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THE SYMMETRIZATION PROBLEM FOR MULTIPLE ORTHOGONAL POLYNOMIALS

AMÍLCAR BRANQUINHO AND EDMUNDO J. HUERTAS

ABSTRACT: We analyze the effect of symmetrization in the theory of multiple orthogonal polynomials. For a symmetric sequence of type II multiple orthogonal polynomials satisfying a high–term recurrence relation, we fully characterize the Weyl function associated to the corresponding block Jacobi matrix as well as the Stieltjes matrix function. Next, from an arbitrary sequence of type II multiple orthogonal polynomials with respect to a set of d linear functionals, we obtain a total of $d+1$ sequences of type II multiple orthogonal polynomials, which can be used to construct a new sequence of symmetric type II multiple orthogonal polynomials. Finally, we prove a Favard-type result for certain sequences of matrix multiple orthogonal polynomials satisfying a matrix four–term recurrence relation with matrix coefficients.

KEYWORDS: Matrix orthogonal polynomials, linear functional, recurrence relation, operator theory, matrix Sylvester differential equations, full Kostant-Toda systems.
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1. Introduction

In recent years an increasing attention has been paid to the notion of multiple orthogonality. Multiple orthogonal polynomials are a generalization of orthogonal polynomials [11], satisfying orthogonality conditions with respect to a number of measures, instead of just one measure. There exists a vast literature on this subject, e.g. the classical works [1], [2], [18, Ch. 23] and [27] among others. A characterization through a vectorial functional equation, where the authors call them *d–orthogonal polynomials* instead of multiple orthogonal polynomials, was done in [14]. Their asymptotic behavior have been studied in [4], also continued in [12], and properties for their zeros have been analyzed in [17].

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Bcklund transformations resulted from the symmetrization process in the usual (standard) orthogonality, which allows one to jump, from one hierarchy to another, in the whole Toda lattices hierarchy (see [19]). That is, they allow reinterpretations inside the hierarchy. In [8], the authors have found certain Bcklund–type transformations (also known as Miura–type transformations) which allow to reduce problems in a given full Konstant–Toda hierarchy to another. Also, in [5], where Peherstorfer's work [28] is extended to the whole Toda hierarchy, it is shown how this system can be described with the evolution of only one parameter instead of two, using exactly this kind of transformations. Other application to the Toda systems appear in [3], [6], and [7], where the authors studied Bogoyavlenskii systems which were modeled by certain symmetric multiple orthogonal polynomials.

In this paper, we are interested in analyze the effect of symmetrization in systems of multiple orthogonality measures. Our viewpoint seeds some new light on the subject, and we prove that the symmetrization process in multiple orthogonality is a model to define the aforementioned Bcklund–type transformations, as happens in the scalar case with the Bcklund transformations (see [11], [24], [28]). Furthermore, we solve the so called *symmetrization problem* in the theory of multiple orthogonal polynomials. We apply certain *Darboux transformations*, already described in [8], to a $(d+2)$–banded matrix, associated to a $(d+2)$–term recurrence relation satisfied by an arbitrary sequence of type II multiple orthogonal polynomials, to obtain a total of $d+1$ sequences of not necessarily symmetric multiple orthogonal polynomials, which we use to construct a new sequence of symmetric multiple orthogonal polynomials.

On the other hand, following the ideas in [24] (and the references therein) for standard sequences of orthogonal polynomials, in [26] (see also [13]) the authors provide a cubic decomposition for sequences of polynomials, multiple orthogonal with respect to a two different linear functionals. Concerning the symmetric case, in [25] this cubic decomposition is analyzed for a 2-symmetric sequence of polynomials, which is called a *diagonal cubic decomposition (CD)* by the authors. Here, we also extend this notion of diagonal decomposition to a more general case, considering symmetric sequences of polynomials multiple orthogonal with respect to $d > 3$ linear functionals.

The structure of the manuscript is as follows. In Section 2 we summarize without proofs the relevant material about multiple orthogonal polynomials, and a basic background about the matrix interpretation of the type II multi-orthogonality conditions with respect to the a regular system of d linear functionals $\{u^1, \ldots, u^d\}$ and diagonal multi–indices. In Section 3 we fully characterize the Weyl function $\mathcal{R}_J$ and the Stieltjes matrix function $\mathcal{F}$ associated to the block Jacobi matrix J corresponding to a $(d+2)$–term recurrence relation satisfied by a symmetric sequence of type II multiple orthogonal polynomials. In Section 4, starting from an arbitrary sequence of type II multiple polynomials satisfying a $(d+2)$–term recurrence relation, we state the conditions to find a total of $d+1$ sequences of type II multiple orthogonal polynomials, in general non–symmetric, which can be used to construct a new sequence of *symmetric* type II multiple orthogonal polynomials. Moreover, we also deal with the converse problem, i.e., we propose a decomposition of a given *symmetric* type II multiple orthogonal polynomial sequence, which allows us to find a set of other (in general non–symmetric) $d+1$ sequences of type II multiple orthogonal polynomials, satisfying in turn $(d+2)$–term recurrence relations. Finally, in Section 5, we present a Favard-type result, showing that certain 3×3 matrix decomposition of a type II multiple 2–orthogonal polynomials, satisfy a *matrix four–term recurrence relation*, and therefore it is type II multiple 2–orthogonal (in a matrix sense) with respect to a certain system of matrix measures.

2. Definitions and matrix interpretation of multiple orthogonality

Let $\boldsymbol{n} = (n_1, ..., n_d) \in \mathbb{N}^d$ be a multi–index with length $|\mathbf{n}| := n_1 + \cdots + n_d$ and let $\{u^j\}_{j=1}^d$ be a set of linear functionals, i.e. $u^j : \mathbb{P} \to \mathbb{C}$. Let $\{P_{\mathbf{n}}\}$ be a sequence of polynomials, with $\deg P_{\mathbf{n}}$ is at most $|\mathbf{n}|$. $\{P_{\mathbf{n}}\}$ is said to be *type II multiple orthogonal* with respect to the set of linear functionals $\{u_j\}_{j=1}^d$ and multi–index $\boldsymbol{n}$ if

$$u^j(x^k P_{\mathbf{n}}) = 0\,, \quad k = 0, 1, \ldots, n_j - 1\,, \quad j = 1, \ldots, d\,. \tag{1}$$

A multi–index $\boldsymbol{n}$ is said to be *normal* for the set of linear functionals $\{u_j\}_{j=1}^d$, if the degree of $P_{\mathbf{n}}$ is exactly $|\mathbf{n}| = n$. When all the multi–indices of a given family are normal, we say that the set of linear functionals $\{u_j\}_{j=1}^d$ is *regular*. In the present work, we will restrict our attention ourselves to the so called *diagonal multi–indices* $\boldsymbol{n} = (n_1, ..., n_d) \in \mathcal{I}$, where

$$\begin{aligned}\mathcal{I} = \{&(0, 0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1),\\ &\qquad\qquad (2, 1, \ldots, 1), \ldots, (2, 2, \ldots, 2), \ldots\}.\end{aligned}$$

Notice that there exists a one to one correspondence, $\mathbf{i}$, between the above set of diagonal multi–indices $\mathcal{I} \subset \mathbb{N}^d$ and $\mathbb{N}$, given by $\mathbf{i}(\mathbb{N}^d) = |\mathbf{n}| = n$. Therefore, to simplify the notation, we write in the sequel $P_{\mathbf{n}} \equiv P_{|\mathbf{n}|} = P_n$. The left–multiplication of a linear functional $u : \mathbb{P} \to \mathbb{C}$ by a polynomial $p \in \mathbb{P}$ is given by the new linear functional $p\,u : \mathbb{P} \to \mathbb{C}$ such that

$$p\,u(x^k) = u(p(x)x^k)\,, \quad k \in \mathbb{N}\,.$$

Next, we briefly review a matrix interpretation of type II multiple orthogonal polynomials with respect to a system of d regular linear functionals and a family of diagonal multi–indices. Throughout this work, we will use this matrix interpretation as a useful tool to obtain some of the main results of the manuscript. For a recent and deeper account of the theory (in a more general framework, considering quasi–diagonal multi–indices) we refer the reader to [10].

Let us consider the family of vector polynomials

$$\mathbb{P}^d = \{\begin{bmatrix} P_1 & \cdots & P_d \end{bmatrix}^T, \quad d \in \mathbb{N},\, P_j \in \mathbb{P}\},$$

and $\mathcal{M}_{d\times d}$ the set of $d \times d$ matrices with entries in $\mathbb{C}$. Let $\{\mathcal{X}_j\}$ be the family of vector polynomials $\mathcal{X}_j \in \mathbb{P}^d$ defined by

$$\mathcal{X}_j = \begin{bmatrix} x^{jd} & \cdots & x^{(j+1)d-1} \end{bmatrix}^T, \quad j \in \mathbb{N}, \tag{2}$$

where $\mathcal{X}_0 = \begin{bmatrix} 1 & \cdots & x^{d-1} \end{bmatrix}^T$. By means of the shift $n \to nd$, associated with $\{P_n\}$, we define the sequence of vector polynomials $\{\mathcal{P}_n\}$, with

$$\mathcal{P}_n = \begin{bmatrix} P_{nd}(x) & \cdots & P_{(n+1)d-1}(x) \end{bmatrix}^T, \quad n \in \mathbb{N},\, \mathcal{P}_n \in \mathbb{P}^d. \tag{3}$$

Let $u^j : \mathbb{P} \to \mathbb{C}$ with $j = 1, \ldots, d$ a system of linear functionals as in (1). From now on, we define the *vector of functionals* $\mathcal{U} = \begin{bmatrix} u^1 & \cdots & u^d \end{bmatrix}^T$ acting in $\mathbb{P}^d \to \mathcal{M}_{d\times d}$, by

$$\mathcal{U}(\mathcal{P}) = \left(\mathcal{U}\cdot\mathcal{P}^T\right)^T = \begin{bmatrix} u^1(P_1) & \cdots & u^d(P_1) \\ \vdots & \ddots & \vdots \\ u^1(P_d) & \cdots & u^d(P_d) \end{bmatrix}.$$

