Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 14–07

ZEROS OF ORTHOGONAL POLYNOMIALS GENERATED BY THE GERONIMUS PERTURBATION OF MEASURES

AMÍLCAR BRANQUINHO, EDMUNDO J. HUERTAS AND FERNANDO R. RAFAELI

Abstract: This paper deals with monic orthogonal polynomial sequences (MOPS in short) generated by a Geronimus canonical spectral transformation of a positive Borel measure μ, i.e.,

$$\frac{1}{(x-c)}d\mu(x) + N\delta(x-c),$$

for some free parameter $N \in \mathbb{R}_+$ and shift c. We analyze the behavior of the corresponding MOPS. In particular, we obtain such a behavior when the mass N tends to infinity as well as we characterize the precise values of N such the smallest (respectively, the largest) zero of these MOPS is located outside the support of the original measure μ. When μ is semi-classical, we obtain the ladder operators and the second order linear differential equation satisfied by the Geronimus perturbed MOPS, and we also give an electrostatic interpretation of the zero distribution in terms of a logarithmic potential interaction under the action of an external field. We analyze such an equilibrium problem when the mass point of the perturbation c is located outside the support of μ.

Keywords: Orthogonal polynomials, Canonical spectral transformations of measures, Zeros, Interlacing, Monotonicity, Laguerre and Jacobi polynomials, Asymptotic behavior, Electrostatic interpretation, Logarithmic potential.
AMS Subject Classification (2000): 30C15.

1. Introduction

1.1. Geronimus perturbation of a measure. In the last years some attention has been paid to the so called canonical spectral transformations of measures. Some authors have analyzed them from the point of view of Stieltjes functions associated with such a kind of perturbations (see [27]) or

Received February 25, 2014.
The work of the first author was partially supported by Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.
The work of the second author was supported by Fundação para a Ciência e a Tecnologia (FCT) of Portugal, ref. SFRH/BPD/91841/2012, and partially supported by Dirección General de Investigación Científica y Técnica, Ministerio de Economía y Competitividad of Spain, grant MTM2012-36732-C03-01.
The third author wishes to thank FAPESP and CNPq for their financial support.

from the relation between the corresponding Jacobi matrices (see [28]). The present contribution is focused on the behavior of zeros of monic orthogonal polynomial sequences (MOPS in the sequel) associated with a particular transformation of measures called the *Geronimus canonical transformation on the real line*. Let μ be an absolutely continuous measure with respect to the Lebesgue measure supported on a finite or infinite interval $E = \operatorname{supp}(\mu)$, such that $C_0(E) = [a, b] \subseteq \mathbb{R}$. The basic Geronimus perturbation of μ is defined as

$$d\nu_N(x) = \frac{1}{(x-c)} d\mu(x) + N\delta(x-c), \tag{1}$$

with $N \in \mathbb{R}_+$, $\delta(x-c)$ the Dirac delta function in $x = c$, and the shift of the perturbation verifies $c \notin E$. Observe that it is given simultaneously by a rational modification of μ by a positive linear polynomial whose real zero c is the point of transformation (also known as the *shift* of the transformation) and the addition of a Dirac mass at the point of transformation as well.

This transformation was introduced by Geronimus in the seminal papers [11] and [12] devoted to provide a procedure of constructing new families of orthogonal polynomials from other orthogonal families, and also was studied by Shohat (see [22]) concerning about mechanical quadratures. The problem was revisited by Maroni in [19], into a more general algebraic frame, who gives an expression for the MOPS associated with (1) in terms of the so called co-recursive polynomials of the classical orthogonal polynomials. In the past decade, Bueno and Marcellán reinterpreted this perturbation in the framework of the so called discrete Darboux transformations, LU and UL factorizations of shifted Jacobi matrices [5]. This interpretation as Darboux transformations, together with other canonical transformations (Christoffel and Uvarov), provide a link between orthogonal polynomials and discrete integrable systems (see [1], [23] and [24]). More recently, in [4] the authors present a new computational algorithm for computing the Geronimus transformation with large shifts, and [7] concerns about a new revision of the Geronimus transformation in terms of symmetric bilinear forms in order to include certain Sobolev and Sobolev–type orthogonal polynomials into the scheme of Darboux transformations.

In order to justify the relevance of this contribution, we point out that the behavior of the zeros of orthogonal polynomials is extensively studied because of their applications in many areas of mathematics, physics and engineering. Following this premise, the purpose of this paper is twofold. First, using a similar approach as was done in [14], we provide a new connection formula for the Geronimus perturbed MOPS, which will be crucial to obtain sharp limits (and the speed of convergence to them) of their zeros. We provide a comprehensive study of the zeros in terms of the free parameter of the perturbation N, which somehow determines how important the perturbation on the classical measure μ is. Notice that this work also concerns with the behavior of the eigenvalues of the monic Jacobi matrices associated to certain Darboux transformations with shift c and free parameter N studied in [4]. Second, from the aforementioned new connection formula we recover (from an alternative point of view) a connection formula already known in the literature (see [19]) in terms of two consecutive polynomials of the original measure μ. We also obtain explicit expressions for the ladder operators and the second order differential equation satisfied by the Geronimus perturbed MOPS. When the measure μ is semi-classical, we also obtain the corresponding electrostatic model for the zeros of the Geronimus perturbed MOPS, showing that they are the electrostatic equilibrium points of positive unit charges interacting according to a logarithmic potential under the action of an external field (see, for example, Szegő's book [25, Section 6.7], Ismail's book [17, Ch. 3] and the references therein).

The structure of the paper is as follows. The rest of this Section is devoted to introduce without proofs some relevant material about modified inner products and their corresponding MOPS. In Section 2 we provide our main results. We obtain a new connection formula for orthogonal polynomials generated by a basic Geronimus transformation of a positive Borel measure μ, sharp bounds and speed of convergence to them for their real zeros, and the ladder operators and the second linear differential equation that they satisfy. The results about the zeros follows from a lemma concerning the behavior of the zeros of a linear combination of two polynomials. In Section 3, we proof all the result provided in the former Section. Finally, in Section 4, we explore these results for the Geronimus perturbed Laguerre and Jacobi MOPS. For μ being semi-classical, we obtain the corresponding electrostatic model for the zeros of the Geronimus perturbed MOPS as equilibrium points in a logarithmic potential interaction of positive unit charges under the presence of an external field. We analyze such an equilibrium problem when the mass point is located outside the support of the measure μ, and we provide explicit formulas for the Laguerre and Jacobi weight cases.

1.2. Modified inner products and notation. Let μ be a positive Borel measure μ, with existing moments of all orders, and supported on a subset $E \subseteq \mathbb{R}$ with infinitely many points. Given such a measure, we define the standard inner product $\langle \cdot, \cdot \rangle_{\mu} : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ by

$$\langle f, g \rangle_{\mu} = \int_{E} f(x)g(x)d\mu(x), \quad f, g \in \mathbb{P}, \tag{2}$$

where $\mathbb{P}$ is the linear space of the polynomials with real coefficients, and the corresponding norm $|| \cdot ||_{\mu} : \mathbb{P} \to [0, +\infty)$ is given, as usual, by

$$||f||_{\mu} = \sqrt{\int_{E} |f(x)|^2 d\mu(x)}, \quad f \in \mathbb{P}.$$

Let $\{P_n\}_{n\geq 0}$ be the MOPS associated with μ. It is very well known that the former MOPS satisfy the three term recurrence relation

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \\ P_{-1}(x) &= 0, \quad P_0(x) = 1. \end{aligned} \tag{3}$$

If

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{||P_k||_{\mu}^2} \tag{4}$$

denotes the corresponding n-th kernel polynomial, according to the Christoffel-Darboux formula, for every $n \in \mathbb{N}$ we have

$$K_n(x, y) = \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{(x - y)} \frac{1}{||P_n||_{\mu}^2}.$$

Notice that this structures satisfy the well-known “reproducing property” of the n-th kernel polynomial

$$\int_{E} K_n\left(x, y\right) f\left(x\right) d\mu(x) = f\left(y\right)$$

for any polynomial $f \in \mathbb{P}$ with $\deg\left(f\right) \leq n$.

