(X,f) = \frac{1}{2}\langle NX, l_1(f)\rangle. \tag{63}$$

For any $X, Y \in \Gamma(E)$, we have

$$\begin{aligned}[\mathcal{N},\mu]^2_{RN}(X,Y) &= ([\lambda,l_2]_{RN} + [N,l_2]_{RN} + [\alpha,l_1]_{RN})(X,Y) \\ &= l_2(NX,Y) + l_2(X,NY) - Nl_2(X,Y) + l_1\alpha(X,Y).\end{aligned} \tag{64}$$

According to Lemma 5.9, if $[\mathcal{N},\mu]_{RN}$ is a Lie 2-algebra associated to a Courant structure, then we must have:

$$[\mathcal{N},\mu]^2_{RN}(X,fY) = f[\mathcal{N},\mu]^2_{RN}(X,Y) + 2[\mathcal{N},\mu]^2_{RN}(X,f).Y - \frac{1}{2}\langle X,Y\rangle[\mathcal{N},\mu]^1_{RN}(f). \tag{65}$$

Using (60), (63) and (64), we get

$$\begin{aligned}&[\mathcal{N},\mu]^2_{RN}(X,fY) = l_2(NX,fY) + l_2(X,NfY) - Nl_2(X,fY) + l_1\alpha(X,fY) \\ &= fl_2(NX,Y) + 2l_2(NX,f)Y - \frac{1}{2}\langle NX,Y\rangle l_1(f) \\ &\quad + fl_2(X,NY) + 2l_2(X,f)NY - \frac{1}{2}\langle X,NY\rangle l_1(f) \\ &\quad - fNl_2(X,NY) - 2l_2(X,f)NY + \frac{1}{2}\langle X,Y\rangle Nl_1(f) \\ &\quad + fl_1\alpha(X,Y) + \alpha(X,Y)l_1(f) \\ &= f(l_2(NX,Y) + l_2(X,NY) - Nl_2(X,Y) + l_1\alpha(X,Y)) + 2l_2(NX,f)Y \\ &\quad - \frac{1}{2}\langle X,(N+N^*)Y\rangle l_1(f) + \frac{1}{2}\langle X,Y\rangle Nl_1(f) + \alpha(X,Y)l_1(f)\end{aligned} \tag{66}$$

and

$$f[\mathcal{N},\mu]^2_{RN}(X,Y) + 2[\mathcal{N},\mu]^2_{RN}(X,f).Y - \frac{1}{2}\langle X,Y\rangle[\mathcal{N},\mu]^1_{RN}(f) \tag{67}$$

$$\begin{aligned}&= f(l_2(NX,Y) + l_2(X,NY) - Nl_2(X,NY) + l_1\alpha(X,Y)) + 2l_2(NX,f).Y \\ &- \frac{1}{2}\langle X,Y\rangle(\lambda\,\mathrm{Id}_{\Gamma(E)} - N)l_1(f).\end{aligned}$$

Now, Equations (65), (66) and (67) show that

$$\frac{1}{2}\langle X,(N+N^* - \lambda\,\mathrm{Id}_{\Gamma(E)})Y\rangle l_1(f) = \alpha(X,Y)l_1(f),$$

for all $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$. Since α is skew-symmetric, $\langle ., (N+N^* - \lambda\,\mathrm{Id}).\rangle$ is symmetric on $\Gamma(E) \times \Gamma(E)$ and the anchor is not zero everywhere, which implies that $l_1(f)$ is not always zero, we have $\alpha = 0$ and $N + N^* - \lambda\,\mathrm{Id}_{\Gamma(E)} = 0$. ■

Corollary 5.16. *Let $(E, \circ, \rho, \langle ., .\rangle)$ be a Courant algebroid with anchor ρ being different from zero on a dense subset of E and let μ be the associated Lie 2-algebra structure on the graded vector space $\mathcal{C}^\infty(M) \oplus \Gamma(E)$. Then, there is a one to one correspondence between:*

(i) quadruples (N, K, λ, γ) with N, K being $(1,1)$-tensors on E and λ, γ being Casimir functions satisfying the following conditions:

$$\begin{cases} \circ^{N,N} = \circ^K, \\ NK - KN = 0, \\ N + N^* = \lambda\,\mathrm{Id}_{\Gamma(E)}, \\ K + K^* = \gamma\,\mathrm{Id}_{\Gamma(E)}, \\ (\Gamma(E), \circ^N) \ \textit{and}\ (\Gamma(E), \circ^K) \ \textit{are Leibniz algebras.} \end{cases}$$

(ii) Nijenhuis vector valued forms $\mathcal{N}$ with respect to μ, with square $\mathcal{K}$, such that the deformed brackets $[\mathcal{N}, \mu]_{RN}$ and $[\mathcal{K}, \mu]_{RN}$ are Lie 2-algebras associated to Courant structures with the same scalar product.

Proof: Given a quadruple (N, K, λ, γ) satisfying conditions in item (i), we define vector valued 1-forms $\mathcal{N}$ and $\mathcal{K}$ on the graded vector space $\mathcal{C}^\infty(M) \oplus \Gamma(E)$ as $\mathcal{N}(f) = \lambda f$, $\mathcal{K}(f) = \gamma f$, $\mathcal{N}(X) = NX$ and $\mathcal{K}(X) = KX$, for all $X \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$. We prove that $\mathcal{N}$ is a Nijenhuis vector valued form with respect to μ, with square $\mathcal{K}$. First, notice that using Corollary 5.3, the assumption $\circ^{N,N} = \circ^K$ implies that $(E, \circ^{N,N}, \rho^{N,N}, \langle ., .\rangle)$ and $(E, \circ^K, \rho^K, \langle ., .\rangle)$ are the same pre-Courant algebroid, hence, they have the same associated pre-Lie 2-algebras. On the other hand, using Lemma 5.12, the pre-Lie 2-algebra associated to the pre-Courant algebroid $(E, \circ^{N,N}, \rho^{N,N}, \langle ., .\rangle)$ is $[\mathcal{N}, [\mathcal{N}, \mu]_{RN}]_{RN}$ and the pre-Lie 2-algebra associated to the pre-Courant algebroid $(E, \circ^K, \rho^K, \langle ., .\rangle)$ is $[\mathcal{K}, \mu]_{RN}$. Hence,

$$[\mathcal{N}, [\mathcal{N}, \mu]_{RN}]_{RN} = [\mathcal{K}, \mu]_{RN}. \tag{68}$$

Also, using the assumption $NK - KN = 0$, we get

$$[\mathcal{N}, \mathcal{K}]_{RN} = 0. \tag{69}$$

Equations (68) and (69) show that $\mathcal{N}$ is a Nijenhuis vector valued 1-form with respect to μ, with square $\mathcal{K}$. By Corollary 5.13, $[\mathcal{N},\mu]_{_{RN}}$ is a Lie 2-algebra associated to the Courant algebroid $(E,\circ^N,\rho,\langle.,.\rangle)$ and $[\mathcal{K},\mu]_{_{RN}}$ is a Lie 2-algebra associated to the Courant algebroid $(E,\circ^K,\rho,\langle.,.\rangle)$.

Conversely, assume that $\mathcal{N}$ is a Nijenhuis vector valued form with respect to μ, with square $\mathcal{K}$, such that $[\mathcal{N},\mu]_{_{RN}}$ and $[\mathcal{K},\mu]_{_{RN}}$ are Lie 2-algebras associated to Courant algebroids. Then, by Theorem 5.15, $\mathcal{N}$ is of the form $\lambda+N$ with $N+N^*=\lambda\,\mathrm{Id}_{\Gamma(E)}$ and $\mathcal{K}$ is of the form $\gamma+K$, with $K+K^*=\gamma\,\mathrm{Id}_{\Gamma(E)}$. Moreover, the Courant algebroid which is associated to the Lie 2-algebra $[\mathcal{N},\mu]_{_{RN}}$ (respectively, $[\mathcal{K},\mu]_{_{RN}}$) is $(E,\circ^N,\rho^N,\langle.,.\rangle)$ (respectively, $(E,\circ^K,\rho^K,\langle.,.\rangle)$). From this, we get that $(\Gamma(E),\circ^N)$ and $(\Gamma(E),\circ^K)$ are Leibniz algebras. Since $\mathcal{N}$ is a Nijenhuis vector valued form with respect to μ, with square $\mathcal{K}$, we have

$$[\mathcal{N},[\mathcal{N},\mu]_{_{RN}}]_{_{RN}}=[\mathcal{K},\mu]_{_{RN}} \tag{70}$$

and

$$[\mathcal{N},\mathcal{K}]_{_{RN}}=0. \tag{71}$$

Applying both sides of Equation (70) on a pair of sections $X,Y\in\Gamma(E)$ we get $X\circ^{N,N}Y=X\circ^K Y$, which implies $\circ^{N,N}=\circ^K$. Lastly, Equation (71) implies $KN-NK=0$. ∎

Using Lemma 5.9 and Remark 5.10, and also taking into account the fact that the operator $\mathcal{D}$ associated to a pre-Courant algebroid $(E,\circ,\rho,\langle.,.\rangle)$, given by (48), is a derivation, we may restate Theorem 5.15.