Let

$$A_\ell(x) = \sum_{k=0}^{\ell} A_k^\ell\, x^k,$$

be a matrix polynomial of degree ℓ, where $A_k^\ell \in \mathcal{M}_{2\times 2}$, and $\mathcal{U}$ a vector of functional. We define the new vector of functionals called *left multiplication* *of $\mathcal{U}$ by a matrix polynomial A_ℓ*, and we denote it by $A_\ell\mathcal{U}$, to the map of $\mathbb{P}^d$ into $\mathcal{M}_{d\times d}$, described by

$$\left(A_\ell\mathcal{U}\right)(\mathcal{P}) = \sum_{k=0}^{\ell} \left(x^k\mathcal{U}\right)(\mathcal{P})\left(A_k^n\right)^T. \tag{4}$$

From (4) we introduce the notion of *moments of order $j \in \mathbb{N}$*, associated with the vector of functionals $x^k\mathcal{U}$, which will be in general the following $d\times d$ matrices

$$\mathcal{U}_j^k = \left(x^k\mathcal{U}\right)(\mathcal{X}_j) = \begin{bmatrix} u^1(x^{jd+k}) & \cdots & u^d(x^{jd+k}) \\ \vdots & \ddots & \vdots \\ u^1(x^{(j+1)d-1+k}) & \cdots & u^d(x^{(j+1)d-1+k}) \end{bmatrix},$$

with $j, k \in \mathbb{N}$, and from this moments, we construct the *block Hankel matrix of moments*

$$\mathcal{H}_n = \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^n \\ \vdots & \ddots & \vdots \\ \mathcal{U}_n^0 & \cdots & \mathcal{U}_n^n \end{bmatrix}, \quad n \in \mathbb{N}.$$

We say that the vector of functionals $\mathcal{U}$ is *regular*, if the determinants of the principal minors of the above matrix are non-zero for every $n \in \mathbb{N}$. Having in mind (2) it is obvious that $\mathcal{X}_j = (x^d)^j\mathcal{X}_0,\ \ j \in \mathbb{N}$. Thus, from (3) we can express $\mathcal{P}_n(x)$ in the alternative way

$$\mathcal{P}_n(x) = \sum_{j=0}^{n} P_j^n \mathcal{X}_j\,, \quad P_j^n \in \mathcal{M}_{d\times d}\,, \tag{5}$$

where the matrix coefficients $P_j^n,\ j = 0, 1, \ldots, n$ are uniquely determined. Thus, it also occurs

$$\mathcal{P}_n(x) = W_n(x^d)\mathcal{X}_0\,, \tag{6}$$

where W_n is a matrix polynomial (i.e., W_n is a $d\times d$ matrix whose entries are polynomials) of degree n and dimension d, given by

$$W_n(x) = \sum_{j=0}^{n} P_j^n x^j\,, \quad P_j^n \in \mathcal{M}_{d\times d}. \tag{7}$$

Notice that the matrices $P_j^n \in \mathcal{M}_{d\times d}$ in (7) are the same as in (5). Within this context, we can now describe the matrix interpretation of multiple orthogonality for diagonal multi–indices. Let $\{\mathcal{P}_n\}$ be a sequence of vector polynomials with polynomial entries as in (3), and a vector of functionals $\mathcal{U}$

as described above. $\{\mathcal{P}_n\}$ is said to be a *type II vector multiple orthogonal polynomial sequence* with respect to the vector of functionals $\mathcal{U}$, and a set of diagonal multi–indices, if

$$\left.\begin{array}{rl} i) & (x^k\mathcal{U})(\mathcal{P}_n) = 0_{d\times d}\,, \quad k = 0,1,\ldots,n-1\,, \\ ii) & (x^n\mathcal{U})(\mathcal{P}_n) = \Delta_n\,, \end{array}\right\} \tag{8}$$

where Δ_n is a regular upper triangular $d \times d$ matrix (see [10, Th. 3] considering diagonal multi–indices).

Next, we introduce a few aspects of the duality theory, which will be useful in the sequel. We denote by $\mathbb{P}^*$ the dual space of $\mathbb{P}$, i.e. the linear space of linear functionals defined on $\mathbb{P}$ over $\mathbb{C}$. Let $\{P_n\}$ be a sequence of monic polynomials. We call $\{\ell_n\}$, $\ell_n \in \mathbb{P}^*$, the *dual sequence* of $\{P_n\}$ if $\ell_i(P_j) = \delta_{i,j}$, i, $j \in \mathbb{N}$ holds. Given a sequence of linear functionals $\{\ell_n\} \in \mathbb{P}^*$, by means of the shift $n \to nd$, the vector sequence of linear functionals $\{\mathcal{L}_n\}$, with

$$\mathcal{L}_n = \begin{bmatrix} \ell_{nd} & \cdots & \ell_{(n+1)d-1} \end{bmatrix}^T, \quad n \in \mathbb{N}, \tag{9}$$

is said to be the *vector sequence of linear functionals* associated with $\{\ell_n\}$.

It is very well known (see [14]) that a given sequence of type II polynomials $\{P_n\}$, simultaneously orthogonal with respect to a d linear functionals, or simply *d–orthogonal* polynomials, satisfy the following $(d+2)$–term order recurrence relation

$$xP_{n+d}(x) = P_{n+d+1}(x) + \beta_{n+d}P_{n+d}(x) + \sum_{\nu=0}^{d-1} \gamma_{n+d-\nu}^{d-1-\nu} P_{n+d-1-\nu}(x)\,, \tag{10}$$

$\gamma_{n+1}^0 \neq 0$ for $n \geq 0$, with the initial conditions $P_0(x) = 1$, $P_1(x) = x - \beta_0$, and

$$P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x)\,, \quad 2 \leq n \leq d\,.$$

E.g., if $d = 2$, the sequence of monic type II multiple orthogonal polynomials $\{P_n\}$ with respect to the regular system of functionals $\{u^1, u^2\}$ and normal multi–index satisfy, for every $n \geq 0$, the following four term recurrence relation (see [10, Lemma 1-a], [22])

$$xP_{n+2}(x) = P_{n+3}(x) + \beta_{n+2}P_{n+2}(x) + \gamma_{n+2}^1 P_{n+1}(x) + \gamma_{n+1}^0 P_n(x)\,, \tag{11}$$

where $\beta_{n+2}, \gamma_{n+2}^1, \gamma_{n+1}^0 \in \mathbb{C}$, $\gamma_{n+1}^0 \neq 0$, $P_0(x) = 1$, $P_1(x) = x - \beta_0$ and $P_2(x) = (x - \beta_1)P_1(x) - \gamma_1^1 P_0(x)$.

We follow [14, Def. 4.1.] in assuming that a monic system of polynomials $\{S_n\}$ is said to be *d–symmetric* when it verifies

$$S_n(\xi_k x) = \xi_k^n S_n(x)\,, \quad n \geq 0\,, \tag{12}$$

where $\xi_k = \exp(2k\pi i/(d+1))$, $k = 1,\ldots,d$, and $\xi_k^{d+1} = 1$. Notice that, if $d = 1$, then $\xi_k = -1$ and therefore $S_n(-x) = (-1)^n S_n(x)$ (see [11]). We also assume (see [14, Def. 4.2.]) that the vector of linear functionals $\mathcal{L}_0 = \begin{bmatrix}\ell_0 & \cdots & \ell_{d-1}\end{bmatrix}^T$ is said to be *d–symmetric* when the moments of its entries satisfy, for every $n \geq 0$,

$$\ell_\nu(x^{(d+1)n+\mu}) = 0\,, \quad \nu = 0,1,\ldots,d-1\,, \quad \mu = 0,1,\ldots,d\,, \quad \nu \neq \mu\,. \tag{13}$$

Observe that if $d = 1$, this condition leads to the well known fact $\ell_0(x^{2n+1}) = 0$, i.e., all the odd moments of a symmetric moment functional are zero (see [11, Def. 4.1, p.20]).

Under the above assumptions, we have the following

Theorem 1 (cf. [14, Th. 4.1])**.** *For every sequence of monic polynomials $\{S_n\}$, d–orthogonal with respect to the vector of linear functionals $\mathcal{L}_0 = \begin{bmatrix}\ell_0 & \cdots & \ell_{d-1}\end{bmatrix}^T$, the following statements are equivalent:*

(a) The vector of linear functionals $\mathcal{L}_0$ is d–symmetric.
(b) The sequence $\{S_n\}$ is d–symmetric.
(c) The sequence $\{S_n\}$ satisfies

$$xS_{n+d}(x) = S_{n+d+1}(x) + \gamma_{n+1} S_n(x)\,, \quad n \geq 0, \tag{14}$$

with $S_n(x) = x^n$ for $0 \leq n \leq d$.

Notice that (14) is a particular case of the $(d+2)$–term recurrence relation (10). Continuing the same trivial example above for $d = 2$, it directly implies that the sequence of polynomials $\{S_n\}$, satisfy the particular case of (10), i.e. $S_0(x) = 1$, $S_1(x) = 1$, $S_2(x) = x^2$ and

$$xS_{n+2}(x) = S_{n+3}(x) + \gamma_{n+1} S_n(x)\,, \quad n \geq 0.$$

Notice that the coefficients β_{n+2} and γ_{n+2}^1 of polynomials S_{n+2} and S_{n+1} respectively, on the right hand side of (11) are zero.

On the other hand, the $(d+2)$–term recurrence relation (14) can be rewritten in terms of vector polynomials (3), and then we obtain what will be referred to as the *symmetric type II vector multiple orthogonal polynomial* *sequence* $\mathcal{S}_n = \begin{bmatrix} S_{nd} & \cdots & S_{(n+1)d-1} \end{bmatrix}^T$. For $n \to dn + j$, $j = 0, 1, \dots d-1$ and $n \in \mathbb{N}$, we have the following matrix three term recurrence relation

$$x\mathcal{S}_n = A\mathcal{S}_{n+1} + B\mathcal{S}_n + C_n\mathcal{S}_{n-1}\,, \quad n = 0, 1, \dots \tag{15}$$

with $\mathcal{S}_{-1} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}^T$, $\mathcal{S}_0 = \begin{bmatrix} S_0 & \cdots & S_{d-1} \end{bmatrix}^T$, and matrix coefficients A, B, $C_n \in \mathcal{M}_{d\times d}$ given

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \quad \text{and} \\ C_n &= \operatorname{diag}[\gamma_1,\, \gamma_2, \dots, \gamma_{nd}]. \end{aligned} \tag{16}$$

Note that, in this case, one has

$$\mathcal{S}_0 = \mathcal{X}_0 = \begin{bmatrix} 1 & x & \cdots & x^{d-1} \end{bmatrix}^T.$$

Since $\{S_n\}$ satisfies (14), it is clear that this d–symmetric type II multiple polynomial sequence will be orthogonal with respect to certain system of d linear functionals, say $\{v^1, \dots, v^d\}$. Hence, according to the matrix interpretation of multiple orthogonality, the corresponding type II vector multiple polynomial sequence $\{\mathcal{S}_n\}$ will be orthogonal with respect to a *symmetric vector of functionals* $\mathcal{V} = \begin{bmatrix} v^1 & \cdots & v^d \end{bmatrix}^T$. The corresponding matrix orthogonality conditions for $\{\mathcal{S}_n\}$ and $\mathcal{V}$ are described in (8).