Here and subsequently, $\{P_n^{c,[k]}\}_{n\geq 0}$ denotes the MOPS with respect to the modified inner product

$$\langle f, g \rangle_{\mu,[k]} = \int_{E} f(x)g(x)(x - c)^k d\mu(x), \tag{5}$$

where $c \notin E = \operatorname{supp}(\mu)$. The polynomials $\{P_n^{c,[k]}\}_{n\geq 0}$ are orthogonal with respect to a polynomial modification of the measure μ called the *k-iterated*

Christoffel perturbation. If $k = 1$ we have the *Christoffel canonical transformation of the measure* μ (see [27] and [28]). It is well known that $P_n^{c,[1]}(x)$ is the monic kernel polynomial which can be represented as (see [6, (7.3)])

$$P_n^{c,[1]}(x) = \frac{1}{(x-c)}\left(P_{n+1}(x) - \pi_n\, P_n(x)\right) = \frac{\|P_n\|_\mu^2}{P_n(c)} K_n(x,c), \tag{6}$$

with

$$\pi_n = \pi_n(c) = \frac{P_{n+1}(c)}{P_n(c)}. \tag{7}$$

Notice that $P_n^{c,[1]}(c) \neq 0$. We will denote

$$||P_n^{c,[k]}||_{\mu,[k]}^2 = \int_E |P_n^{c,[k]}(x)|^2 (x-c)^k d\mu.$$

Next, let us consider the basic Geronimus perturbation of μ given in (1). Let $\{Q_n^c\}_{n\geq 0}$ be the MOPS associated with $d\nu_N(x)$ when the $N = 0$. That is, they are orthogonal with respect to the measure

$$d\nu_{N=0}(x) = d\nu(x) = \frac{1}{(x-c)} d\mu(x). \tag{8}$$

This constitutes a linear rational modification of μ, and the corresponding MOPS $\{Q_n^c\}_{n\geq 0}$ with respect to

$$\langle f, g\rangle_\nu = \int_E f(x)g(x)d\nu(x) = \int_E f(x)g(x)\frac{1}{(x-c)}d\mu(x) \tag{9}$$

has been extensively studied in the literature (see, among others, [3], [10, 2.4.2], [17, 2.7], [26], and [27]). It is also well known that $Q_n^c(x)$ can be represented as

$$Q_n^c(x) = P_n(x) - r_{n-1}\, P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{10}$$

where $Q_0^c(x) = 1$,

$$r_{n-1} = r_{n-1}(c) = \frac{F_n(c)}{F_{n-1}(c)}, \quad c \notin E, \tag{11}$$

and $F_{-1}(c) = 1$. The functions

$$F_n(s) = \int_E \frac{P_n(x)}{x-s} d\mu(x), \quad s \in \mathbb{C}\setminus E,$$

are the Cauchy integrals of $\{P_n\}_{n\geq 0}$, or functions of the second kind associated with the monic polynomials $\{P_n\}_{n\geq 0}$. For a proper way to compute the above Cauchy integrals, we refer the reader to [10, 2.3].

It is clear that

$$K_n^c(x,y) = \sum_{k=0}^{n} \frac{Q_k^c(x)Q_k^c(y)}{||Q_k^c||_\nu^2} = \frac{Q_{n+1}^c(x)Q_n^c(y) - Q_{n+1}^c(y)Q_n^c(x)}{(x-y)} \frac{1}{||Q_n^c||_\nu^2} \tag{12}$$

are the kernel polynomials corresponding to the MOPS $\{Q_n^c\}_{n\geq 0}$, which also satisfies the corresponding reproducing property of polynomial kernels with respect to the measure $d\nu$

$$\int_E f(x)\, K_n^c(x,c)\, d\nu(x) = f(c), \tag{13}$$

for any polynomial $f \in \mathbb{P}$ with $\deg f \leq n$. The so called confluent form of (12) is given by (see [6])

$$K_n^c(c,c) = \frac{[Q_{n+1}^c]'(c)Q_n^c(c) - [Q_n^c]'(c)Q_{n+1}^c(c)}{||Q_n^c||_\nu^2}, \tag{14}$$

which is always a positive quantity

$$K_n^c(c,c) = \sum_{k=0}^{n} \frac{[Q_k^c(c)]^2}{\|Q_k^c\|_\nu^2} > 0. \tag{15}$$

The key concept to find several of our results is that the polynomials $\{P_n\}_{n\geq 0}$ are the monic kernel polynomials of parameter c of the sequence $\{Q_n^c\}_{n\geq 0}$. According to this argument, the following expressions

$$P_n(x) = \frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} K_n^c(x,c) = \frac{1}{(x-c)} \left(Q_{n+1}^c(x) - \frac{Q_{n+1}^c(c)}{Q_n^c(c)} Q_n^c(x) \right) \tag{16}$$

hold.

Finally, let $\{Q_n^{c,N}\}_{n\geq 0}$ be the MOPS associated to $d\nu_N$ when $N > 0$. That is, $\{Q_n^{c,N}\}_{n\geq 0}$ are the Geronimus perturbed polynomials orthogonal with respect to the the inner product

$$\langle f, g\rangle_{\nu_N} = \int_E f(x)g(x)\frac{1}{(x-c)} d\mu(x) + N f(c)g(c). \tag{17}$$

Note that this is a standard inner product in the sense that, for every $f, g \in \mathbb{P}$, we have $\langle xf, g\rangle_{\nu_N} = \langle f, xg\rangle_{\nu_N}$. From (9) and (17), a trivial verification shows that

$$\langle f, g\rangle_{\nu_N} = \langle f, g\rangle_{\nu} + Nf(c)g(c). \tag{18}$$

Is the aim of this contribution to find and analyze the asymptotic behavior of the zeros of $Q_n^{c,N}(x)$ with the parameter N, present in the Geronimus perturbation (1), and provide as well an electrostatic model for these zeros when the original measure μ is semiclassical. To this end, we will use some remarkable facts, which are straightforward consequences of the inner products (2), (5), (9) and (17). Taking into account that the multiplication operator by $(x-c)$ is a symmetric operator with respect to (9), for any $f(x), g(x) \in \mathbb{P}$ we have

$$\langle (x-c)f, g\rangle_{\nu} = \langle f, (x-c)g\rangle_{\nu} = \langle f, g\rangle_{\mu}$$

If we consider the polynomials $(x-c)f(x)$ or $(x-c)g(x)$ in the above expression, we deduce

$$(x-c)f(x)|_{x=c} = (x-c)g(x)|_{x=c} = 0,$$

which makes it obvious that $(x-c)$ is also a symmetric operator with respect to the Geronimus inner product (17), i.e.,

$$\langle (x-c)f, g\rangle_{\nu_N} = \langle f, (x-c)g\rangle_{\nu_N} = \langle (x-c)f, g\rangle_{\nu}. \tag{19}$$

Finally, another useful consequence of the above relations is

$$\langle (x-c)f, (x-c)g\rangle_{\nu_N} = \langle f, g\rangle_{c,[1]}. \tag{20}$$

2. Statement of the main results

2.1. Connection formulas. Next, we provide a new connection formula for the Geronimus perturbed orthogonal polynomials $Q_n^{c,N}(x)$, in terms of the polynomials $Q_n^{c}(x)$ and the monic Kernel polynomials $P_n^{c,[1]}(x)$. This representation will allow us to obtain the results about monotonicity, asymptotics, and speed of convergence (presented below in this Section) for the zeros of $Q_n^{c,N}(x)$ in terms of the parameter N present in the perturbation (1).

Theorem 1 (connection formula)**.** *The MOPS* $\{\tilde{Q}_n^{c,N}\}_{n\geq 0}$ *can be represented as*

$$\tilde{Q}_n^{c,N}(x) = Q_n^{c}(x) + NB_n^{c}(x-c)P_{n-1}^{c,[1]}(x), \tag{21}$$

with $\tilde{Q}_n^{c,N}(x) = \kappa_n Q_n^{c,N}(x)$, $\kappa_n = 1 + NB_n^c$ *and*

$$B_n^c = \frac{-Q_n^c(c)P_{n-1}(c)}{\|P_{n-1}\|_{\mu}^2} = K_{n-1}^c(c,c) > 0. \tag{22}$$

Observe that one can even give another alternative expression for B_n^c, which only involves polynomials and functions of the second kind relative to the original measure μ, evaluated at the point of tranformation c. Combining (10) with (22), we deduce that

$$B_n^c = K_{n-1}^c(c,c) = \frac{r_{n-1}\, P_{n-1}^2 - P_n(c)P_{n-1}(c)}{\|P_{n-1}\|_\mu^2}. \tag{23}$$

As a direct consequence of the above theorem, we can express $Q_n^{c,N}(x)$ in terms of only two consecutive elements of the initial sequence $\{P_n\}_{n\geq 0}$. This expression of $Q_n^{c,N}(x)$ was already studied in the literature (see [19, formula (1.4)] and [7, Sec. 1]). In fact, the original aim of Geronimus in its pioneer works on the subject was to find necessary and sufficient conditions for the existence of a sequence of coefficients Λ_n, such that the linear combination of monic polinomials

$$P_n(x) + \Lambda_n P_{n-1}(x), \quad \Lambda_n \neq 0,\ n = 1, 2, \ldots,$$

were, in turn, orthogonal with respect to some measure supported on $\mathbb{R}$. Here we rewrite the value of Λ_n in several new equivalent ways. Substituting (10) and (6) into (21) yields

$$\tilde{Q}_n^{c,N}(x) = \kappa_n Q_n^{c,N}(x) = P_n(x) - r_{n-1}P_{n-1}(x) + NB_n^c\left(P_n(x) - \pi_{n-1}P_{n-1}(x)\right).$$

Thus, having in mind that $\kappa_n = 1 + NB_n^c$, after some trivial computations we can state the following result.

Proposition 1. *The monic Geronimus perturbed orthogonal polynomials of the sequence $\{Q_n^{c,N}\}_{n\geq 0}$ can be represented as*

$$Q_n^{c,N}(x) = P_n(x) + \Lambda_n^c\, P_{n-1}(x), \tag{24}$$

with

$$\Lambda_n^c = \Lambda_n^c(N) = \frac{\pi_{n-1} - r_{n-1}}{1 + NB_n^c} - \pi_{n-1}, \tag{25}$$

and π_{n-1}, r_{n-1} given respectively in (7) and (11) respectively. Notice that Λ_n^c is independent of the variable x.

