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ON CLOSURE PROPERTIES OF RISK AVERSION MEASURES

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ABSTRACT: Increasing relative risk aversion (IRRA) is a common assumption in economics, but this property often is difficult to verify. We provide alternative characterizations of IRRA utility functions in terms of increasing absolute risk aversion (IARA) utility functions. We also investigate whether the decreasing generalized reversed failure rate (DGRFR) property is preserved under common transformations of random variables, in particular, we prove that DGRFR property is preserved under convolution.

KEYWORDS: utility theory, risk aversion, economics, closure properties, elasticity function.
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1. Introduction

It is well known that utility functions, $u(x)$, measure or represent the risk preferences of a decision maker. The assumption commonly made in economics is that the utility function is an increasing function. Therefore, a cumulative distribution function may be convenient for describing one's utilities (see [7] and references therein). To investigate the behavior of a person from the utility function, Meyer [18] proposed to use the marginal utility, $u'(x)$, for a measure of risk preferences. Note that, using marginal utilities is analogous to using density functions. Other functions used in the literature as measures of risk aversion are the absolute risk aversion (ARA) and the relative risk aversion (RRA) measures defined by Arrow [2] and Pratt [23]. The ARA measure, is defined by

$$A(x) = -\frac{u''(x)}{u'(x)}.$$

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Note that, if the utility function is increasing then it is possible to recover the utility function from an ARA measure using the following formula:

$$u(x) = \int e^{-\int A(x)dx}.$$

The RRA measure is defined by

$$R(x) = -x\frac{u''(x)}{u'(x)} = xA(x).$$

It is worth noting that there exist some links among risk measures and functions used to characterize random variables. Let X be a random variable with distribution function $F(x)$, survival function $\bar{F}(x)$ and density function $f(x)$. Its failure rate, reversed failure rate, odds and Glaser's functions (see [10]) are defined respectively by:

$$r(x) = \frac{f(x)}{\bar{F}(x)},\ \widetilde{r}(x) = \frac{f(x)}{F(x)},\ O(x) = \frac{F(x)}{\bar{F}(x)} \text{ and } \eta(x) = -\frac{f'(x)}{f(x)}.$$

Lariviere and Porteus [14] defined the generalized failure rate (called length-biased failure rate in [24]) of X as

$$G(x) = xr(x).$$

Analogously, we define the generalized reversed failure rate of X as

$$\widetilde{G}(x) = x\widetilde{r}(x).$$

Following Foschi and Spizzichino [9], we consider the following affine transformation of the distribution function:

$$u(x) = cF(x), c > 0. \tag{1}$$

The function $u(x)$ can be seen as an utility function associated with the distribution function of a random variable X and analogously, the function $u'(x)$ can be seen as a marginal utility associated with the density function of X. Note that, in this case, the absolute risk aversion and the relative risk aversion measures can be written as:

$$A(x) = -\frac{f'(x)}{f(x)} = \eta(x) \text{ and } R(x) = -x\frac{f'(x)}{f(x)} = x\eta(x),$$

respectively. The relative risk aversion measure, $R(x)$, has well known difficulties with negative values, and therefore, throughout the paper, only non-negative random variables are considered.

Some of the above concepts include one important notion in economic theory. In fact, elasticity of a positive differentiable transformation g defined on all or part of the positive real axis is defined as $d\log g(x)/d\log x$. Thus, it is the ratio of the relative change in the function g with respect to the relative change in x (see [20]). For example, if x is a price of a commodity and $D(x)$ denotes the demand on that commodity, then the elasticity of the demand is defined by

$$e_D(x) = -x\frac{D'(x)}{D(x)}, x \geq 0.$$

This concept was recently applied by Lariviere [13] to the supply management and showed the close relationship that exists between this elasticity and the generalized failure rate when $D(x) = \bar{F}(x)$. As another example, the elasticity of the distribution of a random variable defined in [27] is the generalized reversed failure rate $\widetilde{G}(x)$. Moreover, the RRA measure is the elasticity of the marginal utility, and is also the elasticity of the density function when (1) holds.

The study of monotonicity of the different risk measures is an important topic in different fields, such as economic analysis, demography, actuarial science, among others. Foschi and Spizzichino [9] pointed out some interactions between notions of risk and notions of ageing. Next, we recall some well known ageing and risk notions and also we define a new ageing notion related with the generalized reversed failure rate function.

Definition 1. *Let X be a random variable with distribution function $F(x)$, survival function $\bar{F}(x)$ and density function $f(x)$.*

a) *X has an increasing (decreasing) failure rate, IFR (DFR), and $F(x)$ is an IFR (DFR) distribution if $r(x)$ is increasing (decreasing) for all x.*
b) *X has an increasing (decreasing) generalized failure rate, IGFR (DGFR), and $F(x)$ is an IGFR (DGFR) distribution if $G(x)$ is increasing (decreasing) for all x.*
c) *X has an increasing (decreasing) reversed failure rate, IRFR (DRFR), and $F(x)$ is an IRFR (DRFR) distribution if $\widetilde{r}(x)$ is increasing (decreasing) for all x.*
d) *X has an increasing (decreasing) generalized reversed failure rate, IGRFR (DGRFR), and $F(x)$ is an IGRFR (DGRFR) distribution if $\widetilde{G}(x)$ is increasing (decreasing) for all x.*

Note that nonnegative random variables cannot have distributions with increasing reverse failure rate (IRFR) (see p. 179 in [16]). However, there are some nonnegative random variables with increasing generalized reverse failure rate (IGRFR) (see example 25).