One of the main goals of this manuscript is to analyze symmetric sequences of type II vector multiple polynomials, orthogonal with respect to a symmetric vector of functionals. The remainder of this section will be devoted to the proof of one of our main results concerning the moments of the d functional entries of such symmetric vector of functionals $\mathcal{V}$. The following theorem states that, under certain conditions, the moment of each functional entry in $\mathcal{V}$ can be given in terms of the moments of other functional entry in the same $\mathcal{V}$.

Lemma 1. *If $\mathcal{V} = \begin{bmatrix} v^1 & \cdots & v^d \end{bmatrix}^T$ is a symmetric vector of functionals, the moments of each functional entry v^j, $j = 1, 2, \dots, d$ in $\mathcal{V}$, can be expressed for all $n \geq 0$ as*

(*i*) *If $\mu = 0, 1, \dots, j-2$*

$$v^j(x^{(d+1)n+\mu}) = \frac{v_{j,\mu}}{v_{\mu+1,\mu}} v^{\mu+1}(x^{(d+1)n+\mu}), \tag{17}$$

where $v_{k,l} = v^k(S_l)$.

(ii) *If* $\mu = j-1$*, the value* $v^j(x^{(d+1)n+\mu})$ *depends on* v^j*, and it is different from zero.*

(iii) *If* $\mu = j, j+1, \dots, d$

$$v^j(x^{(d+1)n+\mu}) = 0.$$

Proof: In the matrix framework of multiple orthogonality, the type II vector polynomials $\mathcal{S}_n$ are multiple orthogonal with respect to the symmetric vector moment of functionals $\mathcal{V} : \mathbb{P}^d \to \mathcal{M}_{d\times d}$, with $\mathcal{V} = \begin{bmatrix} v^1 & \cdots & v^d \end{bmatrix}^T$. If we multiply (15) by x^{n-1}, together with the linearity of $\mathcal{V}$, we get, for $n = 0, 1, \dots$

$$\mathcal{V}\left(x^n \mathcal{S}_n\right) = A\mathcal{V}\left(x^{n-1}\mathcal{S}_{n+1}\right) + B\mathcal{V}\left(x^{n-1}\mathcal{S}_n\right) + C_n\mathcal{V}\left(x^{n-1}\mathcal{S}_{n-1}\right), \quad n = 0, 1, \dots.$$

By the orthogonality conditions (8) for $\mathcal{V}$, we have

$$A\mathcal{V}\left(x^{n-1}\mathcal{S}_{n+1}\right) = B\mathcal{V}\left(x^{n-1}\mathcal{S}_n\right) = 0_{d\times d}, \quad n = 0, 1, \dots,$$

and iterating the remain expression

$$\mathcal{V}\left(x^n \mathcal{S}_n\right) = C_n\mathcal{V}\left(x^{n-1}\mathcal{S}_{n-1}\right), \quad n = 0, 1, \dots$$

we obtain

$$\mathcal{V}\left(x^n \mathcal{S}_n\right) = C_n C_{n-1} \cdots C_1 \mathcal{V}\left(\mathcal{S}_0\right), \quad n = 0, 1, \dots.$$

The above matrix $\mathcal{V}\left(\mathcal{S}_0\right)$ is given by

$$\mathcal{V}\left(\mathcal{S}_0\right) = \mathcal{V}_0^0 = \begin{bmatrix} v^1(S_0) & \cdots & v^d(S_0) \\ \vdots & \ddots & \vdots \\ v^1(S_{d-1}) & \cdots & v^d(S_{d-1}) \end{bmatrix}.$$

To simplify the notation, in the sequel $v_{i,j-1}$ denotes $v^i(S_{j-1})$. Notice that (8) leads to the fact that the above matrix is an upper triangular matrix, which in turn means that $v_{i,j-1} = 0$, for every $i, j = 1, \dots, d-1$, that is

$$\mathcal{V}_0^0 = \begin{bmatrix} v_{1,0} & v_{2,0} & \cdots & v_{d,0} \\ & v_{2,1} & \cdots & v_{d,1} \\ & & \ddots & \vdots \\ & & & v_{d,d-1} \end{bmatrix}. \tag{18}$$

Let $\mathcal{L}_0$ be a d–symmetric vector of linear functionals as in Theorem 1. We can express $\mathcal{V}$ in terms of $\mathcal{L}_0$ as $\mathcal{V} = G_0\mathcal{L}_0$. Thus, we have

$$(G_0^{-1}\mathcal{V})(\mathcal{S}_0) = \mathcal{L}_0(\mathcal{S}_0) = I_d.$$

From (4) we have

$$\left(G_0^{-1}\,\mathcal{V}\right)(\mathcal{S}_0) = \mathcal{V}\,(\mathcal{S}_0)\,(G_0^{-1})^T = \mathcal{V}_0^0(G_0^{-1})^T = I_d\,.$$

Therefore, taking into account (18), we conclude

$$G_0 = (\mathcal{V}_0^0)^T = \begin{bmatrix} v_{1,0} & & & \\ v_{2,0} & v_{2,1} & & \\ \vdots & \vdots & \ddots & \\ v_{d,0} & v_{d,1} & \cdots & v_{d,d-1} \end{bmatrix}. \tag{19}$$

Observe that the matrix $(\mathcal{V}_0^0)^T$ is lower triangular, and every entry in their main diagonal is different from zero, so G_0 always exists and is a lower triangular matrix. Since $\mathcal{V} = G_0\mathcal{L}_0$, we finally obtain the expressions

$$\begin{aligned} v^1 &= v_{1,0}\ell_0, \\ v^2 &= v_{2,0}\ell_0 + v_{2,1}\ell_1, \\ v^3 &= v_{3,0}\ell_0 + v_{3,1}\ell_1 + v_{3,2}\ell_2, \\ \cdots& \\ v^d &= v_{d,0}\ell_0 + v_{d,1}\ell_1 + \cdots + v_{d,d-1}\ell_{d-1}, \end{aligned} \tag{20}$$

between the entries of $\mathcal{L}_0$ and $\mathcal{V}$.

Next, from (20), (13), together with the crucial fact that every value in the main diagonal of $\mathcal{V}_0^0$ is different from zero, it is a simple matter to check that the three statements of the lemma follow. We conclude the proof only for the functionals v^2 and v^1. The other cases can be deduced in a similar way.

From (20) we get $v^1(x^{(d+1)n+\mu}) = v_{1,0}\cdot\ell_0(x^{(d+1)n+\mu}) \neq 0$. Then, from (13) we see that for every $\mu \neq 0$ we have $v^1(x^{(d+1)n+\mu}) = 0$ (statement (iii)). If $\mu = 0$, we have $v^1(x^{(d+1)n}) = v_{1,0}\cdot\ell_0(x^{(d+1)n}) \neq 0$ (statement (ii)). Next, from (20) we get $v^2(x^{(d+1)n+\mu}) = v_{2,0}\cdot\ell_0(x^{(d+1)n+\mu}) + v_{2,1}\cdot\ell_1(x^{(d+1)n+\mu})$. Then, from (13) we see that for every $\mu \neq 1$, we have $v^2(x^{(d+1)n+\mu}) = 0$ (statement (iii)). If $\mu = 0$ we have

$$v^2(x^{(d+1)n}) = \frac{v_{2,0}}{v_{1,0}}v^1(x^{(d+1)n}) \ \text{(statement } (i)).$$

If $\mu = 1$ then $v^2(x^{(d+1)n+1}) = v_{2,1}\cdot\ell_1(x^{(d+1)n+1}) \neq 0$ (statement (ii)).

Thus, the lemma follows. ∎

3. Representation of the Stieltjes and Weyl functions

Let $\mathcal{U}$ a vector of functionals $\mathcal{U} = \begin{bmatrix} u^1 & \cdots & u^d \end{bmatrix}^T$. We define the *Stieltjes matrix function* associated to $\mathcal{U}$ (or *matrix generating function* associated to $\mathcal{U}$), $\mathcal{F}$ by (see [10])

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} \frac{(x^n \mathcal{U})\,(\mathcal{X}_0(x))}{z^{n+1}}.$$

In this Section we find the relation between the Stieltjes matrix function $\mathcal{F}$, associated to a certain d–symmetric vector of functionals $\mathcal{V}$, and certain interesting function associated to the corresponding block Jacobi matrix J. Here we deal with d–symmetric sequences of type II multiple orthogonal polynomials $\{S_n\}$, and hence J is a $(d+2)$–banded matrix with only two extreme non-zero diagonals, which is the block–matrix representation of the three–term recurrence relation with $d \times d$ matrix coefficients, satisfied by the vector sequence of polynomials $\{\mathcal{S}_n\}$ (associated to $\{S_n\}$), orthogonal with respect to $\mathcal{V}$.