Remark 1. *The coefficient $\Lambda_n^c(N)$ can also be expressed only in terms of quantities relative to the original non-perturbed measure μ, the point of transformation c and the mass N. Thus, from (23) and (25), we obtain*

$$\Lambda_n^c(N) = \left(\frac{1}{\pi_{n-1} - r_{n-1}} - N\frac{P_{n-1}^2(c)}{\|P_{n-1}\|_\mu^2}\right)^{-1} - \pi_{n-1}.$$

Also, observe that for $N = 0$, the coefficient $\Lambda_n^c(0)$ reduces to r_{n-1}, and we recover the connection formula (10).

2.2. Asymptotic behavior and sharp limits of the zeros. Let $x_{n,s}$, $x_{n,s}^{c,[k]}$, $y_{n,s}^c$, and $y_{n,s}^{c,N}$, $s = 1, \ldots, n$ be the zeros of $P_n(x)$, $P_n^{c,[k]}(x)$, $Q_n^c(x)$, and $Q_n^{c,N}(x)$, respectively, all arranged in an increasing order, and assume that $C_0(E) = [a, b]$. Next, we analyze the behavior of zeros $y_{n,s}^{c,N}$ as a function of the mass N in (1). We obtain such a behavior when N tends from zero to infinity as well as we characterize the exact values of N such the smallest (respectively, the largest) zero of $\{Q_n^{c,N}\}_{n\geq 0}$ is located outside of $E = \operatorname{supp}(\mu)$.

In order to do that, we use a technique developed in [2, Lemma 1] and [8, Lemmas 1 and 2] concerning the behavior and the asymptotics of the zeros of linear combinations of two polynomials $h, g \in \mathbb{P}$ with interlacing zeros, such that $f(x) = h_n(x) + cg_n(x)$. From now on, we will refer to this technique as the *Interlacing Lemma*, and for the convenience of the reader we include its statement in the final Appendix.

Taking into account that the positive constant B_n^c does not depend on N, we can use the connection formula (21) to obtain results about monotonicity, asymptotics, and speed of convergence for the zeros of $Q_n^{c,N}(x)$ in terms of the mass N. Indeed, let assume that $y_{n,k}^{c,N}$, $k = 1, 2, ..., n$, are the zeros of $Q_n^{c,N}(x)$. Thus, from (21), the positivity of B_n^c, and Theorem 2, we are in the hypothesis of the Interlacing Lemma, and we immediately conclude the following results.

Theorem 2. *If $C_0(E) = [a, b]$ and $c < a$, then*

$$c < y_{n,1}^{c,N} < y_{n,1}^c < x_{n-1,1}^{c,[1]} < y_{n,2}^{c,N} < y_{n,2}^c < \cdots < x_{n-1,n-1}^{c,[1]} < y_{n,n}^{c,N} < y_{n,n}^c.$$

Moreover, each $y_{n,k}^{c,N}$ is a decreasing function of N and, for each $k = 1, \ldots, n-1$,

$$\lim_{N\to\infty} y_{n,1}^{c,N} = c, \quad \lim_{N\to\infty} y_{n,k+1}^{c,N} = x_{n-1,k}^{c,[1]},$$

as well as

$$\lim_{N\to\infty} N[y_{n,1}^{c,N} - c] = \frac{-Q_n^c(c)}{B_n^c P_{n-1}^{c,[1]}(c)},$$

$$\lim_{N\to\infty} N[y_{n,k+1}^{c,N} - x_{n-1,k}^{c,[1]}] = \frac{-Q_n^c(x_{n-1,k}^{c,[1]})}{B_n^c(x_{n-1,k}^{c,[1]} - c)[P_{n-1}^{c,[1]}]'(x_{n-1,k}^{c,[1]})}.$$

Theorem 3. *If $C_0(E) = [a, b]$ and $c > b$, then*

$$y^{c}_{n,1} < y^{c,N}_{n,1} < x^{c,[1]}_{n-1,1} < \cdots < y^{c}_{n,n-1} < y^{c,N}_{n,n-1} < x^{c,[1]}_{n-1,n-1} < y^{c}_{n,n} < y^{c,N}_{n,n} < c.$$

Moreover, each $y^{c,N}_{n,k}$ is an increasing function of N and, for each $k = 1, \dots, n-1$,

$$\lim_{N\to\infty} y^{c,N}_{n,n} = c, \qquad \lim_{N\to\infty} y^{c,N}_{n,k} = x^{c,[1]}_{n-1,k},$$

and

$$\lim_{N\to\infty} N[c - y^{c,N}_{n,n}] = \frac{Q^{c}_{n}(c)}{B^{c}_{n} P^{c,[1]}_{n-1}(c)},$$

$$\lim_{N\to\infty} N[x^{c,[1]}_{n-1,k} - y^{c,N}_{n,k}] = \frac{Q^{c}_{n}(x^{c,[1]}_{n-1,k})}{B^{c}_{n}(x^{c,[1]}_{n-1,k} - c)[P^{c,[1]}_{n-1}]'(x^{c,[1]}_{n-1,k})}.$$

Notice that the mass point c attracts one zero of $Q^{c,N}_{n}(x)$, i.e. when $N \to \infty$, it captures either the smallest or the largest zero, according to the location of the point c with respect to the support of the measure μ. When either $c < a$ or $c > b$, at most one of the zeros of $Q^{c,N}_{n}(x)$ is located outside of $[a, b]$. Next, give explicitly the value N_0 of the mass N, such that for $N > N_0$ one of the zeros is located outside $[a, b]$.

Corollary 1 (minimum mass)**.** *If $C_0(E) = [a, b]$ and $c \notin [a, b]$, the following expressions hold.*

(*a*) *If $c < a$, then the smallest zero $y^{c,N}_{n,1}$ satisfies*

$$\begin{aligned} y^{c,N}_{n,1} &> a, \quad \text{for} \quad N < N_0, \\ y^{c,N}_{n,1} &= a, \quad \text{for} \quad N = N_0, \\ y^{c,N}_{n,1} &< a, \quad \text{for} \quad N > N_0, \end{aligned}$$

where

$$N_0 = N_0(n, c, a) = \frac{-Q^{c}_{n}(a)}{K^{c}_{n-1}(c, c)\,(a - c) P^{c,[1]}_{n-1}(a)} > 0.$$

(*b*) *If $c > b$, then the largest zero $y^{c,N}_{n,n}$ satisfies*

$$\begin{aligned} y^{c,N}_{n,n} &< b, \quad \text{for} \quad N < N_0, \\ y^{c,N}_{n,n} &= b, \quad \text{for} \quad N = N_0, \\ y^{c,N}_{n,n} &> b, \quad \text{for} \quad N > N_0, \end{aligned}$$

where

$$N_0 = N_0(n, c, b) = \frac{-Q_n^c(b)}{K_{n-1}^c\,(c, c)\,(b - c)P_{n-1}^{c,[1]}(b)} > 0.$$

Proof: (a) In order to deduce the location of $y_{n,1}^{c,N}$ with respect to the point $x = a$, it is enough to observe that $Q_n^{c,N}(a) = 0$ if and only if $N = N_0$.

(b) Also, in order to find the location of $y_{n,n}^{c,N}$ with respect to the point $x = b$, notice that $Q_n^{c,N}(b) = 0$ if and only if $N = N_0$. ■

2.3. Ladder operators and second order linear differential equation. Our next result concerns the *ladder (creation and annihilation) operators*, and the *second order linear differential equation* (also known as the *holonomic equation*) corresponding to $\{Q_n^{c,N}\}_{n\geq 0}$. We restrict ourselves to the case in which μ is a classical or semi-classical measure, and therefore satisfying a structure relation (see [9] and [20]) as

$$\sigma(x)[P_n(x)]' = a(x;n)P_n(x) + b(x;n)P_{n-1}(x) \tag{26}$$

where $a(x;n)$ and $b(x;n)$ are polynomials in the variable x, whose fixed degree do not depend on n.

In order to obtain these results, we will follow a different approach as in [17, Ch. 3]. Our technique is based on the connection formula (24) given in Proposition 1, the three term recurrence relation (3) satisfied by $\{P_n\}_{n\geq 0}$, and the structure relation (26). The results are presented here and will be proved in Section 3.

