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# MONOIDS AND POINTED $S$-PROTOMODULAR CATEGORIES

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Abstract: We investigate the notion of pointed $S$-protomodular category, with respect to a suitable class $S$ of split epimorphisms, and we prove that these categories satisfy, relatively to the class $S$, many partial aspects of the properties of Mal'tsev and protomodular categories, like the split short five lemma for the $S$-split exact sequences, or the fact that a reflexive $S$-relation is transitive. The main examples of $S$-protomodular categories are the category of monoids and, more generally, any category of monoids with operations, where the class $S$ is the class of Schreier split epimorphisms.

Keywords: fibration of points, Mal'tsev and protomodular categories, monoids with operations, Schreier split epimorphisms, pointed $S$-protomodular categories.
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## 1. Introduction

The notion of semi-abelian category [10] allowed to describe intrinsically many classical properties and results in group theory (see, for example, [1]), and to point out the similarities with other algebraic structures, like rings, associative algebras, Lie algebras and many others.

From a categorical point of view, much less is known for other algebraic structures, like monoids. However, as Mac Lane observed in the preface of [11], the notion of monoid is fundamental in category theory. Until now, the most important categorical property of monoids that has been pointed out is unitality [2]; this property allows to describe the algebraic notion of commutativity of subobjects and, more generally, of morphisms.

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In the recent paper [12], the three last authors introduced the algebraic context of *monoids with operations*, inspired by the analogous notion, introduced by Porter [14], of groups with operations. This new context includes, among other examples, monoids, commutative monoids, semirings, join-semilattices with a bottom element and distributive lattices with a bottom element (or a top one). The study of the semidirect products in this setting allowed to identify a class of split epimorphisms, called *Schreier split epimorphisms* (the name is inspired by the notion of Schreier internal category introduced by Patchkoria [13] in the category of monoids). Schreier split epimorphisms correspond to actions via the semidirect product construction, as it is proved in [12].

In the monograph [7], and in the paper [6], the present authors observed moreover that, in the case of monoids, Schreier split epimorphisms satisfy some important properties that are classically known to be satisfied by all split epimorphisms of groups (but not by all split epimorphisms of monoids), like the split short five lemma. Defining a *Schreier reflexive relation* as a reflexive relation such that the split epimorphism, given by the first projection and the reflexivity morphism, is a Schreier one, it was proved that any Schreier reflexive relation is transitive. Moreover, many others interesting properties of Schreier split epimorphisms of monoids were studied, and they were extended to the case of semirings. In particular, it was shown that special Schreier extensions with a fixed abelian kernel form an abelian group, as it happens for all extensions with abelian kernel in the category of groups.

All these results gave evidence of the need of a conceptual notion which captures this algebraic context; it was introduced, in the pointed case, in [7], under the name of *S-protomodularity*, where $S$ is a suitable class of split epimorphisms. The aim of the present paper is to investigate the properties of this intrinsic setting and to show how it conceptually allows to recover many partial aspects of the properties of Mal'tsev [8] and protomodular [1] categories. By [7], the main examples of such a situation are the category $Mon$ of monoids and $SRng$ of semirings with the class $S$ of Schreier split epimorphisms. We show here that the same is true for the general context of monoids with operations.

Among other things, the definition of $S$-special morphism (see Section 6) allows to associate with any $S$-protomodular category $\mathbb{C}$ a protomodular subcategory $S^\sharp\mathbb{C}$, called the *protomodular core* of $\mathbb{C}$ relatively to $S$. If $\mathbb{C}$ is the category of monoids, equipped with the class of Schreier split epimorphisms, its protomodular core is the category of groups. This gives then a characterization of groups among monoids. In the same way, we prove that the protomodular core of the category of semirings is the category of rings, and we generalize this result to any category of monoids with operations. The notion of $S$-special morphism permits also a characterization of reflexive graphs (relatively to the class $S$) that are internal groupoids. This characterization is completely analogous to the one known for Mal'tsev categories (see [9]).

The paper is organized as follows. In Section 2 we recall from [2] the notion of unital category, and from [7] a generalization of it, namely the notion of $\mathbb{C}'$-unital category, which will be used later to describe some Mal'tsev-type properties of $S$-protomodular categories. In Section 3 we define $S$-protomodular categories and we study their first properties. In Section 4 we recall the notion of monoids with operations and of the class $S$ of Schreier split epimorphisms, and we show that they are examples of $S$-protomodular categories. In Section 5 we prove that a $S$-reflexive graph has at most one structure of internal category, and that any $S$-reflexive relation is transitive, relating $S$-protomodular categories with Mal'tsev ones. In Section 6 we define $S$-special morphisms and we use them to characterize internal groupoids among internal $S$-categories and equivalence relations among $S$-reflexive relations. Moreover, we define the protomodular core of a $S$-protomodular category, and describe it in the examples of monoids with operations. In Section 7 we describe other Mal'tsev aspects of $S$-protomodular categories, mainly related with the centrality for reflexive relations. Moreover, we show that a $S$-reflexive graph such that the domain morphism is $S$-special is an internal groupoid if and only if the kernel pairs of the domain and the codomain centralize each other.

## 2. Unital and $\mathbb{C}'$-unital categories

We start by recalling from [2] the following definition.

**Definition 2.1.** *Let $\mathbb{C}$ be a pointed category with finite products. Given two objects $A$ and $B$ in $\mathbb{C}$, consider the following diagram*

$$A \underset{\langle 1_A, 0\rangle}{\overset{\pi_A}{\leftrightarrows}} A \times B \underset{\langle 0, 1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B. \tag{1}$$

*The category $\mathbb{C}$ is said to be unital if, for every pair of objects $A, B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ are jointly strongly epimorphic.*

If moreover $\mathbb{C}$ is finitely complete, then any pair $(\langle 1_A, 0\rangle, \langle 0, 1_B\rangle)$ is jointly epimorphic. Hence finitely complete unital categories are a setting where it is possible to express a categorical notion of commutativity.

**Definition 2.2** ([3])**.** *Let $\mathbb{C}$ be a finitely complete unital category. Two morphisms with the same codomain $f\colon X \to Z$ and $g\colon Y \to Z$ are said to cooperate (or to commute) if there exists a morphism $\varphi\colon X \times Y \to Z$ such that both triangles in the following diagram commute:*

$$\begin{tikzcd}
X \arrow[r, "\langle 1_X, 0\rangle"] \arrow[dr, "f"'] & X \times Y \arrow[d, dotted, "\varphi"] & Y \arrow[l, "\langle 0, 1_Y\rangle"'] \arrow[dl, "g"] \\
 & Z. &
\end{tikzcd}$$

*The morphism $\varphi$ is necessarily unique, because $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly epimorphic, and it is called the* cooperator *of $f$ and $g$.*

The uniqueness of the cooperator makes commutativity a property and not an additional structure in the category $\mathbb{C}$.

**Definition 2.3** ([3])**.** *An object $A$ of a finitely complete unital category $\mathbb{C}$ is said to be* commutative *if the identity $1_A$ cooperates with itself.*

A generalization of the notion of unital category, that we shall need later on, is given by the following definition, that we recall from [7].

**Definition 2.4.** *Let $\mathbb{C}'$ be a full subcategory of any pointed category $\mathbb{C}$ with finite products. The category $\mathbb{C}$ is said to be* $\mathbb{C}'$-unital *when, for any object $A \in \mathbb{C}'$ and any object $B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ in the following diagram are jointly strongly epimorphic:*

$$A \underset{\langle 1_A, 0\rangle}{\overset{\pi_A}{\leftrightarrows}} A \times B \underset{\langle 0, 1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B. \tag{2}$$

In a finitely complete $\mathbb{C}'$-unital category we can still speak of cooperating pairs $(f, g)$ of morphisms, provided that the domain $X$ of $f$ belongs to $\mathbb{C}'$. More generally, $X \times Y$ being isomorphic to $Y \times X$, we can speak of cooperating pair as soon as the domain of one of the two maps is in $\mathbb{C}'$. Accordingly, we can still speak of commutative objects in $\mathbb{C}'$.