Thus, the shape of a Jacobi matrix J, associated with the $(d+2)$–term recurrence relation (14) satisfied by a d–symmetric sequence of type II multiple orthogonal polynomials $\{S_n\}$ is

$$J = \begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ \vdots & & \ddots & \ddots & & & \\ 0 & & & 0 & 1 & & \\ \gamma_1 & 0 & & & 0 & 1 & \\ & \gamma_2 & 0 & & & 0 & \ddots \\ & & \ddots & \ddots & & & \ddots \end{bmatrix}. \tag{21}$$

We can rewrite J as the block tridiagonal matrix

$$J = \begin{bmatrix} B & A & & \\ C_1 & B & A & \\ & C_2 & B & A \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

associated to a three term recurrence relation with matrix coefficients, satisfied by the sequence of type II vector multiple orthogonal polynomials $\{\mathcal{S}_n\}$ associated to $\{S_n\}$. Here, every block matrix A, B and C_n has $d \times d$ size, and they are given in (16). When J is a bounded operator, it is possible to define the *resolvent operator* by

$$(zI - J)^{-1} = \sum_{n=0}^{\infty} \frac{J^n}{z^{n+1}}, \quad |z| > ||J||,$$

(see [9]) and we can put in correspondence the following block tridiagonal analytic function, known as the *Weyl function associated with* J

$$\mathcal{R}_J(z) = \sum_{n=0}^{\infty} \frac{\mathbf{e}_0^T \, J^n \, \mathbf{e}_0}{z^{n+1}}, \quad |z| > ||J||, \tag{22}$$

where $\mathbf{e}_0 = \begin{bmatrix} I_d & 0_{d\times d} & \cdots \end{bmatrix}^T$. If we denote by M_{ij} the $d \times d$ block matrices of a semi-infinite matrix M, formed by the entries of rows $d(i-1)+1, d(i-1)+2, \ldots, di$, and columns $d(j-1)+1, d(j-1)+2, \ldots, dj$, the matrix J^n can be written as the semi-infinite block matrix

$$J^n = \begin{bmatrix} J_{11}^n & J_{12}^n & \cdots \\ J_{21}^n & J_{22}^n & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

We can now formulate our first important result in this Section. For more details we refer the reader to [6, Sec. 1.2] and [3].

Let $\{\mathcal{S}_n\}$ be a symmetric type II vector multiple polynomial sequence orthogonal with respect to the d–symmetric vector of functionals $\mathcal{V}$. Following [10, Th. 7], the matrix generating function associated to $\mathcal{V}$, and the Weyl function associated with J, the block Jacobi matrix corresponding to $\{\mathcal{S}_n\}$, can be put in correspondence by means of the matrix expression

$$\mathcal{F}(z) = R_J(z)\mathcal{V}(\mathcal{X}_0)\,, \tag{23}$$

where, for the d–symmetric case, $\mathcal{V}(\mathcal{X}_0) = \mathcal{V}(\mathcal{S}_0) = \mathcal{V}_0^0$, which is explicitly given in (18).

First we study the case $d = 2$, and next we consider more general situations for $d > 2$ functional entries in $\mathcal{V}$. From Lemma 1, we obtain the entries for the representation of the Stieltjes matrix function $\mathcal{F}(z)$, associated with $\mathcal{V} =$

$\begin{bmatrix} v^1 & v^2 \end{bmatrix}$, as

$$\mathcal{F}(z) = \sum_{n=0}^{\infty} \begin{bmatrix} v^1(x^{3n}) & \frac{v_{2,0} \cdot v^1(x^{3n})}{v_{1,0}} \\ 0 & v^2(x^{3n+1}) \end{bmatrix} / z^{3n+1} + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & v^2(x^{3n+1}) \\ 0 & 0 \end{bmatrix} / z^{3n+2}$$
$$+ \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ v^1(x^{3n+3}) & \frac{v_{2,0} \cdot v^1(x^{3n+3})}{v_{1,0}} \end{bmatrix} / z^{3n+3}. \quad (24)$$

Notice that we have $\mathcal{F}(z) = \mathcal{F}_1(z) + \mathcal{F}_2(z) + \mathcal{F}_3(z)$. The following theorem shows that we can obtain $\mathcal{F}(z)$ in our particular case, analyzing two different situations.

Theorem 2. *Let* $\mathcal{V} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}^T$ *be a symmetric vector of functionals, with* $d = 2$. *Then the Weyl function is given by*

$$\mathcal{R}_J(z) = \sum_{n=0}^{\infty} \frac{\begin{bmatrix} \frac{v^1(x^{3n})}{v_{1,0}} & 0 \\ 0 & \frac{v^2(x^{3n+1})}{v_{2,1}} \end{bmatrix}}{z^{3n+1}} + \sum_{n=0}^{\infty} \frac{\begin{bmatrix} 0 & \frac{v^2(x^{3n+1})}{v_{2,1}} \\ 0 & 0 \end{bmatrix}}{z^{3n+2}} + \sum_{n=0}^{\infty} \frac{\begin{bmatrix} 0 & 0 \\ \frac{v^1(x^{3n+3})}{v_{1,0}} & 0 \end{bmatrix}}{z^{3n+3}}$$

Proof: It is enough to multiply (24) by $(\mathcal{V}_0^0)^{-1}$. The explicit expression for $\mathcal{V}_0^0$ is given in (18). ■

Computations considering $d > 2$ functionals, can be cumbersome but doable. The matrix generating function $\mathcal{F}$ (as well as the Weyl function), will be the sum of $d + 1$ matrix terms, i.e. $\mathcal{F}(z) = \mathcal{F}_1(z) + \cdots + \mathcal{F}_{d+1}(z)$, each of them of size $d \times d$.

Let us now outline the very interesting structure of $\mathcal{R}_J(z)$ for the general case of d functionals. We shall describe the structure of $\mathcal{R}_J(z)$ for $d = 3$, comparing the situation with the general case. Let $*$ denote every non-zero entry in a given matrix. Thus, there will be four J_{11} matrices of size 3×3. Here, and in the general case, the first matrix $[J_{11}^{(d+1)n}]_{d \times d}$ will allways be diagonal, as follows

$$[J_{11}^{4n}]_{3\times 3} = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}.$$

Indeed, observe that for $n=0$, $[J_{11}^{(d+1)n}]_{d\times d}=I_d$. Next, we have

$$[J_{11}^{4n+1}]_{3\times 3}=\begin{bmatrix}0 & * & 0\\ 0 & 0 & *\\ 0 & 0 & 0\end{bmatrix}.$$

From (14) we know that the "distance" between the two extreme non-zero diagonals of J will allways consist of d zero diagonals. It directly implies that, also in the general case, every entry in the last row of $[J_{11}^{(d+1)n}]_{d\times d}$ will always be zero, and therefore the unique non-zero entries in $[J_{11}^{(d+1)n}]_{d\times d}$ will be one step over the main diagonal.

Next we have

$$[J_{11}^{4n+2}]_{3\times 3}=\begin{bmatrix}0 & 0 & *\\ 0 & 0 & 0\\ * & 0 & 0\end{bmatrix}.$$

Notice that for every step, the main diagonal in $[J_{11}^{(d+1)n}]_{d\times d}$ goes "one diagonal" up, but the other extreme diagonal of J is also moving upwards, with exactly d zero diagonals between them. It directly implies that, no matter the number of functionals, only the lowest-left element of $[J_{11}^{(d+1)n+2}]_{d\times d}$ will be different from zero. Finally, we have

$$[J_{11}^{4n+3}]_{3\times 3}=\begin{bmatrix}0 & 0 & 0\\ * & 0 & 0\\ 0 & * & 0\end{bmatrix}.$$

Here, the last non-zero entry of the main diagonal in $[J_{11}^{4n}]_{3\times 3}$ vanishes. In the general case, it will occur exactly at step $[J_{11}^{(d+1)n+(d-1)}]_{d\times d}$, in which only the uppest-right entry is different from zero. Meanwhile, in matrices $[J_{11}^{(d+1)n+3}]_{d\times d}$ up to $[J_{11}^{(d+1)n+(d-2)}]_{d\times d}$ will be non-zero entries of the two extreme diagonals. In this last situation, the non-zero entries of $[J_{11}^{(d+1)n+(d-1)}]_{d\times d}$, will always be exactly one step under the main diagonal.

4. The symmetrization problem for multiple OP

Throughout this section, let $\{A_n^1\}$ be an arbitrary and not necesarily symmetric sequence of type II multiple orthogonal polynomials, satisfying a $(d+2)$–term recurrence relation with known recurrence coefficients, and let J_1 be the corresponding $(d+2)$–banded matrix. Let J_1 be such that the following LU factorization

$$J_1 = LU = L_1 L_2 \cdots L_d U \tag{25}$$

is unique, where U is an upper two–banded, semi–infinite, and invertible matrix, L is a $(d+1)$–lower triangular semi–infinite with ones in the main diagonal, and every L_i, $i = 1, \ldots, d$ is a lower two–banded, semi–infinite, and invertible matrix with ones in the main diagonal. We follow [8, Def. 3], where the authors generalize the concept of Darboux transformation to general Hessenberg banded matrices, in assuming that any circular permutation of $L_1 L_2 \cdots L_d U$ is a Darboux transformation of J_1. Thus we have d possible Darboux transformations of J_1, say J_j, $j = 2, \ldots, d+1$, with $J_2 = L_2 \cdots L_d U L_1$, $J_3 = L_3 \cdots L_d U L_1 L_2, \ldots, J_{d+1} = U L_1 L_2 \cdots L_d$.

Next, we solve the so called *symmetrization problem* in the theory of multiple orthogonal polynomials, i.e., starting with $\{A_n^1\}$, we find a total $d+1$ type II multiple orthogonal polynomial sequences $\{A_n^j\}$, $j = 1, \ldots, d+1$, satisfying $(d+2)$–term recurrence relation with known recurrence coefficients, which can be used to construct a new d–symmetric type II multiple orthogonal polynomial sequence $\{S_n\}$. It is worth pointing out that all the aforesaid sequences $\{A_n^j\}$, $j = 1, \ldots, d+1$ are of the same kind, with the same number of elements in their respectives $(d+2)$–term recurrence relations, and multiple orthogonal with respect to the same number of functionals d.

Theorem 3. *Let $\{A_n^1\}$ be an arbitrary and not necesarily symmetric sequence of type II multiple orthogonal polynomials as stated above. Let J_j, $j = 2, \ldots, d+1$, be the Darboux transformations of J_1 given by the d cyclid permutations of the matrices in the right hand side of (25). Let $\{A_n^j\}$, $j = 2, \ldots, d+1$, d new families of type II multiple orthogonal polynomials satisfying $(d+2)$–term recurrence relations given by the matrices J_j, $j = 2, \ldots, d+1$. Then, the sequence $\{S_n\}$ defined by*

$$\begin{cases} S_{(d+1)n}(x) = A_n^1(x^{d+1}), \\ S_{(d+1)n+1}(x) = x A_n^2(x^{d+1}), \\ \cdots \\ S_{(d+1)n+d}(x) = x^d A_n^{d+1}(x^{d+1}), \end{cases} \tag{26}$$

is a d–symmetric sequence of type II multiple orthogonal polynomials.