In the newsvendor with pricing literature, the IFR and IGFR assumptions are commonly used. It is easy to check that all IFR distributions are IGFR, but the reverse is not true. Recent interest has focused on IGFR distributions, see Lariviere and Porteus [14], Paul [22], Lariviere [13] or Colombo and Labrecciosa [6].

Definition 2. *Let $u(x)$ be a utility function.*

a) *The utility function evinces decreasing (increasing) absolute risk aversion, DARA (IARA) if and only if its absolute risk aversion function $A(x)$ is decreasing (increasing).*
b) *The utility function evinces decreasing (increasing) relative risk aversion, DRRA (IRRA) if and only if its relative risk aversion function $R(x)$ is decreasing (increasing).*

Throughout the paper the terms *increasing* and *decreasing* stand for *non-decreasing* and *non-increasing*, respectively.

Since Arrow [3], who hypothesises that most investors display decreasing absolute risk aversion (DARA) and increasing relative risk aversion (IRRA) with respect to wealth, these assumptions have been used increasingly in the literature. DARA (IARA) assumptions have been used in auction theory (see [17]), in stochastic dominance in complete markets (see [5]), to characterize aversion to an increase in downside risk (see [15]), in newsvendor model under expected utility theory (see [28]). DRRA (IRRA) utilities have been used in Ogaki and Zhang [21], Meyer and Meyer [19], Guiso and Paiella [11], Sévi [25].

The IFR (DFR), DRFR (IRFR) and IARA (DARA) properties are equivalent to log-concavity (log-convexity) of the survival, distribution and density functions, respectively. Bagnoli and Bergstrom [4] listed several commonly-used continuous, univariate probability distributions that have log-concave density functions. Note that, when the utility function is described by a distribution function, IARA (DARA) properties are equivalent to increasing (decreasing) likelihood ratio, ILR (DLR), a very well known ageing notion in reliability theory (see [26]).

Here, we consider that risk preferences are represented by a distribution function as utility function. Under this assumption, the purpose of this work is twofold. Firstly, we provide alternative characterizations for IRRA utilities, since, this property often is difficult to verify. Secondly, we investigate properties of the class of DGRFR distributions and compare this class with the class of DRFR distributions.

The remainder of this paper is organized as follows. Section 2 contains alternative characterizations of IRRA (DRRA) utility functions in terms of IARA (DARA) utility functions. In addition, some closure properties (closure with respect scaling, shifting and raising to a power) of IRRA (DRRA) and IARA (DARA) utilities are given. In Section 3 we investigate whether the DGRFR (IGRFR) property is preserved under common transformations of random variables, in particular, we prove that DGRFR property is preserved under convolution. Section 4 is devoted to present some counterexamples about the lack of closure (closure with respect shifting and mixing) of IRRA and DGRFR distributions. Finally, in Section 5, we summarize the results presented here, as well as, results known in the literature.

2. On absolute and relative risk aversion

In this section, we establish alternative characterizations of IRRA (DRRA) distributions and investigate closure properties of IRRA (DRRA) and IARA (DARA) utilities under fundamental operations on random variables such as scaling, raising to a power and shifting.

Let X and Y be univariate random variables with Glaser's functions η_X and η_Y, respectively. We say that X is smaller than Y in the likelihood ratio order if $\eta_X(x) \geq \eta_Y(x)$ for all x, denoted by $X \leq_{lr} Y$. Note that, when risk preferences are represented by a distribution function as utility function, the likelihood ratio ordering is equivalent to DARA stochastic dominance due to the relation between the Glaser's function and the absolute risk aversion measure. In addition, it is well known that, when the means of the two random variables are the same, DARA stochastic dominance and third degree stochastic dominance (TSD) are equivalent concepts (see [8]).

Theorem 3. *The following statements are equivalent:*

a) X *is IRRA.*
b) $\log X$ *is IARA.*
c) $X \leq_{lr} \lambda X$ *for* $\lambda \geq 1$.

d) $\log X \leq_{lr} a + \log X$ *for* $a \geq 0$.
e) λX *is IRRA for* $\lambda > 0$.

Proof: The proof of the equivalence of parts a) and b) comes from simple reasoning using the fact that the distribution function of $\log X$ is $F_L(x) = F(e^x)$, thus the absolute risk aversion is $A_L(x) = R(e^x) - 1$. Hence, it is clear that if $R(x)$ is increasing, then $A_L(x)$ is also increasing and vice versa. Note that $\log X$ is IARA means that its density function is log-concave. From Theorem 3.3 in [12], we know that parts b) and d) are equivalent. The equivalence of parts c) and d) was established also in [12]. To link parts a) and e), follows directly from the fact that the relative risk aversion of λX is $R_\lambda(x) = R(x/\lambda)$. ■

Remark 4. *Because* $A_L(x) = R(e^x) - 1$*, it is easy to see that* X *is DRRA if and only if* $\log X$ *is DARA.*

Note that part c) in Theorem 3 means that an investor with utility function $u(x) = F(x)$ is more risk averse than an agent with utility function $u_\lambda(x) = F_\lambda(x)$, since the absolute risk aversion of $u(x)$ dominates the absolute risk aversion of $u_\lambda(x)$ pointwise, i.e., $A(x) \geq A_\lambda(x)$, where $F_\lambda(x)$ and $A_\lambda(x)$ are defined analogously to the relative risk aversion $R_\lambda(x)$.