**Proposition 2.5.** *Suppose that $\mathbb{C}$ is $\mathbb{C}'$-unital and that $\mathbb{C}'$ is closed under finite products (in particular, it contains the zero object $0$). Then $\mathbb{C}'$ is unital.*

*Proof*: Straightforward. ∎

## 3. $S$-protomodular categories

From now on, we will denote by $\mathbb{C}$ a pointed finitely complete category. Let $S$ be a class of split epimorphisms in $\mathbb{C}$ which is stable under pullbacks in the following sense: given a downward pullback

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[r, "g"'] \arrow[u, shift right, "s'"'] & Y, \arrow[u, shift right, "s"'] \end{tikzcd}$$

where the two vertical morphisms are split epimorphisms and the upward square commutes (or, in other terms, the pair $(g, g')$ is a morphism of points), if $(f, s)$ belongs to $S$, then $(f', s')$ belongs to $S$, too. Accordingly this class determines a subfibration $\P^S_{\mathbb{C}}$ of the fibration of points $\P_{\mathbb{C}}\colon Pt\mathbb{C} \to \mathbb{C}$. Let us denote by $SPt\mathbb{C}$ the full subcategory of $Pt\mathbb{C}$ whose objects are those which are in $S$:

$$\begin{tikzcd} SPt\mathbb{C} \arrow[rr, tail, "j"] \arrow[dr, "\P^S_{\mathbb{C}}"'] & & Pt\mathbb{C} \arrow[dl, "\P_{\mathbb{C}}"] \\ & \mathbb{C} & \end{tikzcd}$$

Given a split epimorphism $A \underset{f}{\overset{s}{\rightleftarrows}} B$ in $\mathbb{C}$, we say that it is a *strongly split epimorphism* (see [4], and [12], where strongly split epimorphisms were introduced under the name of regular points) if the pair $(k, s)$, where $k$ is a kernel of $f$, is jointly strongly epimorphic.

**Definition 3.1.** *The pointed category $\mathbb{C}$ will be said to be $S$-*protomodular *when:*

(1) *any object in $SPt\mathbb{C}$ is a strongly split epimorphism;*

(2) *$SPt\mathbb{C}$ is closed under finite limits in $Pt\mathbb{C}$ (in particular, it contains the terminal object $0 \rightleftarrows 0$ of $Pt\mathbb{C}$).*

So, $S$ is a class of strongly split epimorphisms. The first part of condition 2 implies that any fiber $SPt_Y\mathbb{C}$ is closed under finite limits in the fiber $Pt_Y\mathbb{C}$ and that any change-of-base functor with respect to $¶_{\mathbb{C}}^S$ is left exact. The fact that $SPt\mathbb{C}$ contains the terminal object implies that the class $S$ contains the isomorphisms (because $S$ is stable under pullbacks, and any isomorphism can be seen as a pullback of the terminal object $0 \rightleftarrows 0$ of $Pt\mathbb{C}$). Hence, any fiber $SPt_Y\mathbb{C}$ is pointed.

**Theorem 3.2.** *Let $\mathbb{C}$ be a pointed finitely complete category and $S$ a class of split epimorphisms stable under pullbacks. Then:*

(1) *when $S$ satisfies the condition 1 of Definition 3.1, any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital;*

(2) *when $\mathbb{C}$ is $S$-protomodular, any fiber $SPt_Y\mathbb{C}$ is unital;*

(3) *when $\mathbb{C}$ is $S$-protomodular, any change of base functor with respect to the fibration $¶_{\mathbb{C}}^S$ is conservative.*

*Proof*: (1) Consider the following left hand side downward pullback of split epimorphisms, where the split epimorphism $(f, s)$ is in the fiber $SPt_Y\mathbb{C}$:

$$\begin{tikzcd} X' \arrow[r, shift left, "g'"'] \arrow[d, shift right, "f'"'] & X \arrow[l, shift left, "t'"'] \arrow[d, shift right, "f"'] \\ Y' \arrow[u, shift right, "s'"'] \arrow[r, shift left, "g"'] & Y \arrow[l, shift left, "t"'] \arrow[u, shift right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} K[f] \arrow[r, tail, "k"] & X \arrow[r, "t'"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ & Y \arrow[u, shift right, "s"'] \arrow[r, "t"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd} \tag{3}$$

Then $(f', s')$ belongs to $SPt_{Y'}\mathbb{C}$, since $¶_{\mathbb{C}}^S$ is a subfibration of $¶_{\mathbb{C}}$. So the split epimorphism $(f', s')$ is a strongly split epimorphism. On the other hand, the right hand side square is still a pullback, so the map $t'k$ (where $k$ is a kernel of $f$) is a kernel of $f'$. Accordingly the pair $(t'k, s')$ is jointly strongly epimorphic. So this is equally the case for the pair $(t', s')$. Accordingly the fibre $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital.

(2) This comes immediately from 1, because, as we already observed, if $\mathbb{C}$ is $S$-protomodular, then $SPt_Y\mathbb{C}$ is closed under finite products in $Pt_Y\mathbb{C}$.

(3) Since any change-of-base functor with respect to $¶_{\mathbb{C}}^S$ is left exact, it is enough to prove that it is conservative on monomorphisms (see

Lemma 3.3 below). Let us consider the following diagram, where all the quadrangles are pullbacks and all the split epimorphisms are in $SPt\mathbb{C}$:

$$\begin{tikzcd}
K[f'] \arrow[rr, "k_{f'}"] \arrow[dr, "K(m')"'] \arrow[dd] & & X' \arrow[rr, "x"] \arrow[dr, "m'"] \arrow[dd, shift right, "f'"'] & & X \arrow[dr, "m"] \arrow[dd, shift right, "f"'] & \\
& K[\bar{f}'] \arrow[rr, "k_{\bar{f}'}"] \arrow[dl] & & \bar{X}' \arrow[rr, "\bar{x}"] \arrow[dl, shift right, "\bar{f}'"'] & & \bar{X} \arrow[dl, shift right, "\bar{f}"'] \\
1 \arrow[rr] & & Y' \arrow[uu, shift right, "s'"'] \arrow[ur, shift right, "\bar{s}'"'] \arrow[rr, "y"'] & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "\bar{s}"'] &
\end{tikzcd}$$

Suppose moreover that $m'$, and consequently $K(m')$, are isomorphisms. Since $K(m) \simeq K(m')$ is an isomorphism, and the pairs $(k_f, s)$ and $(k_{\bar{f}}, \bar{s})$ are jointly strongly epimorphic, then $m$ is a strong epimorphism, and hence it is an isomorphism.

■

**Lemma 3.3.** *Suppose that $U\colon \mathbb{C} \to \mathbb{D}$ is a left exact functor such that, for any monomorphism $m$ in $\mathbb{C}$, if $Um$ is an isomorphism in $\mathbb{D}$ then $m$ is an isomorphism. Then $U$ is conservative.*

*Proof*: Given any morphism $f$ in $\mathbb{C}$, consider the kernel pair of $f$:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X \arrow[l, "s_0" description] \arrow[r, "f"] & Y.
\end{tikzcd}$$

Since $U$ is left exact, we have that $UR[f]$ is the kernel pair of $Uf$:

$$\begin{tikzcd}
UR[f] = R[Uf] \arrow[r, shift left=2, "Up_0"] \arrow[r, shift right=2, "Up_1"'] & UX \arrow[l, "Us_0" description] \arrow[r, "Uf"] & UY.
\end{tikzcd}$$

Suppose that $Uf$ is an isomorphism. Then $Us_0$ is an isomorphism. Since $s_0$ is a monomorphism, our hypothesis implies that $s_0$ is an isomorphism. But then $f$ is a monomorphism, hence an isomorphism by our hypothesis. ■

**Proposition 3.4.** *Let $\mathbb{C}$ be a pointed finitely complete category, and $S$ a class of strongly split epimorphisms which is stable under pullbacks. Given* *any commutative square of split epimorphisms:*

$$\begin{tikzcd}
X' \arrow[r, shift right, "f'"'] \arrow[d, shift right, "g'"'] & X \arrow[l, shift right, "s'"'] \arrow[d, shift right, "g"'] \\
Y' \arrow[u, shift right, "t'"'] \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \arrow[u, shift right, "t"']
\end{tikzcd}$$

*where the split epimorphism $(g, t)$ is in $S$, the induced factorization to the pullback of $(g, t)$ along $f$ is an extremal epimorphism.*