Proof: Let $\{A^1_n\}$ satify the $(d+2)$–term recurrence relation given by (10)

$$xA^1_{n+d}(x) = A^1_{n+d+1}(x) + b^{[1]}_{n+d}A^1_{n+d}(x) + \sum_{\nu=0}^{d-1} c^{d-1-\nu,[1]}_{n+d-\nu} A^1_{n+d-1-\nu}(x)\,,$$

$c^{0,[1]}_{n+1} \neq 0$ for $n \geq 0$, with the initial conditions $A^1_0(x) = 1$, $A^1_1(x) = x - b^{[1]}_0$, and

$$A^1_n(x) = (x - b^{[1]}_{n-1})A^1_{n-1}(x) - \sum_{\nu=0}^{n-2} c^{d-1-\nu,[1]}_{n-1-\nu} A^1_{n-2-\nu}(x)\,, \quad 2 \leq n \leq d\,,$$

with known recurrence coefficients. Hence, in a matrix notation, we have

$$x\underline{\mathbf{A}}^1 = J_1\,\underline{\mathbf{A}}^1, \tag{27}$$

where

$$J_1 = \begin{bmatrix} b^{[1]}_d & 1 & & & & & \\ c^{d-1,[1]}_{d+1} & b^{[1]}_{d+1} & 1 & & & & \\ c^{d-2,[1]}_{d+1} & c^{d-1,[1]}_{d+2} & b^{[1]}_{d+2} & 1 & & & \\ \vdots & c^{d-2,[1]}_{d+2} & c^{d-1,[1]}_{d+3} & b^{[1]}_{d+3} & 1 & & \\ c^{0,[1]}_{d+1} & \cdots & c^{d-2,[1]}_{d+3} & c^{d-1,[1]}_{d+4} & b^{[1]}_{d+4} & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Following [8] (see also [23]), the unique LU factorization for the square $(d+2)$–banded semi-infinite Hessenberg matrix J_1 is such that

$$U = \begin{bmatrix} \gamma_1 & 1 & & & \\ & \gamma_{d+2} & 1 & & \\ & & \gamma_{2d+3} & 1 & \\ & & & \gamma_{3d+4} & \ddots \\ & & & & \ddots \end{bmatrix}$$

is an upper two-banded, semi-infinite, and invertible matrix, and L is a $(d+1)$–lower triangular, semi–infinite, and invertible matrix with ones in the main diagonal. It is clear that the entries in L and U depend entirely on the known entries of J_1. Thus, we rewrite (27) as

$$x\underline{\mathbf{A}}^1 = LU\,\underline{\mathbf{A}}^1. \tag{28}$$

Next, we define a new sequence of polynomials $\{A^{d+1}_n\}$ by $x\underline{\mathbf{A}}^{d+1} = U\underline{\mathbf{A}}^1$. Multiplying both sides of (28) by the matrix U, we have

$$x\left(U\,\underline{\mathbf{A}}^1\right) = UL\left(U\,\underline{\mathbf{A}}^1\right) = x(x\underline{\mathbf{A}}^{d+1}) = UL(x\underline{\mathbf{A}}^{d+1}),$$

and pulling out x we get

$$x\underline{\mathbf{A}}^{d+1} = UL\,\underline{\mathbf{A}}^{d+1}, \tag{29}$$

which is the matrix form of the $(d+2)$–term recurrence relation satisfied by the new type II multiple polynomial sequence $\{A_n^{d+1}\}$.

Since L is given by (see [8])

$$L = L_1 L_2 \cdots L_d,$$

where every L_j is the lower two-banded, semi-infinite, and invertible matrix

$$L_j = \begin{bmatrix} 1 & & & & \\ \gamma_{d-j} & 1 & & & \\ & \gamma_{2d+1-j} & 1 & & \\ & & \gamma_{3d+2-j} & 1 & \\ & & & \ddots & \ddots \end{bmatrix}$$

with ones in the main diagonal, it is also clear that the entries in L_j will depend on the known entries in J_1. Under the same hypotheses, we can define new $d-1$ polinomial sequences starting with $\underline{\mathbf{A}}^1$ as follows: $\underline{\mathbf{A}}^2 = L_1^{-1}\underline{\mathbf{A}}^1$, $\underline{\mathbf{A}}^3 = L_2^{-1}\underline{\mathbf{A}}^2$, …, $\underline{\mathbf{A}}^d = L_{d-1}^{-1}\underline{\mathbf{A}}^{d-1}$ up to $\underline{\mathbf{A}}^{d+1} = L_d^{-1}\underline{\mathbf{A}}^d$. That is, $\underline{\mathbf{A}}^{j+1} = L_j^{-1}\underline{\mathbf{A}}^j$, $j = 1, \ldots d-1$. Combining this fact with (29) we deduce

$$\begin{aligned} x\underline{\mathbf{A}}^d &= L_d U L_1 L_2 \cdots L_{d-1}\underline{\mathbf{A}}^d, \\ x\underline{\mathbf{A}}^{d-1} &= L_{d-1} L_d U L_1 L_2 \cdots L_{d-2}\underline{\mathbf{A}}^{d-1}, \end{aligned}$$

up to the known expression (27)

$$x\underline{\mathbf{A}}^1 = L_1 L_2 \cdots L_d U \underline{\mathbf{A}}^1.$$

The above expresions mean that all these $d+1$ sequences $\{A^j\}$, $j = 1, \ldots, d+1$ are in turn type II multiple orthogonal polynomials. Finally, from these $d+1$ sequences, we construct the type II polynomials in the sequence $\{S_n\}$ as (26). Note that, according to (12), it directly follows that $\{S_n\}$ is a d–symmetric type II multiple orthogonal polynomial sequence, which proves our assertion. ∎

Next, we state the converse of the above theorem. That is, given a sequence of type II d–symmetric multiple orthogonal polynomials $\{S_n\}$ satisfying the high–term recurrence relation (14), we find a set of $d+1$ polynomial families $\{A_n^j\}$, $j = 1, \ldots, d+1$ of not necessarily symmetric type II multiple orthogonal polynomials, satisfying in turn $(d+2)$–term recurrence relations, so o they are themselves sequences of type II multiple orthogonal polynomials. When $d = 2$, this construction goes back to the work of Douak and Maroni (see [13]).

Theorem 4. *Let $\{S_n\}$ be a d–symmetric sequence of type II multiple orthogonal polynomials satisfiyng the corresponding high–order recurrence relation (14), and $\{A_n^j\}$, $j = 1, \ldots, d+1$, the sequences of polynomials given by (26). Then, each sequence $\{A_n^j\}$, $j = 1, \ldots, d+1$, satisfies the $(d+2)$–term recurrence relation*

$$xA_{n+d}^{j}(x) = A_{n+d+1}^{j}(x) + b_{n+d}^{[j]} A_{n+d}^{j}(x) + \sum_{\nu=0}^{d-1} c_{n+d-\nu}^{d-1-\nu,[j]} A_{n+d-1-\nu}^{j}(x)\,,$$

$c_{n+1}^{0,[j]} \neq 0$ *for* $n \geq 0$, *with initial conditions* $A_0^j(x) = 1$, $A_1^j(x) = x - b_0^{[j]}$,

$$A_n^j(x) = (x - b_{n-1}^{[j]})A_{n-1}^j(x) - \sum_{\nu=0}^{n-2} c_{n-1-\nu}^{d-1-\nu,[j]} A_{n-2-\nu}^j(x)\,, \quad 2 \leq n \leq d\,,$$

and therefore they are type II multiple orthogonal polynomial sequences.

Proof: Since $\{S_n\}$ is a d–symmetric multiple orthogonal sequence, it satisfies (14) with $S_n(x) = x^n$ for $0 \leq n \leq d$. Shifting, for convenience, the multi–index in (14) as $n \to (d+1)n - d + j$, $\quad j = 0, 1, 2, \ldots. d$, we obtain the equivalent system of $(d+1)$ equations

$$\left\{\begin{array}{rlr} xS_{(d+1)n}(x) &= S_{(d+1)n+1}(x) + \gamma_{(d+1)n-d+1} S_{(d+1)n-d}(x)\,, & j = 0, \\ xS_{(d+1)n+1}(x) &= S_{(d+1)n+2}(x) + \gamma_{(d+1)n-d+2} S_{(d+1)n-d+1}(x)\,, & j = 1, \\ & \vdots & \vdots \\ xS_{(d+1)n+d-1}(x) &= S_{(d+1)n+d}(x) + \gamma_{(d+1)n} S_{(d+1)n-1}(x)\,, & j = d-1, \\ xS_{(d+1)n+d}(x) &= S_{(d+1)n+(d+1)}(x) + \gamma_{(d+1)n+1} S_{(d+1)n}(x)\,, & j = d. \end{array}\right. .$$

Substituting (26) into the above expressions, and replacing $x^{d+1} \to x$, we get the following system of $(d+1)$ equations

$$\left\{\begin{array}{ll} 1) & A_n^1(x) = A_n^2(x) + \gamma_{(d+1)n-(d-1)} A_{n-1}^2(x)\,, \\ 2) & A_n^2(x) = A_n^3(x) + \gamma_{(d+1)n-(d-2)} A_{n-1}^3(x)\,, \\ \vdots & \qquad\qquad \vdots \\ d) & A_n^d(x) = A_n^{d+1}(x) + \gamma_{(d+1)n} A_{n-1}^{d+1}(x), \\ d+1) & xA_n^{d+1}(x) = A_{n+1}^1(x) + \gamma_{(d+1)n+1} A_n^1(x)\,. \end{array}\right. \tag{30}$$

