Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–50

# CHARACTERISTIC SUBOBJECTS IN SEMI-ABELIAN CATEGORIES

ALAN S. CIGOLI AND ANDREA MONTOLI

ABSTRACT: We extend to semi-abelian categories the notion of characteristic subobject, which is widely used in group theory and in the theory of Lie algebras. Moreover, we show that many of the classical properties of characteristic subgroups of a group hold in the general semi-abelian context, or in stronger ones.

KEYWORDS: characteristic subobject, semi-abelian categories, commutators, centralisers.

## 1. Introduction

The notion of characteristic subgroup (which means a subgroup that is invariant under all automorphisms of the bigger group) is widely used in group theory. Examples of characteristic subgroups are the centre and the derived subgroup of any group. The main properties of characteristic subgroups are the following: if $H$ is a characteristic subgroup of $K$ and $K$ is a characteristic subgroup of $G$, then $H$ is a characteristic subgroup of $G$; moreover, if $H$ is characteristic in $K$ and $K$ is normal in $G$, then $H$ is normal in $G$. These transitivity properties of characteristic subgroups imply, for example, that the derived series and the central series of a group are normal series, and this fact is very useful in order to deal with solvable and nilpotent groups.

An analogous notion exists for Lie algebras (over a commutative ring $R$): a characteristic ideal of a Lie algebra is a subalgebra which is invariant under all derivations of the bigger one. The two transitivity properties mentioned above hold also in this context, and again this allows to easily describe solvable and nilpotent Lie algebras.

Received October 31, 2013.
The first author was partially supported by FSE, Regione Lombardia. The second author was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2013 and grants number PTDC/MAT/120222/2010 and SFRH/BPD/69661/2010.

The strong parallelism between these two contexts is explained by the fact that automorphisms represent group actions, as well as derivations represent actions of Lie algebras in the following sense. An action of a group $B$ on a group $G$ can be described simply as a group homomorphism $B \to \mathrm{Aut}(G)$; in the same way, an action of a Lie algebra $B$ on a Lie algebra $G$ is a homomorphism of Lie algebras $B \to \mathrm{Der}(G)$.

The aim of this paper is to extend the definition and the main properties of characteristic subobjects to a categorical context. In order to do this, we will use the notion of *internal action* introduced in [3]. In [9] it is proved that, in semi-abelian categories [18], internal actions are equivalent to split extensions, via a semidirect product construction which generalises the classical one known for groups. Examples of semi-abelian categories are groups, rings, associative algebras, Lie algebras and, in general, any variety of $\Omega$-groups.

We define a characteristic subobject as a subobject $H$ of an object $G$ which is invariant under all (internal) actions over $G$. In the semi-abelian context, we can use the equivalence between actions and split extensions mentioned above in order to deduce properties of characteristic subobjects from properties of the kernel functor which associates with any split epimorphism its kernel.

The paper is organized as follows: in Section 2 we give the definition of characteristic subobject and we prove some properties that hold in any semi-abelian category, like the transitivity properties mentioned at the beginning, or the fact that the intersection of characteristic subobjects is characteristic. Then we study properties that hold in stronger contexts, such as:

- the join of two characteristic subobjects is characteristic (Section 3);
- the commutator of two characteristic subobjects is characteristic (Section 4);
- the centraliser of a characteristic subobject is characteristic (Section 5).

Some properties about actors of characteristic subobjects are studied in Section 6 in the context of action representative categories [4, 2] and analogous results are proved in action accessible categories [10], replacing actors with suitable objects.

## 2. Definition and basic properties

A characteristic subgroup of a group $G$ is classically defined as a subgroup $H$ of $G$ which is invariant under all the automorphisms of $G$. This means that any automorphism of $G$ restricts to an automorphism of $H$. Since the automorphism group $\mathrm{Aut}(G)$ of a group $G$ classifies all the group actions on $G$, a subgroup $H$ of a group $G$ is characteristic if and only if any group action on $G$ restricts to an action on $H$.

In other algebraic contexts it is no longer true that automorphisms classify actions, hence the notions of invariance under automorphisms and under actions are different. As already explained in the introduction, here we are interested in the latter. In order to study it in a categorical setting, we are going to use the notion of *internal action*, introduced in [3]. We briefly recall the definition.

Let $\mathcal{C}$ be a pointed category with finite limits and finite coproducts. For any object $B$ in $\mathcal{C}$, we can define the category $\mathsf{Pt}_B(\mathcal{C})$ of *points* over $B$, whose objects are split epimorphisms $(A, p, s)$ with codomain $B$ and whose arrows are commutative triangles of the following form, with $p'f = p$ and $fs = s'$:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, shift left, "p"] & & A' \arrow[dl, shift left, "p'"] \\ & B \arrow[ul, shift left, "s"] \arrow[ur, shift left, "s'"] & \end{tikzcd}$$

We then get the two following functors:

$$\mathrm{Ker}_B \colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}\,,$$

given by $\mathrm{Ker}_B(A, p, s) = \mathrm{Ker}\ p$, and

$$B + (-) \colon \mathcal{C} \to \mathsf{Pt}_B(\mathcal{C})\,,$$

where $B + (X)$ is the point $\begin{tikzcd} B + X \arrow[r, shift left, "{[1,0]}"] & B \arrow[l, shift left, "\iota_B"] \end{tikzcd}$.

These functors give rise to an adjunction. The corresponding monad on $\mathcal{C}$ is denoted by $B\flat(-)$. For any object $X \in \mathcal{C}$, we have that $B\flat X$ is the kernel of the morphism $[1, 0] \colon B + X \to B$. The algebras for this monad are called *internal actions*. The comparison functor associates with every point $(A, p, s)$ an action $\xi$ as described in the following diagram (where $X$ is the kernel of $p$ and $\xi$ is induced by the universal property of $X$):

$$\begin{tikzcd}
B\flat X \arrow[r, "{\ker[1,0]}"] \arrow[d, "\xi"'] & B+X \arrow[r, shift left, "{[1,0]}"] \arrow[d, "{[s,k]}"] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\
X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"]
\end{tikzcd}$$

When $\mathcal{C}$ is the category $\mathsf{Gp}$ of groups, the elements of $B\flat X$ are generated by formal sequences of type $(b; x; b^{-1})$ with $b \in B$ and $x \in X$, and the internal action $\xi$ is nothing but the realisation of these sequences in $X$, that is $\xi(b; x; b^{-1}) = bxb^{-1}$, or more properly $\xi(b; x; b^{-1}) = k^{-1}(s(b)k(x)s(b^{-1}))$ since the product is actually computed in $A$.

Vice versa, given a group action $\xi$ of $B$ over $K$, we can always associate with it the semidirect product $K \rtimes_\xi B$ and a point as in the following diagram where the left hand side square is constructed as a pushout:

$$\begin{tikzcd}
B\flat X \arrow[r, "{\ker[1,0]}"] \arrow[d, "\xi"'] & B+X \arrow[r, shift left, "{[1,0]}"] \arrow[d] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\
X \arrow[r, "i_X"'] & X \rtimes_\xi B \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"]
\end{tikzcd}$$

We can repeat the same construction in the categorical context mentioned above. However, in general, the bottom row is not always a split short exact sequence. This is the case when the comparison functor is an equivalence, as, for example, in any semi-abelian category [18, 1], as shown in [9], where the categorical notion of semi-direct product is introduced.

We are now ready to give the following definition:

**Definition 2.1.** *Let $\mathcal{C}$ be a pointed category with finite limits and finite coproducts. Let $G$ be an object in $\mathcal{C}$ and $h\colon H \rightarrowtail G$ a subobject. We say that $H$ is* characteristic *in $G$, and we write $H \underset{\text{char}}{<} G$, if, for all pairs $(B, \xi)$, with $B$ an object of $\mathcal{C}$ and $\xi$ an internal action of $B$ on $G$, the action $\xi$ restricts to the subobject $H$. In other words, there exists a (unique) action $\overline{\xi}$ of $B$ on*

*$H$ which makes the following diagram commute:*

$$\begin{tikzcd} B\flat H \arrow[r, "1\flat h"] \arrow[d, "\overline{\xi}"'] & B\flat G \arrow[d, "\xi"] \\ H \arrow[r, tail, "h"'] & G \end{tikzcd}$$

When $\mathcal{C}$ is a semi-abelian category, the above mentioned equivalence between actions and points allows us to reformulate the definition of characteristic subobject.