Theorem 5.17. *Let $(\circ,\rho,\langle.,.\rangle)$ be a Courant structure on a vector bundle $E\to M$, with the associated symmetric Lie 2-algebra structure $l_1+l_2+l_3$ on the graded vector space $V=E_{-2}\oplus E_{-1}$, where $E_{-2}:=\mathcal{C}^\infty(M)$ and $E_{-1}:=\Gamma(E)$. Let $\mathcal{N}=\lambda+N+\alpha$ be a $C^\infty(M)$-linear vector valued form of degree zero on V. Assume also that ρ is not equal to zero on a dense subset of the base manifold. If $[\mathcal{N},l_1+l_2+l_3]_{_{RN}}=l'_1+l'_2+l'_3$, where the vector valued forms l'_1,l'_2,l'_3 are obtained from a pre-Courant algebroid, with the same scalar product, by the construction given in (52), then*

(1) λ is a Casimir,
(2) $\alpha=0$,
(3) $N+N^*=\lambda\,\mathrm{Id}_{\Gamma(E)}$.

In this case, the Courant structure that $[\mathcal{N},l_1+l_2+l_3]_{_{RN}}$ is associated to, is $(\circ^N,\rho^N,\langle.,.\rangle)$.

And this leads to the next result:

Corollary 5.18. *Let $(E, \circ, \rho, \langle ., . \rangle)$ be a Courant algebroid with anchor ρ being different from zero on a dense subset of E, with the associated Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$ on the graded vector space $\mathcal{C}^\infty(M) \oplus \Gamma(E)$. Then, there is a one to one correspondence between:*

(i) quadruples (N, K, λ, γ) with N, K being $(1,1)$-tensors and λ, γ being Casimir functions satisfying the following conditions:

$$\begin{cases} \circ^{N,N} = \circ^K, \\ NK - KN = 0, \\ N + N^* = \lambda \operatorname{Id}_{\Gamma(E)}, \\ K + K^* = \gamma \operatorname{Id}_{\Gamma(E)}. \end{cases} \tag{72}$$

(ii) Nijenhuis vector valued forms $\mathcal{N}$ with respect to μ, with square $\mathcal{K}$, such that the deformed bracket is of the form $[\mathcal{N}, \mu]_{RN} = l_1' + l_2' + l_3'$ and l_1', l_2', l_3' are constructed by the procedure in (52) obtained from a pre-Courant algebroid, with the same scalar product.

Proof: Let $\mathcal{N}$ be a Nijenhuis vector valued form with respect to the Lie 2-algebra structure $\mu = l_1 + l_2 + l_3$, with square $\mathcal{K}$, and assume that $[\mathcal{N}, \mu]_{RN}$ is obtained from a pre-Courant algebroid. Let $\mathcal{N}|_{\Gamma(E)} = N, \mathcal{N}|_{\mathcal{C}^\infty(M)} = \lambda \operatorname{Id}_{\mathcal{C}^\infty(M)}$, $\mathcal{K}|_{\Gamma(E)} = K$ and $\mathcal{K}|_{\mathcal{C}^\infty(M)} = \gamma \operatorname{Id}_{\mathcal{C}^\infty(M)}$. By Theorem 5.17, $N + N^* = \lambda \operatorname{Id}_{\Gamma(E)}$ and $(E, \circ^N, \rho^N, \langle ., . \rangle)$ is a pre-Courant algebroid (it is, in fact, the pre-Courant algebroid which $[\mathcal{N}, \mu]_{RN}$ is obtained from). Hence, by Lemma 5.6, $(E, \circ^{N,N}, \rho^{N,N}, \langle ., . \rangle)$ is a pre-Courant algebroid. Now, Lemma 5.12 implies that $[\mathcal{K}, \mu]_{RN} = \left[\mathcal{N}, [\mathcal{N}, \mu]_{RN}\right]_{RN}$ is obtained from the pre-Courant algebroid $(E, \circ^{N,N}, \rho^{N,N}, \langle ., . \rangle)$, by the construction given in (52). Therefore, by Theorem 5.17, $K + K^* = \gamma \operatorname{Id}_{\Gamma(E)}$. The assumption $[\mathcal{N}, \mathcal{K}]_{RN} = 0$ implies that $NK - KN = 0$, while $\left[\mathcal{N}, [\mathcal{N}, \mu]_{RN}\right]_{RN} = [\mathcal{K}, \mu]_{RN}$ implies that $\circ^{N,N} = \circ^K$.

Conversely, assume that we are given a quadruple (N, K, λ, γ) satisfying the properties in (72). By Lemma 5.6, $(E, \circ^N, \rho^N, \langle ., . \rangle)$ is a pre-Courant and by Lemma 5.12, the pre-Lie 2-algebra structure associated to the pre-Courant algebroid $(E, \circ^N, \rho^N, \langle ., . \rangle)$ is $[\mathcal{N}, \mu]_{RN}$. Similar arguments prove that the pre-Lie 2-algebra structure associated to the pre-Courant algebroid $(E, \circ^{N,N}, \rho^{N,N}, \langle ., . \rangle)$ is $\left[\mathcal{N}, [\mathcal{N}, \mu]_{RN}\right]_{RN}$ and the pre-Lie 2-algebra structure associated to the pre-Courant algebroid $(E, \circ^K, \rho^K, \langle ., . \rangle)$ is $[\mathcal{K}, \mu]_{_{RN}}$. Now, the assumption $\circ^{N,N} = \circ^K$ and Lemma 5.3 imply that $(E, \circ^{N,N}, \rho^{N,N}, \langle ., . \rangle)$ and $(E, \circ^K, \rho^K, \langle ., . \rangle)$ are the same pre-Courant algebroid; therefore, we have $\left[\mathcal{N}, [\mathcal{N}, \mu]_{_{RN}}\right]_{_{RN}} = [\mathcal{K}, \mu]_{_{RN}}$. It follows from the assumption $NK - KN = 0$ that $[\mathcal{N}, \mathcal{K}] = 0$. Hence, $\mathcal{N}$ is a Nijenhuis vector valued form with respect to the Lie 2-algebra structure μ, with square $\mathcal{K}$. ■

6. Multiplicative L_∞-structures

Adapting the notion of P_∞-structure on a graded vector space [5] to the symmetric graded case, we define, in this section, multiplicative L_∞-structures. We classify all multiplicative L_∞-structures on $\Gamma(\wedge A)[2]$, for $A \to M$ an arbitrary vector bundle over a manifold M. When $A \to M$ is equipped with a Lie algebroid structure, given a $(1,1)$-tensor N on A, we define the extension of N by derivation, which is a symmetric vector valued 1-form on $\Gamma(\wedge A)[2]$, of degree zero. For a k-form on the Lie algebroid, we also define its extension by derivation, yielding a symmetric vector valued form k-form of degree $k-2$. These multi-derivations will be used in the next section to construct examples of Nijenhuis forms.

There is an important graded Lie subalgebra of $(\tilde{S}(E^*) \otimes E,\ [., .]_{RN})$, when E itself is equipped with a graded commutative associative algebra structure on $E[2]$, denoted by $\wedge$, that is, a bilinear operation such that for all $X \in E_i, Y \in E_j, Z \in E_k$

- $X \wedge Y \in E_{i+j+2}$,
- $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$,
- $X \wedge Y = (-1)^{|X||Y|} Y \wedge X$,

where $|X| = i + 2$ and $|Y| = j + 2$.

Definition 6.1. Let E be a graded vector space equipped with an associative graded commutative algebra structure, that is a graded symmetric bilinear map $\wedge$ of degree zero which is associative. An element $D \in S^d(E^*) \otimes E$ is called a *multi-derivation vector valued d-form*, if

$$\begin{aligned} & D(X_1, \cdots, X_{i-1}, Y \wedge Z, X_{i+1}, \cdots, X_d) \\ & = (-1)^{|Z|(|X_{i+1}| + \cdots + |X_d|)} D(X_1, \cdots, X_{i-1}, Y, X_{i+1}, \cdots, X_d) \wedge Z \\ & +(-1)^{|Y|(|X_1| + \cdots + |X_{i-1}| + \bar{D})} Y \wedge D(X_1, \cdots, X_{i-1}, Z, X_{i+1}, \cdots, X_d), \end{aligned} \tag{73}$$

for all $X_1, \cdots, X_d, Y, Z \in E$, where $\bar{D}$ is the degree of D as a graded map.

Remark 6.2. The graded commutativity of the product $\wedge$ implies that Equation (73) is equivalent to

$$\begin{aligned} & D(X_1, \cdots, X_{d-1}, Y \wedge Z) \\ & = D(X_1, \cdots, X_{d-1}, Y) \wedge Z + (-1)^{|Y||Z|} D(X_1, \cdots, X_{d-1}, Z) \wedge Y. \end{aligned}$$

We denote the space of all multi-derivation vector valued forms by $MultiDer(E)$. Elements of $S^1(E^*) \otimes E$ are simply called *derivations*. By definition, $E \subset MultiDer(E)$ and we have the following:

Proposition 6.3. *$MultiDer(E)$ is a graded Lie subalgebra of $(\tilde{S}(E^*) \otimes E, [.,.]_{_{RN}})$.*

We shall use the following lemmas in the proof of Proposition 6.3.