Theorem 4 (ladder operators)**.** *Let* $\mathfrak{a}_n$ *and* $\mathfrak{a}_n^\dagger$ *be the differential operators*

$$\begin{aligned} \mathfrak{a}_n &= -\xi_1^c(x;n)\mathrm{I} + \mathrm{D}_x, \\ \mathfrak{a}_n^\dagger &= -\eta_2^c(x;n)\mathrm{I} + \mathrm{D}_x, \end{aligned}$$

where I, D_x *are the identity and* x *derivative operators respectively, satisfying*

$$\mathfrak{a}_n[Q_n^{c,N}(x)] = \eta_1^c(x;n)\,Q_{n-1}^{c,N}(x), \tag{27}$$

$$\mathfrak{a}_n^\dagger[Q_{n-1}^{c,N}(x)] = \xi_2^c(x;n)\,Q_n^{c,N}(x), \tag{28}$$

with, for $k=1,2$

$$\begin{aligned}\xi_k^c(x;n) &= \frac{C_k(x;n)B_2(x;n)\,\gamma_{n-1}+D_k(x;n)\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}},\\ \eta_k^c(x;n) &= \frac{D_k(x;n)-C_k(x;n)\Lambda_n^c}{\Delta(x;n)}.\end{aligned}$$

In turn, all the above expressions are given only in terms of the coefficients in (3), (26), and (24) as follows

$$\begin{aligned}B_2(x;n) &= \Lambda_{n-1}^c\left(\frac{1}{\Lambda_{n-1}^c}+\frac{(x-\beta_{n-1})}{\gamma_{n-1}}\right),\\ C_1(x;n) &= \frac{1}{\sigma(x)}\left(a(x;n)-\Lambda_n^c\frac{b(x;n)}{\gamma_{n-1}}\right),\\ D_1(x;n) &= \frac{1}{\sigma(x)}\left(b(x;n)+\Lambda_n^c\,b(x;n-1)\left(\frac{a(x;n-1)}{b(x;n-1)}+\frac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\right),\\ C_2(x;n) &= \frac{-\Lambda_{n-1}^c}{\sigma(x)}\left(\frac{a(x;n)}{\gamma_{n-1}}+\frac{b(x;n-1)}{\gamma_{n-1}}\left(\frac{1}{\Lambda_{n-1}^c}+\frac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\right),\\ D_2(x;n) &= \frac{\Lambda_{n-1}^c}{\sigma(x)}\left[\frac{\sigma(x)-b(x;n)}{\gamma_{n-1}}+b(x;n-1)\times\right.\\ &\qquad \left.\left(\frac{a(x;n-1)}{b(x;n-1)}+\frac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\left(\frac{1}{\Lambda_{n-1}^c}+\frac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\right],\\ \Delta(x;n) &= B_2(x;n)+\frac{\Lambda_n^c\,\Lambda_{n-1}^c}{\gamma_{n-1}},\quad \deg\Delta(x;n)=1.\end{aligned}$$

Thus, $\mathfrak{a}_n$ and $\mathfrak{a}_n^\dagger$ are respectively lowering and raising operators associated to the Geronimus perturbed MOPS $\{Q_n^{c,N}\}_{n\geq 0}$.

For a deeper discusion of raising and lowering operators we refer the reader to [17, Ch. 3]. We next provide the second order linear differential equation satisfied by the MOPS $\{Q_n^{c,N}\}_{n\geq 0}$ when the measure μ is semi-classical (for definition of a semi-classical measure see [20]). This is the main tool for the further electrostatic interpretation of zeros.

Theorem 5 (holonomic equation)**.** *The MOPS $\{Q_n^{c,N}\}_{n\geq 0}$ satisfies the holonomic equation (second order linear differential equation)*

$$[Q_n^{c,N}(x)]''+\mathcal{R}(x;n)[Q_n^{c,N}(x)]'+\mathcal{S}(x;n)Q_n^{c,N}(x)=0, \tag{29}$$

where

$$\begin{aligned}\mathcal{R}(x;n) &= -\left(\xi_1^c(x;n)+\eta_2^c(x;n)+\frac{[\eta_1^c(x;n)]'}{\eta_1^c(x;n)}\right),\\ \mathcal{S}(x;n) &= \xi_1^c(x;n)\eta_2^c(x;n)-\eta_1^c(x;n)\xi_2^c(x;n)\\ &\quad+\frac{\xi_1^c(x;n)[\eta_1^c(x;n)]'-[\xi_1^c(x;n)]'\eta_1^c(x;n)}{\eta_1^c(x;n)}.\end{aligned}$$

3. Proofs of the main results.

3.1. Proof of Theorem 1 and the positivity of B_n^c. First, we need to prove the following lemma concerning a first way to represent the Geronimus perturbed polynomials $Q_n^{c,N}(x)$, using the kernels (12).

Lemma 1. *Let $\{Q_n^{c,N}\}_{n\geq 0}$ and $\{Q_n^c\}_{n\geq 0}$ be the MOPS corresponding to the measures $d\nu_N$ and $d\nu(x)$ respectively. Then, the following connection formula holds*

$$Q_n^{c,N}(x)=Q_n^c(x)-NQ_n^{c,N}(c)K_{n-1}^c(x,c), \tag{30}$$

where

$$Q_n^{c,N}(c)=\frac{Q_n^c(c)}{1+NK_{n-1}^c(c,c)}=\kappa_n^{-1}Q_n^c(c), \tag{31}$$

$\kappa_n=1+NB_n^c$, *and* $K_{n-1}^c(c,c)$ *is given in (14).*

Proof: From (18) it is trivial to express $Q_n^{c,N}(x)$ in terms of the polynomials $Q_n^c(x)$

$$Q_n^{c,N}(x)=\sum_{k=0}^{n} b_{n,k}Q_n^c(x), \tag{32}$$

having

$$b_{n,k}=\frac{\langle Q_i^c(x),Q_n^{c,N}(x)\rangle_\nu}{\|Q_k^c\|_\nu^2},\quad 0\leq k\leq n-1.$$

Thus, (32) becomes

$$Q_n^{c,N}(x)=Q_n^c(x)-NQ_n^{c,N}(c)\sum_{k=0}^{n-1}\frac{Q_k^c(x)Q_k^c\,(c)}{\|Q_k^c\|_\nu^2}.$$

Next, taking into account (4) for the sequence $\{Q_k^c\}_{n\geq 0}$, we get

$$Q_n^{c,N}(x)=Q_n^c(x)-NQ_n^{c,N}(c)K_{n-1}^c(x,c).$$

In order to find $Q_n^{c,N}(c)$, we evaluate (30) in $x=c$. Thus

$$Q_n^{c,N}(c)=\frac{Q_n^c(c)}{1+NK_{n-1}^c(c,c)}. \tag{33}$$

This completes the proof. ■

Next, in order to prove the orthogonality of the polynomials defined by (21), we deal with the basis $\mathcal{B}^n=\{1,(x-c),(x-c)^2,\ldots,(x-c)^n\}$ of the space of polynomials of degree at most n. We prove that there exist a positive constant B_n^c such that every element in this basis is orthogonal to every polynomial of the sequence $\{\tilde{Q}_n^{c,N}\}_{n\geq 0}$ with respect to the inner product (17). Thus, from (18), (21) and $\tilde{Q}_n^{c,N}(c)=\kappa_n Q_n^{c,N}(c)$ we have

$$\langle 1,\tilde{Q}_n^{c,N}\rangle_{\nu_N}=\langle 1,Q_n^c(x)\rangle_\nu+NB_n^c\langle 1,(x-c)P_{n-1}^{c,[1]}(x)\rangle_\nu+N\kappa_n Q_n^{n,N}(c)=0.$$

Notice that to get $\langle 1,\tilde{Q}_n^c(x)\rangle_\nu=0$ for every $n>1$ we need

$$B_n^c=\frac{-\kappa_n Q_n^{c,N}(c)}{\langle 1,(x-c)P_{n-1}^{c,[1]}(x)\rangle_\nu}. \tag{34}$$

Next, we prove the orthogonality with respect to the other elements of $\mathcal{B}^n$. From (21), (19), (20) and the orthogonality with respect to $d\nu(x)$ and $d\mu^{[1]}(x)$ we get

$$\langle (x-c),Q_n^{c,N}(x)\rangle_{\nu_N}=\langle (x-c),Q_n^c(x)\rangle_\nu+NB_n^c\langle 1,P_{n-1}^{c,[1]}(x)\rangle_{\mu,[1]}=0,\quad n>1.$$

We continue in this fashion, verifying that

$$\langle (x-c)^{n-1},\tilde{Q}_n^{c,N}(x)\rangle_{\nu_N}=$$

$$\begin{aligned}&\langle (x-c)^{n-1},Q_n^c(x)\rangle_{\nu_N}+NB_n^c\langle (x-c)^{n-1},(x-c)P_{n-1}^{c,[1]}(x)\rangle_{\nu_N}\\ =\;&\langle (x-c)^{n-1},Q_n^c(x)\rangle_\nu+NB_n^c\langle (x-c)^{n-2},P_{n-1}^{c,[1]}(x)\rangle_{\mu,[1]}=0,\end{aligned}$$

and finally

$$\begin{aligned}\langle (x-c)^n,\tilde{Q}_n^{c,N}\rangle_{\nu_N}&=\|\tilde{Q}_n^{c,N}\|_{\nu_N}^2\\ &=\langle (x-c)^n,Q_n^c(x)\rangle_\nu+NB_n^c\langle (x-c)^{n-1},P_{n-1}^{c,[1]}(x)\rangle_{\mu,[1]}\\ &=\|Q_n^c\|_\nu^2+NB_n^c\|P_{n-1}^{c,[1]}\|_{\mu,[1]}^2.\end{aligned}$$