Notice that having $x = 0$, we define the γ as

$$\gamma_{(d+1)n+1} = \frac{-A^1_{n+1}(0)}{A^1_n(0)}.$$

In the remainder of this section we deal with the matrix representation of each equation in (30). Notice that the first d equations of (30), namely $A^j_n = A^{j+1}_n + \gamma_{(d+1)n+(j-d)} A^{j+1}_{n-1}$, $j = 1, \ldots, d$, can be written in the matrix way

$$\underline{\mathbf{A}}^j = L_j \, \underline{\mathbf{A}}^{j+1}, \tag{31}$$

where L_j is the lower two-banded, semi-infinite, and invertible matrix

$$L_j = \begin{bmatrix} 1 & & & & \\ \gamma_{d-j} & 1 & & & \\ & \gamma_{2d+1-j} & 1 & & \\ & & \gamma_{3d+2-j} & 1 & \\ & & & \ddots & \ddots \end{bmatrix},$$

and

$$\underline{\mathbf{A}}^j = \begin{bmatrix} A^j_0(x) & A^j_1(x) & A^j_2(x) & \cdots \end{bmatrix}^T.$$

Similarly, the $d+1)$–th equation in (30) can be expressed as

$$x\underline{\mathbf{A}}^{d+1} = U \, \underline{\mathbf{A}}^1, \tag{32}$$

where U is the upper two-banded, semi-infinite, and invertible matrix

$$U = \begin{bmatrix} \gamma_1 & 1 & & & \\ & \gamma_{d+2} & 1 & & \\ & & \gamma_{2d+3} & 1 & \\ & & & \gamma_{3d+4} & \ddots \\ & & & & \ddots \end{bmatrix}.$$

It is clear that the entries in the above matrices L_j and U are given in terms of the recurrence coefficients γ_{n+1} for $\{S_n\}$ given (14). From (31) we have $\underline{\mathbf{A}}^1$ defined in terms of $\underline{\mathbf{A}}^2$, as $\underline{\mathbf{A}}^1 = L_1 \underline{\mathbf{A}}^2$. Likewise $\underline{\mathbf{A}}^2$ in terms of $\underline{\mathbf{A}}^3$ as $\underline{\mathbf{A}}^2 = L_2 \underline{\mathbf{A}}^3$, and so on up to $j = d$. Thus, it is easy to see that,

$$\underline{\mathbf{A}}^1 = L_1 L_2 \cdots L_d \underline{\mathbf{A}}^{d+1}.$$

Next, we multiply by x both sides of the above expression, and we apply (32) to obtain

$$x\underline{\mathbf{A}}^1 = L_1 L_2 \cdots L_d U \, \underline{\mathbf{A}}^1.$$

Since each L_j and U are lower and upper two-banded semi-infinite matrices, it follows easily that $L_1L_2\cdots L_d$ is a lower triangular $(d+1)$–banded matrix with ones in the main diagonal, so the above decomposition is indeed a LU decomposition of certain $(d+2)$–banded Hessenberg matrix $J_1 = L_1L_2\cdots L_dU$ (see for instance [8] and [23, Sec. 3.2 and 3.3]). The values of the entries of J_1 come from the usual definition for matrix multiplication, matching every entry in $J_1 = L_1L_2\cdots L_dU$, with $L_j\,U$ given in (31) and (32) respectively.

On the other hand, starting with $\underline{\mathbf{A}}^2 = L_2\underline{\mathbf{A}}^3$ instead of $\underline{\mathbf{A}}^1$, and proceeding in the same fashion as above, we can reach

$$\begin{aligned} x\underline{\mathbf{A}}^2 &= L_2\cdots L_dUL_1\,\underline{\mathbf{A}}^2,\\ x\underline{\mathbf{A}}^3 &= L_3\cdots L_dUL_1L_2\,\underline{\mathbf{A}}^3,\\ &\vdots\\ x\underline{\mathbf{A}}^{d+1} &= UL_1L_2\cdots L_d\,\underline{\mathbf{A}}^{d+1}. \end{aligned}$$

Observe that J_j denotes a particular circular permutation of the matrix product $L_1L_2\cdots L_dU$. Thus, we have $J_1 = L_1L_2\cdots L_dU$, $J_2 = L_2\cdots L_dUL_1$, $\ldots$, $J_{d+1} = UL_1L_2\cdots L_d$. Using this notation, J_j is the matrix representation of the operator of multiplication by x in

$$x\underline{\mathbf{A}}^j = J_j\underline{\mathbf{A}}^j, \tag{33}$$

which from (10) directly implies that each polynomial sequence $\{A^j\}$, $j = 1,\ldots d+1$, satisfies a $(d+2)$–term recurrence relation as in the statement of the theorem, with coefficients given in terms of the recurrence coefficients γ_{n+1} , from the high–term recurrence relation (14) satisfied by the symmetric sequence $\{S_n\}$.

This completes the proof. ■

5. Matrix multiple orthogonality

For an arbitrary system of type II vector multiple polynomials $\{P_n\}$ orthogonal with respect to certain vector of functionals $\mathcal{U} = \begin{bmatrix} u^1 & u^2 \end{bmatrix}^T$, with

$$\mathcal{P}_n = \begin{bmatrix} P_{3n}(x) & P_{3n+1}(x) & P_{3n+2}(x) \end{bmatrix}^T, \tag{34}$$

there exists a matrix decomposition

$$\mathcal{P}_n = W_n(x^3)\mathcal{X}_n \to \begin{bmatrix} P_{3n}(x)\\ P_{3n+1}(x)\\ P_{3n+2}(x) \end{bmatrix} = W_n(x^3)\begin{bmatrix} 1\\ x\\ x^2 \end{bmatrix}, \tag{35}$$

with W_n being the matrix polynomial (see [25])

$$W_n(x) = \begin{bmatrix} A_n^1(x) & A_{n-1}^2(x) & A_{n-1}^3(x) \\ B_n^1(x) & B_n^2(x) & B_{n-1}^3(x) \\ C_n^1(x) & C_n^2(x) & C_n^3(x) \end{bmatrix}. \tag{36}$$

Throughout this Section, for simplicity of computations, we assume $d = 2$ for the vector of functionals $\mathcal{U}$, but the same results can be easily extended for an arbitrary number of functionals. We first show that, if a sequence of type II multiple 2–orthogonal polynomials $\{P_n\}$ satisfy a recurrence relation like (11), then there exists a sequence of matrix polynomials $\{W_n\}$, $W_n(x) \in \mathbb{P}^{3\times 3}$ associated to $\{P_n\}$ by (34) and (35), satisfying a matrix four term recurrence relation with matrix coefficients.

Theorem 5. *Let $\{P_n\}$ be a sequence of type II multiple polynomials, 2–orthogonal with respect to the system of functionals $\{u^1, u^2\}$, i.e., satisfying the four–term type recurrence relation (11). Let $\{W_n\}$, $W_n(x) \in \mathbb{P}^{3\times 3}$ associated to $\{P_n\}$ by (34) and (35). Then, the matrix polynomials W_n satisfy a matrix four term recurrence relation with matrix coefficients.*

Proof: We first prove that the sequence of vector polynomials $\{W_n\}$ satisfy a four term recurrence relation with matrix coefficients, starting from the fact that $\{P_n\}$ satisfy (11). In order to get this result, we use the matrix interpretation of multiple orthogonality described in Section 2. We know that the sequence of type II multiple 2–orthogonal polynomials $\{P_n\}$ satisfy the four term recurrence relation (11). From (34), using the matrix interpretation for multiple orthogonality, the above expression can be seen as the matrix three term recurrence relation

$$x\mathcal{P}_n = A\mathcal{P}_{n+1} + B_n\mathcal{P}_n + C_n\mathcal{P}_{n-1}$$

or, equivalently

$$x\begin{bmatrix} P_{3n} \\ P_{3n+1} \\ P_{3n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{3n+3} \\ P_{3n+4} \\ P_{3n+5} \end{bmatrix}$$

$$+ \begin{bmatrix} b_{3n} & 1 & 0 \\ c_{3n+1} & b_{3n+1} & 1 \\ d_{3n+2} & c_{3n+2} & b_{3n+2} \end{bmatrix} \begin{bmatrix} P_{3n} \\ P_{3n+1} \\ P_{3n+2} \end{bmatrix} + \begin{bmatrix} 0 & d_{3n} & c_{3n} \\ 0 & 0 & d_{3n+1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{3n-3} \\ P_{3n-2} \\ P_{3n-1} \end{bmatrix}$$

Multiplying the above expression by x we get

$$\begin{aligned} x^2\mathcal{P}_n &= Ax\mathcal{P}_{n+1} + B_n x\mathcal{P}_n + C_n x\mathcal{P}_{n-1} \\ &= A\left[x\mathcal{P}_{n+1}\right] + B_n\left[x\mathcal{P}_n\right] + C_n\left[x\mathcal{P}_{n-1}\right] \\ &= AA\mathcal{P}_{n+2} + \left[AB_{n+1} + B_nA\right]\mathcal{P}_{n+1} \\ &\quad + \left[AC_{n+1} + B_nB_n + C_nA\right]\mathcal{P}_n \\ &\quad + \left[B_nC_n + C_nB_{n-1}\right]\mathcal{P}_{n-1} + C_nC_{n-1}\mathcal{P}_{n-2} \end{aligned} \tag{37}$$

The matrix A is nilpotent, so AA is the zero matrix of size 3×3. Having

$$\begin{aligned} A_n^{\langle 1\rangle} &= AB_{n+1} + B_nA, & B_n^{\langle 1\rangle} &= AC_{n+1} + B_nB_n + C_nA, \\ C_n^{\langle 1\rangle} &= B_nC_n + C_nB_{n-1}, & D_n^{\langle 1\rangle} &= C_nC_{n-1}, \end{aligned}$$

where the entries of $A_n^{\langle 1\rangle}$, $B_n^{\langle 1\rangle}$, and $C_n^{\langle 1\rangle}$ can be easily obtained using a computational software as Mathematica® or Maple®, from the entries of A_n, B_n, and C_n. Thus, we can rewrite (37) as

$$x^2\mathcal{P}_n = A_n^{\langle 1\rangle}\mathcal{P}_{n+1} + B_n^{\langle 1\rangle}\mathcal{P}_n + C_n^{\langle 1\rangle}\mathcal{P}_{n-1} + D_n^{\langle 1\rangle}\mathcal{P}_{n-2}.$$