Lemma 6.4. *Let D_1 and D_2 be two derivations. Then, $[D_1, D_2]_{_{RN}}$ is also a derivation.*

Proof: We have

$$\begin{aligned} [D_1, D_2]_{_{RN}} &= D_2 \circ D_1 - (-1)^{\bar{D}_1 \bar{D}_2} D_1 \circ D_2 \\ &= -(-1)^{\bar{D}_1 \bar{D}_2} [D_1, D_2], \end{aligned}$$

where $[.,.]$ is the graded commutator on the space of derivations of the graded associative commutative algebra $(E, \wedge)$. Since $[D_1, D_2]$ is a derivation, so is $[D_1, D_2]_{_{RN}}$. ■

Lemma 6.5. *If $D \in S^d(E^*) \otimes E$ is a multi-derivation vector valued d-form, then for all $X \in E$, $[X, D]_{_{RN}}$ is a multi-derivation vector valued $(d-1)$-form.*

Proof: It is a direct consequence of

$$[X, D]_{_{RN}}(X_1, \cdots, X_{d-2}, Y \wedge Z) = D(X, X_1, \cdots, X_{d-2}, Y \wedge Z),$$

which holds for all elements $Y, Z, X_1, \cdots, X_{d-2} \in E$. ■

Proof: (of Proposition 6.3) Let D, D' be two multi-derivation vector valued d-form and d'-form, respectively. We show that $[D, D']_{_{RN}}$ is a multi-derivation vector valued $(d+d'-1)$-form, using induction on the number $n = d+d'-1$. Lemmas 6.4 and 6.5 prove the case $n = 1$. Assume, by induction, that $[D, D']_{_{RN}}$ is a multi-derivation vector valued $(d+d'-1)$-form and let D_1 and

D_2 be two multi-derivation vector valued d_1- and d_2-forms respectively, such that $d_1 + d_2 - 1 = n + 1$. From (3) we have

$$\begin{aligned}&[D_1, D_2]_{RN}(X_1, \cdots, X_{d_1+d_2-2}, Y \wedge Z)\\&= [Y \wedge Z, [X_{d_1+d_2-2}, \cdots, [X_1, [D_1, D_2]_{RN}]_{RN} \cdots]_{RN}]_{RN},\end{aligned}$$

or, using the graded Jacobi identity of $[.,.]_{RN}$,

$$\begin{aligned}&[D_1, D_2]_{RN}(X_1, \cdots, X_{d_1+d_2-2}, Y \wedge Z)\\&= [Y \wedge Z, [X_{d_1+d_2-2}, \cdots, [[X_1, D_1]_{RN}, D_2]_{RN} \cdots]_{RN}]_{RN}\\&+(-1)^{\bar{D}_1\bar{X}_1}[Y \wedge Z, [X_{d_1+d_2-2}, \cdots, [D_1, [X_1, D_2]_{RN}]_{RN} \cdots]_{RN}]_{RN},\end{aligned}$$

for all $X_1, \cdots, X_{d_1+d_2-2}, Y, Z \in E$. By Lemma 6.5, $[X_1, D_1]_{RN}$ and $[X_1, D_2]_{RN}$ are multi-derivation vector valued (d_1-1)- and (d_2-1)-forms respectively, and hence using the assumption of induction, $[[X_1, D_1]_{RN}, D_2]_{RN}$ and $[D_1, [X_1, D_2]_{RN}]_{RN}$ are multi-derivation vector valued n-forms. Therefore,

$$\begin{aligned}&[D_1, D_2]_{RN}(X_1, \cdots, X_{d_1+d_2-2}, Y \wedge Z)\\&= [[X_1, D_1]_{RN}, D_2]_{RN}(X_2, \cdots, X_{d_1+d_2-2}, Y \wedge Z)\\&+(-1)^{\bar{D}_1\bar{X}_1}[D_1, [X_1, D_2]_{RN}]_{RN}(X_2, \cdots, X_{d_1+d_2-2}, Y \wedge Z)\\&= [[X_1, D_1]_{RN}, D_2]_{RN}(X_2, \cdots, X_{d_1+d_2-2}, Y) \wedge Z\\&+(-1)^{|Y||Z|}[[X_1, D_1]_{RN}, D_2]_{RN}(X_2, \cdots, X_{d_1+d_2-2}, Z) \wedge Y\\&+(-1)^{\bar{D}_1\bar{X}_1}[D_1, [X_1, D_2]_{RN}]_{RN}(X_2, \cdots, X_{d_1+d_2-2}, Y) \wedge Z\\&+(-1)^{\bar{D}_1\bar{X}_1}(-1)^{|Y||Z|}[D_1, [X_1, D_2]_{RN}]_{RN}(X_2, \cdots, X_{d_1+d_2-2}, Z) \wedge Y\\&= [D_1, D_2]_{RN}(X_1, \cdots, X_{d_1+d_2-2}, Y) \wedge Z\\&+(-1)^{|Y||Z|}[D_1, D_2]_{RN}(X_1, \cdots, X_{d_1+d_2-2}, Z) \wedge Y,\end{aligned}$$

which completes the induction and also the proof (see Remark 6.2). ■

Let us now define multiplicative L_∞-algebra.

Definition 6.6. An L_∞-structure $\mu = \sum_{i=1}^{\infty} l_i$ on a graded vector space E equipped with a graded commutative product $\wedge : E_i \times E_j \to E_{i+j}$ is called *multiplicative* if all the multi-linear brackets l_i are multi-derivations.

Next, we discuss the relation between multiplicative L_∞-structures and Lie algebroids.

A *pre-Lie algebroid* structure on a vector bundle $A \to M$ over a manifold M is a pair $(\rho, [.,.])$, with $\rho : A \to TM$ a vector bundle morphism over the identity of M, called *anchor map*, and $[.,.]$ a skew-symmetric bilinear endomorphism of $\Gamma(A)$ subject to the so-called Leibniz identity:

$$[X, fY] = f[X,Y] + (\rho(X)f)\,Y,$$

for all $X, Y \in \Gamma(A)$ and all $f \in C^\infty(M)$. When, moreover, $[.,.]$ is a Lie algebra bracket, the pair $([.,.]\,,\rho)$ is called a *Lie algebroid structure on* $A \to M$. We denote by $[.,.]_{_{SN}}$ the *Schouten-Nijenhuis* bracket on the the space of multivectors of the (pre-)Lie algebroid A and by d^A the (pre-)differential of A. We recall that

$$\begin{array}{rcl} [X,f]_{_{SN}} &=& \rho(X)f \\ [P,Q]_{_{SN}} &=& -(-1)^{pq}\,[Q,P]_{_{SN}} \\ [P,Q\wedge R]_{_{SN}} &=& [P,Q]_{_{SN}}\wedge R + (-1)^{qr}\,[P,R]_{_{SN}}\wedge Q, \end{array} \tag{74}$$

for all $X \in \Gamma(A)$, $P \in \Gamma(\wedge^{p+1}A)$, $Q \in \Gamma(\wedge^{q+1}A)$,$R \in \Gamma(\wedge^{r+1}A)$ and $f \in \mathcal{C}^\infty(M)$ and that

$$\mathsf{d}^A\omega(X_0,\ldots,X_k) := \sum_{i=0}^{k}(-1)^i\rho(X_i)\omega(\widehat{X_i}) + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],\widehat{X_{i,j}}),$$

for all $X_0,\ldots,X_k \in \Gamma(A), \omega \in \Gamma(\wedge^k A^*)$, where $\widehat{X_i}$ and $\widehat{X_{i,j}}$ stand for

$$X_1,\cdots,X_{i-1},X_{i+1},\cdots,X_k \text{ and } X_1,\cdots,X_{i-1},X_{i+1},\cdots,X_{j-1},X_{j+1},\cdots,X_k$$

respectively. Notice that in the above expression, we have implicitly identified elements of $\Gamma(\wedge^k A^*)$ with skew-symmetric k-linear maps from $\Gamma(A)\times \overset{k}{\cdots} \times\Gamma(A)$ to $\mathcal{C}^\infty(M)$.

Let $([.,.]\,,\rho)$ be a pre-Lie algebroid structure on a vector bundle $A \to M$. Set $E_i := \Gamma(\wedge^{i+1}A)$ and $E = \oplus_{i\ge -1}E_i$, with $E_{-1} = \Gamma(\wedge^0 A) = \mathcal{C}^\infty(M)$. The Schouten-Nijenhuis bracket is a graded skew-symmetric bracket of degree zero on $E = \oplus_{i\ge-1}E_i$ and it is known that a pre-Lie algebroid structure $(\rho,[.,.])$ is a Lie algebroid structure on the vector bundle $A \to M$, if and only if $[.,.]_{_{SN}}$ is a graded Lie algebra bracket on $E = \Gamma(\wedge A)[1]$. It is also well known that the pre-differential d^A is a derivation of $\Gamma(\wedge A^*)$ and that d^A squares to zero if and only if $(A,\,[.,.]\,,\rho)$ is Lie algebroid.

