OPTIMAL PAIR OF TWO LINEAR VARIETIES

ARMANDO GONÇALVES, M. A. FACAS VICENTE AND JOSÉ VITÓRIA

Abstract: The optimal pair of two linear varieties is considered as a best approximation problem, namely the distance between a point and the difference set of two linear varieties. The Gram determinant allows to get the optimal pair in closed form.

Keywords: linear varieties, best approximation pair, approximation theory.


1. Introduction

In this paper we deal with the problem of finding the optimal points of two linear varieties in a finite dimensional real linear space.

The distance between two linear varieties has been dealt with in several papers: [2], [3] and [5]. In [5], only the distance is considered and the points that realize the distance are not exhibited. In [2] the best approximation points are found.

In [2] projecting equations were called into play; in [3] the difference set of two closed convex sets in $\mathbb{R}^m$ was considered.

In this paper, we formulate the problem as suggested by [3, page 196] and we use Gram theory [9, page 74], [4, page 65] to solve it.

We use results on existence and uniqueness of optimal points by considering a least norm problem of the difference set of two closed convex sets in $\mathbb{R}^m$ [3]. For the concepts, results and motivation on the study of the distance between convex sets see [3].

We endow the space $\mathbb{R}^m$ with the usual inner product $\cdot$:

$$\vec{p} \cdot \vec{q} := p_1q_1 + p_2q_2 + \cdots + p_mq_m,$$

Received December 18, 2013.

The second named author is supported by Instituto de Engenharia de Sistemas e Computadores—Coimbra, Rua Antero de Quental, 199, 3000-033 Coimbra, Portugal.
where $\vec{p} = [p_1 \ p_2 \ \cdots \ p_m]^T$ and $\vec{q} = [q_1 \ q_2 \ \cdots \ q_m]^T$ and with the Euclidean norm
$$\|\vec{p}\| = \sqrt{\vec{p} \cdot \vec{p}}.$$

2. The Result

We are looking for the optimal pair of two linear varieties $V_\vec{b}$ and $V_\vec{c}$ in $\mathbb{R}^m$ defined by
$$V_\vec{b} := \{\vec{b} + B\vec{u} : \vec{u} \in \mathbb{R}^{l_1}\} \text{ and } V_\vec{c} := \{\vec{c} - C\vec{v} : \vec{v} \in \mathbb{R}^{l_2}\},$$
with $\vec{b}$ and $\vec{c}$ given vectors in $\mathbb{R}^m$.

Following [3, page 196] the distance $d(V_\vec{b}, V_\vec{c})$ between the two linear varieties $V_\vec{b}$ and $V_\vec{c}$ is obtained through the minimization of $\|A\vec{x} - \vec{d}\|$, where $\vec{d} = \vec{c} - \vec{b}$; $A = [B \ C]$ is a real $m \times (l_1 + l_2)$ matrix; and $\vec{x} = [\vec{u}^T \ \vec{v}^T]^T \in \mathbb{R}^{l_1+l_2}$.

So the distance $d(V_\vec{b}, V_\vec{c})$ between the varieties $V_\vec{b}$ and $V_\vec{c}$, may be studied using the shortest distance between a point $\vec{d}$ and the subspace $\text{Range}(A)$, the column space of matrix $A$.

Besides getting the distance $d(V_\vec{b}, V_\vec{c})$, we also find the points $\vec{b}^* \in V_\vec{b}$ and $\vec{c}^* \in V_\vec{c}$ such that $d(V_\vec{b}, V_\vec{c}) = \|\vec{b}^* - \vec{c}^*\|$, that is to say $\vec{b}^*$ and $\vec{c}^*$ are the best approximation points.

In this new setting, and using the Euclidean norm, Gram theory [9] can play an important role.

Some definitions are needed.