**Proposition 2.2.** *Let $\mathcal{C}$ be a semi-abelian category. A subobject $h\colon H \rightarrowtail G$ is characteristic in $G$ if and only if for every split extension of kernel $G$*

$$\begin{tikzcd} G \arrow[r, "\triangleright"] & X \arrow[r, shift left] & B \arrow[l, shift left] \end{tikzcd}$$

*there exist a split extension $H \triangleright\!\!\longrightarrow Y \longrightarrow B$ and a morphism of split extensions inducing $h$ on kernels and $1_B$ on cokernels (it is necessarily a monomorphism thanks to the split short five lemma):*

$$\begin{tikzcd} H \arrow[r, "\triangleright"] \arrow[d, tail, "h"'] & Y \arrow[r, shift left] \arrow[d, tail] & B \arrow[l, shift left] \arrow[d, equal] \\ G \arrow[r, "\triangleright"] & X \arrow[r, shift left] & B \arrow[l, shift left] \end{tikzcd}$$

As we will see afterwards, this reformulation makes the notion of characteristic subobject much easier to handle. Moreover, the translation in terms of points reveals that, when actions are equivalent to points, many properties of characteristic subobjects are strictly related with the properties of the fibration of points (see [1]) or, to be more precise, of the kernel functors:

$$\mathrm{Ker}_B\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$$

For these reasons, in our investigation, we will focus on contexts which are at least semi-abelian, possibly with additional requirements. The behaviour of characteristic subobjects in weaker contexts is material for future work.

**Proposition 2.3.** *If $H$ is a characteristic subobject of $K$, and $K$ is a characteristic subobject of $G$, then $H$ is characteristic in $G$.*

*Proof*: The result is a straightforward consequence of Definition 2.1. ■

**Proposition 2.4.** *If $H$ is a characteristic subobject of $K$, and $K$ is a normal subobject of $G$, then $H$ is normal in $G$.*

*Proof*: It suffices to observe that, in the semi-abelian context, normal subobjects are exactly those closed under the conjugation action (i.e. clots, see for example [19]). Indeed, the conjugation action of $G$ on itself restricts to $K$ by normality, and then to $H$, since $H \underset{\text{char}}{<} K$, thus proving that $H \lhd G$. ∎

**Corollary 2.5.** *If $H$ is a characteristic subobject of $G$, then $H$ is normal in $G$.*

**Proposition 2.6.** *Let $I$ be a set of indices. If $\{H_i\}_{i \in I}$ is a family of characteristic subobjects of $G$, then the intersection $\bigwedge_{i \in I} H_i$ is characteristic in $G$.*

*Proof*: If $h_i \colon H_i \rightarrowtail G$ is a characteristic subobject, then, for every action $\xi \colon B \flat G \to G$, there is a morphism in $\mathsf{Pt}_B(\mathcal{C})$:

$$\begin{tikzcd}
H_i \arrow[r, triangle] \arrow[d, tail, "h_i"'] & Y_i \arrow[r, shift left, "p_i"] \arrow[d, tail] & B \arrow[l, shift left, "s_i"] \arrow[d, equal] \\
G \arrow[r, triangle] & G \rtimes_\xi B \arrow[r, shift left] & B \arrow[l, shift left]
\end{tikzcd}$$

Since the kernel functor $\mathrm{Ker}_B \colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ has a left adjoint, it preserves intersections, so the object $\bigwedge_{i \in I} H_i$ in $\mathcal{C}$ is the kernel of the intersection $\bigwedge_{i \in I}(Y_i, p_i, s_i)$ in $\mathsf{Pt}_B(\mathcal{C})$. ∎

When the category $\mathcal{C}$ is not only semi-abelian, but also strongly protomodular [7], internal actions behave well with respect to quotients. More precisely, in [20] the following result is proved.

**Proposition 2.7.** *A semi-abelian category is strongly semi-abelian (i.e. semi-abelian and strongly protomodular) if and only if the following property holds:*

- *for every normal subobject $H \lhd G$ and every action $\xi \colon B \flat G \to G$, if $\xi$ restricts to $H$, then $\xi$ also induces a (unique) action $\widetilde{\xi}$ on the quotient $G/H$:*

$$\begin{tikzcd}
B \flat H \arrow[r] \arrow[d, "\overline{\xi}"'] & B \flat G \arrow[r] \arrow[d, "\xi"] & B \flat (G/H) \arrow[d, "\widetilde{\xi}"] \\
H \arrow[r, triangle, "h"'] & G \arrow[r, "q"', two heads] & G/H
\end{tikzcd}$$

In terms of split extensions, this means that if a kernel $h$ is the restriction of some $\phi$ in $\mathsf{Pt}_B(\mathcal{C})$, then $q = \mathrm{coker}(h)$ is the restriction of $\gamma = \mathrm{coker}(\phi)$ in

$\mathsf{Pt}_B(\mathcal{C})$:

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, tail, "h"'] & Y \arrow[r, shift left] \arrow[d, tail, "\phi"] & B \arrow[l, shift left] \arrow[d, equal] \\
G \arrow[r, tail] \arrow[d, "q"'] & X \arrow[r, shift left] \arrow[d, "\gamma"] & B \arrow[l, shift left] \arrow[d, equal] \\
G/H \arrow[r, tail] & Z \arrow[r, shift left] & B \arrow[l, shift left]
\end{tikzcd} \tag{1}$$

In fact, it turns out that, for the special class of characteristic subobjects, strong protomodularity is not needed in order to transfer actions to the quotient.

**Proposition 2.8.** *If $H$ is a characteristic subobject of $G$, then every action on $G$ induces an action on the quotient $G/H$, as in the diagram of Proposition 2.7.*

*Proof*: By Proposition 2.4, for every action $\xi\colon B\flat G \to G$, $H$ is a normal subobject of $G \rtimes_\xi B$. Then the arrow $Y \rightarrowtail G \rtimes_\xi B$, induced by the restriction of $\xi$ to $H$, is a normal monomorphism in $\mathsf{Pt}_B(\mathcal{C})$, according to [1, Proposition 6.2.1]:

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, tail, "h"] & Y \arrow[r, shift left] \arrow[d, tail] & B \arrow[l, shift left] \arrow[d, equal] \\
G \arrow[r, tail] & G \rtimes_\xi B \arrow[r, shift left] & B \arrow[l, shift left]
\end{tikzcd}$$

By taking its cokernel, we get an exact sequence as in diagram (1). ∎

**Proposition 2.9.** *If $H \leq K \leq G$, $H$ is characteristic in $G$ and $K/H$ is characteristic in $G/H$, then $K$ is characteristic in $G$.*

*Proof*: Let us consider the following diagram

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, equal] & K \arrow[r] \arrow[d, tail, "k"] & K/H \arrow[d, tail, "\widetilde{k}"] \\
H \arrow[r, tail] & G \arrow[r, "q"'] & G/H
\end{tikzcd}$$

The right hand side square is a pullback (this comes from the fact that the category $\mathcal{C}$, being semi-abelian, is protomodular [5]). By Proposition 2.8 every action of some $B$ on $G$ induces an action on $G/H$. By assumption, the same action restricts to $K/H$. In terms of points, we have a cospan in $\mathsf{Pt}_B(\mathcal{C})$ whose restriction to the kernels is the pair $(q, \widetilde{k})$. Now, since the kernel functors preserve pullbacks, $K$ is the kernel of the pullback in $\mathsf{Pt}_B(\mathcal{C})$ of the same cospan, hence the action of $B$ on $G$ restricts to $K$. ∎

**Proposition 2.10.** *If $H$ is characteristic in $G$, then its corresponding equivalence relation $R$ on $G$ is closed under actions on $G$, i.e. there exists an action $R(\xi)$ of $B$ on $R$ which makes the following diagram commute:*

$$\begin{tikzcd} B\flat R \arrow[r, shift left, "1\flat r_1"] \arrow[r, shift right, "1\flat r_2"'] \arrow[d, "R(\xi)"'] & B\flat G \arrow[d, "\xi"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & G \end{tikzcd}$$

*Proof*: By Proposition 2.8 every action of some $B$ on $G$ induces an action on $G/H$. Now, since kernel functors preserve pullbacks, $R$ is the kernel of the kernel pair in $\mathsf{Pt}_B(\mathcal{C})$ of the morphism $\gamma$ of diagram (1):

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, shift right, "r_2"'] \arrow[d, shift left, "r_1"] & R_\gamma \arrow[r, shift left] \arrow[d, shift right] \arrow[d, shift left] & B \arrow[l, shift left] \arrow[d, equal] \\ G \arrow[r, tail] \arrow[d, "q"'] & X \arrow[r, shift left] \arrow[d, "\gamma"] & B \arrow[l, shift left] \arrow[d, equal] \\ G/H \arrow[r, tail] & Z \arrow[r, shift left] & B \arrow[l, shift left] \end{tikzcd}$$

∎

We can make explicit the previous proposition in the category $\mathsf{Gp}$. It says that for all pairs $(x, y) \in R$ and for all $b \in B$, the pair $({}^b x,\ {}^b y) \in R$.