The discussion above leads to the conclusion that there are two ways to see Lie algebroids as L_∞-structures: the first one will make it an L_∞-structure on $\Gamma(\wedge A)$, and the second one will make it an L_∞-structure on $\Gamma(\wedge A^*)$ [13]. More precisely:

Proposition 6.7. *Let $A \to M$ be a vector bundle and $A^* \to M$ its dual. There is a one to one correspondence between:*

(i) pre-Lie algebroid structures $(\rho, [.,.])$ on $A \to M$,
(ii) binary multi-derivations of $\Gamma(\wedge A)[2]$ of degree 1,
(iii) unary multi-derivations of $\Gamma(\wedge A^*)[2]$ of degree 1.

The one to one correspondence above restricts to a one to one correspondence between:

(i′) Lie algebroid structures $(\rho, [.,.])$ on $A \to M$,
(ii′) multiplicative L_∞-structures on $\Gamma(\wedge A)[2]$ given by a binary bracket,
(iii′) multiplicative L_∞-structures on $\Gamma(\wedge A^*)[2]$ given by a unary bracket.

Given a $(1,1)$-tensor N on a Lie algebroid $(A, [.,.], \rho)$, we define a linear map $\underline{N}$ on the graded vector space $\Gamma(\wedge A)[2]$, by setting

$$\underline{N}(f) := 0,$$

for all $f \in \mathcal{C}^\infty(M)$, and

$$\underline{N}(P) := \sum_{i=1}^{p} P_1 \wedge \cdots \wedge P_{i-1} \wedge N(P_i) \wedge P_{i+1} \wedge \cdots \wedge P_p,$$

for all monomial multi-sections $P = P_1 \wedge \cdots \wedge P_p \in \Gamma(\wedge A)[2]$. The map $\underline{N}$ is called the *extension of N by derivation* on the graded vector space $\Gamma(\wedge A)[2]$. It is a multi-derivation on the graded vector space $\Gamma(\wedge A)[2]$, hence a symmetric vector valued 1-form on $\Gamma(\wedge A)[2]$, and has degree zero.

For a k-form on a Lie algebroid, we also consider its extension by derivation. More precisely, if $\kappa \in \Gamma(\wedge^k A^*)$, the extension of κ by derivation is a k-linear map, denoted by $\underline{\kappa}$, given by

$$\underline{\kappa}(P_1, \cdots, P_k) := \sum_{i_1, \cdots, i_k = 1}^{p_1, \cdots, p_k} (-1)^{\spadesuit} \kappa(P_{1,i_1}, \cdots, P_{k,i_k}) \widehat{P_{1,i_1}} \wedge \cdots \wedge \widehat{P_{k,i_k}},$$

for all homogeneous multi-sections $P_i = P_{i,1} \wedge \cdots \wedge P_{i,p_i} \in \Gamma(\wedge^{p_i} A)$, with $i = 1, \cdots, k$, where $1 \leq i_j \leq p_j$ for all $1 \leq j \leq k$,

$$\widehat{P_{j,i_j}} = P_{j,1} \wedge \cdots \wedge P_{j,i_j - 1} \wedge P_{j,i_j + 1} \wedge \cdots \wedge P_{j,p_j} \in \Gamma(\wedge^{p_j - 1} A)$$

and

$$\spadesuit = 2p_1 + 3p_2 + \cdots + (k+1)p_k + i_1 + \cdots + i_k + 1.$$

It follows from its definition that $\underline{\kappa}$ is a multi-derivation on the graded vector space $\Gamma(\wedge A)[2]$ and that it is a symmetric vector valued k-form of degree $k-2$ on $\Gamma(\wedge A)[2]$.

Lemma 6.8. *Let $(A, [.,.], \rho)$ be a Lie algebroid, $\alpha \in \Gamma(\wedge^k A^*)$ be a k-form and $\beta \in \Gamma(\wedge^l A^*)$ be an l-form. Then,*

$$\left[\underline{\alpha}, \underline{\beta}\right]_{RN} = 0.$$

Proof: The fact that $\underline{\alpha}$ (respectively $\underline{\beta}$) is a vector valued k-form (respectively l-form) of degree $k-2$ (respectively $l-2$), imply that $\left[\underline{\alpha}, \underline{\beta}\right]_{RN}$ is a vector valued $(k+l-1)$-form of degree $k+l-4$ on the graded vector space $\Gamma(\wedge A) = \oplus_{i\geq 0}\Gamma(\wedge^i A)$. Therefore, for all $l, k \geq 0$ the restriction of $\left[\underline{\alpha}, \underline{\beta}\right]_{RN}$ to the space of sections is zero and hence we have $\left[\underline{\alpha}, \underline{\beta}\right]_{RN} = 0$, because $\left[\underline{\alpha}, \underline{\beta}\right]_{RN}$ is a multi-derivation and it is uniquely determined on the space of sections. ■

According to Proposition 6.7, for a given Lie algebroid $(A, [.,.], \rho)$, the bracket $l_2^{[.,.]}$ given by

$$l_2^{[.,.]}(P, Q) = (-1)^{p-1}[P, Q]_{SN}, \quad P \in \Gamma(\wedge^p A), Q \in \Gamma(\wedge^q A), \tag{75}$$

defines a multiplicative graded Lie algebra structure on $\Gamma(\wedge A)[2]$. When we deform the bracket $[.,.]$ by N as

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y],$$

for all $X, Y \in \Gamma(A)$, of course we may consider $l_2^{[.,.]_N}$ using Equation (75) and we may take the Schouten-Nijenhuis bracket $[.,.]_{SN}^N$ corresponding to the deformed bracket $[.,.]_N$. Note that the bracket $l_2^{[.,.]_N}$ is not necessarily a multiplicative graded Lie algebra structure. On the other hand, since $l_2^{[.,.]}$ is a symmetric vector valued 2-form of degree 1 and $\underline{N}$ is a (symmetric) vector valued 1-form of degree zero, we can consider the deformation of $l_2^{[.,.]}$ by $\underline{N}$. The following lemma shows the relation between $\left[\underline{N}, l_2^{[.,.]}\right]_{RN}$ and $l_2^{[.,.]_N}$.

Lemma 6.9. *Let N be a $(1,1)$-tensor on a Lie algebroid $(A, [.,.], \rho)$. Then, we have*

$$\left[\underline{N}, l_2^{[.,.]}\right]_{RN} = l_2^{[.,.]_N}.$$

Proof: The proof follows directly from the fact that the Schouten-Nijenhuis bracket on $\Gamma(\wedge A)$ associated to the bracket $[.,.]_N$ is given by

$$[P,Q]^N_{SN} = [\underline{N}P,Q]_{SN} + [P,\underline{N}Q]_{SN} - \underline{N}\,[P,Q]_{SN}\,,$$

for all $P,Q \in \Gamma(\wedge A)$, see [13]. ∎

We will need the following lemma for our next purpose.

Lemma 6.10. *Let $(A,[.,.],\rho)$ be a Lie algebroid, with differential d^A and associated multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on $\Gamma(\wedge A)[2]$. Then,*

$$\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN} = \underline{\mathsf{d}^A\alpha},$$

for all $\alpha \in \Gamma(\wedge^n A^*)$.

Proof: We shall prove the statement for $n=2$. A similar proof can be done for any $n \geq 1$. First note that $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}$ is a vector valued 3-form of degree 1 on the graded vector space $\Gamma(\wedge A)[2]$. This implies that the restriction of $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}$ to $\Gamma(A)$ is of the form:

$$\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}|_{\Gamma(A)\times\Gamma(A)\times\Gamma(A)} : \Gamma(A)\times\Gamma(A)\times\Gamma(A) \to \mathcal{C}^\infty(M)$$

and, by degree reasons, any other restriction of $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}$ is zero. On the other hand, by Proposition 6.3, $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}$ is a multi-derivation, so that its restriction to the sections $\Gamma(A)$ is a $\mathcal{C}^\infty(M)$-linear map. Therefore, $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN} \in \Gamma(\wedge^3 A^*)$. Next, we show that

$$\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}|_{\Gamma(A)\times\Gamma(A)\times\Gamma(A)} = \mathsf{d}^A\alpha$$

and this together with the fact that $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}$ is a multi-derivation will imply that $\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN} = \underline{\mathsf{d}^A\alpha}$, by the uniqueness of extension by derivation of $\mathsf{d}^A\alpha$ to the graded vector space $\Gamma(\wedge A)[2]$. A direct computation shows that

$$\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}(X,Y,Z) = [\alpha(X,Y),Z]_{SN} - \alpha([X,Y]_{SN},Z) + c.p.,$$

for all $X, Y, Z \in \Gamma(A)$. Hence, Equation (74) together with the definition of d^A imply that

$$\left[\underline{\alpha}, l_2^{[.,.]}\right]_{RN}(X,Y,Z) = \rho(Z)\alpha(X,Y) - \alpha([X,Y],Z) + c.p. = \mathsf{d}\alpha^A(X,Y,Z).$$

This completes the proof. ■

7. Nijenhuis forms on multiplicative L_∞-structures associated to Lie algebroids

In this section we consider several structures defined by tensors and pairs of tensors on a Lie algebroid and, by using their extensions by derivations, we construct Nijenhuis forms (weak Nijenhuis and co-boundary Nijenhuis, in some cases) with respect to the graded Lie algebra associated to the Lie algebroid structure.