The equivalence of parts a) and e) means that IRRA distributions are closed under positive scale transformations. Analogously, one can see DRRA distributions are also closed under positive scale transformations.

Let us mention that Theorem 3 also simplifies verifying the IRRA property as we show in the following example.

Example 5. *For instance, if* X *has a lognormal distribution, it is easy to check that its absolute risk aversion is nonmonotone and therefore one cannot immediately conclude that* X *is IRRA. However,* $\log X$ *is normally distributed and hence IARA (its density function is log-concave). Thus, from Theorem 3,* X *is IRRA.*

Proposition 6. *For* $\lambda \geq 1$*, if* X *is IRRA then* $R(x) \geq R_\lambda(x)$*. If* X *is DRRA then* $R(x) \leq R_\lambda(x)$*.*

Proof: By definition, X is IRRA if and only if $R(x)$ is increasing in x. Then, $R(x) \geq R(x/\lambda)$ since $x \geq x/\lambda$. Note that the relative risk aversion of λX is $R_\lambda(x) = R(x/\lambda)$. Hence, $R(x) \geq R_\lambda(x)$. For the second part, the proof is similar reverting the inequalities. ■

Corollary 7. *For $\lambda \geq 1$, if X is IRRA then $X \leq_{lr} \lambda X$. If X is DRRA then $X \geq_{lr} \lambda X$.*

Proof: We know from Proposition 6 a) that if X is IRRA then $R(x) \geq R_\lambda(x)$. By definition, $R(x) = xA(x)$, then $A(x) \geq A_\lambda(x)$ and this condition is equivalent to $X \leq_{lr} \lambda X$. Again, the proof of the second part is similar reverting the inequalities. ∎

The following result shows that the class of IRRA (DRRA) distributions is closed under the operation of taking arbitrary positive powers of the underlying random variable.

Proposition 8. *X is IRRA (DRRA) if and only if X^b is IRRA (DRRA) for all $b > 0$.*

Proof: This proof follows from simple reasoning using the fact that the distribution function of X^b is $F_b(x) = F(x^{1/b})$, thus the relative risk aversion is $R_b(x) = (b-1)/b + R(x^{1/b})/b$. Hence, it is clear that if $R(x)$ is increasing, then $R_b(x)$ is also increasing and vice versa. For DRRA distributions, the proof is similar changing increasing by decreasing. ∎

As we prove in the following result, DARA distributions are closed under the operation of taking powers greater than one. However, in general, IARA distributions are not closed under this transformation. In the following result, we show that for IARA distributions it is necessary an additional condition.

Proposition 9.

a) *If X is IARA and its distribution function is concave, then X^b is also IARA for $0 < b \leq 1$.*
b) *If X is DARA, then X^b is also DARA for $b \geq 1$.*

Proof:

a) By definition, we know that the IARA property is equivalent to the density function being log-concave. The density function of X^b is $f_b(x) = x^{-1+1/b} f(x^{1/b})/b$. It is easy to see that $x^{-1+1/b}$ is log-concave and $x^{1/b}$ is convex if $0 < b \leq 1$. By the assumptions, we know that the density function of X is decreasing, since its distribution function is concave. Now, from Proposition A5 (p. 689) in [16] we have that $f(x^{1/b})$ is log-concave. Then $f_b(x)$ is also log-concave since the product of log-concave functions is log-concave.

b) X^b is DARA if and only if $f_b(x)$ is log-convex. By the assumptions, X is DARA, then its distribution function is concave and its density function decreasing. Since $x^{1/b}$ is concave, then $f(x^{1/b})$ is log-convex. Now, $x^{-1+1/b}$ is log-convex if $b \geq 1$, then $f_b(x)$ is also log-convex since the product of log-convex functions is log-convex.

■

Proposition 10. *IARA and DARA distributions are closed under positive scale transformations and shifting.*

The proofs follow directly from the definition of IARA and DARA distributions.

Proposition 11. *DRRA distributions are closed under right-shifting.*

Proof: The distribution and density functions of the right-shifted random variable $X + a$, $a > 0$, are $F_{+a}(x) = F(x - a)$ and $f_{+a}(x) = f(x - a)$, respectively. Therefore, the absolute risk aversion is $A_{+a}(x) = A(x - a)$ and the relative risk aversion is

$$R_{+a}(x) = xA(x - a) = R(x - a) + aA(x - a).$$

Since DRRA property implies DARA property, then if X is DRRA, it is clear that $X + a$ is also DRRA. ■

Note that, in general, IRRA distributions are not closed under right-shifting (see the Counterexample 26). For left-shifting random variables $X - a$, $a > 0$, it can be shown that if X is DARA and also IRRA, then $X - a$ is IRRA, since

$$R_{-a}(x) = xA(x + a) = R(x + a) - aA(x + a).$$

It is well know that IARA (DARA) distributions are closed under convolution (see [16]). From this fact and Theorem 3 we have the following preservation property.

Corollary 12. *If both X and Y are IRRA (DRRA), then XY is also IRRA (DRRA).*

Another interesting criterion to establish characterizations of IRRA (DRRA) utility functions is derived from a simple manipulation of the risk functions as we show in the following result.

Theorem 13. *Assume the non negative random variable X has a twice differentiable density function. X is IRRA if*

$$Q(x) = x\left(\left(\frac{f'(x)}{f(x)}\right)^2 - \frac{f''(x)}{f(x)}\right) - \frac{f'(x)}{f(x)} \geq 0. \tag{2}$$

The random variable is DRRA is $Q(x) \leq 0$, $x \geq 0$.