More in general, whenever $B$ acts on $G$, there is an induced action on $G \times G$ (simply computing the product in $\mathsf{Pt}_B(\mathcal{C})$), and the inclusion $R \rightarrowtail G \times G$ is compatible with the corresponding actions. However, this does not mean that $R$ is a characteristic subobject of $G \times G$.

## 3. Joins

While the outcomes listed in Section 2 hold in the very general case of semi-abelian categories, the property that finite joins of characteristic subobjects are characteristic (which is true in the category of groups, for example) seems to hold only in stronger contexts.

An additional requirement, for a semi-abelian category, which turns out to be crucial in this sense, is to ask that kernel functors preserve jointly strongly epimorphic pairs. This is equivalent to the fact that, for all pairs $((Y, p_1, s_1), (Z, p_2, s_2))$ of objects in $\mathsf{Pt}_B(\mathcal{C})$, the canonical arrow in $\mathcal{C}$:

$$\mathrm{Ker}_B(Y, p_1, s_1) + \mathrm{Ker}_B(Z, p_2, s_2) \to \mathrm{Ker}_B((Y, p_1, s_1) + (Z, p_2, s_2))$$

is a regular epimorphism.

**Lemma 3.1** ([15])**.** *Let $\mathcal{C}$ be a semi-abelian category. For any object $B \in \mathcal{C}$ the kernel functor $\mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ preserves jointly strongly epimorphic pairs if and only if it preserves binary joins.*

It is worth noting that the previous lemma does not say, in particular, that, under the assumption, kernel functors preserve coproducts. A counterexample to this fact is given in the proof of Proposition 6.2 in [16] for the category of commutative (not necessarily unitary) rings.

**Proposition 3.2.** *Let $\mathcal{C}$ be a semi-abelian category where kernel functors preserve jointly strongly epimorphic pairs. If $H$ and $K$ are characteristic subobjects of $G$, then $H \vee K$ is characteristic in $G$.*

*Proof*: Being $H$ and $K$ characteristic, for every action of $B$ on $G$, the cospan $H \rightarrowtail^{h} G \leftarrowtail^{k} K$ is the restriction to kernels of a cospan in $\mathsf{Pt}_B(\mathcal{C})$. By Lemma 3.1, $H \vee K$ is the kernel of a point over $B$. ∎

A context in which the property of preservation of binary joins by the kernel functor holds is the one of *locally algebraically cartesian closed* categories [8]. A semi-abelian category $\mathcal{C}$ is said locally algebraically cartesian closed (or simply LACC) if, for any morphism $p \colon E \to B$ in $\mathcal{C}$, the change of base functor

$$p^* \colon \mathsf{Pt}_B(\mathcal{C}) \to \mathsf{Pt}_E(\mathcal{C}),$$

defined by taking pullbacks along $p$, has a right adjoint. Examples of this situation are the categories $\mathsf{Gp}$ of groups and $\mathsf{Lie}$ of Lie algebras over a fixed commutative ring $R$. In this context the kernel functors (which are change of base functors with $E = 0$), having right adjoints, preserve all finite colimits, and hence the canonical arrow

$$\mathrm{Ker}_B(Y, p_1, s_1) + \mathrm{Ker}_B(Z, p_2, s_2) \to \mathrm{Ker}_B((Y, p_1, s_1) + (Z, p_2, s_2))$$

mentioned above is an isomorphism.

Another context in which preservation of binary joins holds is given by *categories of interest* [22]. We recall that a category of interest is a category $\mathcal{C}$

whose objects are groups with a set of operation $\Omega$ and with a set of equalities $\mathbb{E}$, such that $\mathbb{E}$ includes the group laws and the following conditions hold. If $\Omega_i$ is the set of $i$-ary operations in $\Omega$, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(b) the group operations (written additively: $0, -, +$, even if the group is not necessarily abelian) are elements of $\Omega_0$, $\Omega_1$ and $\Omega_2$ respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $* \in \Omega_2'$, then $\Omega_2'$ contains $*^\circ$ defined by $x *^\circ y = y * x$. Assume further that $\Omega_0 = \{0\}$;
(c) for any $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;
(d) for any $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x * y)$;
(e) **Axiom 1** $\quad x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$ for any $* \in \Omega_2'$;
(f) **Axiom 2** $\quad$ for any ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word $W$ such that

$$(x_1 * x_2)\overline{*}x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,$$

$$x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in $\Omega_2'$.

Examples of categories of interest are groups, Lie algebras, rings, associative algebras, Leibniz algebras, Poisson algebras and many others. Also in this context the kernel functors preserve binary joins, as follows from [12] and Lemma 3.1 herein.

Since it will be useful later, we give here a description of internal actions in *categories of interest* (called *derived actions* in [11]). In a *category of interest* $\mathcal{C}$, an action of an object $B$ on an object $X$ is a set of functions:

$$f_*\colon B \times X \to X\,,$$

one for each operation $*$ in $\Omega_2$ (we will write $b \cdot x$ for $f_+(b, x)$ and $b * x$ for $f_*(b, x)$, with $* \in \Omega_2'$), such that the one corresponding to the group operation $+$ satisfies the usual axioms for a group action, the others are bilinear with respect to $+$ and moreover the following axioms are satisfied (for all $b, b_i \in B$, $x, x_i \in X$ and $*, \overline{*} \in \Omega_2'$):

(1) $b \cdot (x_1 * x_2) = x_1 * x_2$;
(2) $x_1 + (b * x_2) = (b * x_2) + x_1$;
(3) $(b_1 * b_2) \cdot x = x$;
(4) $b_1 \cdot (b_2 * x) = b_2 * x$;

$$\begin{array}{llcl}
(5) & (b * x_1)\overline{*}x_2 & = & W(b(x_1x_2), b(x_2x_1), (x_1x_2)b, (x_2x_1)b, \\
 & & & x_1(bx_2), x_1(x_2b), (bx_2)x_1, (x_2b)x_1); \\
(6) & (x_1 * x_2)\overline{*}b & = & W(x_1(x_2b), x_1(bx_2), (x_2b)x_1, (bx_2)x_1, \\
 & & & x_2(x_1b), x_2(bx_1), (x_1b)x_2, (bx_1)x_2); \\
(7) & (b_1 * b_2)\overline{*}x & = & W(b_1(b_2x), b_1(xb_2), (b_2x)b_1, (xb_2)b_1, \\
 & & & b_2(b_1x), b_2(xb_1), (b_1x)b_2, (xb_1)b_2); \\
(8) & (b_1 * x)\overline{*}b_2 & = & W(b_1(xb_2), b_1(b_2x), (xb_2)b_1, (b_2x)b_1, \\
 & & & x(b_1b_2), x(b_2b_1), (b_1b_2)x, (b_2b_1)x);
\end{array}$$

where $W$ indicates the same word in **Axiom 2** corresponding to the choice of $*$ and $\overline{*}$.

Observe that axioms 1–4 above come from **Axiom 1**, while axioms 5–8 come from **Axiom 2** by replacing each operation with the corresponding action (notice that the group action replaces the conjugation and not the group operation). These axioms are nothing but the translation of the condition that one obtains by considering the equivalence between actions and points and expressing the action as the conjugation into the semidirect product. More explicitly, given a split extension:

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

the corresponding action is given by:

$$b \cdot x = k^{-1}(s(b) + k(x) - s(b));$$
$$b * x = k^{-1}(s(b) * k(x)).$$

A wider class of semi-abelian varieties is given by *groups with operations* introduced by Porter in [23]. In that class, the description of internal actions is similar to the one given above; axioms 1–8 are replaced by suitable ones coming from the identities of the corresponding algebraic theory.