*Proof*: Consider the following diagram:

$$\begin{tikzcd}
X' \arrow[rr, shift right, "f'"'] \arrow[dd, shift right, "g'"'] \arrow[dr, "\theta"] & & X \arrow[ll, shift right, "s'"'] \arrow[dd, shift right, "g"'] \arrow[dr, equal] & \\
& \bar{X} \arrow[rr, shift right, "\bar{f}"'] \arrow[dl, shift right, "\bar{g}"'] & & X \arrow[ll, shift right, "\bar{s}"'] \arrow[dl, shift right, "g"'] \\
Y' \arrow[uu, shift right, "t'"'] \arrow[ur, shift right, "\bar{t}"'] \arrow[rr, shift right, "f"'] & & Y \arrow[ll, shift right, "s"'] \arrow[uu, shift right, "t"'] \arrow[ur, shift right, "t"'] &
\end{tikzcd}$$

where the square $f\bar{g} = g\bar{f}$ is a pullback. Since $S$ is stable under pullbacks, $(\bar{g}, \bar{t})$ belongs to $S$. Moreover, since the category $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital (by Theorem 3.2) and the pullback considered above is actually the product of the two split epimorphisms $Y' \underset{f}{\overset{s}{\leftrightarrows}} Y$ and $X \underset{g}{\overset{t}{\leftrightarrows}} Y$ in the category $Pt_Y\mathbb{C}$, the pair $(\bar{t}, \bar{s})$ is jointly strongly epimorphic. Now let $\theta$ be the factorization in question. Suppose $j : \bar{U} \rightarrowtail \bar{X}$ is a monomorphism such that $\theta$ factors through it by a map $\theta'$: $j\theta' = \theta$. Consider the following diagram:

$$\begin{tikzcd}
K[\bar{g}j] \arrow[rr] \arrow[dr, "K(j)"] \arrow[dd, shift right] & & \bar{U} \arrow[dr, tail, "j"] \arrow[dd, shift right, "\bar{g}j"'] & \\
& K[\bar{g}] \arrow[rr] \arrow[dl, shift right] & & \bar{X} \arrow[dl, shift right, "\bar{g}"'] \\
0 \arrow[uu, shift right] \arrow[ur, shift right] \arrow[rr] & & Y' \arrow[uu, shift right, "\theta' t'"'] \arrow[ur, shift right, "\bar{t}"'] &
\end{tikzcd}$$

We have that $(\bar{g}j, \theta' t')$ is a split epimorphism, because

$$\bar{g}j\theta' t' = \bar{g}\theta t' = \bar{g}\bar{t} = 1.$$

$K(j)$ is an isomorphism, because, being $j$ a monomorphism, $K[\bar{g}j] \simeq K[\bar{g}]$. Using the same argument as in the proof point 3 of Theorem 3.2, we can conclude that $j$ is a strong epimorphism, and hence an isomorphism. ■

## 4. Schreier split epimorphisms in monoids with operations

The aim of this section is to introduce an important class of examples of the situation described in the previous one. We start by recalling from [12] the following definition, which was inspired by the analogous one of *groups with operations* introduced by Porter in [14].

**Definition 4.1.** *Let $\Omega$ be a set of finitary operations such that the following conditions hold: if $\Omega_i$ is the set of i-ary operations in $\Omega$, then:*

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(2) *There is a binary operation $+ \in \Omega_2$ (not necessarily commutative) and a constant $0 \in \Omega_0$ satisfying the usual axioms for monoids;*
(3) $\Omega_0 = \{0\}$;
(4) *Let $\Omega_2' = \Omega_2 \backslash \{+\}$; if $* \in \Omega_2'$, then $*^\circ$ defined by $x *^\circ y = y * x$ is also in $\Omega_2'$;*
(5) *Any $* \in \Omega_2'$ is left distributive w.r.t. $+$, i.e.:*
$$a * (b + c) = a * b + a * c;$$
(6) *For any $* \in \Omega_2'$ we have $b * 0 = 0$;*
(7) *Any $\omega \in \Omega_1$ satisfies the following conditions:*
 - $\omega(x + y) = \omega(x) + \omega(y)$;
 - *for any $* \in \Omega_2'$, $\omega(a * b) = \omega(a) * b$.*

*Let moreover $E$ be a set of axioms including the ones above. We will denote by $\mathbb{C}$ the category of $(\Omega, E)$-algebras. We call the objects of $\mathbb{C}$* monoids with operations.

**Remark.** The definition above does not include the case of groups, or more generally, the one of groups with operations. Indeed, the unary operation given by the group inverses, denoted by $-$, does not satisfy Condition 7. However, in order to recover all these structures, it suffices to add another condition: if the base monoid structure (given by the operations $+$ and $0$) is a group, then the operation $-$ should be distinguished from the other unary operations. In other terms, Condition 7 should be satisfied only by operations in $\Omega_1' = \Omega_1 \backslash \{-\}$. In this way, our definition becomes a generalization of the concept of groups with operations.

**Example 4.2.** Apart from the known structures covered by Porter's definition, such as groups, rings, associative algebras, Lie algebras and many others, our definition includes the cases of monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), join-semilattices with a bottom element, distributive lattices with a bottom element (or a top one).

Let us observe that, if $\mathbb{C}$ is a category of monoids with operations, then it is pointed, complete and unital.

We now introduce the split epimorphisms that will form the desired class $S$. For the rest of the section, $\mathbb{C}$ will denote a category of monoids with operations.

**Definition 4.3** ([12]). *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *in* $\mathbb{C}$ *is said to be a* Schreier split epimorphism *when, for any* $a \in A$*, there exists a unique* $\alpha$ *in the kernel* $K[f]$ *of* $f$ *such that* $a = \alpha + sf(a)$*.*

As it is shown in [12], in the category $Mon$ of monoids Schreier split epimorphisms are equivalent to monoid actions, where an action of a monoid $B$ on a monoid $X$ is a monoid homomorphism $B \to End(X)$, being $End(X)$ the monoid of endomorphisms of $X$.

**Proposition 4.4.** *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *is a Schreier split epimorphism if and only if there exists a set-theoretical map* $q\colon A \dashrightarrow K[f]$ *such that:*

$$\begin{aligned} q(a) + sf(a) &= a \\ q(\alpha + s(b)) &= \alpha \end{aligned}$$

*for every* $a \in A$, $\alpha \in K[f]$ *and* $b \in B$*.*

*Proof*: Suppose that for every $a \in A$, there exists a unique $\alpha \in K[f]$ such that $a = \alpha + sf(a)$. This property defines a map $q\colon A \to K[f]$, by $q(a) = \alpha$ such that $a = q(a) + sf(a)$, for every $a \in A$. In order to prove that $q(\alpha + s(b)) = \alpha$ for any $\alpha \in K[f]$, it suffices to observe that $sf(\alpha + s(b)) = s(b)$.

Conversely, given a set-theoretical map $q\colon A \to B$ satisfying the asserted identities, we can choose $\alpha = q(a)$ for every $a \in A$ by the first identity; suppose now that $a = \alpha' + sf(a)$, then we get:

$$q(a) = q(\alpha' + sf(a)) = \alpha'$$

by the second identity. ∎

We shall call the following diagram:

$$\begin{tikzcd} K[f] \arrow[r, tail, "k"'] & A \arrow[l, bend right, dotted, "q"'] \arrow[r, shift right, "f"'] & B, \arrow[l, tail, shift right, "s"'] \end{tikzcd}$$

the canonical *Schreier split sequence* associated with the Schreier split epimorphism and $q$ the associated *Schreier retraction*. The following properties of the retraction $q$ will be useful later. For the sake of simplicity, we consider $k$ just as an inclusion.

**Proposition 4.5.** *Given a Schreier split epimorphism $(A, B, f, s)$, we have:*

(a) $qk = 1_{K[f]}$*;*
(b) $qs = 0$*;*
(c) $q(0) = 0$*;*
(d) *if $b \in B$ and $\alpha \in K[f]$, then $q(s(b) + \alpha) + s(b) = s(b) + \alpha$;*
(e) *for every $a, a' \in A$ $q(a + a') = q(a) + q(sf(a) + q(a'))$.*

*Proof*: (a) is a straightforward consequence of the second identity in Proposition 4.4.

(b) for $b \in B$ we have that $s(b) = 0 + sf(s(b))$, and the uniqueness of $q$ gives that $qs(b) = 0$ for every $b \in B$.

(c) obviously we have $0 = 0 + sf(0)$.

(d) for any $b \in B$ and any $\alpha \in K[f]$ we have:

$$s(b) + \alpha = q(s(b) + \alpha) + sf(s(b) + \alpha) =$$
$$= q(s(b) + \alpha) + sfs(b) + sf(\alpha) = q(s(b) + \alpha) + s(b).$$

(e) $q(a + a')$ is the unique element of $K[f]$ such that

$$a + a' = q(a + a') + sf(a + a') = q(a + a') + sf(a) + sf(a'),$$

so it suffices to prove that

$$q(a) + q(sf(a) + q(a')) + sf(a) + sf(a') = a + a'.$$

By point (d), we have that

$$q(sf(a) + q(a')) + sf(a) = sf(a) + q(a')$$

and hence

$$q(a)+q(sf(a)+q(a'))+sf(a)+sf(a') = q(a)+sf(a)+q(a')+sf(a') = a+a'.$$

∎

We are now going to show that any category $\mathbb{C}$ of monoids with operations is $S$-protomodular, where $S$ is the class of Schreier split epimorphisms. The results below were already proved in [7] for the particular cases of monoids and semirings.