Proof: Differentiating the absolute and the relative risk aversion measures we obtain

$$A'(x) = A^2(x) - \frac{f''(x)}{f(x)},$$

and

$$R'(x) = A(x) + xA'(x) = xA^2(x) + A(x) - x\frac{f''(x)}{f(x)}.$$

Clearly, if $R'(x)$ has constant sign then $R(x)$ is monotone. Note that the above expression for $R'(x)$ is quadratic in $A(x)$, therefore to find a criterion for the sign of $R'(x)$ to be constant, it is enough to study the zeros of the quadratic function that characterizes $R'(x)$. Solving $R'(x) = 0$ leads to

$$A(x) = \frac{-1 \pm \sqrt{1 + 4x^2\frac{f''(x)}{f(x)}}}{2x} \quad \Leftrightarrow \quad R(x) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + x^2\frac{f''(x)}{f(x)}}.$$

Thus, a criterion of monotonicity of $R(x)$ is to check that one of the two inequalities holds, for all $x \in \mathbb{R}$,

$$\left|R(x) + \frac{1}{2}\right| < \sqrt{\frac{1}{4} + x^2\frac{f''(x)}{f(x)}} \qquad \text{or} \qquad \left|R(x) + \frac{1}{2}\right| > \sqrt{\frac{1}{4} + x^2\frac{f''(x)}{f(x)}}. \tag{3}$$

The above criterion can be rewritten as

$$\left(R(x) + \frac{1}{2}\right)^2 - \left(\frac{1}{4} + x^2\frac{f''(x)}{f(x)}\right) = xQ(x) \tag{4}$$

has constant sign. Thus, since $x \geq 0$, if (4) is positive (negative) then the utility function is IRRA (DRRA). ■

Next, we present a particular case to illustrate the criterion for IRRA (DRRA) established above.

Example 14. *Assume that*

$$f(x) \sim p(x)e^{-q(x)}, \text{ with } p(x) = x^{\beta} \text{and } q(x) = dx^{\alpha}, \tag{5}$$

with $d \geq 0$. In this case, (2) is equal to $d\alpha^2 x^{\alpha-1}$ which is nonnegative since $x \geq 0$. Note that Weibull and gamma distributions are included in the proposed model. By using this new criterion, we obtain that both distributions (Weibull and gamma) are IRRA. In addition, this criterion is useful to study whether a right-shifted random variable $X+a$, $a > 0$ is IRRA (DRRA). Note that, in this case, (2) is

$$x\left(\left(\frac{f'(x-a)}{f(x-a)}\right)^2 - \frac{f''(x-a)}{f(x-a)}\right) - \frac{f'(x-a)}{f(x-a)}, \tag{6}$$

since the density function of $X + a$ is $f_{+a}(x) = f(x-a)$. After some computations, (6) is equal to

$$\frac{T(x)}{(x-a)^2} := \frac{a\beta + d\alpha(x-a)^{\alpha}(\alpha x - a)}{(x-a)^2},$$

for the particular case (5) . Clearly, the function $T(x)$ determines the sign of the above expression. Assume $\beta < 0$. In this case, $T(a) = a\beta < 0$, thus the distribution can not be IRRA. Note that if $d = 1$ and $\beta = \alpha - 1$, then the random variable X has a Weibull distribution. Therefore, if $0 \leq \alpha < 1$ the right-shifted random variable from Weibull distributions can not be IRRA (see counterexample 26). Let $\beta \geq 0$. If $\alpha \geq a$ then $T(x) \geq 0$ for all $x \geq a$ since $\alpha x - a \geq 0$. Hence the right-shifted random variable is IRRA. The conclusion also holds if $\alpha < 0$, because in this case $d\alpha(\alpha x - a) = d\alpha^2 x - ad\alpha \geq 0$.

3. On generalized (reversed) failure rate

Recently, Veres-Ferrer and Pavía [27] studied the relationship between the reversed hazard rate and the elasticity of the distribution of a random variable (here, we call this elasticity as generalized reversed failure rate). Since the elasticity is a concept broadly used in economics, we investigate closure properties of DGRFR (IGRFR) utilities. In particular, we prove that DGRFR distributions are closed under convolutions. In addition, we relate this new ageing notion with IFR (DFR), IGFR (DGFR) and the risk measures of Section 2.

Let X and Y be univariate random variables with reversed failure rate functions $\widetilde{r}_X$ and $\widetilde{r}_Y$, respectively. We say that X is smaller than Y in the reversed hazard rate order if $\widetilde{r}_X(x) \leq \widetilde{r}_Y(x)$ for all x, denoted by $X \leq_{rh} Y$.

Theorem 15. *The following statements are equivalent:*

a) X *is DGRFR.*
b) $\log X$ *is DRFR.*
c) $X \leq_{rh} \lambda X$ *for* $\lambda \geq 1$.
d) $\log X \leq_{rh} a + \log X$ *for* $a \geq 0$.
e) λX *is DGRFR for* $\lambda > 0$.