## 4. Commutators

Another classical property of characteristic subgroups of a group is the fact that the commutator of two characteristic subgroups is characteristic as well. In order to study this property in a categorical setting, we will use an intrinsic definition of the commutator of two subobjects. There are different possible definitions. The first we consider is the so-called Huq commutator [17]. It can be constructed in the following way (see [1] and [19]): given two subobjects $h\colon H \rightarrowtail G$ and $k\colon K \rightarrowtail G$ of an object $G$, the Huq commutator

$[H,K]_G$ of $H$ and $K$ is given by the following diagram:

$$\begin{tikzcd} & H+K \arrow[r, "\Sigma_{H,K}"] \arrow[d, "{[h,k]}"'] & H\times K \arrow[d] \\ {[H,K]_G} \arrow[r, tail] & G \arrow[r, "\pi"'] & \frac{G}{[H,K]_G}, \end{tikzcd}$$

where $\Sigma_{H,K}$ is the canonical map

$$\Sigma_{H,K} = \langle [1,0],[0,1]\rangle = [\langle 1,0\rangle, \langle\ 0,1\rangle] \colon H+K \to H\times K$$

from the coproduct to the product and the commutative square is a pushout. Then the Huq commutator appears as the kernel of the morphism $\pi$. Being a kernel, the Huq commutator is always a normal subobject, even if $H$ and $K$ are not.

Another possible way to define the commutator is via the so-called Higgins commutator [19]. Given two subobjects $h\colon H \rightarrowtail G$ and $k\colon K \rightarrowtail G$ of an object $G$, let us denote by $\sigma_{H,K}\colon H\diamond K \to H+K$ the kernel of the canonical morphism $\Sigma_{H,K}\colon H+K \to H\times K$. The Higgins commutator $[H,K]$ of $H$ and $K$ is the regular image of $H\diamond K$ under the morphism $[h,k]\sigma_{H,K}$, as in the following diagram:

$$\begin{tikzcd} H\diamond K \arrow[r, two heads] \arrow[d, tail, "\sigma_{H,K}"'] & {[H,K]} \arrow[d, tail] \\ H+K \arrow[r, "{[h,k]}"'] & G. \end{tikzcd}$$

The Higgins commutator of $H$ and $K$ is not necessarily a normal subobject of $G$, even when $H$ and $K$ are. In fact, its normalisation in $G$ is the Huq commutator. A category $\mathcal{C}$ is said to satisfy the (NH) property when the Higgins commutator of two normal subobjects is normal, or, in other words, when Higgins and Huq commutators of normal subobjects coincide. The (NH) property is satisfied both by (LACC) categories and by categories of interest (see [13]).

Let us observe that in the special case where $H = K = G$ and $h = k = 1_G$, $[G,G]$ is always normal in $G$, since the map $[1,1]\colon G+G \to G$ is a regular epimorphism and in the semi-abelian context regular images of normal subobjects along regular epimorphisms are normal.

Let us now start the study of the Huq commutator of two characteristic subobjects.

**Proposition 4.1.** *Let $\mathcal{C}$ be a semi-abelian category satisfying the following properties:*

(1) *the kernel functors* $\mathrm{Ker}\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ *preserve jointly strongly epimorphic pairs;*

(2) *the kernel functors* $\mathrm{Ker}\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ *preserve cokernels.*

*If $H$ and $K$ are characteristic subobjects of $G$, then the Huq commutator $[H,K]_G$ is a characteristic subobject of $G$.*

*Proof*: If $H$ and $K$ are characteristic subobjects of $G$, then, for every action $\xi\colon B\flat G \to G$, there is a cospan in $\mathsf{Pt}_B(\mathcal{C})$:

$$\begin{tikzcd}
H \arrow[r, "k_1"] \arrow[d, tail, "h"'] & Y \arrow[r, shift left, "p_1"] \arrow[d, tail] & B \arrow[l, shift left, "s_1"] \arrow[d, equal] \\
G \arrow[r, "i_G"] & G \rtimes_\xi B \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"] \\
K \arrow[r, "k_2"] \arrow[u, tail, "k"] & Z \arrow[r, shift left, "p_2"] \arrow[u, tail] & B \arrow[l, shift left, "s_2"] \arrow[u, equal]
\end{tikzcd}$$

The product $(Y,p_1,s_1)\times(Z,p_2,s_2)$ in $\mathsf{Pt}_B(\mathcal{C})$ has $H\times K$ as kernel. As already explained in the proof of the Lemma 3.1, the kernel $N$ of the coproduct $(Y,p_1,s_1)+(Z,p_2,s_2)$ is different, in general, from $H+K$; however, under the assumption 1, the canonical map $u\colon H+K\to N$ is a regular epimorphism. Now, consider the following commutative diagram, where $\alpha$ is the arrow induced on kernels by the canonical morphism $(Y,p_1,s_1)+(Z,p_2,s_2)\to(Y,p_1,s_1)\times(Z,p_2,s_2)$ in $\mathsf{Pt}_B(\mathcal{C})$, $\beta$ is induced by $(Y,p_1,s_1)+(Z,p_2,s_2)\to(G\rtimes B,p_B,i_B)$, and $j=\ker(\alpha)$:

$$\begin{tikzcd}
H \diamond K \arrow[r, "\sigma_{H,K}"] \arrow[d, "v"'] & H+K \arrow[r, "\Sigma_{H,K}"] \arrow[d, "u"] & H\times K \arrow[d, equal] \\
M \arrow[r, "j"] \arrow[d] & N \arrow[r, "\alpha"] \arrow[d, "\beta"] \arrow[ur, phantom, "\ulcorner"] & H\times K \arrow[d] \\
{[H,K]_G} \arrow[r] & G \arrow[r] \arrow[ur, phantom, "\ulcorner"] & G/[H,K]_G
\end{tikzcd}$$

The arrow $v\colon H\diamond K\to M$ is a regular epimorphism, thanks to the short five lemma. The Huq commutator $[H,K]_G$ is defined as the kernel of the pushout of $\Sigma_{H,K}$ along $\beta u=[h,k]$. Moreover, $G/[H,K]_G=\mathrm{Coker}(\beta u\sigma_{H,K})$

by composition of pushouts, and, as $v$ is a regular epimorhism, we also have $G/[H,K]_G = \mathrm{Coker}(\beta j)$.

Remembering that kernel functors preserve kernels, $M$ is the kernel of the object in $\mathsf{Pt}_B(\mathcal{C})$ defined as the kernel of the arrow $(Y,p_1,s_1)+(Z,p_2,s_2) \to (Y,p_1,s_1)\times(Z,p_2,s_2)$, so $\beta j$ is the arrow induced on kernels by an arrow in $\mathsf{Pt}_B(\mathcal{C})$. Now, by hypothesis 2, the kernel functors preserve cokernels, so that $G/[H,K]_G$ turns out to be the kernel of a cokernel in $\mathsf{Pt}_B(\mathcal{C})$. In particular, this means that there is an action of $B$ on $G/[H,K]_G$ induced by the one on $G$. As a consequence, we also have an action of $B$ on $[H,K]_G$, again because the kernel functors preserve kernels. ∎

**Corollary 4.2.** *Let $\mathcal{C}$ be a semi-abelian category satisfying the following properties:*

(1) *the kernel functors* $\mathrm{Ker}\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ *preserve jointly strongly epimorphic pairs;*
(2) *the kernel functors* $\mathrm{Ker}\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ *preserve cokernels.*

*The derived subobject $[G,G]$ is characteristic in $G$.*

**Corollary 4.3.** *Let $\mathcal{C}$ be either a semi-abelian* (LACC) *category or a* category of interest. *If $H$ and $K$ are characteristic subobjects of $G$, then the Huq commutator $[H,K]_G$ is a characteristic subobject of $G$.*

*Proof*: This depends on the fact that both classes of categories satisfy the conditions of Proposition 4.1.

This is obvious in the case of (LACC) categories. For *categories of interest*, it is proved in [12]. ∎

An analogous result can be stated for the Higgins commutator.