Sumarizing

$$\begin{aligned}
\langle 1, \tilde{Q}_n^{c,N}\rangle_{\nu_N} &= \langle 1, Q_n^c\rangle_\nu + NB_n^c\langle 1, P_{n-1}^{c,[1]}\rangle_\mu + NQ_n^c(c) = 0,\\
\langle (x-c), \tilde{Q}_n^{c,N}\rangle_{\nu_N} &= \langle (x-c), Q_n^c\rangle_\nu + NB_n^c\langle 1, P_{n-1}^{c,[1]}\rangle_{\mu,[1]} = 0,\\
&\vdots\\
\langle (x-c)^{n-1}, \tilde{Q}_n^{c,N}\rangle_{\nu_N} &= \langle (x-c)^{n-1}, Q_n^c\rangle_\nu + NB_n^c\langle (x-c)^{n-2}, P_{n-1}^{c,[1]}\rangle_{\mu,[1]}\\
&= 0,\\
\langle (x-c)^n, \tilde{Q}_n^{c,N}\rangle_{\nu_N} &= \|Q_n^c\|_\nu^2 + NB_n^c\|P_{n-1}^{c,[1]}\|_{\mu,[1]}^2.
\end{aligned}$$

Next, we briefly analyze the value of B_n^c in (34) and we prove (22). From (4) and (6) we have

$$\begin{aligned}
\langle 1, (x-c)P_{n-1}^{c,[1]}\rangle_\nu &= \langle 1, P_{n-1}^{c,[1]}(x)\rangle_\mu = \frac{\|P_{n-1}\|_\mu^2}{P_{n-1}(c)}\langle 1, K_{n-1}(x,c)\rangle_\mu\\
&= \frac{\|P_{n-1}\|_\mu^2}{P_{n-1}(c)}\sum_{k=0}^{n-1}\frac{P_k(c)}{||P_k||_\mu^2}\langle 1, P_k(x)\rangle_\mu.
\end{aligned}$$

Because the orthogonality, the only term which survive in the above sum is for $k=0$, hence

$$\sum_{k=0}^{n-1}\frac{P_k(c)}{||P_k||_\mu^2}\langle 1, P_k(x)\rangle_\mu = 1,$$

and therefore

$$\langle 1, (x-c)P_{n-1}^{c,[1]}\rangle_\nu = \frac{\|P_{n-1}\|_\mu^2}{P_{n-1}(c)}.$$

Thus, taking into account (31)

$$B_n^c = \frac{-Q_n^c(c)P_{n-1}(c)}{\|P_{n-1}\|_\mu^2}. \tag{35}$$

In order to prove (22), from (6), (16), (13) we get

$$\begin{aligned}
\langle (x-c), P_{n-1}^{c,[1]}(x)\rangle_{\nu} &= \int_E (x-c)\,\frac{1}{(x-c)}\left(P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x)\right) d\nu(x) \\
&= \int_E P_n(x)d\nu(x) - \frac{P_n(c)}{P_{n-1}(c)}\int_E P_{n-1}(x)d\nu(x) \\
&= \frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)}\int_E K_n^c(x,c)d\nu(x) - \frac{Q_{n-1}^c(c)}{\|Q_{n-1}^c\|_\nu^2}\frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)}\cdot \qquad (36)\\
&\qquad \frac{K_n^c(c,c)}{K_{n-1}^c(c,c)}\frac{\|Q_{n-1}^c\|_\nu^2}{Q_{n-1}^c(c)}\int_E K_{n-1}^c(x,c)d\nu(x) \\
&= \frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)} - \frac{Q_{n-1}^c(c)}{\|Q_{n-1}^c\|_\nu^2}\frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)}\frac{K_n^c(c,c)}{K_{n-1}^c(c,c)}\frac{\|Q_{n-1}^c\|_\nu^2}{Q_{n-1}^c(c)} \\
&= \frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)}\left(1 - \frac{K_n^c(c,c)}{K_{n-1}^c(c,c)}\right).
\end{aligned}$$

A general property for kernels is, from (12)

$$K_n^c(x,c) = \sum_{k=0}^{n}\frac{Q_k^c(x)Q_k^c(c)}{||Q_k^c||_\nu^2} = \frac{Q_n^c(x)Q_n^c(c)}{||Q_n^c||_\nu^2} + K_{n-1}^c(x,c)$$

and therefore

$$\left(1 - \frac{K_n^c(c,c)}{K_{n-1}^c(c,c)}\right) = \frac{-[Q_n^c(c)]^2}{||Q_n^c||_\nu^2 K_{n-1}^c(c,c)} \qquad (37)$$

Replacing in (36)

$$\langle (x-c), P_{n-1}^{c,[1]}(x)\rangle_\nu = \frac{\|Q_n^c\|_\nu^2}{Q_n^c(c)}\left(\frac{-[Q_n^c(c)]^2}{||Q_n^c||_\nu^2 K_{n-1}^c(c,c)}\right) = \frac{-Q_n^c(c)}{K_{n-1}^c(c,c)}.$$

Thus, from (33) and (34)

$$B_n^c = \frac{-\kappa_n Q_n^{c,N}(c)}{\langle 1, (x-c)P_{n-1}^{c,[1]}(x)\rangle_\nu} = \frac{-\kappa_n \frac{Q_n^c(c)}{1+NK_{n-1}^c(c,c)}}{\frac{-Q_n^c(c)}{K_{n-1}^c(c,c)}} = K_{n-1}^c(c,c).$$

Finally, being $c \notin \operatorname{supp}(\nu)$, from (15) we can conclude that B_n^c is always positive

$$B_n^c = K_{n-1}^c(c,c) > 0.$$

3.2. Proofs of Theorems 2 and 3. To apply the Interlacing Lemma and get the results of Theorems 2 and 3, we need to show that we satisfy the hypotheses of the Interlacing Lemma. To do this, we first prove that the zeros of $Q_n^c(x)$ and $(x-c)P_{n-1}^{c,[1]}(x)$ interlace.

Lemma 2. *Let $y_{n,k}^c$ and $x_{n,k}^{c,[1]}$ be the zeros of $Q_n^c(x)$ and $P_n^{c,[1]}(x)$, respectively, all arranged in an increasing order. The inequalities*

$$y_{n+1,1}^c < x_{n,1}^{c,[1]} < y_{n+1,2}^c < x_{n,2}^{c,[1]} < \cdots < y_{n+1,n}^c < x_{n,n}^{c,[1]} < y_{n+1,n+1}^c$$

hold for every $n \in \mathbb{N}$.

Proof: Combining (16) with (6) yields

$$(x-c)^2 P_n^{c,[1]}(x) = Q_{n+2}^c(x) - d_n^c Q_{n+1}^c(x) + e_n^c Q_n^c(x), \tag{38}$$

where

$$\begin{aligned} e_n^c &= \frac{P_{n+1}(c)}{P_n(c)} \frac{Q_{n+1}^c(c)}{Q_n^c(c)} \\ &= \frac{\|Q_{n+1}^c\|_\nu^2}{\|Q_n^c\|_\nu^2} \frac{K_{n+1}^c(c,c)}{K_n^c(c,c)} > 0, \end{aligned}$$

and

$$\begin{aligned} d_n^c &= \frac{Q_{n+2}^c(c)}{Q_{n+1}^c(c)} + \frac{P_{n+1}(c)}{P_n(c)} \\ &= \frac{Q_{n+2}^c(c)}{Q_{n+1}^c(c)} + \frac{Q_n^c(c)}{Q_{n+1}^c(c)} \frac{\|Q_{n+1}^c\|_\nu^2}{\|Q_n^c\|_\nu^2} \frac{K_{n+1}^c(c,c)}{K_n^c(c,c)} \\ &= \frac{Q_{n+2}^c(c) + Q_n^c(c) e_n^c}{Q_{n+1}^c(c)}. \end{aligned}$$

On the other hand, the sequence $\{Q_n^c\}_{n\geq 0}$ satisfies the three term recurrence relation

$$Q_n^c(x) = (x - \beta_n^c) Q_{n-1}^c(x) - \gamma_n^c Q_{n-2}^c(x), \quad n = 1, 2, \ldots \tag{39}$$

The coefficients β_n^c, and γ_n^c are given in several works. A particularly clear discussion about how to obtain β_n^c, γ_n^c from those β_n, γ_n of the initial μ is given in [10, 2.4.4]. From (11), for $n \geq 1$, the modified coefficients are given by

$$\begin{aligned}\beta_n^c &= \beta_n + r_n - r_{n-1},\\ \gamma_n^c &= \gamma_{n-1}\frac{r_{n-1}}{r_{n-2}},\end{aligned}$$

with the initial convention, for $n = 0$,

$$\begin{aligned}\beta_0^c &= \beta_0 + r_0,\\ \gamma_0^c &= \int_E d\nu(x) = \int_E \frac{1}{x-c}d\mu(x) = -F_0(c).\end{aligned}$$

Combining (38) with (39) yields

$$(x-c)^2 P_n^{c,[1]}(x) = (x - \beta_{n+2}^c - d_n^c)\, Q_{n+1}^c(x) + (e_n^c - \gamma_{n+2}^c)\, Q_n^c(x). \tag{40}$$

Being μ a positive definite measure, the modified measure $d\nu$ is also positive definite, because $c \notin E = \text{supp}(\mu)$ and therefore $(x-c)^{-1}$ do not change sign in E. Hence, by [6, Th. 4.2 (a)] and (37), the coefficient of $Q_n^c(x)$ in (40) can be expressed by

$$\begin{aligned} e_n^c - \gamma_{n+2}^c &= \frac{||Q_{n+1}^c||_\nu^2}{||Q_n^c||_\nu^2}\frac{K_{n+1}^c(c,c)}{K_n^c(c,c)} - \frac{||Q_{n+1}^c||_\nu^2}{||Q_n^c||_\nu^2}\\ &= \frac{||Q_{n+1}^c||_\nu^2}{||Q_n^c||_\nu^2}\left(\frac{K_{n+1}^c(c,c)}{K_n^c(c,c)} - 1\right)\\ &= \frac{1}{||Q_n^c||_\nu^2}\frac{[Q_{n+1}^c(c)]^2}{K_n^c(c,c)} > 0, \end{aligned} \tag{41}$$

which is positive for every $n \geq 0$, no matter the position of c with respect to the interval E.