Let $(A, [.,.], \rho)$ be a Lie algebroid and $N : A \to A$ an endomorphim. Then, as in the case of Lie algebras, the Nijenhuis torsion of N with respect to the Lie bracket $[.,.]$, denoted by $T_{[.,.]}N$, is defined by Equation (8) and again a direct computation shows that

$$T_{[.,.]}N(X,Y) = \frac{1}{2}\left([X,Y]_{N,N} - [X,Y]_{N^2}\right),$$

for all $X, Y \in \Gamma(A)$. A $(1,1)$-tensor N on a Lie algebroid $(A, [.,.], \rho)$ is said to be *Nijenhuis* if the Nijenhuis torsion of N, with respect to the Lie algebroid bracket $[.,.]$, vanishes. As a consequence of Lemma 6.9, we have the following proposition:

Proposition 7.1. *For every Nijenhuis tensor N on a Lie algebroid $(A, [.,.], \rho)$, the extension $\underline{N}$ of N by derivation is a Nijenhuis vector valued 1-form with respect to the multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$, with square $\underline{(N^2)}$.*

Proof: Applying Lemma 6.9 twice, for the tensor N and the bracket $l_2^{[.,.]}$, we get $\left[\underline{N}, \left[\underline{N}, l_2^{[.,.]}\right]_{RN}\right]_{RN} = l_2^{[.,.]_{N,N}}$. The same lemma gives $\left[\underline{N^2}, l_2^{[.,.]}\right]_{RN} = l_2^{[.,.]_{N^2}}$. Since N is a Nijenhuis $(1,1)$-tensor on A, we have $l_2^{[.,.]_{N,N}} = l_2^{[.,.]_{N^2}}$, which implies that $\left[\underline{N}, \left[\underline{N}, l_2^{[.,.]}\right]_{RN}\right]_{RN} = \left[\underline{N^2}, l_2^{[.,.]}\right]_{RN}$. Also, $\underline{(N^2)}$ and $\underline{N}$ commute with respect to the Richardson-Nijenhuis bracket. ■

In the next proposition we obtain a Nijenhuis vector valued form which is the sum of a vector valued 1-form with a vector valued 2-form.

Proposition 7.2. *Let $(A, [.,.], \rho)$ be a Lie algebroid, with differential d^A and associated multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on $\Gamma(\wedge A)[2]$. Then, for every section $\alpha \in \Gamma(\wedge^2 A^*)$, $S + \underline{\alpha}$ is a Nijenhuis vector valued form with respect to $l_2^{[.,.]}$, with square $S + 2\,\underline{\alpha}$. The deformed structure is $l_2^{[.,.]} + \underline{\mathsf{d}^A \alpha}$.*

Proof: As a direct consequence of Lemma 6.10, we have

$$\left[S + \underline{\alpha}, l_2^{[.,.]}\right]_{RN} = l_2^{[.,.]} + \underline{\mathsf{d}^A \alpha}.$$

A simple computation gives

$$\left[S + \underline{\alpha}, \left[S + \underline{\alpha}, l_2^{[.,.]}\right]_{RN}\right]_{RN} = l_2^{[.,.]} + 2\,\underline{\mathsf{d}^A \alpha} = \left[S + 2\,\underline{\alpha}, l_2^{[.,.]}\right]_{RN}$$

and the fact that $[S + \underline{\alpha}, S + 2\underline{\alpha}]_{RN} = 0$ completes the proof. ■

Our next purpose is to use well-known structures on a Lie algebroid defined by pairs of compatible tensors, such as ΩN-, Poisson-Nijenhuis and $P\Omega$-structures [14, 1, 3], to construct Nijenhuis forms on the multiplicative graded Lie algebra associated to the Lie algebroid. We start by recalling what an ΩN-structure is.

Definition 7.3. [1, 14] Let $(A, [.,.], \rho)$ be a Lie algebroid, with differential d^A, N be a $(1,1)$-tensor on A and $\alpha \in \Gamma(\wedge^2 A^*)$ a 2-form. Let $\alpha_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ be a bilinear map, defined as

$$\alpha_N(X, Y) = \alpha(NX, Y). \tag{76}$$

Then, the pair (α, N) is an ΩN-*structure* on the Lie algebroid A if $\alpha(NX, Y) = \alpha(X, NY)$ for all $X, Y \in \Gamma(A)$ (which amounts to α_N being skew-symmetric and therefore a 2-form on A), and α and α_N are d^A-closed.

Lemma 7.4. *Let $(A, [.,.], \rho)$ be a Lie algebroid, with differential d^A and with the associated multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$. Let N be a $(1,1)$-tensor on the Lie algebroid and $\alpha \in \Gamma(\wedge^2 A^*)$ be a 2-form such that $\alpha_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ given by (76) is skew-symmetric and therefore a 2-form on A. Then,*

i) $[\underline{N}, \underline{\alpha}]_{RN} = 2\underline{\alpha_N}$,

ii) $\left[\underline{N} + \underline{\alpha}, l_2^{[.,.]}\right]_{RN} = l_2^{[.,.]_N} + \underline{\mathsf{d}^A \alpha}$

iii) If N is Nijenhuis, then

$$\left[\underline{N}+\underline{\alpha},\left[\underline{N}+\underline{\alpha},l_2^{[.,.]}\right]_{RN}\right]_{RN}=\left[\underline{N^2},l_2^{[.,.]}\right]_{RN}-2\,\underline{\mathsf{d}^A\alpha_N}+2\left[\underline{N},\underline{\mathsf{d}^A\alpha}\right]_{RN}.$$

Proof: i) First notice that for all $X,Y\in\Gamma(A)$ we have

$$\left[\underline{N},\underline{\alpha}\right]_{RN}(X,Y)=\alpha(NX,Y)-\alpha(NY,X)=2\alpha_N(X,Y).$$

Since $\underline{N}$ and $\underline{\alpha}$ are both derivations, by Lemma 6.4 $\left[\underline{N},\underline{\alpha}\right]_{RN}$ is a derivation and hence it is the unique extension of $2\alpha_N$ by derivation.

ii) It is a direct consequence of Lemma 6.9 together with Lemma 6.10.

iii) Using item (ii) and Lemma 6.9, we have

$$\begin{aligned}\left[\underline{N}+\underline{\alpha},\left[\underline{N}+\underline{\alpha},l_2^{[.,.]}\right]_{RN}\right]_{RN}&=\left[\underline{N}+\underline{\alpha},l_2^{[.,.]_N}+\underline{\mathsf{d}^A\alpha}\right]_{RN}\\&=l_2^{[.,.]_{N,N}}+\left[\underline{N},\underline{\mathsf{d}^A\alpha}\right]_{RN}+\left[\underline{\alpha},l_2^{[.,.]_N}\right]_{RN}+\left[\underline{\alpha},\underline{\mathsf{d}^A\alpha}\right]_{RN}.\end{aligned}$$

But, using Lemma 6.9 and the graded Jacobi identity we have

$$\begin{aligned}\left[\underline{\alpha},l_2^{[.,.]_N}\right]_{RN}&=\left[\underline{\alpha},\left[\underline{N},l_2^{[.,.]}\right]_{RN}\right]_{RN}\\&=\left[\left[\underline{\alpha},\underline{N}\right]_{RN},l_2^{[.,.]}\right]_{RN}+\left[\underline{N},\left[\underline{\alpha},l_2^{[.,.]}\right]_{RN}\right]_{RN}\\&=\left[-2\underline{\alpha_N},l_2^{[.,.]}\right]_{RN}+\left[\underline{N},\underline{\mathsf{d}^A\alpha}\right]_{RN}\end{aligned}$$

and, by Lemma 6.8, $\left[\underline{\alpha},\underline{\mathsf{d}^A\alpha}\right]_{RN}=0$. Hence, since N is Nijenhuis, we get

$$\begin{aligned}[\underline{N}+\underline{\alpha},[\underline{N}+\underline{\alpha},l_2^{[.,.]}]_{RN}]_{RN}&=\left[\underline{N^2}-2\underline{\alpha_N},l_2^{[.,.]}\right]_{RN}+2\left[\underline{N},\underline{\mathsf{d}^A\alpha}\right]_{RN}\\&=\left[\underline{N^2},l_2^{[.,.]}\right]_{RN}-2\,\underline{\mathsf{d}^A\alpha_N}+2\left[\underline{N},\underline{\mathsf{d}^A\alpha}\right]_{RN}.\end{aligned}$$

■

The next proposition is now immediate.

Proposition 7.5. *Let $(A,[.,.],\rho)$ be a Lie algebroid, with differential d^A and with associated multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$. If (α,N) is an ΩN-structure on the Lie algebroid A, then $\underline{N}+\underline{\alpha}$ is a Nijenhuis vector valued form, with respect to $l_2^{[.,.]}$, with square $\underline{N^2}+\underline{\alpha_N}$.*

Proof: Let (α, N) be an ΩN-structure on the Lie algebroid A. Then, $\mathsf{d}^A\alpha_N = 0$ and, by Lemma 6.10, we have $\left[\underline{\alpha_{_N}}, l_2^{[.,.]}\right]_{_{RN}} = 0$. It follows from item (iii) in Lemma 7.4, that

$$\left[\underline{N}+\underline{\alpha}, \left[\underline{N}+\underline{\alpha}, l_2^{[.,.]}\right]_{_{RN}}\right]_{_{RN}} = \left[\underline{N^2}+\underline{\alpha_{_N}}, l_2^{[.,.]}\right]_{_{RN}}.$$

Since

$$\left[\underline{N}+\underline{\alpha}, \underline{N^2}+\underline{\alpha_{_N}}\right]_{_{RN}} = \left[\underline{N}, \underline{\alpha_{_N}}\right]_{_{RN}} + \left[\underline{\alpha}, \underline{N^2}\right]_{_{RN}} = 2\underline{(\alpha_{_N})_{_N}} - 2\underline{\alpha_{_{N^2}}} = 0,$$

the proof is complete. ∎

We are now going to see how to include Poisson-Nijenhuis structures among our examples of Nijenhuis structures on L_∞-algebras. Let us first fix and recall some notations and notions.