**Proposition 4.4.** *Let $\mathcal{C}$ be a semi-abelian category where the kernel functors* $\mathrm{Ker}\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ *preserve jointly strongly epimorphic pairs. If $H$ and $K$ are characteristic subobjects of $G$, then the Higgins commutator $[H,K]$ is a characteristic subobject of $G$.*

*Proof*: The result is a straightforward consequence of Proposition 6.2 in [13]. ∎

As a consequence, we have:

**Corollary 4.5.** *Under the assumptions of the previous proposition, if $H$ and $K$ are characteristic subobjects of an object $X$ in $\mathcal{C}$ then the Huq commutator and the Higgins commutator of $H$ and $K$ coincide.*

In the category of (not necessarily unitary) rings, given a ring $X$ and two subrings $H$ and $K$, the commutator $[H, K]$ is nothing but the subring $HK$ of $X$ generated by $H$ and $K$. Hence Proposition 4.4 says that, if $H$ and $K$ are characteristic, $HK$ also is. The same happens in the category of Lie algebras (over a commutative ring $R$), where the commutator $[H, K]$ of two subalgebras is the Lie subalgebra generated by $H$ and $K$.

**Remark 4.6.** When the category $\mathcal{C}$ satisfies (NH), Propositions 4.1 and 4.4 are both consequences of Proposition 3.3 in [13], where it is shown that, in the semi-abelian context, the property

$$H, K \text{ characteristic in } X \quad \Rightarrow \quad [H, K] \text{ characteristic in } X$$

can be deduced directly from (NH).

The fact that the Huq (or the Higgins) commutator of two characteristic subobjects is characteristic is not true in a general semi-abelian category. Not even the derived subobject of an object (which is the same in the Higgins or in the Huq sense) is characteristic in general, as the following example shows.

**Example 4.7.** Let us consider the category $\mathsf{NARng}$ of not necessarily associative rings, i.e. abelian groups with a binary operation which is distributive w.r.t. the group operation. Let $G$ be the object in $\mathsf{NARng}$ given by the free abelian group on two generators $G = \mathbb{Z}x + \mathbb{Z}y$, endowed with a distributive binary operation, defined on generators as:

$$\begin{array}{c|cc} * & x & y \\ \hline x & x & 0 \\ y & 0 & 0 \end{array}$$

Then the derived subobject $[G, G] = \mathbb{Z}x$ is an ideal (i.e. a normal subobject) of $G$, but it is not characteristic in $G$. Indeed, if we consider the object given by the abelian group $\mathbb{Z}$ with trivial multiplication, $[G, G]$ is not stable under the following action of $\mathbb{Z}$ over $G$:

$$\begin{aligned} \mathbb{Z} \times G \to G\,, &\quad z * (\alpha x + \beta y) = (z\beta)x + (z\alpha)y\,, \\ G \times \mathbb{Z} \to G\,, &\quad (\alpha x + \beta y) * z = (z\beta)x + (z\alpha)y\,. \end{aligned}$$

We emphasize that $G$ is, in fact, an associative ring, but the present is not a counterexample for the category $\mathsf{Rng}$ of rings, since the one described above is an action in $\mathsf{NARng}$ but not in $\mathsf{Rng}$. Indeed, according to the explicit description of actions recalled at the end of Section 3, an action of $\mathbb{Z}$ over $G$ in $\mathsf{NARng}$ is just a pair of bilinear maps $\mathbb{Z} \times G \to G$ and $G \times \mathbb{Z} \to G$, while an action in Rng must also satisfy some "associativity" axioms. In the example above, the axiom

$$z * (xx) = (z * x)x$$

is not satisfied, indeed $z * (xx) = z * x = zy$, while $(z * x)x = (zy)x = 0$.

## 5. Centres and centralisers

Given a characteristic subgroup $H$ of a group $G$, its centraliser $Z_G(H)$ is characteristic, too. In particular, the centre of a group is always a characteristic subgroup. This is not true in any semi-abelian category, as we will show later, so we need to consider further hypotheses on the category in order to get this property. In a semi-abelian category $\mathcal{C}$, given a subobject $H$ of an object $G$, the centraliser of $H$ in $G$ is the largest subobject $Z_G(H)$ of $G$ such that the Huq commutator $[H, Z_G(H)]_G$ vanishes. The centre of an object $G$ is the largest subobject $Z(G)$ of $G$ such that $[G, Z(G)] = 0$.

The centres and centralisers do not always exist in a semi-abelian category, and even when they exist, they can be difficult to handle. Bourn and Janelidze introduced in [10] a categorical context, namely *action accessible categories*, in which the centres and the centralisers have an easy description. We recall now the definition of action accessible categories and their basic properties.

Let $\mathcal{C}$ be a semi-abelian category. Fixed an object $K \in \mathcal{C}$, a split extension with kernel $K$ is a diagram

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd},$$

such that $ps = 1_B$ and $k = \mathrm{Ker}(p)$. We denote such a split extension by $(B, A, p, s, k)$. Given another split extension $(D, C, q, t, l)$ with the same kernel $K$, a morphism of split extensions

$$(g, f) \colon (B, A, p, s, k) \longrightarrow (D, C, q, t, l) \tag{2}$$

is a pair $(g, f)$ of morphisms:

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift left, "p"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\ K \arrow[r, "l"] & C \arrow[r, shift left, "q"] & D \arrow[l, shift left, "t"] \end{tikzcd} \tag{3}$$

such that $l = fk$, $qf = gp$ and $fs = tg$. Let us notice that, since the category $\mathcal{C}$ is protomodular, the pair $(k, s)$ is jointly (strongly) epimorphic, and then the morphism $f$ in (3) is uniquely determined by $g$.

The split extensions with fixed kernel $K$ form a category, denoted by $\mathsf{SplExt}_{\mathcal{C}}(K)$, or simply by $\mathsf{SplExt}(K)$.

**Definition 5.1** ([10]).

- *An object in* $\mathsf{SplExt}(K)$ *is said to be* faithful *if any object in* $\mathsf{SplExt}(K)$ *admits at most one morphism into it.*
- *Split extensions with a morphism into a faithful one are called* accessible.
- *If, for any* $K \in \mathcal{C}$, *every object in* $\mathsf{SplExt}(K)$ *is accessible, we say that the category* $\mathcal{C}$ *is* action accessible.

In the case of groups, faithful extensions are those inducing a group action of $B$ on $K$ (via conjugation in $A$) which is faithful. Every split extension in $\mathsf{Gp}$ is accessible and a morphism into a faithful one can be performed by taking the quotients of $B$ and $A$ over the centraliser $Z_B(K)$, i.e. the (normal) subobject of $A$ given by those elements of $B$ that commute in $A$ with every element of $K$.

In [21] it is proved that any category of interest in the sense of [22] is action accessible. Examples of action accessible categories are then groups, rings, associative algebras, Lie algebras, Leibniz algebras and Poisson algebras, as mentioned before.

In the context of action accessible categories it is easy to describe the centraliser of a normal subobject. We give now a brief description of the construction, without proof (that can be found, for example, in [14]). Let $x\colon X \to A$ be a normal subobject of $A$, and let $R[p]$ be the equivalence relation on $A$ induced by $X$ (i.e. the kernel pair of the quotient $p\colon A \to A/X$). Consider the following morphism of split extensions, where the codomain is a faithful one (it exists because the category is action accessible):

$$
\begin{tikzcd}
X \arrow[r, "\langle x,0\rangle"] \arrow[d, equal] & R[p] \arrow[r, shift left, "r_0"] \arrow[d, "f"'] & A \arrow[l, shift left, "s_0"] \arrow[d, "g"] \\
X \arrow[r, "k"'] & C \arrow[r, shift left, "q"] & D. \arrow[l, shift left, "t"]
\end{tikzcd}
$$

Then the kernel of $g$ is the centraliser $Z_A(X)$ of $X$ in $A$. This implies, in particular, that in an action accessible category the centraliser of a normal subobject is normal [14, Corollary 2.6], which is not always the case in general semi-abelian categories, even when $Z_A(X)$ exists (see examples in [14]).

We are now ready to prove that, in the context of action accessible categories, the centraliser of a characteristic subobject is characteristic.