Proof: The proof of the equivalence of parts a) and b) follows from simple reasoning using the fact that the distribution function of $\log X$ is $F_L(x) = F(e^x)$, thus the reversed failure rate is $\widetilde{r}_L(x) = \widetilde{G}(e^x)$. Hence, it is clear that if $\widetilde{G}(x)$ is increasing, then $\widetilde{r}_L(x)$ is also increasing and vice versa. Note that $\log X$ is DRFR means that its distribution function is log-concave. From Theorem 3.3 in [12], we know that parts b) and d) are equivalent. The equivalence of parts c) and d) was established also in [12]. To link parts a) and e), follows directly from the fact that the generalized reversed failure rate of λX is $\widetilde{G}_\lambda(x) = \widetilde{G}(x/\lambda)$. ■

Remark 16. *Because* $\widetilde{r}_L(x) = \widetilde{G}(e^x)$, *it is easy to see that* X *is IGRFR if and only if* $\log X$ *is IRFR.*

The equivalence of parts a) and e) means that DGRFR distributions are closed under positive scale transformations. Analogously, one can see IGRFR distributions are also closed under positive scale transformations.

In the following result, we investigate the relationship between DGFR (IGFR) and DGRFR (IGRFR) properties, obtaining similar relations to those between DFR (IFR) and DRFR (IRFR) properties, which are well known in the literature.

Proposition 17.

a) *If* X *is DGFR then it is also DGRFR.*
b) *If* X *is IRGFR then it is also IGFR.*

Proof: It is easy to see that the generalized reversed failure rate can be written as $\widetilde{G}(x) = xr(x)(O(x))^{-1}$, so if the failure rate, $r(x)$, is decreasing then $\widetilde{G}(x)$ is also decreasing, since $(O(x))^{-1}$ is decreasing. Analogously, it can be seen that the generalized failure rate can be written as $G(x) = x\widetilde{r}(x)O(x)$, so if the reversed failure rate, $\widetilde{r}(x)$, is increasing then $G(x)$ is also increasing, since $O(x)$ is increasing. ■

Proposition 18. *X is DGRFR (IGRFR) if and only if X^b is DGRFR (IGRFR) for all $b > 0$.*

Proof: This proof relies on the fact that the distribution function of X^b is $F_b(x) = F(x^{1/b})$, thus the generalized reversed failure rate is $\widetilde{G}_b(x) = \widetilde{G}(x^{1/b})/b$. Hence, it is clear that if $\widetilde{G}$ is increasing (decreasing), then $\widetilde{G}_b(x)$ is also increasing (decreasing) and vice versa. ■

Paul [22] proved that IGFR distributions are closed under operations of raising to a power. Analogously, it can be seen that X is DGFR if and only if X^b is DGFR for all positive real numbers b, since the generalized failure rate of X^b is $G_b(x) = G(x^{1/b})/b$.

Proposition 19. *If X is DFR (DRFR) then X^b is DFR (DRFR) for $b \geq 1$.*

Proof: The survival function of X^b is $\bar{F}_b(x) = \bar{F}(x^{1/b})$ and X^b is DFR if and only if $\bar{F}_b(x)$ is log-convex. Clearly, $x^{1/b}$ is concave if $b \geq 1$, $\log \bar{F}(x)$ is decreasing in x and it is convex by the assumption, then from Proposition A5 (p. 689) in [16] we have that $\bar{F}(x^{1/b})$ is log-convex. Analogously, X^b is DRFR if and only if $F_b(x)$ is log-concave. Because $x^{1/b}$ is concave if $b \geq 1$, $\log F(x)$ is increasing in x and it is concave, then $F(x^{1/b})$ is log-concave again from Proposition A5 in [16]. ■

Paul [22] showed with the help of a counterexample that, in general, IFR distributions are not closed under the operation of taking arbitrary positive powers of the underlying random variable. In particular, he considered $b = 2$. However, using an argument similar to those in Proposition 19, it can be seen that X^b is IFR if $0 < b \leq 1$, since, in this case, $x^{1/b}$ is convex.

Proposition 20. *DGRFR and DGFR distributions are closed under right-shifting.*

Proof: Now consider the right-shifted random variable $X + a$, $a > 0$. Remember that the distribution and density functions of $X + a$ are $F_{+a}(x) = F(x - a)$ and $f_{+a}(x) = f(x - a)$, respectively. The reversed failure rate is $\widetilde{r}_{+a}(x) = \widetilde{r}(x - a)$ and the generalized reversed failure rate is

$$\widetilde{G}_{+a}(x) = x\widetilde{r}(x - a) = \widetilde{G}(x - a) + a\widetilde{r}(x - a).$$

Since DGRFR property implies DRFR property, then if X is DGRFR, it is clear that $X + a$ is also DGRFR. Analogously, it can be prove that the generalized failure rate of $X + a$ is

$$G_{+a}(x) = xr(x - a) = G(x - a) + ar(x - a).$$

Then if X is DGFR, it is clear that $X+a$ is also DGFR, since DGFR property implies DFR property. ■

Note that, in general, IGFR distributions are not closed under right-shifting (see counterexample 2 in [22]). For left-shifting random variables $X - a$, $a > 0$, it can be shown that if X is IGRFR and also DRFR, then $X - a$ is IGRFR, since

$$\widetilde{G}_{-a}(x) = x\widetilde{r}(x + a) = \widetilde{G}(x + a) - a\widetilde{r}(x + a). \tag{7}$$

In general, DGRFR distributions are not closed under left-shifting (see counterexample 27). Analogously, it can be prove that the generalized failure rate of $X - a$ is

$$G_{-a}(x) = xr(x + a) = G(x + a) - ar(x + a).$$

Therefore, if X is IGFR and also DFR, then $X - a$ is IGFR. In the counterexample 2 in [22], the author showed that if X is a Pareto random variable with density function $f(x) = \theta x^{-(\theta+1)}$, then $X - a$ is IGFR. Note that, in this case, the failure rate function for X is θ/x and clearly it is decreasing. In general, DGFR distributions are not closed under left-shifting.