**Proposition 4.6.** *Schreier split epimorphisms are stable under pullbacks along any morphism.*

*Proof*: Consider the following diagram, where the lower row is a Schreier split sequence and the right hand side square is a pullback:

$$\begin{tikzcd}
K[\pi_Y] \arrow[r, shift right, "\langle k,0\rangle"'] \arrow[d, "\simeq"'] & A \times_B Y \arrow[l, dotted, shift right, "q'"'] \arrow[r, shift right, two heads, "\pi_Y"'] \arrow[d, "\pi_A"'] & Y \arrow[l, shift right, "\langle sh,1_Y\rangle"'] \arrow[d, "h"] \\
K[f] \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift right, two heads, "f"'] & B. \arrow[l, shift right, "s"']
\end{tikzcd}$$

The map $q'$ defined by $q'(a,y) = (q(a),0)$ satisfies the conditions of Proposition 4.4 since:

$$(a,y) = (q(a),0) + (sf(a),y) = (q(a),0) + (sh(y),y)$$

for any $(a,y) \in A \times_B Y$. Moreover, the elements of $K[\pi_Y]$ are of the form $(\alpha,0)$, with $\alpha \in K[f]$, and then:

$$q'((\alpha,0) + (sh(y),y)) = q'(\alpha + sh(y),y) = (q(\alpha + sh(y)),0) = (\alpha,0)$$

Accordingly the upper row is a Schreier split epimorphism. ∎

**Lemma 4.7.** *A Schreier split epimorphism is a strongly split epimorphism.*

*Proof*: Given a Schreier split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$, the formula $a = q(a)+sf(a)$ proves that the pair $(k,s)$ is jointly strongly epimorphic. ∎

**Proposition 4.8.** *Given any direct product diagram*

$$X \underset{\langle 1_X,0\rangle}{\overset{\pi_X}{\leftrightarrows}} X \times B \underset{\langle 0,1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B,$$

*the canonical split epimorphism* $(X \times B, B, \pi_B, \langle 0, 1_B \rangle)$ *is a Schreier split epimorphism.*

*Proof*: It suffices to observe that $(x, 0) + (0, b) = (x, b)$ for any $x \in X$ and any $b \in B$. Here the Schreier retraction $\pi_X$ is a monoid homomorphism. ■

**Corollary 4.9.** *For any* $X \in \mathbb{C}$*, the identity split epimorphism* $X \underset{1_X}{\overset{1_X}{\rightleftarrows}} X$*, and more generally any isomorphism, is a Schreier split epimorphism.*

**Example 4.10.** We denote by $\mathbb{Z}^*$ the monoid of non-zero integers with the usual multiplication, and by $\mathbb{N}^*$ its submonoid whose elements are the numbers greater than 0. Then the split epimorphism

$$\mathbb{Z}^* \underset{abs}{\overset{i}{\rightleftarrows}} \mathbb{N}^*,$$

where $i$ is the inclusion and $abs$ associates with any integer its absolute value, is a Schreier split epimorphism. In fact $K[abs] = \{\pm 1\}$, and it is immediate to see that any non-zero integer $z$ can be written in a unique way as $z = \pm 1 \cdot |z| = |z| \cdot \pm 1$.

**Proposition 4.11.** *Schreier split epimorphisms are closed under products, i.e. the product of two Schreier split epimorphisms is a Schreier one.*

*Proof*: Consider the two Schreier split exact sequences

$$K[f] \underset{k}{\overset{q}{\rightleftarrows}} A \underset{f}{\overset{s}{\rightleftarrows}} B$$

and

$$K[f'] \underset{k'}{\overset{q'}{\rightleftarrows}} A' \underset{f'}{\overset{s'}{\rightleftarrows}} B',$$

Their term by term product

$$K[f] \times K[f'] \underset{k \times k'}{\overset{q \times q'}{\rightleftarrows}} A \times A' \underset{f \times f'}{\overset{s \times s'}{\rightleftarrows}} B \times B',$$

clearly satisfies the conditions of Proposition 4.4. ■

**Lemma 4.12.** *Consider a (vertical) map $(h, l)$ in $Pt\mathbb{C}$:*

$$\begin{tikzcd}
X \arrow[r, shift left, twoheadrightarrow, "f"'] \arrow[d, "l"'] & Y \arrow[l, shift left, "s"'] \arrow[d, "h"] \\
X' \arrow[r, shift left, twoheadrightarrow, "f'"'] & Y'. \arrow[l, shift left, "s'"']
\end{tikzcd}$$

*Suppose that the two rows are Schreier split epimorphisms, then the Schreier retractions are compatible, i.e. the following leftward left hand side diagram commutes (in the category Set of sets):*

$$\begin{tikzcd}
K[f] \arrow[r, shift right, "k"'] \arrow[d, "K(l)"'] & X \arrow[l, shift right, dotted, "q"'] \arrow[r, shift left, twoheadrightarrow, "f"'] \arrow[d, "l"'] & Y \arrow[l, shift left, "s"'] \arrow[d, "h"] \\
K[f'] \arrow[r, shift right, "k'"'] & X' \arrow[l, shift right, dotted, "q'"'] \arrow[r, shift left, twoheadrightarrow, "f'"'] & Y'. \arrow[l, shift left, "s'"']
\end{tikzcd}$$

*Proof*: We have to show that $q'l(x) = lq(x)$ for any $x$ in $X$. It is true since we have:

$$lq(x) + s'f'l(x) = lq(x) + lsf(x) = l(q(x) + sf(x)) = l(x) = q'l(x) + s'f'l(x).$$

■

**Proposition 4.13.** *Schreier split epimorphisms are closed also under equalizers, i.e. the equalizer of two parallel morphisms between Schreier split epimorphisms is a Schreier one. Hence Schreier split epimorphisms are closed under finite limits.*

*Proof*: Given two parallel morphisms of Schreier split epimorphisms:

$$\begin{tikzcd}
A \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "f"'] & A' \arrow[d, shift right, "f'"'] \\
B \arrow[r, shift left, "h'"] \arrow[r, shift right, "g'"'] \arrow[u, shift right, "s"'] & B', \arrow[u, shift right, "s'"']
\end{tikzcd}$$

consider the following diagram:

$$
\begin{tikzcd}
K[\phi] \arrow[r, tail, "K(j)"] \arrow[d, shift right, "k_\phi"'] & K[f] \arrow[r, shift left, "K(h)"] \arrow[r, shift right, "K(g)"'] \arrow[d, shift right, "k_f"'] & K[f'] \arrow[d, shift right, "k_{f'}"'] \\
E \arrow[r, tail, "j"] \arrow[u, shift right, dotted, "q"'] \arrow[d, shift right, "\phi"'] & A \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] \arrow[u, shift right, dotted, "q_f"'] \arrow[d, shift right, "f"'] & A' \arrow[u, shift right, dotted, "q_{f'}"'] \arrow[d, shift right, "f'"'] \\
E' \arrow[r, tail, "j'"'] \arrow[u, shift right, "\sigma"'] & B \arrow[r, shift left, "h'"] \arrow[r, shift right, "g'"'] \arrow[u, shift right, "s"'] & B', \arrow[u, shift right, "s'"']
\end{tikzcd}
$$

where $j$ is an equalizer of $h$ and $g$ and $j'$ is an equalizer of $h'$ and $g'$ in $\mathbb{C}$. Then the lower part of the diagram is an equalizer diagram in $Pt\mathbb{C}$. Since the kernel functor preserves equalizers, $K(j)$ is an equalizer of $K(h)$ and $K(g)$ in $\mathbb{C}$, and hence in the category $Set$ of sets. By the previous lemma, the set-theoretical Schreier retractions $q_f$ and $q_{f'}$ make the upward right hand side square commute; hence we get a factorization $q$ which satisfies the conditions of a Schreier retraction for the split epimorphism $(\phi, \sigma)$ and makes it a Schreier split epimorphism. ■

## 5. Internal $S$-reflexive relations and $S$-categories

We recall that an internal reflexive graph in a category $\mathbb{C}$ is a diagram of the form

$$
\begin{tikzcd}
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}
$$

such that $d_0 s_0 = 1_{X_0} = d_1 s_0$. A reflexive relation is a reflexive graph such that the pair $(d_0, d_1)$ is jointly monomorphic.