**Lemma 5.2** ([13]). *Consider a split extension as in the bottom row of the diagram*

$$\begin{tikzcd}
K \arrow[r, dashed] \arrow[d, tail, "k"'] & K' \arrow[r, dashed, shift left] \arrow[d, dashed] & Z \arrow[l, dashed, shift left] \arrow[d, equal] \\
X \arrow[r, tail, "x"'] & Y \arrow[r, shift left, "f"] & Z \arrow[l, shift left, "s"]
\end{tikzcd}$$

*such that $xk$ is normal. Then this split extension lifts along $k\colon K \to X$ to yield a normal monomorphism of split extensions.*

*Proof*: The needed lifting is obtained via the pullback of split extensions in the diagram

$$\begin{tikzcd}
& K \arrow[rr, dashed, tail] \arrow[dl, equal] \arrow[dd, tail, "k"] & & K' \arrow[rr, dashed, shift left] \arrow[dl] \arrow[dd, dashed] & & Z \arrow[ll, dashed, shift left] \arrow[dl, "s"'] \arrow[dd, dashed, equal] \\
K \arrow[rr, tail] \arrow[dd, tail, "xk"'] & & R \arrow[rr, shift left, "r_1"] \arrow[dd, "\langle r_1{,}r_2\rangle"'] & & Y \arrow[ll, shift left] \arrow[dd, equal] & \\
& X \arrow[rr, tail, "x"'] \arrow[dl, tail, "x"] & & Y \arrow[rr, shift left, "f"] \arrow[dl, "\langle sf{,}1_Y\rangle"] & & Z \arrow[ll, shift left, "s"] \arrow[dl, "s"] \\
Y \arrow[rr, tail, "\langle 0{,}1_Y\rangle"'] & & Y \times Y \arrow[rr, shift left, "\pi_1"] & & Y \arrow[ll, shift left, "\langle 1_Y{,}1_Y\rangle"] &
\end{tikzcd}$$

where $R$ is the denormalisation ([6, 1]) of $xk$. ■

**Lemma 5.3.** *Let $\mathcal{C}$ be a semi-abelian category where, for every normal subobject $H \lhd G$, the centraliser $Z_G(H)$ of $H$ in $G$ is normal in $G$. Then if $G$ is a normal subobject of $G'$, $Z_G(H)$ is also normal in $G'$.*

*Proof*: By definition of centraliser, $Z_G(H)$ is the largest subobject of $G$ such that $[H, Z_G(H)]_{G'} = 0$. Hence, it is contained in both $G$ and $Z_{G'}(H)$, and it is the largest with this property, so it is defined by the following pullback:

$$\begin{tikzcd} Z_G(H) \arrow[r, tail] \arrow[d, tail] & Z_{G'}(H) \arrow[d, tail] \\ G \arrow[r, tail] & G' \end{tikzcd}$$

In other words, $Z_G(H) = Z_{G'}(H) \wedge G$ and it is normal in $G$ as intersection of two normal subobjects. ∎

**Proposition 5.4.** *Let $\mathcal{C}$ be a semi-abelian category where, for every normal subobject $H \lhd G$, the centraliser $Z_G(H)$ of $H$ in $G$ is normal in $G$. Then if $H$ is a characteristic subobject of $G$, $Z_G(H)$ is also characteristic in $G$.*

*Proof*: Consider an object $B$ and an action $\xi \colon B \flat G \to G$. $G$ is a normal subobject of $G \rtimes_\xi B$; so, being characteristic in $G$, $H$ is normal in $G \rtimes_\xi B$ by Proposition 2.4. Hence, by Lemma 5.3, $Z_G(H)$ is a normal subobject of $G \rtimes_\xi B$. Now, we can apply Lemma 5.2 to the following situation:

$$\begin{tikzcd} & Z_G(H) \arrow[d, tail] \arrow[dr, tail] & & & \\ 0 \arrow[r] & G \arrow[r, tail, "i_G"'] & G \rtimes_\xi B \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"] \arrow[r] & 0 \end{tikzcd}$$

thus obtaining a morphism of split extensions:

$$\begin{tikzcd} 0 \arrow[r] & Z_G(H) \arrow[r, tail] \arrow[d, tail] & Z_{G \rtimes_\xi B}(G) \arrow[r, shift left] \arrow[d, tail] & B \arrow[l, shift left] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & G \arrow[r, tail, "i_G"'] & G \rtimes_\xi B \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"] \arrow[r] & 0 \end{tikzcd}$$

which gives the desired action $\xi' \colon B \flat Z_G(H) \to Z_G(H)$ as a restriction of the action $\xi$. ∎

**Corollary 5.5.** *Let $\mathcal{C}$ be a semi-abelian category where, for every normal subobject $H \lhd G$, the centraliser $Z_G(H)$ of $H$ in $G$ is normal in $G$. Then the centre $Z(G)$ is a characteristic subobject of $G$.*

In the category of (not necessarily unitary) rings, given an ideal $H$ of a ring $G$, the centraliser $Z_G(H)$ is the annihilator of $H$ in $G$, i.e.

$$Z_G(H) = \{g \in G \mid gh = hg = 0 \text{ for all } h \in H\}.$$

Hence, if $H$ is characteristic in $G$, then the annihilator of $H$ in $G$ is characteristic, as well. In particular, for any ring $G$, the annihilator of $G$ is a characteristic ideal of $G$. The same happens in the category of Lie algebras over a commutative ring $R$.

Proposition 5.4 and Corollary 5.5 are true, in particular, in semi-abelian action accessible categories. However, they do not hold in any semi-abelian category. The following is a counterexample.

**Example 5.6.** Let us consider again the category $\mathsf{NARng}$ of not necessarily associative rings and the object $G$ in $\mathsf{NARng}$ described in Example 4.7. The centre $Z(G) = \mathbb{Z}y$ is an ideal (i.e. a normal subobject) of $G$, but it is not characteristic in $G$, since it is not stable under the action of $\mathbb{Z}$ over $G$ described in the same example.

## 6. Induced morphisms between actors

In the category $\mathsf{Gp}$ of groups, if $H$ is a characteristic subgroup of $G$, then there are induced morphisms $\mathrm{Aut}(G) \to \mathrm{Aut}(H)$ and $\mathrm{Aut}(G) \to \mathrm{Aut}(G/H)$. This comes from the fact that actions on $G$ (which are equivalent to split extensions with kernel $G$, as already observed) are represented by the automorphism group $\mathrm{Aut}(G)$, in the sense that an action of a group $B$ on $G$ can be described simply as a group homomorphism $B \to \mathrm{Aut}(G)$. We are going to show that the same induced morphisms exist in a context in which internal actions (which are equivalent to split extensions in a semi-abelian category) are representable. Categories in which this happens are called *action representative* [4, 2]. We now recall the definition of an action representative category.

**Definition 6.1** ([4]). *A semi-abelian category $\mathcal{C}$ is action representative if, for any object $X \in \mathcal{C}$, there exists an object* $\mathrm{Act}(X)$*, called the* actor *of $X$, and a split extension*

$$X \longrightarrow X \rtimes \mathrm{Act}(X) \rightleftarrows \mathrm{Act}(X) \ ,$$

*called the* split extension classifier *of $X$, such that, for any split extension with kernel $X$:*

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

*there exists a unique morphism $\varphi\colon B \to \mathrm{Act}(X)$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift left, "p"] \arrow[d, "\varphi_1"'] & B \arrow[l, shift left, "s"] \arrow[d, "\varphi"] \\ X \arrow[r] & X \rtimes \mathrm{Act}(X) \arrow[r, shift right] & \mathrm{Act}(X) \arrow[l, shift right] \end{tikzcd}$$

*where the morphism $\varphi_1$ is uniquely determined by $\varphi$ and the identity on $X$ (since $k$ and $s$ are jointly strongly epimorphic).*

Examples of action representative categories are the category $\mathsf{Gp}$ of groups, where the actor is the group of automorphisms, and the category $\mathsf{Lie}$ of Lie algebras over a commutative ring $R$, where the actor of a Lie algebra $X$ is the Lie algebra $\mathrm{Der}(X)$ of derivations of $X$.

It is well-known that the assignment $G \mapsto \mathrm{Act}(G)$ is not functorial. Nevertheless, it behaves well with respect to characteristic subobjects.

