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OPTIMAL PAIR OF TWO LINEAR VARIETIES

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ABSTRACT: The optimal pair of two linear varieties is considered as a best approximation problem, namely the distance between a point and the difference set of two linear varieties. The Gram determinant allows to get the optimal pair in closed form.

KEYWORDS: linear varieties, best approximation pair, approximation theory.

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1. Introduction

In this paper we deal with the problem of finding the the optimal points of two linear varieties in a finite dimensional real linear space.

The distance between two linear varieties has been dealt with in several papers: [2], [3] and [5]. In [5], only the distance is considered and the points that realize the distance are not exhibited. In [2] the best approximation points are found.

In [2] projecting equations were called into play; in [3] the difference set of two closed convex sets in $\mathbb{R}^m$ was considered.

In this paper, we formulate the problem as suggested by [3, page 196] and we use Gram theory [9, page 74], [4, page 65] to solve it.

We use results on existence and uniqueness of optimal points by considering a least norm problem of the difference set of two closed convex sets in $\mathbb{R}^m$ [3]. For the concepts, results and motivation on the study of the distance between convex sets see [3].

We endow the space $\mathbb{R}^m$ with the usual inner product $\bullet$:

$$\overrightarrow{p} \bullet \overrightarrow{q} := p_1 q_1 + p_2 q_2 + \cdots p_m q_m \,,$$

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where $\overrightarrow{p} = \left[\begin{array}{cccc} p_1 & p_2 & \cdots & p_m \end{array}\right]^T$ and $\overrightarrow{q} = \left[\begin{array}{cccc} q_1 & q_2 & \cdots & q_m \end{array}\right]^T$ and with the Euclidean norm

$$\|\overrightarrow{p}\| = \sqrt{\overrightarrow{p} \bullet \overrightarrow{p}}.$$

2. The Result

We are looking for the optimal pair of two linear varieties $V_{\vec{b}}$ and $V_{\vec{c}}$ in $\mathbb{R}^m$ defined by

$$V_{\vec{b}} := \{\vec{b} + B\vec{u} : \vec{u} \in \mathbb{R}^{l_1}\} \text{ and } V_{\vec{c}} := \{\vec{c} - C\vec{v} : \vec{v} \in \mathbb{R}^{l_2}\},$$

with $\vec{b}$ and $\vec{c}$ given vectors in $\mathbb{R}^m$.

Following [3, page 196] the distance $d(V_{\vec{b}}, V_{\vec{c}})$ between the two linear varieties $V_{\vec{b}}$ and $V_{\vec{c}}$ is obtained through the minimization of $\|A\vec{x} - \vec{d}\|$, where $\vec{d} = \vec{c} - \vec{b}$; $A = [B\ C]$ is a real $m \times (l_1 + l_2)$ matrix; and $\vec{x} = [\vec{u}^T\ \vec{v}^T]^T \in \mathbb{R}^{l_1+l_2}$.

So the distance $d(V_{\vec{b}}, V_{\vec{c}})$ between the varieties $V_{\vec{b}}$ and $V_{\vec{c}}$, may be studied using the shortest distance between a point $\vec{d}$ and the subspace Range(A), the column space of matrix A.

Besides getting the distance $d(V_{\vec{b}}, V_{\vec{c}})$, we also find the points $\vec{b}^* \in V_{\vec{b}}$ and $\vec{c}^* \in V_{\vec{c}}$ such that $d(V_{\vec{b}}, V_{\vec{c}}) = \|\vec{b}^* - \vec{c}^*\|$, that is to say $\vec{b}^*$ and $\vec{c}^*$ are the best approximation points.

In this new setting, and using the Euclidean norm, Gram theory [9] can play an important role.

Some definitions are needed.