**Definition 5.1.** *An internal reflexive graph (resp. category, groupoid) in a $S$-protomodular category $\mathbb{C}$*

$$
\begin{tikzcd}
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}
$$

*is said to be a $S$-*reflexive graph *(resp. $S$-category, $S$-groupoid) if the split epimorphism $(d_0, s_0)$ is in $S$.*

As a consequence of condition 2 of the definition of an $S$-protomodular category, $S$-reflexive graphs are closed under finite limits inside the category of internal reflexive graphs. The same is true for $S$-categories and $S$-groupoids. Let us recall that an internal category $\underline{X}_1$ in $\mathbb{C}$ is a reflexive graph:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

such that the following pullback of split epimorphisms, which defines $X_2$ as the internal object of the composable pairs

$$\begin{tikzcd} X_2 \arrow[r, shift right, "d_2"'] \arrow[d, shift right, "d_0"'] & X_1 \arrow[l, shift right, "s_1"'] \arrow[d, shift right, "d_0"'] \\ X_1 \arrow[r, shift right, "d_1"'] \arrow[u, shift right, "s_0"'] & X_0, \arrow[l, shift right, "s_0"'] \arrow[u, shift right, "s_0"'] \end{tikzcd} \tag{4}$$

is endowed with a composition map $d_1 \colon X_2 \to X_1$ satisfying the remaining simplicial identities:
(1) $d_0d_1 = d_0d_0$, $d_1d_1 = d_1d_2$ (incidence axioms)
(2) $d_1s_0 = 1_{X_1}$, $d_1s_1 = 1_{X_1}$ (composition with identities)
This composition must satisfy the associativity axiom; for that consider the following pullback of split epimorphisms (where $X_3$ is the object of composable triples):

$$\begin{tikzcd} X_3 \arrow[r, shift right, "d_3"'] \arrow[d, shift right, "d_0"'] & X_2 \arrow[l, shift right, "s_2"'] \arrow[d, shift right, "d_0"'] \\ X_2 \arrow[r, shift right, "d_2"'] \arrow[u, shift right, "s_0"'] & X_1 \arrow[l, shift right, "s_1"'] \arrow[u, shift right, "s_0"'] \end{tikzcd} \tag{5}$$

The composition map $d_1$ induces a couple of maps $(d_1, d_2) \colon X_3 \rightrightarrows X_2$ such that $d_0d_1 = d_0d_0$, $d_2d_1 = d_1d_3$ and $d_0d_2 = d_1d_0$, $d_2d_2 = d_2d_3$. The associativity is given by the remaining simplicial axiom:
(3) $d_1d_1 = d_1d_2$.

**Proposition 5.2.** *Let $\mathbb{C}$ be a $S$-protomodular category. On a $S$-reflexive graph there is at most one structure of internal category. It is sufficient to have the composition map $d_1 \colon X_2 \to X_1$ with axiom* (2)*, the axioms* (1) *and* (3) *come for free.*

*Proof*: Let us consider the following $S$-reflexive graph:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

Consider now the diagram (4):

$$\begin{tikzcd} X_2 \arrow[r, shift right, "d_2"'] \arrow[d, shift right, "d_0"'] & X_1 \arrow[l, shift right, "s_1"'] \arrow[d, shift right, "d_0"'] \\ X_1 \arrow[r, shift right, "d_1"'] \arrow[u, shift right, "s_0"'] & X_0 \arrow[l, shift right, "s_0"'] \arrow[u, shift right, "s_0"'] \end{tikzcd}$$

Since the rightward horizontal square is a pullback, and the right hand side split epimorphism is in $S$, the left hand side one is in $S$, too. Moreover, since the category $Pt_{X_0}\mathbb{C}$ is $SPt_{X_0}\mathbb{C}$-unital (by Theorem 3.2) and the pullback above is actually the product of the two split epimorphisms $X_1 \underset{d_0}{\overset{s_0}{\rightleftarrows}} X_0$ and $X_1 \underset{d_1}{\overset{s_0}{\rightleftarrows}} X_0$ in the category $Pt_{X_0}\mathbb{C}$, the pair $(s_0, s_1)$ is jointly strongly epimorphic. Hence there is at most one map $d_1$ satisfying axiom (2). Axiom (1) can be also verified by composition with the pair $(s_0, s_1)$. Axiom (3) comes by composition with the pair $(s_0, s_2)$ of diagram (5), which is jointly strongly epimorphic as well. ■

Let a reflexive graph be given:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

Let us recall that its simplicial kernel is the upper part of the universal 2-simplicial object associated with it:

$$\begin{tikzcd} K[d_0, d_1] \arrow[r, "p_1"'] \arrow[r, bend left=50, "p_0"] \arrow[r, bend right=50, "p_2"'] & X_1 \arrow[l, shift right=3, "s_0"'] \arrow[l, shift left=3, "s_1"] \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd} \tag{6}$$

In a finitely complete category $\mathbb{C}$, it is obtained by the following pullback of reflexive graphs:

$$\begin{tikzcd}
K[d_0,d_1] \arrow[rr, "p_2"] \arrow[dr, "(p_0{,}p_1)"'] \arrow[dd, shift right=2, "p_0"'] \arrow[dd, shift left=2, "p_1"] & & X_1 \arrow[dr, "(d_0{,}d_1)"] \arrow[dd, shift right=2, "d_0"'] \arrow[dd, shift left=2, "d_1"] & \\
& R[d_0] \arrow[rr, "(d_1{,}d_1)"] \arrow[dl, shift right, "d_0"'] \arrow[dl, shift left, "d_1"] & & X_0 \times X_0 \arrow[dl, shift right, "p_0"'] \arrow[dl, shift left, "p_1"] \\
X_1 \arrow[uu] \arrow[ur] \arrow[rr, "d_1"'] & & X_0 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

In set-theoretical terms, $K[d_0, d_1]$ is the set of triples $(x_0, x_1, x_2) \in X_1$ whose incidence conditions are given by the following drawing:

$$\begin{tikzcd}
\bullet \arrow[rr, "x_2"] & & \bullet \\
& \bullet \arrow[ul, "x_0"] \arrow[ur, "x_1"'] &
\end{tikzcd}$$

**Proposition 5.3.** *Let $\mathbb{C}$ be a $S$-protomodular category. Any $S$-reflexive relation is transitive.*

*Proof*: Let us consider the following $S$-reflexive relation:

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

The square $d_1 p_0 = d_0 p_2$ in the diagram above determines a factorization $(p_0, p_2)\colon K[d_0, d_1] \to X_2$ to the following vertical pullback:

$$\begin{tikzcd}
K[d_0,d_1] \arrow[dr] \arrow[drr, "p_2"] \arrow[ddr, "p_0"] & & \\
& X_2 \arrow[r, shift right, "d_2"'] \arrow[d, shift right, "d_0"'] & X_1 \arrow[l, shift right, "s_1"'] \arrow[d, shift right, "d_0"'] \arrow[ull, dotted, bend right, "s_1"'] \\
& X_1 \arrow[r, shift right, "d_1"'] \arrow[u, shift right, "s_0"'] \arrow[uul, dotted, bend left, "s_0"] & X_0 \arrow[l, shift right, "s_0"'] \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

Since $(d_0, d_1)\colon X_1 \rightarrowtail X_0 \times X_0$ is a relation, and hence $d_0$ and $d_1$ are jointly monomorphic, the factorization $(p_0, p_2)$ is a monomorphism. In order to prove this fact, it suffices to observe that it is true in set-teoretical terms, and that it is invariant under the Yoneda embedding. The left hand side split epimorphism $(d_0, s_0)$ is in $S$, because $X_1 \underset{d_1}{\overset{d_0}{\underset{\leftarrow s_0 -}{\rightrightarrows}}} X_0$ is a $S$-reflexive relation and the rightward horizontal square is a pullback. According to Proposition 3.4, the factorization $(p_0, p_2) \colon K[d_0, d_1] \rightarrowtail X_2$ is an extremal epimorphism, and hence an isomorphism; accordingly the morphism $X_2 \xrightarrow{(p_0,p_2)^{-1}} K[d_0, d_1] \xrightarrow{p_2} X_1$ produces the desired transitivity map. ■

A $S$-reflexive relation doesn't need to be an equivalence relation, because symmetry can fail. The following is a concrete counterexample in the category $Mon$ of monoids, equipped with the class of Schreier split epimorphisms described in section 4.

**Example 5.4** ([6], Example 5.3). *The internal order in $Mon$ given by the usual order between natural numbers:*

$$\mathcal{O}_{\mathbb{N}} \underset{p_1}{\overset{p_0}{\underset{\leftarrow s_0 -}{\rightrightarrows}}} \mathbb{N},$$

*where*

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\},$$

*is a Schreier order relation.*

## 6. $S$-special morphims and internal $S$-groupoids

**Definition 6.1.** *Let $\mathbb{C}$ be a $S$-protomodular category. A morphism $f \colon X \to Y$ in $\mathbb{C}$ will be called* $S$-special *when the kernel equivalence relation $R[f]$ is a $S$-equivalence relation. An object $X$ will be called* $S$-special *when the terminal morphism $\tau_X \colon X \to 1$ is $S$-special.*

In a $S$-protomodular category, the $S$-special morphisms are stable under pullbacks (because the class $S$ is stable under pullbacks). Moreover, the full subcategory $S^\sharp\mathbb{C} \subseteq \mathbb{C}$ of $S$-special objects is closed under finite limits in $\mathbb{C}$ (this comes from Condition 2 of Definition 3.1).