We now continue in this fashion,multiplying again by x

$$\begin{aligned} x^3\mathcal{P}_n &= A_n^{\langle 1\rangle}x\mathcal{P}_{n+1} + B_n^{\langle 1\rangle}x\mathcal{P}_n + C_n^{\langle 1\rangle}x\mathcal{P}_{n-1} + D_n^{\langle 1\rangle}x\mathcal{P}_{n-2} \\ &= A_n^{\langle 1\rangle}A\mathcal{P}_{n+2} + \left[A_n^{\langle 1\rangle}B_{n+1} + B_n^{\langle 1\rangle}A\right]\mathcal{P}_{n+1} \\ &\quad + \left[A_n^{\langle 1\rangle}C_{n+1} + B_n^{\langle 1\rangle}B_n + C_n^{\langle 1\rangle}A\right]\mathcal{P}_n \\ &\quad + \left[B_n^{\langle 1\rangle}C_n + C_n^{\langle 1\rangle}B_{n-1} + D_n^{\langle 1\rangle}A\right]\mathcal{P}_{n-1} \\ &\quad + \left[C_n^{\langle 1\rangle}C_{n-1} + D_n^{\langle 1\rangle}B_{n-2}\right]\mathcal{P}_{n-2} + D_n^{\langle 1\rangle}C_{n-2}\mathcal{P}_{n-3} \end{aligned}$$

The matrix products $A_n^{\langle 1\rangle}A$ and $D_n^{\langle 1\rangle}C_{n-2}$ both give the zero matrix of size 3×3, and the remaining matrix coefficients are

$$\begin{aligned} A_n^{\langle 2\rangle} &= A_n^{\langle 1\rangle}B_{n+1} + B_n^{\langle 1\rangle}A, & B_n^{\langle 2\rangle} &= A_n^{\langle 1\rangle}C_{n+1} + B_n^{\langle 1\rangle}B_n + C_n^{\langle 1\rangle}A, \\ C_n^{\langle 2\rangle} &= B_n^{\langle 1\rangle}C_n + C_n^{\langle 1\rangle}B_{n-1} + D_n^{\langle 1\rangle}A, & D_n^{\langle 2\rangle} &= C_n^{\langle 1\rangle}C_{n-1} + D_n^{\langle 1\rangle}B_{n-2}. \end{aligned} \tag{38}$$

Using the expressions stated above, the matrix coefficients (38) can be easily obtained as well. Beyond the explicit expression of their respective entries, the key point is that they are structured matrices, namely $A_n^{\langle 2\rangle}$ is lower-triangular with one's in the main diagonal, $D_n^{\langle 2\rangle}$ is upper-triangular, and

$B_n^{\langle 2\rangle}$, $C_n^{\langle 2\rangle}$ are full matrices. Therefore, the sequence of type II vector multiple orthogonal polynomials satisfy the following matrix four term recurrence relation

$$x\mathcal{P}_n = A_n^{\langle 2\rangle}\mathcal{P}_{n+1} + B_n^{\langle 2\rangle}\mathcal{P}_n + C_n^{\langle 2\rangle}\mathcal{P}_{n-1} + D_n^{\langle 2\rangle}\mathcal{P}_{n-2}, \quad n = 2, 3, \dots , \tag{39}$$

Next, combining (35) with (39), we can assert that

$$\begin{aligned} x^3 W_n(x^3)\begin{bmatrix}1\\x\\x^2\end{bmatrix} &= A_n^{\langle 2\rangle} W_{n+1}(x^3)\begin{bmatrix}1\\x\\x^2\end{bmatrix} + B_n^{\langle 2\rangle} W_n(x^3)\begin{bmatrix}1\\x\\x^2\end{bmatrix} \\ &\quad + C_n^{\langle 2\rangle} W_{n-1}(x^3)\begin{bmatrix}1\\x\\x^2\end{bmatrix} + D_n^{\langle 2\rangle} W_{n-2}(x^3)\begin{bmatrix}1\\x\\x^2\end{bmatrix}, \quad n = 2, 3, \dots . \end{aligned}$$

The vector $\begin{bmatrix}1 & x & x^2\end{bmatrix}^T$ can be removed, and after the shift $x^3 \to x$ the above expression may be simplified as

$$xW_n(x) = A_n^{\langle 2\rangle} W_{n+1}(x) + B_n^{\langle 2\rangle} W_n(x) + C_n^{\langle 2\rangle} W_{n-1}(x) + D_n^{\langle 2\rangle} W_{n-2}(x), \tag{40}$$

where $W_{-1} = 0_{3\times 3}$ and W_0 is certain constant matrix, for every $n = 1, 2, \dots$, which is the desired matrix four term recurrence relation for $W_n(x)$. ■

This kind of matrix high–term recurrence relation completely characterizes certain type of orthogonality. Hence, we are going prove a Favard type Theorem which states that, under the assumptions of Theorem 5, the matrix polynomials $\{W_n\}$ are *type II matrix multiple orthogonal* with respect to a system of two matrices of measures $\{d\mathbf{M}^1, d\mathbf{M}^2\}$.

Next, we briefly review some of the standard facts on the theory of *matrix orthogonality*, or *orthogonality with respect to a matrix of measures* (see [15], [16] and the references therein). Let $W, V \in \mathbb{P}^{3\times 3}$ be two matrix polynomials, and let

$$\mathbf{M}(x) = \begin{bmatrix}\mu_{11}(x) & \mu_{12}(x) & \mu_{13}(x)\\ \mu_{21}(x) & \mu_{22}(x) & \mu_{23}(x)\\ \mu_{31}(x) & \mu_{32}(x) & \mu_{33}(x)\end{bmatrix},$$

be a matrix with positive Borel measures $\mu_{i,j}(x)$. Let $\mathbf{M}(E)$ be positive definite for any Borel set $E \subset \mathbb{R}$, having finite moments

$$\bar{\mu}_k = \int_E d\mathbf{M}(x)x^k, \quad k = 0, 1, 2, \dots$$

of every order, and satisfying that

$$\int_E V(x) d\mathbf{M}(x) V^*(x),$$

where $V^* \in \mathbb{P}^{3\times 3}$ is the adjoint matrix of $V \in \mathbb{P}^{3\times 3}$, is non-singular if the matrix leading coefficient of the matrix polynomial V is non-singular. Under these conditions, it is possible to associate to a weight matrix $\mathbf{M}$, the Hermitian sesquilinear form

$$\langle W, V \rangle = \int_E W(x) d\mathbf{M}(x) V^*(x).$$

We then say that a sequence of matrix polynomials $\{W_n\}$, $W_n \in \mathbb{P}^{3\times 3}$ with degree n and nonsingular leading coefficient, is orthogonal with respect to $\mathbf{M}$ if

$$\langle W_m, W_n \rangle = \Delta_n \delta_{m,n} , \tag{41}$$

where $\Delta_n \in \mathcal{M}_{3\times 3}$ is a positive definite upper triangular matrix, for $n \geq 0$.

We can define the *matrix moment functional* M acting in $\mathbb{P}^{3\times 3}$ over $\mathcal{M}_{3\times 3}$, in terms of the above matrix inner product, by $\mathrm{M}(WV) = \langle W, V \rangle$. This construction is due to Jódar *et al.* (see [20], [21]) where the authors extend to the matrix framework the linear moment functional approach developed by Chihara in [11]. Hence, the moments of $\mathbf{M}(x)$ and (41) can be written

$$\begin{aligned} &\mathrm{M}(x^k) = \bar{\mu}_k , \quad k = 0, 1, 2, \ldots , \\ &\mathrm{M}(W_m V_n) = \Delta_n \delta_{m,n} , \quad m, n = 0, 1, 2, \ldots . \end{aligned}$$

Let $\boldsymbol{m} = (m_1, m_2) \in \mathbb{N}^2$ be a multi–index with length $|\mathbf{m}| := m_1 + \cdots + m_2$ and let $\{\mathrm{M}^1, \mathrm{M}^2\}$ be a set of matrix moment functionals as defined above. Let $\{W_{\mathbf{m}}\}$ be a sequence of matrix polynomials, with $\deg W_{\mathbf{m}}$ is at most $|\mathbf{m}|$. $\{W_{\mathbf{m}}\}$ is said to be a *type II* multiple orthogonal with respect to the set of linear functionals $\{\mathrm{M}^1, \mathrm{M}^2\}$ and multi–index $\boldsymbol{m}$ if it satisfy the following orthogonality conditions

$$\mathrm{M}^j(x^k W_{\mathbf{n}}) = 0_{3\times 3} , \quad k = 0, 1, \ldots , n_j - 1 , \quad j = 1, 2. \tag{42}$$

A multi–index $\boldsymbol{m}$ is said to be *normal* for the set of matrix moment functionals $\{\mathrm{M}^1, \mathrm{M}^2\}$, if the degree of $W_{\mathbf{m}}$ is exactly $|\mathbf{m}| = m$. Thus, in what follows we will write $W_{\mathbf{m}} \equiv W_{|\mathbf{m}|} = W_m$.

In this framework, let consider the sequence of *vector of matrix polynomials* $\{\mathcal{B}_n\}$ where

$$\mathcal{B}_n = \begin{bmatrix} W_{2n} \\ W_{2n+1} \end{bmatrix}. \tag{43}$$

We define the vector of matrix–functionals $\mathfrak{M} = \begin{bmatrix} \mathrm{M}^1 & \mathrm{M}^2 \end{bmatrix}^T$, with $\mathfrak{M} : \mathbb{P}^{6\times 3} \to \mathcal{M}_{6\times 6}$, by means of the action $\mathfrak{M}$ on $\mathcal{B}_n$, as follows

$$\mathfrak{M}\left(\mathcal{B}_n\right) = \begin{bmatrix} \mathrm{M}^1(W_{2n}) & \mathrm{M}^2(W_{2n}) \\ \mathrm{M}^1(W_{2n+1}) & \mathrm{M}^2(W_{2n+1}) \end{bmatrix} \in \mathcal{M}_{6\times 6}\,. \tag{44}$$

where

$$\mathrm{M}^i(W_j) = \int W_j(x) d\mathbf{M}^i = \Delta_j^i, \quad i = 1, 2, \text{ and } j = 0, 1, \tag{45}$$

with $\{d\mathbf{M}^1, d\mathbf{M}^2\}$ being a system of two matrix of measures as described above. Thus, we say that a sequence of vectors of matrix polynomials $\{\mathcal{B}_n\}$ is orthogonal with respect to a vector of matrix functionals $\mathfrak{M}$ if

$$\left.\begin{array}{rl} i) & \mathfrak{M}(x^k\mathcal{B}_n) = 0_{6\times 6}\,, \quad k = 0, 1, \ldots, n-1\,, \\ ii) & \mathfrak{M}(x^n\mathcal{B}_n) = \Omega_n\,, \end{array}\right\} \tag{46}$$

where Ω_n is a regular block upper triangular 6×6 matrix, holds.