Let $(A, \mu = [.,.], \rho)$ be a Lie algebroid, $\pi \in \Gamma(\wedge^2 A)$ a bivector and $N : A \to A$ a vector bundle morphism. We denote by N^* the morphism $N^* : A^* \to A^*$ given by $\langle N^*\alpha, X\rangle = \langle \alpha, NX\rangle$, for all $X, Y \in \Gamma(A)$. We consider the morphism induced by π, $\pi^\# : A^* \to A$, given by $\langle \beta, \pi^\#\alpha\rangle = \pi(\alpha, \beta)$, and we denote by $\pi_{_N}$ the bivector defined by

$$\pi_{_N}(\alpha, \beta) = \langle \beta, N\pi^\#\alpha\rangle = \langle N^*\beta, \pi^\#\alpha\rangle, \tag{77}$$

for all $\alpha, \beta \in \Gamma(A^*)$. A bracket $\{\cdot,\cdot\}_{_\pi}^{\mu}$ can be defined on $\Gamma(A^*)$, the space of 1-forms on the Lie algebroid $(A, \mu = [.,.], \rho)$, as follows:

$$\{\alpha, \beta\}_{_\pi}^{\mu} = \mathcal{L}^A_{\pi^\#(\alpha)}\beta - \mathcal{L}^A_{\pi^\#(\beta)}\alpha - \mathsf{d}^A(\pi(\alpha, \beta)),$$

for all $\alpha, \beta \in \Gamma(A^*)$. It is well known that if π is a Poisson bivector on the Lie algebroid $(A, \mu = [.,.], \rho)$, that is $[\pi, \pi]_{_{SN}} = 0$, then $(\Gamma(A^*), \{.,.\}_{_\pi}^{\mu})$ is a Lie algebra and if this is the case, then $\pi^\#$ is a Lie algebra morphism form the Lie algebra $(\Gamma(A^*), \{.,.\}_{_\pi}^{\mu})$ to the Lie algebra $(\Gamma(A), \mu)$.

For every Poisson structure π on a Lie algebroid A, the triple $(\Gamma(\wedge A)[1], [.,.]_{_{SN}}, [\pi, .]_{_{SN}})$ is a skew-symmetric differential graded Lie algebra, so that the pair $(l_1^{[.,.],\pi}, l_2^{[.,.]})$ given by

$$l_1^{[.,.],\pi}(P) = [\pi, P]_{_{SN}} \quad \text{and} \quad l_2^{[.,.]}(P, Q) := (-1)^{(p-1)}\, [P, Q]_{_{SN}},$$

where $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$, is an L_∞-structure on the graded vector space $\Gamma(\wedge A)[2]$, which is clearly multiplicative. This L_∞-structure is called *the L_∞-structure associated to the Poisson structure π and the Lie algebroid A.*

Now, we recall the notion of Poisson-Nijenhuis structure on a Lie algebroid.

Definition 7.6. [13] Let $(A, \mu = [.,.], \rho)$ be a Lie algebroid, $\pi \in \Gamma(\wedge^2 A)$ a bivector and N a $(1,1)$-tensor on A. Then, the pair (π, N) is a *Poisson-Nijenhuis structure* on the Lie algebroid $(A, \mu = [.,.], \rho)$ if

i) N is a Nijenhuis $(1,1)$-tensor with respect to the Lie bracket μ,
ii) π is a Poisson bivector,
iii) $N \circ \pi^\# = \pi^\# \circ N^*$,
iv) $(\{\alpha, \beta\}_\pi^\mu)_{N^*} = \{\alpha, \beta\}_\pi^{\mu^N}$,

for all $\alpha, \beta \in \Gamma(A^*)$, where $(\{.,.\}_\pi^\mu)_{N^*}$ is the deformation of the Lie bracket $\{\cdot,\cdot\}_\pi^\mu$ by N^* and $\{.,.\}_\pi^{\mu^N}$ is the bracket determined by the pair $(\pi, \mu^N = [.,.]_N)$ according to formula (77).

Notice that $\pi_N^\# = N\pi^\# = \pi^\# N^*$ and hence,

$$\underline{N}(\pi) = 2\pi_N. \tag{78}$$

Recall from [13] that if (π, N) is a Poisson-Nijenhuis structure on a Lie algebroid $(A, \mu = [.,.], \rho)$, then $\left(A, \mu^N = [.,.]_N, \rho \circ N\right)$ and $\left(A^*, \{.,.\}_\pi^\mu, \rho \circ \pi^\#\right)$ are Lie algebroids. Also,

$$\left((\{.,.\}_\pi^\mu)_{N^*}, \rho \circ \pi^\# \circ N^*\right), \left(\{.,.\}_\pi^{\mu^N}, \rho \circ N \circ \pi^\#\right) \text{ and } \left(\{.,.\}_{\pi_N}^\mu, \rho \circ \pi_N^\#\right)$$

define the same Lie algebroid structure on A^*. Moreover, identifying the graded vector spaces $\Gamma(\wedge A^{**})$ and $\Gamma(\wedge A)$, the differential $\mathsf{d}^{A^*}_{(\{.,.\}_\pi^\mu)}$ coincide with the linear map $[\pi, .]_{SN}$. Hence, we have

$$\mathsf{d}^{A^*}_{(\{.,.\}_\pi^{\mu^N})} = \mathsf{d}^{A^*}_{(\{.,.\}_{\pi_N}^\mu)},$$

which is equivalent to

$$[\pi, .]_{SN}^N = [\pi_N, .]_{SN}, \tag{79}$$

where $[.,.]_{SN}^N$ is the Schouten-Nijenhuis bracket with respect to the Lie bracket $[.,.]_N$.

Lemma 7.7. *Let (π, N) be a Poisson-Nijenhuis structure on a Lie algebroid $(A, [.,.], \rho)$. Then,*

$$\left[\underline{N}, l_1^{[.,.],\pi}\right]_{RN}(P) = [\pi, \underline{N}(P)]_{SN} - \underline{N}[\pi, P]_{SN} = [-\pi_N, P]_{SN},$$

for all $P \in \Gamma(\wedge A)$.

Proof: The first equality follows directly from the definition of $l_1^{[.,.],\pi}$. For the second equality, observe that for all $P \in \Gamma(\wedge A)$ we have

$$[\pi, P]_{SN}^{N} = [\underline{N}(\pi), P]_{SN} + [\pi, \underline{N}(P)]_{SN} - \underline{N}\,[\pi, P]_{SN},$$

where $[.,.]_{SN}^{N}$ stands for the Schouten-Nijenhuis bracket with respect to the Lie bracket $[.,.]_N$. Hence, using (78) and (79), we have

$$\begin{aligned}
[\pi, \underline{N}(P)]_{SN} - \underline{N}\,[\pi, P]_{SN} &= [\pi, P]_{SN}^{N} - [\underline{N}(\pi), P]_{SN} = [\pi, P]_{SN}^{N} - 2\,[\pi_N, P]_{SN} \\
&= \left([\pi, P]_{SN}^{N} - [\pi_N, P]_{SN}\right) - [\pi_N, P]_{SN} \\
&= -\,[\pi_N, P]_{SN}.
\end{aligned}$$

■

Proposition 7.8. *Let (π, N) be a Poisson-Nijenhuis structure on a Lie algebroid $(A, [.,.], \rho)$. Then, the derivation $\underline{N}$ is a weak Nijenhuis tensor for the L_∞-structure associated to the Poisson structure π and the Lie algebroid $(A, [.,.], \rho)$.*

In this case, the deformed structure $[\underline{N}, l_1^{[.,.],\pi} + l_2^{[.,.]}]_{RN}$ is the L_∞-structure associated to the Poisson structure $-\pi_N$ and the Lie algebroid $(A, [.,.]_N, \rho \circ N)$.