Finally, evaluating $P_n^{c,[1]}(x)$ at the zeros $y_{n+1,k}$, from (40) and (41), we get

$$(x-c)^2 P_n^{c,[1]}(y_{n+1,k}) = (e_n^c - \gamma_{n+2}^c)\, Q_n^c(y_{n+1,k}), \quad k = 1, \ldots, n+1,$$

so it is clear that

$$sign(P_n^{c,[1]}(y_{n+1,k})) = sign(Q_n^c(y_{n+1,k})), \quad k = 1, \ldots, n+1. \tag{42}$$

Thus, from (42) and very well known fact that the zeros of $Q_{n+1}^c(x)$ interlace with the zeros of $Q_n^c(x)$, we conclude that $P_n^{c,[1]}(x)$ has at least one zero in every interval $(y_{n+1,k}, y_{n+1,k+1})$ for every $k = 1, \ldots n$. This completes the proof. ■

3.3. Proofs of Theorems 4 and 5. We begin by proving several lemmas that are needed for the proof of Theorem 4.

Lemma 3. *For the MOPS* $\{Q_n^{c,N}\}_{n\geq 0}$ *and* $\{P_n\}_{n\geq 0}$ *we have*

$$[Q_n^{c,N}(x)]' = C_1(x;n)P_n(x) + D_1(x;n)P_{n-1}(x), \tag{43}$$

where

$$\begin{aligned} C_1(x;n) &= \frac{1}{\sigma(x)}\left(a(x;n) - \Lambda_n^c \frac{b(x;n)}{\gamma_{n-1}}\right), \quad (44)\\ D_1(x;n) &= \frac{1}{\sigma(x)}\left(b(x;n) + \Lambda_n^c\, b(x;n-1)\left(\frac{a(x;n-1)}{b(x;n-1)} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}}\right)\right). \end{aligned}$$

The coefficient Λ_n^c *is given in (25),* β_{n-1}*,* γ_{n-1} *are given in (3) and* $\sigma(x)$*,* $a(x;n)$*,* $b(x;n)$ *come from the structure relation (26) satisfied by* $\{P_n\}_{n\geq 0}$*.*

Proof: Shifting the index in (26) as $n \to n-1$, and using (3) we obtain

$$\begin{aligned} [P_{n-1}(x)]' &= \frac{-b(x;n-1)}{\sigma(x)\,\gamma_{n-1}} P_n(x) \quad (45)\\ &\quad + \left(\frac{a(x;n-1)}{\sigma(x)} + \frac{b(x;n-1)(x-\beta_{n-1})}{\sigma(x)\,\gamma_{n-1}}\right) P_{n-1}(x). \end{aligned}$$

Next, taking x derivative in both sides of (24), we get

$$[Q_n^{c,N}(x)]' = [P_n(x)]' + \Lambda_n^c\, [P_{n-1}(x)]'.$$

Substituting (26) and (45) into the above expression the Lemma follows. ■

Lemma 4. *The sequences of monic polynomials* $\{Q_n^{c,N}\}_{n\geq 0}$ *and* $\{P_n\}_{n\geq 0}$ *are also related by*

$$\begin{aligned} Q_{n-1}^{c,N}(x) &= A_2(n)P_n(x) + B_2(x;n)P_{n-1}(x), \quad (46)\\ [Q_{n-1}^{c,N}(x)]' &= C_2(x;n)P_n(x) + D_2(x;n)P_{n-1}(x), \quad (47) \end{aligned}$$

where

$$\begin{aligned}
A_2(n) &= \frac{-\Lambda_n^c}{\gamma_{n-1}}, \\
B_2(x;n) &= \Lambda_{n-1}^c \left(\frac{1}{\Lambda_{n-1}^c} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}} \right), \\
C_2(x;n) &= -\frac{\Lambda_{n-1}^c}{\sigma(x)} \left(\frac{a(x;n)}{\gamma_{n-1}} + \frac{b(x;n-1)}{\gamma_{n-1}} \left(\frac{1}{\Lambda_{n-1}^c} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}} \right) \right) \qquad (48) \\
D_2(x;n) &= \frac{\Lambda_{n-1}^c}{\sigma(x)} \left[\frac{\sigma(x) - b(x;n)}{\gamma_{n-1}} + b(x;n-1) \times \right. \\
& \left. \left(\frac{a(x;n-1)}{b(x;n-1)} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}} \right) \left(\frac{1}{\Lambda_{n-1}^c} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}} \right) \right]
\end{aligned}$$

Proof: The proof of (46) and (47) is a straightforward consequence of (24), (26), Lemma 3, and the three term recurrence relation (3) for the MOPS $\{P_n\}_{n\geq 0}$. ■

Remark 2. *Observe that the set of coefficients (44) and (48) can be given strictly in terms of the following known quantities: the coefficient Λ_n^c in (25), the coefficients β_{n-1}, γ_{n-1} of the three term recurrence relation (3) and $\sigma(x)$, $a(x;n)$, $b(x;n)$ of the structure relation (26) satisfied by $\{P_n\}_{n\geq 0}$.*

The following lemma shows the converse relation of (24)–(46) for the polynomials $P_n(x)$ and $P_{n-1}(x)$. That is, we express these two consecutive polynomials of $\{P_n\}_{n\geq 0}$ in terms of only two consecutive Geronimus perturbed polynomials of the MOPS $\{Q_n^{c,N}\}_{n\geq 0}$.

Lemma 5.

$$P_n(x) = \frac{B_2(x;n)}{\Delta(x;n)} Q_n^{c,N}(x) - \frac{\Lambda_n^c}{\Delta(x;n)} Q_{n-1}^{c,N}(x), \qquad (49)$$

$$P_{n-1}(x) = \frac{\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}} Q_n^{c,N}(x) + \frac{1}{\Delta(x;n)} Q_{n-1}^{c,N}(x). \qquad (50)$$

where

$$\Delta(x;n) = \frac{\Lambda_{n-1}^c}{\gamma_{n-1}} \left(x - \beta_{n-1} + \Lambda_n^c + \frac{\gamma_{n-1}}{\Lambda_{n-1}^c} \right), \quad \deg \Delta(x;n) = 1.$$

Proof: Note that (24)–(46) can be interpreted as a system of two linear equations with two polynomial unknowns, namely $P_n(x)$ and $P_{n-1}(x)$, hence from Cramer's rule the lemma follows. ■

The proof of Theorem 4 easily follows from Lemmas 3, 4 and 5. Replacing (49)–(50) in (43) and (47) one obtains the ladder equations

$$\begin{aligned}[Q_n^{c,N}(x)]' &= \frac{C_1(x;n)B_2(x;n)\,\gamma_{n-1}+D_1(x;n)\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}}Q_n^{c,N}(x)\\ &\qquad +\frac{D_1(x;n)-C_1(x;n)\Lambda_n^c}{\Delta(x;n)}Q_{n-1}^{c,N}(x)\end{aligned}$$

and

$$\begin{aligned}[Q_{n-1}^{c,N}(x)]' &= \frac{C_2(x;n)B_2(x;n)\,\gamma_{n-1}+D_2(x;n)\Lambda_{n-1}^c}{\Delta(x;n)\,\gamma_{n-1}}Q_n^{c,N}(x)\\ &\qquad +\frac{D_2(x;n)-C_2(x;n)\Lambda_n^c}{\Delta(x;n)}Q_{n-1}^{c,N}(x),\end{aligned}$$

which are fully equivalent to (27)–(28). This completes the proof of Theorem 4.

Next, the proof of Theorem 5 comes directly from the ladder operators provided in Theorem 4. The usual technique (see, for example [17, Th. 3.2.3]) consists in applying the raising operator to both sides of the equation satistied by the lowering operator, i.e.

$$\mathfrak{a}_n^{\dagger}\left[\frac{1}{\eta_1^c(x;n)}\mathfrak{a}_n[Q_n^{c,N}(x)]\right]=\mathfrak{a}_n^{\dagger}\left[Q_{n-1}^{c,N}(x)\right],$$

which directly implies that

$$\mathfrak{a}_n^{\dagger}\left[\frac{1}{\eta_1^c(x;n)}\mathfrak{a}_n[Q_n^{c,N}(x)]\right]=\xi_2^c(x;n)\,Q_n^{c,N}(x)$$

is a second order differential equation for $Q_n^{c,N}(x)$. After some doable computations, Theorem 5 easily follows.