Now we are in a position to prove the following

Theorem 6 (Favard type)**.** *Let $\{\mathcal{B}_n\}$ a sequence of vectors of matrix polynomials of size 6×3, defined in (43), with W_n matrix polynomials satisfying the four term recurrence relation (40). The following statements are equivalent:*

(a) *The sequence $\{\mathcal{B}_n\}_{n\geq 0}$ is orthogonal with respect to a certain vector of two matrix–functionals.*

(b) *There are sequences of scalar 6×6 block matrices $\{A_n^{\langle 3\rangle}\}_{n\geq 0}$, $\{B_n^{\langle 3\rangle}\}_{n\geq 0}$, and $\{C_n^{\langle 3\rangle}\}_{n\geq 0}$, with $C_n^{\langle 3\rangle}$ block upper triangular non-singular matrix for $n\in\mathbb{N}$, such that the sequence $\{\mathcal{B}_n\}$ satisfy the matrix three term recurrence relation*

$$x\mathcal{B}_n = A_n^{\langle 3\rangle}\mathcal{B}_{n+1} + B_n^{\langle 3\rangle}\mathcal{B}_n + C_n^{\langle 3\rangle}\mathcal{B}_{n-1}, \tag{47}$$

with $\mathcal{B}_{n-1} = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}^T$, $\mathcal{B}_0$ given, and $C_n^{\langle 3\rangle}$ non-singular.

Proof: First we prove that (a) implies (b). Since the sequence of vector of matrix polynomials $\{\mathcal{B}_n\}$ is a basis in the linear space $\mathbb{P}^{6\times 3}$, we can write

$$x\mathcal{B}_n = \sum_{k=0}^{n+1} \widetilde{A}_k^n \mathcal{B}_k, \quad \widetilde{A}_k^n \in \mathcal{M}_{6\times 6}\,.$$

Then, from the orthogonality conditions (46), we get

$$\mathfrak{M}\,(x\mathcal{B}_n) = \mathfrak{M}(\widetilde{A}_k^n \mathcal{B}_k) = 0_{6\times 6}, \quad k = 0, 1, \dots, n-2.$$

Thus,

$$x\mathcal{B}_n = \widetilde{A}_{n+1}^n \mathcal{B}_{n+1} + \widetilde{A}_n^n \mathcal{B}_n + \widetilde{A}_{n-1}^n \mathcal{B}_{n-1}\,.$$

Having $\widetilde{A}_{n+1}^n = A_n^{\langle 3\rangle}$, $\widetilde{A}_n^n = B_n^{\langle 3\rangle}$, and $\widetilde{A}_{n-1}^n = C_n^{\langle 3\rangle}$, the result follows.

To proof that (b) implies (a), we know from Theorem 5 that the sequence of vector polynomials $\{W_n\}$ satisfy a four term recurrence relation with matrix coefficients. We can associate this matrix four term recurrence relation (40) with the block matrix three term recurrence relation (47). Then, it is sufficient to show that $\mathfrak{M}$ is uniquely determined by its orthogonality conditions (46), in terms of the sequence $\{C_n^{\langle 3\rangle}\}_{n\geq 0}$ in that (47).

Next, from (43) we can rewrite the matrix four term recurrence relation (40) into a matrix three term recurrence relation

$$x\mathcal{B}_n = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ A_{2n+1}^{\langle 2\rangle} & 0_{3\times 3} \end{bmatrix} \mathcal{B}_{n+1} + \begin{bmatrix} B_{2n}^{\langle 2\rangle} & A_{2n}^{\langle 2\rangle} \\ C_{2n+1}^{\langle 2\rangle} & B_{2n+1}^{\langle 2\rangle} \end{bmatrix} \mathcal{B}_n + \begin{bmatrix} D_{2n}^{\langle 2\rangle} & C_{2n}^{\langle 2\rangle} \\ 0_{3\times 3} & D_{2n+1}^{\langle 2\rangle} \end{bmatrix} \mathcal{B}_{n-1},$$

where the size of $\mathcal{B}_n$ is 6×3. We give $\mathfrak{M}$ in terms of its block matrix moments, which in turn are given by the matrix coefficients in (47). There is a unique vector moment functional $\mathfrak{M}$ and hence two matrix measures $d\mathbf{M}^1$ and $d\mathbf{M}^2$, such that

$$\mathfrak{M}\,(\mathcal{B}_0) = \begin{bmatrix} \mathrm{M}^1(W_0) & \mathrm{M}^2(W_0) \\ \mathrm{M}^1(W_1) & \mathrm{M}^2(W_1) \end{bmatrix} = C_0^{\langle 3\rangle} \in \mathcal{M}_{6\times 6},$$

where $\mathrm{M}^i(W_j)$ was defined in (45). For the first moment of $\mathfrak{M}_0$ we get

$$\mathfrak{M}_0 = \mathfrak{M}\,(\mathcal{B}_0) = \begin{bmatrix} \Delta_0^1 & \Delta_0^2 \\ \Delta_1^1 & \Delta_1^2 \end{bmatrix} = C_0^{\langle 3\rangle}.$$

Hence, we have $\mathfrak{M}\,(\mathcal{B}_1) = 0_{6\times 6}$, which from (47) also implies $0_{6\times 6} = x\mathfrak{M}\,(\mathcal{B}_0) - B_0^{\langle 3\rangle}\mathfrak{M}\,(\mathcal{B}_0) = \mathfrak{M}_1 - B_0^{\langle 3\rangle}\mathfrak{M}_0$. Therefore

$$\mathfrak{M}_1 = B_0^{\langle 3\rangle} C_0^{\langle 3\rangle}.$$

By a similar argument, we have

$$\begin{aligned}0_{6\times 6} &= \mathfrak{M}\Big((A_0^{\langle 3\rangle})^{-1}x^2\mathcal{B}_0 - (A_0^{\langle 3\rangle})^{-1}B_0^{\langle 3\rangle}x\mathcal{B}_0 - B_1^{\langle 3\rangle}(A_0^{\langle 3\rangle})^{-1}x\mathcal{B}_0 \\ &\qquad + B_1(A_0^{\langle 3\rangle})^{-1}B_0^{\langle 3\rangle}\mathcal{B}_0 - C_1^{\langle 3\rangle}\mathcal{B}_0\Big) \\ &= (A_0^{\langle 3\rangle})^{-1}\mathfrak{M}_2 - \Big((A_0^{\langle 3\rangle})^{-1}B_0^{\langle 3\rangle} + B_1^{\langle 3\rangle}(A_0^{\langle 3\rangle})^{-1}\Big)\mathfrak{M}_1 \\ &\qquad + \Big(B_1^{\langle 3\rangle}(A_0^{\langle 3\rangle})^{-1}B_0^{\langle 3\rangle} - C_1^{\langle 3\rangle}\Big)\mathfrak{M}_0,\end{aligned}$$

which in turn yields the second moment of $\mathfrak{M}$

$$\mathfrak{M}_2 = \Big(B_0^{\langle 3\rangle} + A_0^{\langle 3\rangle}B_1^{\langle 3\rangle}(A_0^{\langle 3\rangle})^{-1}\Big)B_0^{\langle 3\rangle}C_0^{\langle 3\rangle} - A_0^{\langle 3\rangle}\Big(B_1^{\langle 3\rangle}(A_0^{\langle 3\rangle})^{-1}B_0^{\langle 3\rangle} - C_1^{\langle 3\rangle}\Big)C_0^{\langle 3\rangle}.$$

Repeated application of this inductive process, enables us to determine $\mathfrak{M}$ in a unique way through its moments, only in terms of the sequences of matrix coefficients $\{A_n^{\langle 3\rangle}\}_{n\geq 0}$, $\{B_n^{\langle 3\rangle}\}_{n\geq 0}$ and $\{C_n^{\langle 3\rangle}\}_{n\geq 0}$.

On the other hand, because of (44) and (47) we have

$$\mathfrak{M}(x\mathcal{B}_n) = 0_{6\times 6}, \quad n \geq 2.$$

Multiplying by x both sides of (47), from the above result we get

$$\mathfrak{M}\left(x^2\mathcal{B}_n\right) = 0_{6\times 6}, \quad n \geq 3.$$

The same conclusion can be drawn for $0 < k < n$

$$\mathfrak{M}\left(x^k\mathcal{B}_n\right) = 0_{6\times 6}, \quad 0 < k < n, \tag{48}$$

and finally

$$\mathfrak{M}(x^n\mathcal{B}_n) = C_n^{\langle 3\rangle}\mathfrak{M}\left(x^{n-1}\mathcal{B}_{n-1}\right).$$

Notice that the repeated application of the above argument leads to

$$\mathfrak{M}(x^n\mathcal{B}_n) = C_n^{\langle 3\rangle}C_{n-1}^{\langle 3\rangle}C_{n-2}^{\langle 3\rangle}\cdots C_1^{\langle 3\rangle}C_0^{\langle 3\rangle}. \tag{49}$$

From (48), (49) we conclude

$$\begin{aligned}\mathfrak{M}\left(x^k\mathcal{B}_n\right) &= C_n^{\langle 3\rangle}C_{n-1}^{\langle 3\rangle}C_{n-2}^{\langle 3\rangle}\cdots C_1^{\langle 3\rangle}C_0^{\langle 3\rangle}\delta_{n,k} \\ &= \Omega_n\delta_{n,k}, \quad n,k = 0,1,\ldots, \ k \leq n,\end{aligned}$$

which are exactly the desired orthogonality conditions (46) for $\mathfrak{M}$ stated in the theorem. ■

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Amílcar Branquinho
CMUC and Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 COIMBRA, Portugal.

E-mail address: ajplb@mat.uc.pt

Edmundo J. Huertas
CMUC and Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 COIMBRA, Portugal.

E-mail address: ehuertas@mat.uc.pt