Proof: Lemmas 7.7 and 6.9 imply that

$$\left[\underline{N}, l_1^{[.,.],\pi} + l_2^{[.,.]}\right]_{RN} = -l_1^{[.,.],\pi_N} + l_2^{[.,.]_N}.$$

Hence,

$$\begin{aligned}
\left[\underline{N}, \left[\underline{N}, l_1^{[.,.],\pi} + l_2^{[.,.]}\right]_{RN}\right]_{RN} &= l_1^{[.,.],\pi_{N,N}} + l_2^{[.,.]_{N,N}} = l_1^{[.,.],\pi_{N^2}} + l_2^{[.,.]_{N^2}} \\
&= \left[\underline{N^2}, -l_1^{[.,.],\pi} + l_2^{[.,.]}\right]_{RN} \\
&= \left[\underline{N^2}, l_1^{[.,.],\pi} + l_2^{[.,.]}\right]_{RN} - 2\left[\underline{N^2}, l_1^{[.,.],\pi}\right]_{RN}.
\end{aligned} \tag{80}$$

Denoting $\mu = l_1^{[.,.],\pi} + l_2^{[.,.]}$ and using the fact that π_{N^2} is a Poisson bivector on the Lie algebroid $(A, [.,.], \rho)$ and hence $(\Gamma(\wedge A)[2], l_1^{[.,.],\pi_{N^2}} + l_2^{[.,.]})$ is a symmetric differential graded Lie algebra, we have

$$\begin{aligned}
\left[\mu, \left[\underline{N}, [\underline{N}, \mu]_{_{RN}}\right]_{_{RN}}\right]_{_{RN}} &= \left[\mu, \left[\underline{N^2}, \mu\right]_{_{RN}}\right]_{_{RN}} - 2\left[\mu, \left[\underline{N^2}, l_1^{[.,.],\pi}\right]_{_{RN}}\right]_{_{RN}} \\
&= -2\left[\mu, \left[\underline{N^2}, l_1^{[.,.],\pi}\right]_{_{RN}}\right]_{_{RN}} = 2\left[\mu, l_1^{[.,.],\pi_{N^2}}\right]_{_{RN}} \\
&= 2\left[l_1^{[.,.],\pi}, l_1^{[.,.],\pi_{N^2}}\right]_{_{RN}} + 2\left[l_2^{[.,.]}, l_1^{[.,.],\pi_{N^2}}\right]_{_{RN}} \\
&= 2\left[l_1^{[.,.],\pi}, l_1^{[.,.],\pi_{N^2}}\right]_{_{RN}}
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\left[l_1^{[.,.],\pi}, l_1^{[.,.],\pi_{N^2}}\right]_{_{RN}}(P) &= l_1^{[.,.],\pi_{N^2}}(l_1^{[.,.],\pi}(P)) + l_1^{[.,.],\pi}(l_1^{[.,.],\pi_{N^2}}(P)) \\
&= \left[\pi_{_{N^2}}, [\pi, P]_{_{SN}}\right]_{_{SN}} + \left[\pi, \left[\pi_{_{N^2}}, P\right]_{_{SN}}\right]_{_{SN}} \\
&= \left[\left[\pi, \pi_{_{N^2}}\right]_{_{SN}}, P\right]_{_{SN}} \\
&= 0,
\end{aligned} \tag{82}$$

for all $P \in \Gamma(\wedge A)$. Therefore $\left[\mu, \left[\underline{N}, [\underline{N}, \mu]_{_{RN}}\right]_{_{RN}}\right]_{_{RN}} = 0$, which means that $\underline{N}$ is a weak Nijenhuis vector valued form with respect to the symmetric differential graded Lie algebra structure $\mu = l_1^{[.,.],\pi} + l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$. ■

There is a second manner to see Poisson-Nijenhuis structures on a Lie algebroid as a Nijenhuis form.

Proposition 7.9. *Let (π, N) be a Poisson-Nijenhuis structure on a Lie algebroid $(A, [.,.], \rho)$. Then $\underline{N} + \pi$ is a weak Nijenhuis vector valued form with curvature, with respect to the multiplicative differential graded Lie algebra structure $l_1^{[.,.],\pi} + l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$.*

Proof: It follows from Lemma 6.9 that

$$\left[\underline{N} + \pi, l_1^{[.,.],\pi}\right]_{_{RN}} = -l_1^{[.,.],\pi_N} + [\pi, \pi]_{_{SN}} = -l_1^{[.,.],\pi_N},$$

while Lemma 7.7 implies that

$$\left[\underline{N} + \pi, l_2^{[.,.]}\right]_{_{RN}} = l_2^{[.,.]_N} + l_2^{[.,.]}(\pi, .) = l_2^{[.,.]_N} - l_1^{[.,.],\pi}.$$

Hence, we have

$$\begin{aligned}\left[\underline{N}+\pi,\left[\underline{N}+\pi,l_1^{[.,.],\pi}+l_2^{[.,.]}\right]_{RN}\right]_{RN} &= \left[\underline{N}+\pi,-l_1^{[.,.],\pi_N}+l_2^{[.,.]_N}-l_1^{[.,.],\pi}\right]_{RN}\\ = l_1^{[.,.],\pi_{N,N}}+l_1^{[.,.],\pi_N}+l_2^{[.,.]_{N,N}}-l_1^{[.,.],\pi_N}(\pi)-l_1^{[.,.],\pi}(\pi)+l_2^{[.,.]_N}(\pi,.).\end{aligned} \tag{83}$$

But $l_1^{[.,.],\pi}(\pi)=[\pi,\pi]_{SN}=0$, $l_1^{[.,.],\pi_N}(\pi)=[\pi_N,\pi]_{SN}=0$ and $l_1^{[.,.],\pi_N}(P)+l_2^{[.,.]_N}(\pi,P)=[\pi_N,P]_{SN}-[\pi,P]_{SN}^N=0$, for all $P\in\Gamma(\wedge A)$, where $[.,.]_{SN}^N$ is the Schouten-Nijenhuis bracket associated to the Lie bracket $[.,.]_N$. Hence, (83) can be written as

$$\left[\underline{N}+\pi,\left[\underline{N}+\pi,l_1^{[.,.],\pi}+l_2^{[.,.]}\right]_{RN}\right]_{RN}=l_1^{[.,.],\pi_{N,N}}+l_2^{[.,.]_{N,N}}.$$

Similar computations as in (80), (81) and (82) show that $\left[\mu,\left[\underline{N},[\underline{N},\mu]_{RN}\right]_{RN}\right]_{RN}$ $=0$, which means that $\underline{N}$ is weak Nijenhuis vector valued form with respect to the symmetric differential graded Lie algebra structure $\mu=l_1^{[.,.],\pi}+l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$. ■

The next proposition establishes a relation between Poisson-Nijenhuis structures and co-boundary Nijenhuis tensors on a Lie algebroid.

Proposition 7.10. *Let $(A,[.,.],\rho)$ be a Lie algebroid , $\pi\in\Gamma(\wedge^2A)$ a bivector and N a $(1,1)$-tensor on A such that*

$$N\circ\pi^\#=\pi^\#\circ N^*.$$

Then, $\underline{N}+\pi$ is a co-boundary Nijenhuis vector valued form with curvature, with respect to the multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$, with square $\underline{N^2}$, if and only if (π,N) is a Poisson-Nijenhuis structure on the Lie algebroid $(A,[.,.],\rho)$. The deformed structure $[\underline{N},l_2^{[.,.]}]_{RN}$ is the L_∞-structure (indeed a differential graded Lie algebra structure) associated to the Poisson structure π on the Lie algebroid $(A,[.,.]_N,\rho\circ N)$.

Proof: Assume that (π,N) is a Poisson-Nijenhuis structure on the Lie algebroid $(A,[.,.],\rho)$. Then,

$$\left[\underline{N}+\pi,l_2^{[.,.]}\right]_{RN}=l_2^{[.,.]_N}-l_1^{[.,.],\pi}$$

and, by (79), we get

$$\begin{aligned}\left[\underline{N}+\pi\left[\underline{N}+\pi, l_2^{[.,.]}\right]_{RN}\right]_{RN} &= l_2^{[.,.]_{N,N}} + l_1^{[.,.],\pi_N} - l_1^{[.,.]_N,\pi}\\ &= l_2^{[.,.]_{N,N}} = \left[\underline{N^2}, l_2^{[.,.]}\right]_{RN},\end{aligned}$$

which means that $\underline{N}+\pi$ is a co-boundary Nijenhuis with respect to the multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$, with square $\underline{N^2}$.