Proposition 21. *DFR and DRFR distributions are closed under shifting.*

The proofs follow directly from the definition of DFR (DRFR) distributions and from the fact that the failure rate of $X + a$ ($X - a$) is $r_{+a}(x) = r(x - a)$ ($r_{-a}(x) = r(x+a)$) and the reversed failure rate is $\widetilde{r}_{+a}(x) = \widetilde{r}(x-a)$ ($\widetilde{r}_{-a}(x) = \widetilde{r}(x + a)$).

It is well know that DRFR (IRFR) distributions are closed under convolution (see [16]). From this fact and Theorem 15 we have the following preservation property.

Corollary 22. *If both X and Y are DGRFR (IGRFR), then XY is also DGRFR (IGRFR).*

Theorem 23. *DGRFR distributions are closed under convolutions.*

Proof: Let X and Y independent random variables with decreasing generalized reversed failure rate functions. Keilson and Sumita [12] proved that X is DGRFR if and only if $X \leq_{rh} \lambda X$ for $\lambda \geq 1$. Analogously, Y is DGRFR if and only if $Y \leq_{rh} \lambda Y$ for $\lambda \geq 1$. Now, from Lemma 1.B.44

in [26], we know that if X and Y have decreasing reversed failure rate, then $X+Y \leq_{rh} \lambda X+\lambda Y$. Note that, X and Y are DRFR, since they are DGRFR. Again, by Keilson and Sumita [12] we have that $X+Y$ is DGRFR if and only $X+Y \leq_{rh} \lambda\,(X+Y)$. ∎

The next proposition is straightforward from Theorems 3 and 15, Theorem 1 in [13] and Theorem E.2 (p. 134) in [16]. Analogously to the proof of Theorem 1 in [13], it is easy to see that X is DGFR if and only if $\log X$ is DFR, since the failure rate of $\log X$ is $r_L(x) = G(e^x)$.

Proposition 24.

a) *If X is IRRA then is also IGFR and DGRFR.*
b) *If X is DRRA then is also DGFR.*

Proof:

$$X \text{ is IRRA} \Leftrightarrow \log X \text{ is IARA} \Rightarrow \log X \text{ is IFR} \Leftrightarrow X \text{ is IGFR},$$
$$X \text{ is IRRA} \Leftrightarrow \log X \text{ is IARA} \Rightarrow \log X \text{ is DRFR} \Leftrightarrow X \text{ is DGRFR},$$
$$X \text{ is DRRA} \Leftrightarrow \log X \text{ is DARA} \Rightarrow \log X \text{ is DFR} \Leftrightarrow X \text{ is DGFR}.$$

∎

4. Examples and Counterexamples

As we pointed out in the Introduction, there are some nonnegative random variables with IGRFR as we show in the following example.

Example 25. *Let X be a beta random variable with density function*

$$\frac{x^{\alpha-1}\,(1-x)^{\beta-1}}{B(\alpha,\beta)},$$

where $B(\alpha,\beta)$ is the beta function. Note that the generalized reversed failure rate is

$$\widetilde{G}(x) = \frac{x^{\alpha}\,(1-x)^{\beta-1}}{B(x,\alpha,\beta)},$$

where $B(x,\alpha,\beta)$ is the incomplete beta function. Set $\alpha = 2$ and $\beta = 0.5$. It can be seen in Figure 1 that the generalized reversed failure rate is increasing, i.e., X is IGRFR.

As we pointed out in Section 2, the relative risk aversion for $X+a$ is

$$R_{+a}(x) = R(x-a) + aA(x-a).$$

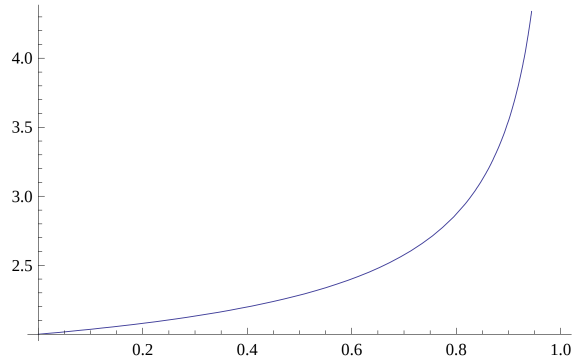

FIGURE 1. Plot of the generalized reversed failure rate for a beta random variable with $\alpha = 2$ and $\beta = 0.5$

In the following counterexample we show that, in general, IRRA distributions are not closed under right-shifting.

Counterexample 26. *Let X be a Weibull random variable with density function*

$$f(x) = \alpha\lambda\,(\lambda x)^{\alpha-1}\exp\{-(\lambda x)^{\alpha}\}\,, x \geq 0.$$

Note that the relative risk aversion of X is $R(x) = 1 + \alpha\left((\lambda x)^{\alpha} - 1\right)$ and it is increasing in x for any positive α, so X is IRRA. Note that the absolute risk aversion of X is $A(x) = R(x)/x$, hence, the relative risk aversion for $X + a$ is

$$R_{+a}(x) = \frac{x}{x-a}\Big(1 + \alpha\left(\lambda^{\alpha}(x-a)^{\alpha} - 1\right)\Big).$$

Set $\lambda = 1$, $\alpha = 0.5$ and $a = 2$, in this case, the relative risk aversion of $X+a$ is not increasing (nor decreasing), i.e., $X + a$ is not IRRA as it can be seen in Figure 2.