**Proposition 6.2.** *Let $\mathcal{C}$ be a semi-abelian action representative category. Every characteristic subobject $h\colon H \rightarrowtail G$ induces a morphism between split extension classifiers:*

$$\begin{tikzcd} G \arrow[r, tail] \arrow[d, "q"', two heads] & G \rtimes \mathrm{Act}(G) \arrow[r, shift left] \arrow[d] & \mathrm{Act}(G) \arrow[l, shift left] \arrow[d] \\ G/H \arrow[r, tail] & G/H \rtimes \mathrm{Act}(G/H) \arrow[r, shift left] & \mathrm{Act}(G/H) \arrow[l, shift left] \end{tikzcd} \tag{4}$$

*and a morphism between actors:* $\mathrm{Act}(G) \to \mathrm{Act}(H)$*.*

*Proof*: As explained in Section 2, if $H$ is a characteristic subobject of $G$, then, for every action $\xi\colon B\flat G \to G$, there exists an exact sequence in $\mathsf{Pt}_B(\mathcal{C})$:

$$\begin{tikzcd} H \arrow[r, tail] \arrow[d, tail, "h"'] & Y \arrow[r, shift left] \arrow[d, tail, "\phi"] & B \arrow[l, shift left] \arrow[d, equal] \\ G \arrow[r, tail] \arrow[d, two heads, "q"'] & X \arrow[r, shift left] \arrow[d, two heads, "\gamma"] & B \arrow[l, shift left] \arrow[d, equal] \\ G/H \arrow[r, tail] & Z \arrow[r, shift left] & B \arrow[l, shift left] \end{tikzcd}$$

Since the category $\mathcal{C}$ is action representative, we can choose, in particular, $B = \mathrm{Act}(G)$ and the middle row to be the split extension classifier of $G$. Thus, thanks to Proposition 2.8, we have a morphism in $\mathsf{Pt}_{\mathrm{Act}(G)}(\mathcal{C})$:

$$\begin{tikzcd}
G \arrow[r, tail] \arrow[d, two heads] & G \rtimes \mathrm{Act}(G) \arrow[r, shift left] \arrow[d] & \mathrm{Act}(G) \arrow[l, shift left] \arrow[d, equal] \\
G/H \arrow[r, tail] & Z \arrow[r, shift left] & \mathrm{Act}(G) \arrow[l, shift left]
\end{tikzcd}$$

By composing with the arrow to the split extension classifier of $G/H$, we get the desired morphism (4).

For the same reason, we also have a morphism:

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, tail, "h"'] & Y \arrow[r, shift left] \arrow[d, tail] & \mathrm{Act}(G) \arrow[l, shift left] \arrow[d, equal] \\
G \arrow[r, tail] & G \rtimes \mathrm{Act}(G) \arrow[r, shift left] & \mathrm{Act}(G) \arrow[l, shift left]
\end{tikzcd}$$

The arrow from the upper split extension to the split extension classifier of $H$ produces the morphism $\mathrm{Act}(G) \to \mathrm{Act}(H)$. ■

It is worth translating the above proposition in terms of internal actions. The first assertion says that there exists a morphism $\widetilde{q}\colon \mathrm{Act}(G) \to \mathrm{Act}(G/H)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Act}(G)\flat G \arrow[r, "\widetilde{q}\flat q"] \arrow[d, "\zeta_G"'] & \mathrm{Act}(G/H)\flat(G/H) \arrow[d, "\zeta_{G/H}"] \\
G \arrow[r, two heads, "q"'] & G/H
\end{tikzcd}$$

where $\zeta_G$ and $\zeta_{G/H}$ are the canonical actions of the actors. On the other hand, the second statement says that there exists a morphism $\overline{h}\colon \mathrm{Act}(G) \to \mathrm{Act}(H)$ making this triangle commute:

$$\begin{tikzcd}
\mathrm{Act}(G)\flat H \arrow[r, "\overline{h}\flat 1"] \arrow[dr, "\overline{\zeta_G}"'] & \mathrm{Act}(H)\flat H \arrow[d, "\zeta_H"] \\
& H
\end{tikzcd}$$

where $\overline{\zeta_G}$ is the action on $H$ induced by $\zeta_G$ and $\zeta_H$ is the canonical action of the actor.

Let us observe that any action representative category is action accessible: indeed, it is easy to see that the split extension classifier is a faithful split extension. On the other hand, the category $\mathsf{Rng}$ of rings is action accessible [10] but not action representative. In the case of action accessible categories, one cannot recover the same properties described above for action representative categories, because there can be many faithful split extensions associated with a given one. However, as observed in [14], there always exists a canonical faithful split extension associated with a given one, and it has properties analogous to the ones described above.

Given a split extension

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

in a regular action accessible category, and a morphism of split extensions with faithful codomain:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift left, "p"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\ X \arrow[r] & C \arrow[r, shift left, "q"] & D \arrow[l, shift left, "t"] \end{tikzcd}$$

the canonical (regular epi, mono) factorization gives rise to another faithful split extension:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift left, "p"] \arrow[d, "e_f"'] & B \arrow[l, shift left, "s"] \arrow[d, "e_g"] \\ X \arrow[r] \arrow[d, equal] & T_1 \arrow[r, shift left] \arrow[d, tail, "m_f"'] & T_0 \arrow[l, shift left] \arrow[d, tail, "m_g"] \\ X \arrow[r] & C \arrow[r, shift left, "q"] & D. \arrow[l, shift left, "t"] \end{tikzcd}$$

The important fact here is that the faithful split extension in the middle of the previous diagram does not depend on the choice of the lower one, so it is a canonical faithful split extension associated with $(A, B, p, s)$. The object $T_0$ is actually the quotient $B/Z_B(X)$ of $B$ over the centraliser of $X$ in $B$ (i.e. the largest subobject of $B$ commuting with $X$ in $A$), while $T_1$ is the quotient $A/Z_B(X)$.

As above, let $H$ be a characteristic subobject of $G$. Then, for every action $\xi\colon B\flat G \to G$, there exists an exact sequence in $\mathsf{Pt}_B(\mathcal{C})$ as in diagram (1).

Let

$$\begin{tikzcd}
G \arrow[r] \arrow[d, equal] & X \arrow[r, shift left] \arrow[d] & B \arrow[l, shift left] \arrow[d] \\
G \arrow[r] & T_1(B,G,\xi) \arrow[r, shift left] & T_0(B,G,\xi) \arrow[l, shift left]
\end{tikzcd}$$

be the morphism onto the canonical faithful split extension (and similarly for the induced split extensions of kernels $H$ and $G/H$).

**Proposition 6.3.** *Let $\mathcal{C}$ be a semi-abelian action accessible category. Every characterstic subobject $h\colon H \rightarrowtail G$ induces a morphism between canonical faithful split extensions:*

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "q"'] & T_1(B,G,\xi) \arrow[r, shift left] \arrow[d] & T_0(B,G,\xi) \arrow[l, shift left] \arrow[d] \\
G/H \arrow[r] & T_1(B,G/H,\widetilde{\xi}) \arrow[r, shift left] & T_0(B,G/H,\widetilde{\xi}) \arrow[l, shift left]
\end{tikzcd} \tag{5}$$

*and a morphism: $T_0(B,G,\xi) \to T_0(B,H,\overline{\xi})$.*

*Proof*: As explained above, the object $T_0(B,G,\xi)$ is nothing but the quotient $B/Z_G(B)$, and $T_1(B,G,\xi) \cong X/Z_G(B)$, and similarly for $T_i(B,H,\overline{\xi})$ and $T_i(B,G/H,\widetilde{\xi})$. The desired morphism (5) will be the bottom rectangle in the following commutative diagram:

$$\begin{tikzcd}
& G \arrow[rr] \arrow[dl, "q"'] \arrow[dd, equal] & & X \arrow[rr, shift left] \arrow[dl, "\gamma"] \arrow[dd] & & B \arrow[ll, shift left] \arrow[dl, equal] \arrow[dd] \\
G/H \arrow[rr] \arrow[dd, equal] & & Z \arrow[rr, shift left] \arrow[dd] & & B \arrow[ll, shift left] \arrow[dd] & \\
& G \arrow[rr] \arrow[dl, "q"] & & T_1(B,G,\xi) \arrow[rr, shift left] \arrow[dl, dashed, "q_1"] & & T_0(B,G,\xi) \arrow[ll, shift left] \arrow[dl, dashed, "q_0"] \\
G/H \arrow[rr] & & T_1(B,G/H,\widetilde{\xi}) \arrow[rr, shift left] & & T_0(B,G/H,\widetilde{\xi}) \arrow[ll, shift left] &
\end{tikzcd}$$