Conversely, assume that $\underline{N}+\pi$ be a co-boundary Nijenhuis with respect to the multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$, with square $\underline{N^2}$, that is,

$$\left[\underline{N}+\pi, \left[\underline{N}+\pi, l_2^{[.,.]}\right]_{RN}\right]_{RN} = \left[\underline{N^2}, l_2^{[.,.]}\right]_{RN}. \tag{84}$$

Decomposing by homogeneous components, we get

$$\begin{aligned}\left[\underline{N}+\pi, \left[\underline{N}+\pi, l_2^{[.,.]}\right]_{RN}\right]_{RN} &= l_2^{[.,.]_{N,N}} + \left(\left[\underline{N}, l_2^{[.,.]}(\pi,.)\right]_{RN} + l_2^{[.,.]_N}(\pi,.)\right)\\ &\quad + l_2^{[.,.]}(\pi,\pi).\end{aligned} \tag{85}$$

From (84) and (85), we get

$$\left[\underline{N^2}, l_2^{[.,.]}\right]_{RN} = l_2^{[.,.]_{N,N}}, \tag{86}$$

$$\left(\left[\underline{N}, l_2^{[.,.]}(\pi,.)\right]_{RN} + l_2^{[.,.]_N}(\pi,.)\right) = 0 \tag{87}$$

and

$$l_2^{[.,.]}(\pi,\pi) = 0. \tag{88}$$

Equation (86) is equivalent to $l_2^{[.,.]_{N,N}} = l_2^{[.,.]_{N^2}}$, or to $[.,.]_{N,N} = [.,.]_{N^2}$, which means that N is a Nijenhuis tensor on A. Equation (88) means that π is Poisson, while Equation (87) gives

$$\left(\left[\underline{N}, l_2^{[.,.]}(\pi,.)\right]_{RN} + l_2^{[.,.]_N}(\pi,.)\right)(P) = 0,$$

or

$$\left[\underline{N}, l_2^{[.,.]}(\pi,.)\right]_{RN}(P) = [\pi, P]_{SN}^N, \tag{89}$$

for all $P \in \Gamma(\wedge A)$. The definition of $[.,.]^N_{SN}$ gives

$$\begin{aligned}
[\pi, P]^N_{SN} &= [\underline{N}(\pi), P]_{SN} + [\pi, \underline{N}(P)]_{SN} - \underline{N}\,[\pi, P]_{SN} \qquad (90)\\
&= 2\,[\pi_N, P]_{SN} + \left[\underline{N}, l_1^{[.,.],\pi}\right]_{RN}(P)\\
&= 2\,[\pi_N, P]_{SN} - \left[\underline{N}, l_2^{[.,.]}(\pi, .)\right]_{RN}(P),
\end{aligned}$$

where in the second equality we used $\underline{N}(\pi) = 2\pi_N$ and the definition of the Richardson-Nijenhuis bracket. From (89) and (90), we get

$$[\pi, P]^N_{SN} = [\pi_N, P]_{SN}\,.$$

and this completes the proof that (π, N) is a Poisson-Nijenhuis structure on the Lie algebroid $(A, [.,.]\,, \rho)$. ■

Last, we shall say a few words about the so-called $P\Omega$-structures [1, 14]. Recall that a *$P\Omega$-structure* on a Lie algebroid $(A, \rho, [.,.])$ is a pair (π, ω) where $\pi \in \Gamma(\wedge^2 A)$ is a Poisson element and $\omega \in \Gamma(\wedge^2 A^*)$ is a 2-form, with $\mathsf{d}^A\alpha = 0$. The 2-form $\omega \in \Gamma(\wedge^2 A^*)$ determines a morphism $\omega^\flat : A \to A^*$, given by $\langle Y, \omega^\flat(X)\rangle = \omega(X, Y)$. Defining a $(1,1)$ tensor $N := \pi^\# \circ \omega^\flat$, it is known that (π, N) is a Poisson-Nijenhuis structure while (ω, N) is an ΩN-structure.

Proposition 7.11. *Let (π, ω) be a $P\Omega$-structure on a Lie algebroid $(A, [.,.]\,, \rho)$. Then, $\mathcal{N} = \underline{\omega} + \pi$ is a co-boundary Nijenhuis form, with curvature, with respect to the multiplicative graded Lie algebra structure $l_2^{[.,.]}$ on the graded vector space $\Gamma(\wedge A)[2]$, with square $\underline{N}$, where $N = \pi^\# \circ \omega^\flat$. The deformed structure is $-l_1^{[.,.],\pi}$.*

Proof: Observe that

$$l_1^{[.,.],\pi}(P) = [\pi, P]_{SN} = -l_2^{[.,.]}(\pi, P) = -\left[\pi, l_2^{[.,.]}\right]_{RN}(P)$$

for all $P \in \Gamma(\wedge^2 A)$. This means that

$$l_1^{[.,.],\pi} = -\left[\pi, l_2^{[.,.]}\right]_{RN}. \qquad (91)$$

Hence,

$$\left[\mathcal{N}, l_2^{[.,.]}\right]_{RN} = -l_1^{[.,.],\pi} + \underline{\mathsf{d}^A\omega} = -l_1^{[.,.],\pi}, \qquad (92)$$

which proves the last claim (and proves that $\mathcal{N}$ is weak Nijenhuis vector valued form with respect to $l_2^{[\cdot,\cdot]}$, since $l_1^{[\cdot,\cdot],\pi}$ is an L_∞-structure on $\Gamma(\wedge A)[2]$). Equations (92) and (91) imply that

$$\begin{aligned}\left[\mathcal{N}, \left[\mathcal{N}, l_2^{[\cdot,\cdot]}\right]_{RN}\right]_{RN} &= -\left[\mathcal{N}, l_1^{[\cdot,\cdot],\pi}\right]_{RN} = -\left[\underline{\omega}, l_1^{[\cdot,\cdot],\pi}\right]_{RN} - [\pi,\pi]_{SN} \\ &= \left[\underline{\omega}, \left[\pi, l_2^{[\cdot,\cdot]}\right]_{RN}\right]_{RN} = \left[[\underline{\omega},\pi]_{RN}, l_2^{[\cdot,\cdot]}\right]_{RN}.\end{aligned}$$

This shows that $\mathcal{N}$ is a co-boundary Nijenhuis vector valued form with respect to the graded Lie algebra structure $l_2^{[\cdot,\cdot]}$, on the graded vector space $\Gamma(\wedge A)[2]$, with square $[\underline{\omega},\pi]_{RN}$. A direct computation shows that $[\pi,\underline{\omega}]_{RN} = \underline{N}$ and completes the proof. ■

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References

[1] P. Antunes, Crochets de Poisson gradués et applications: structures compatibles et généralisations des structures hyperkählériennes, Thèse de doctorat de l'École Polytechnique, March 2010.
[2] P. Antunes, C. Laurent-Gengoux, J. M. Nunes da Costa, Hierarchies and compatibility on Courant algebroids, *Pac. J. Math.* **261** (1) (2013), 1-32.
[3] P. Antunes, J. M. Nunes da Costa, Nijenhuis and compatible tensors on Lie and Courant algebroids, *J. Geom. Phys.* **65** (2013), 66-79.
[4] J. F. Cariñena, J. Grabowski, G. Marmo, Courant algebroid and Lie bialgebroid contractions, *J. Phys. A* **37** (19) (2004), 5189-5202.
[5] A. S. Cataneo, G. Felder, Relative formality theorem and quantization of coisotropic submanifolds, *Adv. Math.* **208** (2007), 521-548.
[6] A. S. Cataneo, F. Schätz, Equivalences of higher derived brackets, *J. Pure and Appl. Algebra* **212** (2008), 2450-2460.
[7] Y. Frégier, C. L. Roger, M. Zambon, Homotopy moment maps, arXiv:1304.2051.
[8] E. Getzler, Higher derived bracket, arXiv:1010.5859v2.
[9] I. Kolar, P. W. Michor, J. Slovak, Natural Operations in Differential Geometry. Springer-Verlag 1993.
[10] M. Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66** (2003), 157-216.
[11] Y. Kosmann-Schwarzbach, Quasi twisted and all that ... in Poisson geometry and Lie algebroid theory, 363-389 in *The breath of symplectic and Poisson geometry*, J. Marsden and T. Ratiu eds., *Prog. Math.* **232**, Birkhaüser, Boston, MA, 2005.
[12] Y. Kosmann-Schwarzbach, Nijenhuis structures on Courant algebroids, *Bull. Braz. Math. Soc. (N.S.)* **42** (4) (2011), 625-649.

[13] Y. Kosmann-Schwarzabch, F. Magri, Poisson-Nijenhuis structures, *Ann. Inst. Henri Poincaré, Série A,* **53** (1990), 35-81.
[14] Y. Kosmann-Schwarzbach, V. Rubtsov, Compatible structures on Lie algebroids and Monge-Ampere operators, *Acta Appl. Math.* **109** (1) (2010), 101-135.
[15] T. Lada, J. Stasheff, Introduction to SH Lie Algebras for Physisists, *Int. J. Theor. Phys.* **32** (7) (1993), 1087-1103.
[16] P. A. B. Lecomte, P. W. Michor, H. Schicketanz, The multigraded Nijenhuis-Richardson algebra, its universal property and applications, *J. Pure Appl. Algebra* **77** (1) (1992), 87-102.
[17] Z. J. Liu, A. Weinstein, P. Xu, Manin triples for Lie for Lie bialgebroids, *J. Diff. Geom.* **45** (1997), 547-574.
[18] Z. J. Liu, Y. Sheng, T. Zhang, Deformations of Lie 2-algebras, arXiv:1306.6225.
[19] C. Rogers, Higher symplectic geometry, Ph.D. thesis (2011), arXiv:1106.4068.
[20] D. Roytenberg, Courant algebroids, derived brakets and even symplectic supermanifolds, Ph.D. thesis, UC Berkeley, 1999.
[21] D. Roytenberg, Quasi-Lie bialgebroids and twisted Poisson manifolds, *Lett. Math. Phys.* **61** (2002), 123-137.
[22] D. Roytenberg, A. Weinstein, Courant algebroids and strongly homotopy algebras, *Lett. Math. Phys.* **46** (1998), 81-93.
[23] T. Voronov, Higher derived brackets and homotopy algebras, *J. Pure and Appl. Algebra* **202** (1-3) (2005), 133-153.
[24] F. Wagemann, On Lie algebra crossed modules, *Comm. Algebra* **34** (2006), 1699-1722.

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