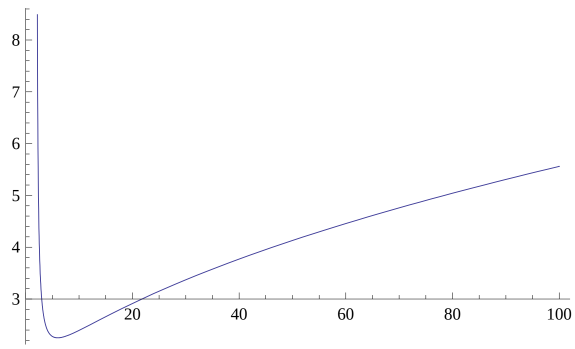

FIGURE 2. Plot of the relative risk aversion for $X+a$ with $a = 2$ when X is a Weibull random variable with $\alpha = 0.5$ and $\lambda = 1$

As we pointed out in Section 3, in general, DGRFR distributions are not closed under left-shifting.

Counterexample 27. *Let X a random variable defined as in the counterexample 26. Then, its generalized reversed failure rate is:*

$$\widetilde{G}(x) = \frac{\alpha(\lambda x)^{\alpha}}{e^{(\lambda x)^{\alpha}} - 1}.$$

It is easy to verify that $\widetilde{G}(x)$ is decreasing in $x \geq 0$ for any α, i.e. X is DGRFR. We compute the generalized reversed failure rate of $X - a$ using (7), which is given by:

$$\widetilde{G}_{-a}(x) = \frac{\alpha\lambda x\,(\lambda(x+a))^{\alpha-1}}{e^{(\lambda(x+a))^{\alpha}} - 1}.$$

Set $\lambda = 1$, $\alpha = 0.5$ and $a = 1$. It can be seen from Figure 3 that the left-shifting of X is not DGRFR.

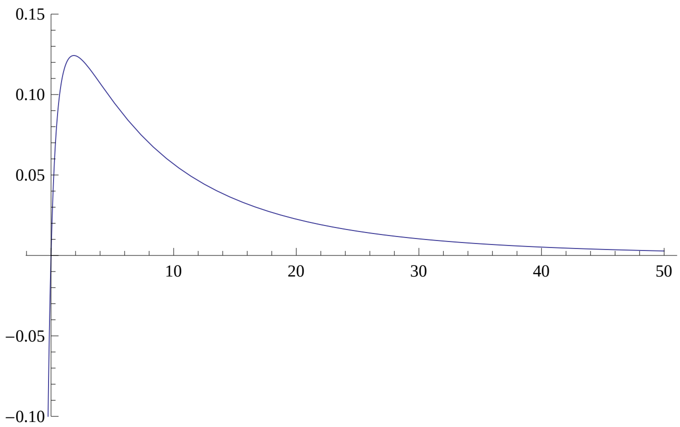

FIGURE 3. Plot of the generalized reversed failure rate for $X - a$ with $a = 1$ when X is a Weibull random variable with $\alpha = 0.5$ and $\lambda = 1$

Another interesting closure property is the mixing property. This closure property for IGFR distributions was considered by Al-Zahrani and Stoyanov [1]. In particular, the class of IGFR distributions do not preserve the mixing property. In the following counterexamples, we focus our attention on mixtures of IRRA and DGRFR distributions. It is well known that, given two random variables X_i with distribution function F_i, survival function $\bar{F}_i(x)$ and density function $f_i(x)$, for $i = 1, 2$, the distribution and the density functions of the mixture distribution are

$$F(x) = pF_1(x) + (1-p)F_2(x) \text{ and } f(x) = pf_1(x) + (1-p)f_2(x),$$

respectively, for any $0 \leq p \leq 1$. Let $A_i(x)$ and $R_i(x)$ the absolute and the relative risk aversion of X_i for $i = 1, 2$. Then, the absolute and the relative risk aversion of the mixture distribution are

$$A(x) = \psi_p(x)A_1(x) + (1-\psi_p(x))A_2(x), \tag{8}$$

and

$$R(x) = \psi_p(x)R_1(x) + (1-\psi_p(x))R_2(x), \tag{9}$$

where

$$\psi_p(x) = \frac{pf_1(x)}{pf_1(x) + (1-p)f_2(x)}.$$

Analogously, let $\widetilde{G}_i(x)$ the generalized reversed failure rate of of X_i for $i = 1, 2$. Then, the generalized reversed failure rate of the mixture distribution is

$$\widetilde{G}(x) = \phi_p(x)\widetilde{G}_1(x) + (1-\phi_p(x))\widetilde{G}_2(x), \tag{10}$$

where

$$\phi_p(x) = \frac{pF_1(x)}{pF_1(x) + (1-p)F_2(x)}.$$

The following counterexamples show that IRRA and DGRFR distributions, in general, are not closed under mixing operation.

Counterexample 28. *Let X_1 be an exponential distribution with hazard rate $\lambda = 4$ and X_2 a gamma distribution with shape parameter $\alpha = 2$ and scale parameter $\lambda = 1$. Then, the relative risk aversion are:*

$$R_1(x) = 4x \text{ and } R_2(x) = x - 1,$$

for $x > 0$, and it is easy to see that $R_1(x)$ and $R_2(x)$ are both increasing in x, i.e., X_1 and X_2 are IRRA distributions. By using (8), (9) and after some computations, we get the relative risk aversion of the mixture distribution, namely

$$R(x) = x - 1 + \frac{4p(1+3x)}{4p + (1-p)xe^{3x}}.$$

Choosing $p = 0.95$, from Figure 4, it is clear that $R(x)$ is not increasing, so the mixture of X_1 and X_2 is not IRRA.