**Proposition 6.2.** *Let $\mathbb{C}$ be a $S$-protomodular category. Any split epimorphism between $S$-special objects is in $S$ and, consequently, is a $S$-special morphism. The subcategory $S^\sharp\mathbb{C}$ of $S$-special objects is protomodular.*

*Proof*: Let us recall that any split epimorphism $(f, s\colon X \rightleftarrows Y)$ produces a kernel diagram in the fibre $Pt_Y\mathbb{C}$:

$$\begin{tikzcd}
X \arrow[r, tail, "{(f,1)}"] \arrow[dr, shift right, "f"'] & Y \times X \arrow[r, shift left, "1\times f"] \arrow[d, shift right, "p_Y"'] & Y \times Y \arrow[l, shift left, tail, "1\times s"] \arrow[dl, shift right, "p_0"'] \\
& Y \arrow[ul, shift right, "s"'] \arrow[u, shift right, "{(1,s)}"'] \arrow[ur, shift right, "s_0"'] &
\end{tikzcd}$$

When $Y$ is in $S^\sharp\mathbb{C}$, the right hand side split epimorphism is in $S$. The following pullback:

$$\begin{tikzcd}
Y \times X \arrow[r, "s\times 1"] \arrow[d, shift right, "p_Y"'] & X \times X \arrow[d, shift right, "p_0"'] \\
Y \arrow[r, "s"'] \arrow[u, shift right, "{(1,s)}"'] & X \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

shows that, when $X$ is in $S^\sharp\mathbb{C}$, the middle split epimorphism is in $S$. Since the fibre $SPt_Y\mathbb{C}$ is closed under finite limits, the kernel $(f, s)$ is in $S$. So, according to Theorem 3.2, the change-of-base functor with respect to the fibration $Pt(S^\sharp\mathbb{C})$ is conservative, and consequently $S^\sharp\mathbb{C}$ is protomodular. On the other hand, since $Pt(S^\sharp\mathbb{C})$ is stable under finite limits, the kernel equivalence relation of $f$ lies in $Pt(S^\sharp\mathbb{C})$, and the split epimorphism $(p_0, s_0)\colon R[f] \rightleftarrows X$ is in $S$. Accordingly $R[f]$ is a $S$-equivalence relation, and $f$ is a $S$-special morphism. ∎

**Definition 6.3.** *Given a $S$-protomodular category $\mathbb{C}$, we will call the subcategory $S^\sharp\mathbb{C}$ the* protomodular core *of $\mathbb{C}$ relatively to $S$.*

We are now going to describe the protomodular core when $\mathbb{C}$ is a category of monoids with operations and $S$ is the class of Schreier split epimorphisms.

**Proposition 6.4.** *Let $\mathbb{C}$ be a category of monoids with operations and $S$ the class of Schreier split epimorphisms. Given an object $X \in \mathbb{C}$, it is $S$-special if and only if $(X, +)$ is a group.*

*Proof*: Suppose that $X$ is $S$-special. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, shift right, "\langle 0,1\rangle"'] & X \times X \arrow[l, shift right, dotted, "q"'] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X, \arrow[l, "s_0" description]
\end{tikzcd}$$

where $q$ is the Schreier retraction associated with the Schreier split epimorphism $(p_0, s_0)$. Let $x \in X$; according to the Schreier condition, the pair $(x, 0) \in X \times X$ can be written as

$$(x, 0) = q(x, 0) + s_0 p_0(x, 0) = q(x, 0) + (x, x).$$

Since $q(x, 0)$ is an element of $\mathrm{Ker}(p_0)$, it is an element of the form $(0, y)$, for some $y \in X$. Hence we have

$$(x, 0) = (0, y) + (x, x) = (x, y + x),$$

and from this equality we get $y + x = 0$. So $y$ is a left inverse for $x$. Doing the same thing for all $x \in X$ we prove that $(X, +)$ is a group.

Conversely, suppose that $(X, +)$ is a group. The needed Schreier retraction is simply given by

$$q(x_1, x_2) = (0, x_2 - x_1).$$

■

As a consequence, we have that, if $\mathbb{C}$ is the category $Mon$ of monoids, its protomodular core is the category $Gp$ of groups. If $\mathbb{C}$ is the category $SRng$ of semirings, the protomodular core is the category $Rng$ of (not necessarily unitary) rings. More generally, given any category $\mathbb{C}$ of monoids with operation, the protomodular core is the corresponding category of groups with operations, obtained from $\mathbb{C}$ by adding the condition that $+$ is a group operation.

**Proposition 6.5.** *Let $\mathbb{C}$ be a $S$-protomodular category. Any split epimorphism $f \colon X \to Y$ which is a $S$-special morphism belongs to $S$ and its kernel is a $S$-special object.*

*Proof*: Let $s$ be the splitting of $f$. Consider the following diagram:

$$\begin{tikzcd}
K[f] \arrow[r, "k_f"] \arrow[d] & X \arrow[r, "s_1"] \arrow[d, shift right, "f"'] & R[f] \arrow[d, shift right, "p_0"'] \\
1 \arrow[r, "\alpha_Y"'] & Y \arrow[r, "s"'] \arrow[u, shift right, "s"'] & X, \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

where $s_1$ is the morphism $(1, sf) \colon X \to R[f]$. The right hand side square is a pullback. If the morphism $f$ is $S$-special then, by definition, the split epimorphism $(p_0, s_0)$ is in $S$. By stability under pullbacks, the split epimorphism $(f, s)$ is in $S$, too. The left hand side square is a pullback as well, so the terminal morphism $K[f] \to 1$ is $S$-special as so is $f$, and then $K[f]$ is a $S$-special object. ■

An internal category $\underline{X}_1$ in a finitely complete category $\mathbb{C}$ is a groupoid when, moreover, the following square determined by the composition map $d_1$ is a pullback:

$$\begin{tikzcd} X_2 \arrow[r, "d_1"] \arrow[d, "d_0"'] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_0"'] & X_0, \end{tikzcd} \tag{7}$$

or, in other words, when the following vertical comparison morphism $j$ is an isomorphism:

$$\begin{tikzcd} X_2 \arrow[d, dotted, "j"'] \arrow[dr, shift left=2, "d_0"] \arrow[dr, "d_1"] & & \\ R[d_0] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X_1 \arrow[ul, shift left=1, "s_0"] \arrow[l, "s_0"] \arrow[r, "d_0"] & X_0 \end{tikzcd}$$

In this case we have a discrete fibration between groupoids:

$$\begin{tikzcd} R[d_0] \arrow[r, "d_2"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & X_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\ X_1 \arrow[u] \arrow[r, "d_1"'] & X_0 \arrow[u] \end{tikzcd}$$

**Proposition 6.6.** *Let $\mathbb{C}$ be a $S$-protomodular category. A $S$-category*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0"] \end{tikzcd}$$

*is a $S$-groupoid if and only if the morphism $d_0 \colon X_1 \to X_0$ is $S$-special.*

*Proof*: Let $\underline{X}_1$ be a $S$-groupoid. Then the split epimorphism $(d_0, s_0) \colon X_1 \rightleftarrows X_0$ is in $S$. By the pullbacks of the previous fibration diagram, the split epimorphism $(d_0, s_0) \colon R[d_0] \rightleftarrows X_1$ is in $S$ and consequently the morphism $d_0 \colon X_1 \to X_0$ is $S$-special. Conversely, suppose that the map $d_0 \colon X_1 \to X_0$ of the $S$-category $\underline{X}_1$ is $S$-special. Consider the following diagram:

$$\begin{tikzcd}
X_1 \arrow[rr, "s_1"] \arrow[dr, equal] \arrow[dd, shift right, "d_0"'] & & X_2 \arrow[dr, "j"] \arrow[dd, shift right, "d_0"'] & \\
& X_1 \arrow[rr, "s_1"] \arrow[dl, shift right, "d_0"'] & & R[d_0] \arrow[dl, shift right, "d_0"'] \\
X_0 \arrow[uu, shift right, "s_0"'] \arrow[ur, shift right, "s_0"'] \arrow[rr, "s_0"'] & & X_1 \arrow[uu, shift right, "s_0"'] \arrow[ur, shift right, "s_0"'] &
\end{tikzcd}$$