It is constructed as follows. By definition, the centraliser $Z_G(B)$ is such that $[G, Z_G(B)]_X = 0$. Composing with $\gamma$, we also have $[G/H, Z_G(B)]_Z = 0$, so that $Z_G(B) \leq Z_{G/H}(B)$, and this induces the arrow $q_0$ between the corresponding cokernels over $B$. On the other hand, $q_1$ is the arrow which completes the following morphism of short exact sequences:

$$\begin{tikzcd}
Z_G(B) \arrow[r, tail] \arrow[d, tail] & X \arrow[r, two heads] \arrow[d, two heads, "\gamma"] & T_1(B, G, \xi) \arrow[d, dashed, two heads, "q_1"] \\
Z_{G/H}(B) \arrow[r, tail] & Z \arrow[r, two heads] & T_1(B, G/H, \widetilde{\xi})
\end{tikzcd}$$

To prove the second assertion, consider the morphism below in $\mathsf{Pt}_B(\mathcal{C})$:

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, tail, "h"'] & Y \arrow[r, shift left] \arrow[d, tail, "\phi"] & B \arrow[l, shift left] \arrow[d, equal] \\
G \arrow[r, tail] & X \arrow[r, shift left] & B \arrow[l, shift left]
\end{tikzcd}$$

By definition, $[G, Z_G(B)]_X = 0$ and, as a consequence, $[H, Z_G(B)]_X = 0$. Since $\phi$ is monomorphic, this implies that $[H, Z_G(B)]_Y = 0$ too, hence $Z_G(B) \leq Z_H(B)$. The morphism

$$T_0(B, G, \xi) \to T_0(B, H, \overline{\xi})$$

is the one induced on the corresponding cokernels over $B$. ■

## 7. Summarising table

We conclude this paper by displaying, in the following table, a list of the properties of characteristic subobjects we proved. In the second column, a categorical context is indicated for each property to hold. In many cases, it is not the most general one; possible extensions to wider contexts are suggested by the proofs.

| Property | True in | Reference |
|---|---|---|
| $H \underset{\text{char}}{<} G \quad \Rightarrow \quad H \lhd G$ | $\mathcal{C}$ semi-abelian | 2.5 |
| $H \underset{\text{char}}{<} K \lhd G \quad \Rightarrow \quad H \lhd G$ | $\mathcal{C}$ semi-abelian | 2.4 |
| $H \underset{\text{char}}{<} K \underset{\text{char}}{<} G \quad \Rightarrow \quad H \underset{\text{char}}{<} G$ | $\mathcal{C}$ semi-abelian | 2.3 |
| $H_i \underset{\text{char}}{<} G \quad (i \in I) \quad \Rightarrow \quad \bigwedge_{i \in I} H_i \underset{\text{char}}{<} G$ | $\mathcal{C}$ semi-abelian | 2.6 |
| $H \underset{\text{char}}{<} G$, $B$ acts on $G \quad \Rightarrow \quad B$ acts on $G/H$ | $\mathcal{C}$ semi-abelian | 2.8 |
| $H \leq K \leq G, \quad H \underset{\text{char}}{<} G, \quad K/H \underset{\text{char}}{<} G/H$ <br> $\Rightarrow \quad K \underset{\text{char}}{<} G$ | $\mathcal{C}$ semi-abelian | 2.9 |
| $H \underset{\text{char}}{<} G, \quad$ ($R$ kernel pair of $G \to G/H$) <br> $\Rightarrow \quad R$ closed under actions on $G$ | $\mathcal{C}$ semi-abelian | 2.10 |
| $H, K \underset{\text{char}}{<} G \quad \Rightarrow \quad H \vee K \underset{\text{char}}{<} G$ | $\mathcal{C}$ (LACC) <br> $\mathcal{C}$ category of interest | 3.2 |
| $[G, G] \underset{\text{char}}{<} G$ | $\mathcal{C}$ (LACC) <br> $\mathcal{C}$ category of interest | 4.2 |
| $H, K \underset{\text{char}}{<} G \quad \Rightarrow \quad [H, K] \underset{\text{char}}{<} G$ | $\mathcal{C}$ (LACC) <br> $\mathcal{C}$ category of interest | 4.1 |
| $Z(G) \underset{\text{char}}{<} G$ | $\mathcal{C}$ action accessible | 5.5 |
| $H \underset{\text{char}}{<} G \quad \Rightarrow \quad Z_G(H) \underset{\text{char}}{<} G$ | $\mathcal{C}$ action accessible | 5.4 |
| $H \underset{\text{char}}{<} G \quad \Rightarrow \quad \begin{cases} \mathrm{Act}(G) \to \mathrm{Act}(G/H) \\ \mathrm{Act}(G) \to \mathrm{Act}(H) \end{cases}$ | $\mathcal{C}$ action representative | 6.2 |

# References


[1] F. Borceux, D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its applications, vol. 566 (2004), Kluwer.

[2] F. Borceux, D. Bourn, *Split extension classifier and centrality*, Contemporary Mathematics, vol. 431 (2007), 85–104.

[3] F. Borceux, G. Janelidze, G.M. Kelly, *Internal object actions*, Commentationes Mathematicae Universitatis Carolinae 46 n.2 (2005), 235–255.

[4] F. Borceux, G. Janelidze, G.M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categories 14 (2005), 244–286.

[5] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, in Lecture Notes in Mathematics, vol. 1488 (1991), Springer-Verlag 43–62.

[6] D. Bourn, *Normal subobjects and abelian objects in protomodular categories*, J. Algebra 228 (2000), 143-164.

[7] D. Bourn, *Normal functors and strong protomodularity*, Theory Appl. Categories 7 (2000) 206-218.

[8] D. Bourn, J.R.A. Gray, *Aspects of algebraic exponentiation*, Bull. Belg. Math. Soc. Simon Stevin 19 (2012), 823–846.

[9] D. Bourn, G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. 4 (1998), No. 2, 37–46.

[10] D. Bourn, G. Janelidze, *Centralizers in action accessible categories*, Cah. Topol. Géom. Différ. Catég. 50 (2009), no. 3, 211–232.
[11] J.M. Casas, T. Datuashvili and M. Ladra, *Universal strict general actors and actors in categories of interest*, Appl. Cat. Structures 18 (2010), 85–114.
[12] A.S. Cigoli, J.R.A. Gray, T. Van der Linden, *Kernel accessible categories and normal cores*, in preparation, 2013.
[13] A.S. Cigoli, J.R.A. Gray, T. Van der Linden, *On the normality of the Higgins commutator*, in preparation, 2013.
[14] A.S. Cigoli, S. Mantovani, *Action accessibility via centralizers*, J. Pure Appl. Algebra 216 (2012), 1852–1865.
[15] A.S. Cigoli, S. Mantovani, G. Metere, *Peiffer product and Peiffer commutator for internal precrossed modules*, in preparation, 2013.
[16] J.R.A. Gray, *Algebraic exponentiation in general categories*, Appl. Cat. Structures 20 (2012), 543–567.
[17] S.A. Huq, *Commutator, nilpotency and solvability in categories*, Quart. J. Math. Oxford 19 (1968), no. 2, 363–389.
[18] G. Janelidze, L. Márki, W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2002), no. 2-3, 367–386.
[19] S. Mantovani, G. Metere, *Normalities and commutators*, J. Algebra 324 (2010), no. 9, 2568–2588.
[20] G. Metere, *A note on strong protomodularity, actions and quotients*, submitted (2013), arXiv:1306.3773v1.
[21] A. Montoli, *Action accessibility for categories of interest*, Theory Appl. Categ. 23 (2010), 7–21.
[22] G. Orzech, *Obstruction theory in algebraic categories, I*, J. Pure and Appl. Algebra 2 (1972), 287–314.
[23] T. Porter, *Extensions, crossed modules and internal categories in categories of groups with operations*, Proceedings of the Edinburgh Math. Society 30 (1987), 373–381.

Alan S. Cigoli
Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, Milano, Italy

*E-mail address*: alan.cigoli@unimi.it

Andrea Montoli
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

*E-mail address*: montoli@mat.uc.pt