Definition 2.1. The optimal pair of two linear varieties $\mathcal{A}$ and $\mathcal{B}$ is the pair $(\vec{a}^*, \vec{b}^*) \in \mathcal{A} \times \mathcal{B}$ satisfying $d(\mathcal{A}, \mathcal{B}) = \|\vec{a}^* - \vec{b}^*\|$, where $d(\mathcal{A}, \mathcal{B})$, the distance between $\mathcal{A}$ and $\mathcal{B}$, is defined as

$$d(\mathcal{A}, \mathcal{B}) = \inf\{\|\vec{a} - \vec{b}\| : \vec{a} \in \mathcal{A}, \vec{b} \in \mathcal{B}\}.$$

Definition 2.2. Let $y_1, y_2, \ldots, y_n$ be elements of $\mathbb{R}^m$. The $n \times n$ matrix

$$G(y_1, y_2, \ldots, y_n) = \begin{bmatrix} y_1 \bullet y_1 & y_1 \bullet y_2 & \cdots & y_1 \bullet y_n \\ y_2 \bullet y_1 & y_2 \bullet y_2 & \cdots & y_2 \bullet y_n \\ \vdots & \vdots & \vdots & \vdots \\ y_n \bullet y_1 & y_n \bullet y_2 & \cdots & y_n \bullet y_n \end{bmatrix}$$

is called the Gram matrix of $y_1, y_2, \ldots, y_n$. The determinant $g(y_1, y_2, \ldots, y_n)$ of the Gram matrix is known as the Gram determinant.

It is known, see for example [8], [9], that

$$g(y_1, y_2, \ldots, y_n) \geq 0 \quad \text{and} \quad g(y_1, y_2, \ldots, y_n) = 0$$

if and only if $y_1, y_2, \ldots, y_n$ are linearly dependent.

Proposition 2.3. *Let be given the linear varieties*

$$V_{\vec{b}} := \{\vec{b} + B\vec{u} : \vec{u} \in \mathbb{R}^{l_1}\}, \, V_{\vec{c}} := \{\vec{c} - C\vec{v} : \vec{v} \in \mathbb{R}^{l_2}\},$$

where $\vec{b}$ and $\vec{c}$ are any vectors in $\mathbb{R}^m$ and $B \in \mathbb{R}^{m \times l_1}$, $C \in \mathbb{R}^{m \times l_2}$ are fixed matrices.

Let consider $\vec{d} = \vec{c} - \vec{b}$ and assume that $A = [B \;\; C] = [\vec{a_1}\, \vec{a_2} \, \ldots \, \vec{a_n}] \in \mathbb{R}^{m \times n}$, $n = l_1 + l_2$, is a full column rank matrix. Then:

(A) *The optimal pair $(\vec{b}^*, \vec{c}^*)$ is obtained as*

$$\vec{b}^* = \vec{b} + B\vec{u}^*, \, \vec{c}^* = \vec{c} - C\vec{v}^*,$$

with

$$\begin{bmatrix} \vec{u}^* \\ \vec{v}^* \end{bmatrix} = -\frac{1}{g(\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n})} \left| \begin{array}{c|c} G(\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}) & \begin{array}{c} \vec{d} \bullet \vec{a_1} \\ \vdots \\ \vec{d} \bullet \vec{a_n} \end{array} \\ \hline \vec{a_1} \quad \vec{a_2} \quad \ldots \quad \vec{a_n} & \vec{0} \end{array} \right|, \qquad (1)$$

where the (formal) determinant is to be expanded by the last row to yield a linear combination of the columns $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ of the matrix A.

(B) *The distance between $V_{\vec{b}}$ and $V_{\vec{c}}$ is given by*

$$d(V_{\vec{b}}, V_{\vec{c}}) = \|\vec{b}^* - \vec{c}^*\|$$

and also

$$d^2(V_{\vec{b}}, V_{\vec{c}}) \;=\; \frac{g(\vec{d}, \vec{a_1}, \vec{a_2}, \ldots, \vec{a_n})}{g(\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n})}. \qquad (2)$$

Proof: From [3, page 196] we must minimize $\left\| A\overrightarrow{x} - \overrightarrow{d} \right\|$, where $\overrightarrow{x} = \begin{bmatrix} \vec{u}^* \\ \vec{v}^* \end{bmatrix}$ and this is equivalent to find the distance from the point $\overrightarrow{d}$ to the column-space Range(A) of A.

By hypothesis, the columns of the matrix A are linearly independent. Then from [9, page 74] we obtain (1) and from [4, page 65] we obtain (2). ■

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