4. Zero behavior and electrostatic model for some examples.

Once we have the second order differential equation satisfied by the MOPS $\{Q_n^{c,N}\}_{n\geq 0}$ it is easy to obtain an electrostatic model for their zeros (see [17, Ch. 3], [16], [14], among others). In this Section we shall derive the electrostatic model for the ceros in case μ is the Laguerre and the Jacobi classical measures.

4.1. The Geronimus perturbed Laguerre case. Let $\{L_n^\alpha\}_{n\geq 0}$ be the monic Laguerre polynomials orthogonal with respect to the Laguerre classical measure $d\mu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, supported on $[0, +\infty)$. We will denote by $\{Q_n^{\alpha,c,N}\}_{n\geq 0}$ and $\{Q_n^{\alpha,c}\}_{n\geq 0}$ the MOPS corresponding to (1) and (8) when μ is the Laguerre classical measure, and $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$, $\{y_{n,s}^{\alpha,c}\}_{s=1}^n$ their corresponding zeros. The structure relation (26) for the monic classical Laguerre polynomials is

$$\sigma(x)[L_n^\alpha(x)]' = a(x;n)L_n^\alpha(x) + a(x;n)L_{n-1}^\alpha(x),$$

and therefore $\sigma(x) = x$, $a(x;n) = n$, and $b(x;n) = n(n+\alpha)$. Their three term recurrence relation is

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + \beta_n L_n^\alpha(x) + \gamma_n L_{n-1}^\alpha(x),$$

with $\beta_n = \beta_n^\alpha = 2n+\alpha+1$, $\gamma_n = \gamma_n^\alpha = n(n+\alpha)$, and the connection formula (24) for $Q_n^{\alpha,c,N}(x)$ in terms of $\{L_n^\alpha\}_{n\geq 0}$ reads

$$Q_n^{\alpha,c,N}(x) = L_n^\alpha(x) + \Lambda_n^{\alpha,c}\, L_{n-1}^\alpha(x).$$

Taking into account exclusively the coefficients in the above three expresions, from Theorems 4 and 5 we obtain the explicit expresions for the ladder operators and the coefficients in the holonomic equation for this first example. After some cumbersome computations, we get the following set of coefficients (44)–(48) for $\Lambda_n^{\alpha,c} = \Lambda_n^{\alpha,c}(N)$ in (24)

$$\begin{aligned}
C_1^\alpha(x;n) &= \frac{n-\Lambda_n^{\alpha,c}}{x},\\
D_1^\alpha(x;n) &= \frac{n(n+\alpha)+(x-(n+\alpha))\Lambda_n^{\alpha,c}}{x},\\
A_2^\alpha(x;n) &= \frac{-\Lambda_n^{\alpha,c}}{(n-1)(n+\alpha-1)},\\
B_2^\alpha(x;n) &= 1+\Lambda_{n-1}^{\alpha,c}\frac{(x+1-2n+\alpha)}{(n-1)(n+\alpha-1)},\\
C_2^\alpha(x;n) &= \frac{-1}{x}-\Lambda_{n-1}^{\alpha,c}\frac{x+1-(n+\alpha)}{x(n-1)(n-1+\alpha)},\\
D_2^\alpha(x;n) &= \frac{x-(n+\alpha)}{x}\\
&\quad +\Lambda_{n-1}^{\alpha,c}\frac{(x+1-2n+\alpha)(x-(n+\alpha))+(x-n(n+\alpha))}{x(n-1)(n-1+\alpha)}.
\end{aligned}$$

Hence, they satisfy the holonomic equation

$$[Q_n^{\alpha,c,N}(x)]'' + \mathcal{R}_L(x;n)[Q_n^{\alpha,c,N}(x)]' + \mathcal{S}_L(x;n)Q_n^{\alpha,c,N}(x) = 0, \tag{51}$$

with coefficients

$$\begin{aligned}
\mathcal{R}_L(x;n) &= -\frac{\Lambda_n^{\alpha,c}}{\Lambda_n^{\alpha,c}x + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})} + \frac{\alpha+1}{x} - 1,\\
\mathcal{S}_L(x;n) &= \frac{\Lambda_n^{\alpha,c}x + (n+\alpha)(n-\Lambda_n^{\alpha,c})}{x\left(\Lambda_n^{\alpha,c}x + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})\right)} + \frac{n-1}{x}.
\end{aligned}$$

Now we evaluate (51) at the zeros $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$, yielding

$$\begin{aligned}
\frac{[Q_n^{\alpha,c,N}(x)]''}{[Q_n^{\alpha,c,N}(x)]'} &= -\mathcal{R}_L(y_{n,s}^{\alpha,c,N};n)\\
&= \frac{\Lambda_n^{\alpha,c}}{\Lambda_n^{\alpha,c}y_{n,s}^{\alpha,c,N} + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})} - \frac{\alpha+1}{y_{n,s}^{\alpha,c,N}} + 1.
\end{aligned}$$

The above equation reads as the electrostatic equilibrium condition for the zeros $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$. Taking $u_L(n;x) = \Lambda_n^{\alpha,c}x + (n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})$, the above condition can be rewritten as (see [13], [17], or [16] for a treatment of more general cases and other examples)

$$\sum_{j=1,\, j\neq k}^{n} \frac{1}{y_{n,j}^{\alpha,c,N} - y_{n,k}^{\alpha,c,N}} + \frac{1}{2}\frac{[u_L]'(n;y_{n,k}^{\alpha,c,N})}{u_L(n;y_{n,k}^{\alpha,c,N})} - \frac{1}{2}\frac{\alpha+1}{y_{n,s}^{\alpha,c,N}} + \frac{1}{2} = 0$$

which means that the set of zeros $\{y_{n,s}^{\alpha,c,N}\}_{s=1}^n$ are the critical points of the gradient of the total energy. Hence, the electrostatic interpretation of the distribution of zeros means that we have an equilibrium position under the presence of an external potential

$$V_L^{ext}(x) = \frac{1}{2}\ln\, u_L(x;n) - \frac{1}{2}\ln\, x^{\alpha+1}e^{-x}, \tag{52}$$

where the first term represents a *short range potential* corresponding to a unit charge located at the unique real zero

$$z_L(n;x) = \frac{-1}{\Lambda_n^{\alpha,c}}(n-\Lambda_n^{\alpha,c})(n+\alpha-\Lambda_n^{\alpha,c})$$

of the linear polynomial $u_L(x;n)$, and the second one is a *long range potential* associated with the Laguerre weight function.

Finally, in order to illustrate the results of Theorem 2, we consider the Geronimus perturbation on the Laguerre measure with $\alpha = 0$ and $c = -1$

$$d\nu_N(x) = \frac{1}{(x+1)} e^{-x} dx + N\delta(x+1), \quad N \geq 0, \tag{53}$$

and we obtain the behavior of the zeros $\{y_{n,s}^{0,-1,N}\}_{s=1}^{n}$ as N increases. We enclose in Figure 1 the graphs of $L_3^0(x)$ (dotted line), $Q_3^{0,-1}(x)$ (dash-dotted line), and $Q_3^{0,-1,N}(x)$ for some N, to show the monotonicity of their zeros as a function of the mass N. Table 1 shows the behavior of the zeros of

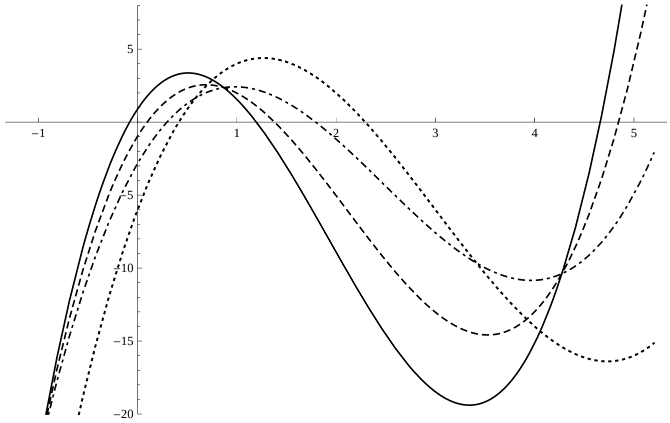

FIGURE 1. The graphs of $L_3^0(x)$ (dotted) and $Q_3^{0,-1,N}(x)$ for some values of N.

$Q_3^{0,-1,N}(x)$ for several choices of N. Observe that the smallest zero converges to $c = -1$ and the other two zeros converge to the zeros of the monic kernel polynomial $L_2^{0,-1,[1]}(x)$, as is shown in Theorem 2. That is, they converge to $x_{2,1}^{0,-1,[1]} = 0.869089$ and $x_{2,2}^{0,-1,[1]} = 4.273768$. Notice that all the zeros decrease as N increases. The zeros outside the interval $[0, +\infty)$, namely the support of the classical Laguerre measure, appear in bold.