Definition 2.1. The optimal pair of two linear varieties $\mathcal{A}$ and $\mathcal{B}$ is the pair $(\vec{a}^*, \vec{b}^*) \in \mathcal{A} \times \mathcal{B}$ satisfying $d(\mathcal{A}, \mathcal{B}) = \|\vec{a}^* - \vec{b}^*\|$, where $d(\mathcal{A}, \mathcal{B})$, the distance between $\mathcal{A}$ and $\mathcal{B}$, is defined as
$$d(\mathcal{A}, \mathcal{B}) = \inf\{\|\vec{a} - \vec{b}\| : \vec{a} \in \mathcal{A}, \vec{b} \in \mathcal{B}\}.$$ 

Definition 2.2. Let $y_1, y_2, \ldots, y_n$ be elements of $\mathbb{R}^m$. The $n \times n$ matrix
$$G(y_1, y_2, \ldots, y_n) = \begin{bmatrix} y_1 \cdot y_1 & y_1 \cdot y_2 & \cdots & y_1 \cdot y_n \\ y_2 \cdot y_1 & y_2 \cdot y_2 & \cdots & y_2 \cdot y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n \cdot y_1 & y_n \cdot y_2 & \cdots & y_n \cdot y_n \end{bmatrix}$$
is called the Gram matrix of \( y_1, y_2, \ldots, y_n \). The determinant \( g(y_1, y_2, \ldots, y_n) \) of the Gram matrix is known as the Gram determinant.

It is known, see for example [8], [9], that
\[
g(y_1, y_2, \ldots, y_n) \geq 0 \quad \text{and} \quad g(y_1, y_2, \ldots, y_n) = 0
\]
if and only if \( y_1, y_2, \ldots, y_n \) are linearly dependent.

**Proposition 2.3.** Let be given the linear varieties
\[
V_{\vec{b}} := \{ \vec{b} + B \vec{u} : \vec{u} \in \mathbb{R}^{l_1} \}, \quad V_{\vec{c}} := \{ \vec{c} - C \vec{v} : \vec{v} \in \mathbb{R}^{l_2} \},
\]
where \( \vec{b} \) and \( \vec{c} \) are any vectors in \( \mathbb{R}^m \) and \( B \in \mathbb{R}^{m \times l_1}, C \in \mathbb{R}^{m \times l_2} \) are fixed matrices.

Let consider \( \vec{d} = \vec{c} - \vec{b} \) and assume that \( A = [B \ C] = [\vec{a}_1 \vec{a}_2 \ldots \vec{a}_n] \in \mathbb{R}^{m \times n}, n = l_1 + l_2 \), is a full column rank matrix. Then:

(A) The optimal pair \((\vec{b}^*, \vec{c}^*)\) is obtained as
\[
\vec{b}^* = \vec{b} + B \vec{u}^*, \quad \vec{c}^* = \vec{c} - C \vec{v}^*,
\]
with
\[
\begin{bmatrix}
\vec{u}^* \\
\vec{v}^*
\end{bmatrix} = -\frac{1}{g(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)} \begin{vmatrix}
G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) & \vec{d} \cdot \vec{a}_1 \\
& \vdots \\
& \vec{d} \cdot \vec{a}_n \\
\vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n & \vec{d} & \vec{0}
\end{vmatrix}
\]
(1)

where the (formal) determinant is to be expanded by the last row to yield a linear combination of the columns \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \) of the matrix \( A \).

(B) The distance between \( V_{\vec{b}} \) and \( V_{\vec{c}} \) is given by
\[
d(V_{\vec{b}}, V_{\vec{c}}) = \|\vec{b}^* - \vec{c}^*\|
\]
and also
\[
d^2(V_{\vec{b}}, V_{\vec{c}}) = \frac{g(\vec{d}, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)}{g(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)}.
\]
Proof: From [3, page 196] we must minimize \( \| A \vec{x} - \vec{d} \| \), where \( \vec{x} = \begin{bmatrix} \vec{u}^* \\ \vec{v}^* \end{bmatrix} \)
and this is equivalent to find the distance from the point \( \vec{d} \) to the column-space \( \text{Range}(A) \) of \( A \).

By hypothesis, the columns of the matrix \( A \) are linearly independent. Then from [9, page 74] we obtain (1) and from [4, page 65] we obtain (2). \( \blacksquare \)

References