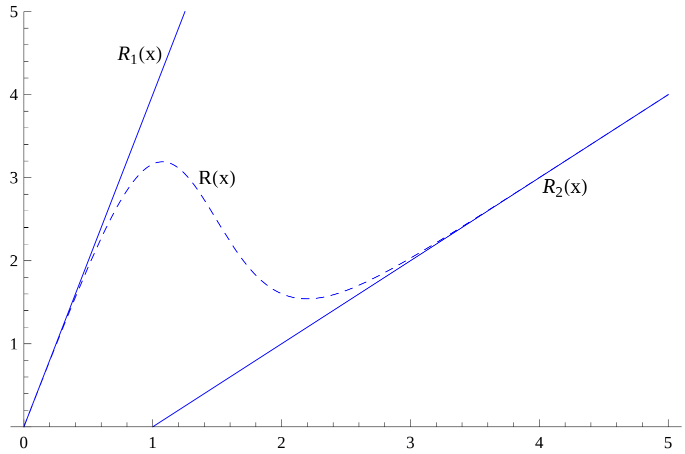

FIGURE 4. Plot of the relative risk aversion of a mixture of one exponential distribution and one gamma distribution

Counterexample 29. *Let X_1 and X_2 two random variables defined as in the counterexample 28. We compute the generalized reversed failure rate of the mixture distribution from (10) and plot $\widetilde{G}(x)$, $\widetilde{G}_1(x)$ and $\widetilde{G}_2(x)$ when $p = 0.45$. It can be seen from Figure 5 that X_1 and X_2 are both DGRFR, however, the mixture of X_1 and X_2 is not DGRFR.*

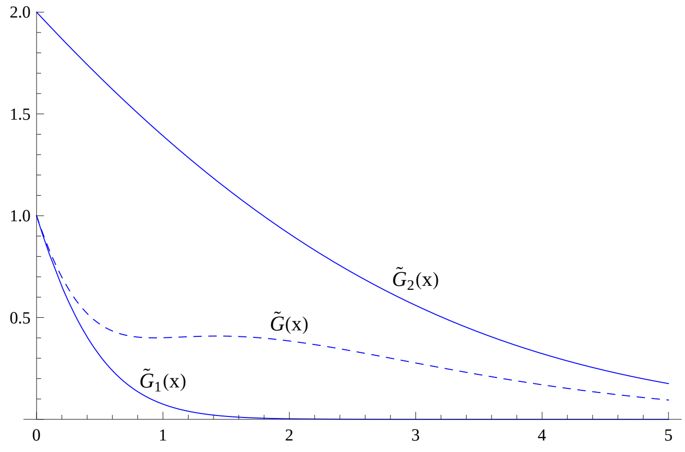

FIGURE 5. Plot of the generalized reversed failure rate of a mixture of one exponential distribution and one gamma distribution

5. Summaries of abbreviations, relationships, and closures

In this last section, we provide a summary of abbreviations of the ageing and risk notions used in this work, as well as, relationships among them, and closures properties.

TABLE 1. Summary of abbreviations

Abbreviation	**Full name**
IFR	Increasing failure rate
DFR	Decreasing failure rate
IRFR	Increasing reversed failure rate
DRFR	Decreasing reversed failure rate
IGFR	Increasing generalized failure rate
DGFR	Decreasing generalized failure rate
IGRFR	Increasing generalized reversed failure rate
DGRFR	Decreasing generalized reversed failure rate
IARA	Increasing absolute risk aversion
DARA	Decreasing absolute risk aversion
IRRA	Increasing relative risk aversion
DRRA	Decreasing relative risk aversion

TABLE 2. Summary of relationships

$$
\begin{array}{ccccccccc}
 & & & & \text{IARA} & \Rightarrow & \text{IFR} & & \\
 & & & & \Downarrow & & \Downarrow & & \\
 & & & & \text{IRRA} & \Rightarrow & \text{IGFR} & & \\
 & & & & \Downarrow & & \Uparrow & & \\
\text{DRRA} & \Rightarrow & \text{DGFR} & \Rightarrow & \text{DGRFR} & & \text{IGRFR} & \Longleftarrow & \text{IRFR} \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \\
\text{DARA} & \Rightarrow & \text{DFR} & \Rightarrow & \text{DRFR} & & & &
\end{array}
$$

Table 3. Summary of closure properties

Class of distribution	Scale Transformation	Power Transformation	Right shifting	Left shifting	Convolution	Mixtures
IARA	closed	not closed*	closed	closed	closed	not closed
IRRA	closed	closed	not closed	not closed*		not closed
IFR	closed	closed ($b \leq 1$)	closed	closed	closed	not closed
IGFR	closed	closed	not closed	not closed*	not closed	not closed
IGRFR	closed	closed	not closed	not closed*		
DARA	closed	closed ($b \geq 1$)	closed	closed	not closed	closed
DRRA	closed	closed	closed	not closed		
DFR	closed	closed ($b \geq 1$)	closed	closed	not closed	closed
DGFR	closed	closed	closed	not closed		
DGRFR	closed	closed	closed	not closed	closed	not closed
DRFR	closed	closed ($b \geq 1$)	closed	closed	closed	not closed

* Property is closed under additional assumptions, but not in general.

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