The two right hand side split epimorphisms are in $S$. The two commutative squares are pullbacks along $s_0\colon X_0 \rightarrowtail X_1$. The diagram means that the image of the map $j$ by the change-of-base functor along $s_0$ is the isomorphism $1_{X_1}$. According to Theorem 3.2, the map $j$ is an isomorphism, and $\underline{X}_1$ is a groupoid. ∎

The previous proposition, according to Proposition 5.3, gives the following

**Corollary 6.7.** *Let $\mathbb{C}$ be a $S$-protomodular category. A $S$-reflexive relation*

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\xleftarrow{\;s_0\;}}{\rightrightarrows}}} X_0$$

*is a $S$-equivalence relation if and only if the morphism $d_0\colon X_1 \to X_0$ is $S$-special.*

When $\mathbb{C}$ is a category of monoids with operations, and $S$ is the class of Schreier split epimorphisms, the converse of Proposition 6.5 holds:

**Proposition 6.8.** *Let $\mathbb{C}$ be a category of monoids with operations, and let $S$ be the class of Schreier split epimorphisms. Given a Schreier split epimorphism*

$$K[f] \underset{k_f}{\overset{q_f}{\leftrightarrows}} X \underset{f}{\overset{s}{\leftrightarrows}} Y,$$

*if $K[f]$ is a $S$-special object (or, in other terms, if $(K[f], +)$ is a group), then $f$ is a $S$-special morphism.*

*Proof*: We have to show that the split epimorphism $R[f] \underset{p_0}{\overset{s_0}{\leftrightarrows}} X$ is a Schreier split epimorphism. Let us define $q_{p_0}(x, x') = (0, q_f(x') - q_f(x))$. We can check that

$q_{p_0}(x, x') + s_0 p_0(x, x') = (0, q_f(x') - q_f(x)) + (x, x) = (x, q_f(x') - q_f(x) + x) =$

$$= (x, q_f(x') + sf(x)) = (x, q_f(x') + sf(x')) = (x, x'),$$

and, thanks to Proposition 4.5,

$$q_{p_0}((0,k)+s_0(x)) = q_{p_0}((0,k)+(x,x)) = q_{p_0}(x, k+x) = (0, q_f(k+x)-q_f(x)) =$$

$$= (0, q_f(k) + q_f(sf(k) + q_f(x)) - q_f(x)) = (0, q_f(k) + q_f(x) - q_f(x)) = (0, k).$$

The thesis follows then from Proposition 4.4. ■

**Example 6.9.** The previous proposition implies that the morphism $abs\colon \mathbb{Z}^* \to \mathbb{N}^*$ of example 4.10 is a special $S$-morphism.

**Corollary 6.10.** *Let $\mathbb{C}$ be a category of monoids with operations, and let $S$ be the class of Schreier split epimorphisms. A $S$-category*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

*is a $S$-groupoid if and only if $K[d_0]$ is a $S$-special object. A $S$-reflexive relation is a $S$-equivalence relation if and only if the kernel $K[d_0]$ of the first projection is a $S$-special object, which is equivalent to say that $(K[d_0], +)$ is a group.*

*Proof*: Thanks to the previous proposition, this is an immediate consequence of Proposition 6.6 and Corollary 6.7. ■

## 7. Mal'tsev aspects of $S$-protomodular categories

**7.1. Mal'tsev categories.** We recall that a category $\mathbb{C}$ is a Mal'tsev category [8, 9] when any internal reflexive relation is an equivalence relation; this is equivalent to the property that any fiber $Pt_Y\mathbb{C}$ of the fibration $\P_{\mathbb{C}}$ is unital (see [2]). The category $Gp$ of groups is a Mal'tsev one. The natural order $\mathcal{O}_{\mathbb{N}}$ of natural numbers (Example 5.4) shows that the category $Mon$ of monoids is not a Mal'tsev one.

On the other hand, in the context of $S$-protomodular categories, any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital and, consequently, any fiber $SPt_Y\mathbb{C}$ is unital. In this section, we shall be interested in what is remaining of the properties of Mal'tsev categories in this new structural context.

**7.2. Elementary observations.** We already observed that, in a $S$-protomodular category, any $S$-reflexive relation $(d_0, d_1)\colon R \rightrightarrows X$ is only necessarily transitive. The natural order on $\mathbb{N}$ gives an example of $S$-reflexive relation (in the category of monoids) which is not an equivalence relation. A $S$-reflexive relation $R$ is an equivalence relation if and only if $d_0$ is $S$-special (Corollary 6.7 above).

In a Mal'tsev category, on a reflexive graph, there is at most one structure of internal category, which is necessarily an internal groupoid. In Section 5 we showed that, on a $S$-reflexive graph, there is again at most one structure of internal category, but there are $S$-categories which are not groupoids. An internal $S$-category is a groupoid if and only if, again, $d_0$ is $S$-special (Proposition 6.6).

In a Mal'tsev category we have also the following useful result (see [2]): given any split epimorphism of reflexive graphs,

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, shift right, "g_1"'] & X_0 \arrow[l, "s_0"' description] \arrow[d, shift right, "g_0"'] \\
X_1' \arrow[r, shift left=2, "d_0'"] \arrow[r, shift right=2, "d_1'"'] \arrow[u, shift right, "t_1"'] & X_0' \arrow[l, "s_0'"' description] \arrow[u, shift right, "t_0"']
\end{tikzcd}$$

the commutative square $g_0 d_1 = d_1' g_1$ is a pullback as soon as so is the square $g_0 d_0 = d_0' g_1$. Here we have:

**Proposition 7.1.** *Let $\mathbb{C}$ be a $S$-protomodular category. Given a split epimorphism of reflexive graphs in $\mathbb{C}$ as in the diagram above, where the split epimorphism $(g_0, t_0)$ is in $S$, the commutative square $g_0 d_1 = d_1' g_1$ is a pullback as soon as so is the square $g_0 d_0 = d_0' g_1$.*

*Proof*: If the square $g_0 d_0 = d_0' g_1$ is a pullback and the split epimorphism $(g_0, t_0)$ is in $S$, so are the split epimorphism $(g_1, t_1)$ and the pullback $(\bar{g}_1, \bar{t}_1)$ of $(g_0, t_0)$ along $d_1'$ in the following diagram:

$$\begin{tikzcd}
X_1 \arrow[rr, shift right, "d_1"'] \arrow[dr, "\theta"] \arrow[dd, shift right, "g_1"'] & & X_0 \arrow[ll, shift right, "s_0"'] \arrow[dr, equal] \arrow[dd, shift right, "g_0"'] & \\
& \bar{X}_1 \arrow[rr, shift right, "\bar{d}_1"'] \arrow[dl, shift right, "\bar{g}_1"'] & & X_0 \arrow[ll, shift right, "\bar{s}_0"'] \arrow[dl, shift right, "g_0"'] \\
X_1' \arrow[rr, shift right, "d_1'"'] \arrow[uu, shift right, "t_1"'] \arrow[ur, shift right, "\bar{t}_1"'] & & X_0' \arrow[ll, shift right, "s_0'"'] \arrow[uu, shift right, "t_0"'] \arrow[ur, shift right, "t_0"'] &
\end{tikzcd}$$

Let $\theta$ be the induced factorization. The two leftward commutative squares are pullbacks; this means that the image of $\theta$ by the change-of-base functor along $s_0$ is the isomorphism $1_{X_0}$. According to Theorem 3.2, the map $\theta$ is itself an isomorphism. ■

**7.3. Centrality with respect to $S$-reflexive relations.** More importantly, the Mal'tsev context fits very well to the notion of centrality between equivalence relations. The equivalence relations $R$ on an object $X$, coinciding with the reflexive relations on $X$, are just the subobjects of the object $(p_0, s_0)\colon X \times X \rightleftarrows X$ in the fibre $Pt_X\mathbb{C}$:

$$\begin{tikzcd} R \arrow[r, tail, "(d_0,d_1)"] \arrow[dr, shift right, "d_0"'] & X \times X \arrow[d, shift right, "p_0"'] \\ & X \arrow[u, shift right, "s_0"'] \arrow[ul, shift right, "s_0"'] \end{tikzcd}$$

Two equivalence relations $R$ and $W$ on $X$ centralize each other in a Mal'tsev category $\mathbb{C}$ when the subobjects $(d_1^R, d_0^R)\colon R \rightarrowtail X \times X$ and $(d_0^W, d_1^W)\colon W \rightarrowtail X \times X$ commute in the unital fiber $Pt_X\mathbb{C}$. In set-theoretical terms, the cooperator $R \times_X W \to X \times X$ in the fiber is necessarily of the form $\phi(xRyWz) = (x, p(xRyWz))$, with the two equations $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphism $p\colon R \times_X W \to X$ satisfying these two equations, which is characteristic of the fact that $R$ and $W$ centralize each other (see [5]), is called the *connector* between the relations $R$ and $W$. It is well known that, in the category $Gp$ of groups, two equivalence relations $R$ and $W$ on a group $G$ centralize each other if and only if the normal subgroups $\overline{1}_R$ and $\overline{1}_W$ given by the equivalence classes of the unit element commute inside the group $G$.

In a $S$-protomodular category $\mathbb{C}$, since any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital, we can keep the same definition provided, now, that, one of the domains, let us choose $W$, is a $S$-reflexive relation:

**Definition 7.2.** *Given a reflexive relation $R$ and a $S$-reflexive relation $W$ on the same object $X$ in a $S$-protomodular category $\mathbb{C}$, we say that $R$ and $W$* centralize each other *when there is a (necessarily unique) morphism*

*$p\colon R\times_X W\to X$, where $R\times_X W$ is defined by the following pullback:*

$$\begin{tikzcd} R\times_X W \arrow[r, shift left, "p_1^R"] \arrow[d, shift right, swap, "p_0^W"] & W \arrow[l, shift left, "\sigma_0^R"] \arrow[d, shift right, swap, "d_0^W"] \\ R \arrow[r, shift left, "d_1^R"] \arrow[u, shift right, swap, "\sigma_0^W"] & X \arrow[l, shift left, "s_0^R"] \arrow[u, shift right, swap, "s_0^W"] \end{tikzcd}$$

*such that $p\sigma_0^R = d_1^W$ and $p\sigma_0^W = d_0^R$. In set-theoretical terms, this means that we have both $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphisms $\sigma_0^R$ and $\sigma_0^W$, defined by the universal property of the pullback, are explicitly given by $\sigma_0^R(yWz) = yRyWz$ and $\sigma_0^W(xRy) = xRyWy$. We denote this situation by $[R, W] = 0$.*

Since $W$ is a $S$-reflexive relation, the split epimorphism $(d_0^W, s_0^W)$ is in $S$, and consequently the pair $(\sigma_0^R, \sigma_0^W)$ is jointly strongly epimorphic. This implies that the connector $p$ is unique.

**Example 7.3.** *Given the order $\mathcal{O}_{\mathbb{N}}$ on $\mathbb{N}$ in $Mon$, with the class $S$ of Schreier split epimorphisms, we have that $[\mathcal{O}_{\mathbb{N}}, \mathcal{O}_{\mathbb{N}}] = 0$; in this case, the connector is the morphism $p$ defined by $p(x \leq y \leq z) = z - y + x$.*

When we have $[R, W] = 0$, we recover a well-known result in Mal'tsev categories:

**Proposition 7.4.** *Let $\mathbb{C}$ be a $S$-protomodular category. Suppose the reflexive relation $R$ and the $S$-reflexive relation $W$ on $X$ centralize each other in $\mathbb{C}$. We have necessarily $xWp(xRyWz)$ and $p(xRyWz)Rz$.*

*Proof*: Let us consider the following pullback:

$$\begin{tikzcd} U \arrow[r, tail] \arrow[d, "j"'] & W \arrow[d, "{(d_0^W, d_1^W)}"] \\ R\times_X W \arrow[r, tail, "{(d_0^R p_0^W, p)}"'] & X\times X \end{tikzcd}$$

It defines $U$ as the subobject of those $xRyWz \in R\times_X W$ such that we have $xWp(xRyWz)$. For any $yWz \in W$, the element $yRyWz \in R\times_X W$ belongs to $U$, since we have $y = p(yRyWz)$ (as we observed in Definition 7.2). This means that $\sigma_0^R$ factors through $U$. In the same way, for any $xRy \in R$, the element $xRyWy \in R\times_X W$ belongs to $U$, since we have $x = p(xRyWy)$. This means that $\sigma_0^W$ factors through $U$. Since the pair

$(\sigma_0^R, \sigma_0^W)$ is jointly strongly epimorphic, the morphism $j$ is an isomorphism, and for every $xRyWz \in R \times_X W$ we have $xWp(xRyWz)$.

We have a similar result concerning the suboject $V \rightarrowtail R \times_X W$ defined by the following pullback:

$$\begin{tikzcd}
V \arrow[r, tail, "j"] \arrow[d] & R \times_X W \arrow[d, "{(p, d_1^W p_1^R)}"] \\
R \arrow[r, tail, "{(d_0^R, d_1^R)}"'] & X \times X
\end{tikzcd}$$

This give us $p(xRyWz)Rz$ for any $xRyWz \in R \times_X W$. ■

In set-theoretical terms, the previous proposition says that, with any triple $xRyWz$, we can associate a square of related elements:

$$\begin{tikzcd}
x \arrow[r, dotted, "W"] \arrow[d, "R"'] & p(x, y, z) \arrow[d, dotted, "R"] \\
y \arrow[r, "W"'] & z.
\end{tikzcd}$$

This says that any connected pair of reflexive relations $(R, W)$ on the object $X$, where $W$ is a $S$-reflexive relation, produces the following diagram of reflexive relations in $\mathbb{C}$:

$$\begin{tikzcd}[row sep=large, column sep=large]
R \times_X W \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "{(d_0^R p_0, p)}"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "{(p, d_1^W p_1)}"] & W \arrow[l] \arrow[d, shift right=2, "d_0^W"'] \arrow[d, shift left=2, "d_1^W"] \\
R \arrow[u] \arrow[r, shift left=2, "d_1^R"] \arrow[r, shift right=2, "d_0^R"'] & X \arrow[l] \arrow[u]
\end{tikzcd}$$

It is called the *centralizing double relation* associated with the connector. When $R$ and $W$ are equivalence relations, all the reflexive relations in this diagram are equivalence relations, and, moreover, any commutative square is a pullback (thanks to Proposition 7.1).

As in the case of Mal'tsev categories (see [5]), in the context of $S$-protomodular categories the existence of a double centralizing relation between a reflexive relation $R$ and a $S$-reflexive relation $W$ characterizes the fact that

$[R, W] = 0$. Indeed, given a double centralizing relation

$$\begin{tikzcd}
C \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & W \arrow[l] \arrow[d, shift right=2, "d_0^W"'] \arrow[d, shift left=2, "d_1^W"] \\
R \arrow[u] \arrow[r, shift left=2, "d_1^R"] \arrow[r, shift right=2, "d_0^R"'] & X \arrow[l] \arrow[u]
\end{tikzcd}$$

i.e. a reflexive relation $C$ both on $R$ and $W$ such that the square $d_1^R p_0 = d_0^W p_1$ is a pullback, the morphism $d_1^R p_0 \colon C \to X$ is the (necessarily unique) connector.

We can now prove the following

**Proposition 7.5.** *Consider a reflexive graph such that $d_0$ is $S$-special:*

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

*The following conditions are equivalent:*

(1) *the graph is underlying a $S$-category;*
(2) *the graph is underlying a $S$-groupoid;*
(3) *the kernel equivalence relations $R[d_0]$ and $R[d_1]$ centralize each other.*

*Proof*: Since $d_0$ is $S$-special, the graph is a $S$-reflexive graph. Moreover $R[d_0]$ is a $S$-equivalence relation, and we can talk about centralization of it with any reflexive relation on $X_1$.

The equivalence between conditions 1 and 2 was already proved (see Proposition 6.6).

To prove the implication $2 \Rightarrow 3$ consider the following diagram:

$$\begin{tikzcd}
R[d_2] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, shift right=1, "R(d_0)"'] & R[d_0] \arrow[l] \arrow[r, dotted, "d_2"] \arrow[d, shift right=1, "d_0"'] & X_1 \arrow[d, shift right=1, "d_0"'] \\
R[d_1] \arrow[u, shift right=1] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & X_1 \arrow[l] \arrow[u, shift right=1] \arrow[r, dotted, "d_1"'] & X_0 \arrow[u, shift right=1, "s_0"']
\end{tikzcd}$$

As we observed in Section 6, the right hand side square is a pullback, and hence the left hand side part of the diagram gives a double centralizing relation, which says that $[R[d_0], R[d_1]] = 0$.

To prove the implication $3 \Rightarrow 1$, suppose that we have $[R[d_0], R[d_1]] = 0$. As we observed in Section 5, in order to equip our $S$-reflexive graph with a structure of internal category, we only need to give the composition map $d_1 \colon X_2 \to X_1$ satisfying the equalities $d_1 s_0 = d_1 s_1 = 1_{X_1}$. The map $d_1$ can be defined as in the case of Mal'tsev categories (Theorem 3.6 in [9]). In set-theoretical terms, $d_1$ is given by:

$$d_1(\alpha, \beta) = p(\beta R[d_0] 1_{d_1(\alpha)} R[d_1] \alpha).$$

It is easy to verify that it satisfies the desired equalities. ∎

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