Remark 3. *Looking at the external potential (52) there are few significant differences with respect to the Uvarov case (see* [14, Sec. 4.2]*). First, the long range potential does not depend on the shift c, as occurs in the Uvarov case, where the long range potential corresponds to a polynomial perturbation of the Laguerre measure by $(x - c)$. Second, in the Uvarov case the polynomial $u_L(x; n)$ has two different real roots when $c < 0$, away from the boundary $[0, +\infty)$, meanwhile in this case there exists only one real root for $u_L(x; n)$. It means that the inclusion of the linear rational modification of the measure*

N	*1st*	*2nd*	*3rd*	*z(N)*
0	0.296771	1.794881	5.327153	−1.27309
0.0125	0.096936	1.381317	4.846199	−0.039345
0.025	**−0.079531**	1.196907	4.66079	−0.015274
0.05	**−0.324373**	1.050055	4.50679	−0.156362
5	**−0.988481**	0.87094	4.276644	−0.700057

TABLE 1. Zeros of $Q_3^{0,-1,N}(x)$ and $z(0,-1,3,N;x)$ for some values of N.

present in the Geronimus transformation has notable dynamical consequences on the electrostatic model.

4.2. The Geronimus perturbed Jacobi case. The electrostatic model in case μ is the Jacobi classical measure is essentially the same as in the former case, but considering the corresponding values and expressions for the Jacobi measure, and the shift c away of its support. Since the exact formulas are cumbersome, we will not write them all down, except the most significant ones.

Let $\{P_n^{\alpha,\beta}\}_{n\geq 0}$ be the monic Jacobi polynomials orthogonal with respect to the Jacobi classical measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$, $\alpha,\beta > -1$, supported on $[-1,1]$. We will denote by $\{Q_n^{\alpha,\beta,c,N}\}_{n\geq 0}$ and $\{Q_n^{\alpha,\beta,c,N}\}_{n\geq 0}$ the MOPS corresponding to (1) and (8) when μ is the Jacobi measure, and $\{y_{n,s}^{\alpha,\beta,c,N}\}_{s=1}^n$, $\{y_{n,s}^{\alpha,\beta,c}\}_{s=1}^n$ their corresponding zeros.

The structure relation for the monic Jacobi polynomials reads

$$\sigma(x)[P_n^{\alpha,\beta}(x)]' = a(x;n)P_n^{\alpha,\beta}(x) + b(x;n)P_{n-1}^{\alpha,\beta}(x),$$

with

$$\begin{aligned} \sigma(x) &= (1-x^2), \\ a(x;n) &= -n(1+x) + \frac{2n(n+\alpha)}{(2n+\alpha+\beta)}, \\ b(x;n) &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2}. \end{aligned}$$

Their three term recurrence relation is

$$xP_n^{\alpha,\beta}(x) = P_{n+1}^{\alpha,\beta}(x) + \beta_n P_n^{\alpha,\beta}(x) + \gamma_n P_{n-1}^{\alpha,\beta}(x),$$

with

$$\begin{aligned}\beta_n &= \beta_n^{\alpha,\beta} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},\\ \gamma_n &= \gamma_n^{\alpha,\beta} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\end{aligned}$$

and the connection formula (24) for $Q_n^{\alpha,c,N}(x)$ in terms of $\{L_n^{\alpha}\}_{n\geq 0}$ is

$$Q_n^{\alpha,\beta,c,N}(x) = P_n^{\alpha,\beta}(x) + \Lambda_n^{\alpha,\beta,c}\, P_{n-1}^{\alpha,\beta}(x).$$

The coefficient of $[Q_n^{\alpha,\beta,c,N}(x)]'$ in the holonomic equation is

$$\mathcal{R}_L(x;n) = -\frac{[u_J]'(n;x)}{u_J(n;x)} - \frac{2x-\beta\,(1-x)+\alpha\,(1+x)}{(1-x)(1+x)},$$

with

$$\begin{aligned}u_J(n;x) &= 4n(n+\alpha)(n+\beta)(n+\alpha+\beta) \qquad (54)\\ &\quad +(2n+\alpha+\beta-1)(2n+\alpha+\beta)\Lambda_n^{\alpha,\beta,c}\cdot\\ &\quad \Big[(2n+\alpha+\beta)^2 x + (\alpha+\beta)\,(\alpha-\beta)\\ &\quad + (2n+\alpha+\beta-1)(2n+\alpha+\beta)\Lambda_n^{\alpha,\beta,c}\Big].\end{aligned}$$

Hence, the electrostatic equilibrium means that the set of zeros $\{y_{n,s}^{\alpha,\beta,c,N}\}_{s=1}^{n}$ have an equilibrium position under the presence of the external potential

$$V_J^{ext}(x) = \frac{1}{2}\ln\, u_J(x;n) - \frac{1}{2}\ln\,(1-x)^{\alpha+1}(1+x)^{\beta+1},$$

where the first term represents a *short range potential* corresponding to a unit charge located at the real root

$$\begin{aligned}z_J(x;n) &= -\frac{(\alpha^2-\beta^2)(2n+\alpha+\beta)+\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)\Lambda_n^{\alpha,\beta,c}}}{(2n+\alpha+\beta)^3}\\ &\quad -\frac{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2\Lambda_n^{\alpha,\beta,c}}{(2n+\alpha+\beta)^3}\end{aligned}$$

of (54), and the second one is a *long range potential* associated with the Jacobi weight function. Observe that, as in the Laguerre case, the long range potential does not depend on the shift c.

Finally, we analyze the consequences of Theorem 2 and 3), for a Geronimus perturbation on the Laguerre measure with $\alpha = 0.5$, $\beta = 1$ and $c = -1.5$

$$d\nu_N(x) = \frac{(1-x)^{0.5}(1+x)}{(x+1.5)}dx + N\delta(x+1.5), \quad N \geq 0,$$

and we obtain the behavior of the zeros $\{y_{n,s}^{0.5,1,-1.5,N}\}_{s=1}^{n}$ as N increases. We provide in Figure 1 the graphs of $P_4^{0.5,1}(x)$ (dotted line), $Q_4^{0.5,1,-1}(x)$ (dash-dotted line), and $Q_4^{0.5,1,-1.5,N}(x)$ for some N, to show the monotonicity of their zeros as a function of the mass N (See Figure 2). Table 2 shows the

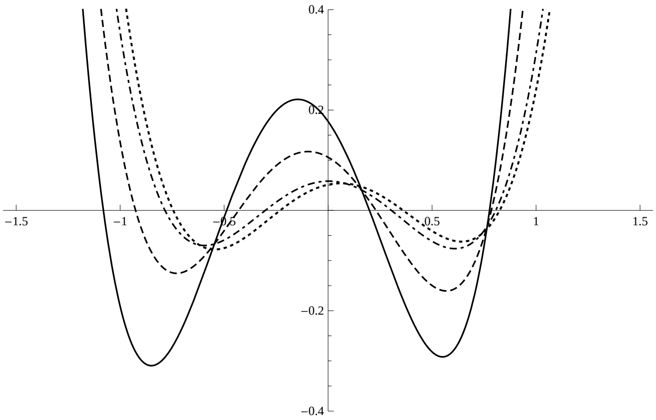

FIGURE 2. The graphs of $P_4^{0.5,1}(x)$ (dotted) and $Q_4^{0.5,1,-1.5,N}(x)$ for some values of N.

zeros $\{y_{4,s}^{\alpha,\beta,c,N}\}_{s=1}^{4}$ of $Q_4^{\alpha,\beta,c,N}(x)$ having $\alpha = 0.5$, $\beta = 1$, and $c = -1.5$ for several values of N. Observe that the smallest zero $y_{4,1}^{0.5,1,-1.5,N}$ converges to $c = -1.5$ and the other three zeros converge to the zeros $\{x_{3,s}^{0.5,1,-1.5,[1]}\}_{s=1}^{3}$ of the kernel polynomial $P_3^{0.5,1,-1.5,[1]}(x)$, as states Theorem 2. That is, they converge respectively to $x_{3,1}^{0.5,1,-1.5,[1]} = -0.546629$, $x_{3,2}^{0.5,1,-1.5,[1]} = 0.161665$, and $x_{3,3}^{0.5,1,-1.5,[1]} = 0.765232$. The zeros outside the interval $[-1, 1]$, namely the support of the classical Jacobi measure, appear in bold.

5. Appendix. The interlacing lemma

Next, we will analyze the behavior of zeros of polynomial of the form $f(x) = h_n(x) + cg_n(x)$. We need the following lemma concerning the behavior and the asymptotics of the zeros of linear combinations of two polynomials with interlacing zeros (see [2, Lemma 1], [8, Lemma 3] for a detailed discussion).

N	*1st*	*2nd*	*3rd*	*4rd*	*z(N)*
0	−0.784545	−0.302212	0.304654	0.806277	−1.61637
0.0008	−0.925906	−0.430453	0.230271	0.784909	−0.97778
0.0020	**−1.080633**	−0.488136	0.199190	0.776221	−1.04893
0.05	**−1.467364**	−0.544057	0.163585	0.765818	−1.35837
5	**−1.499661**	−0.546604	0.161684	0.765238	−1.38587

TABLE 2. Zeros of $Q_4^{0.5,1,-1.5,N}(x)$ and $z_J(x;n)$ for some values of N.

Lemma 6 (Interlacing Lemma). *Let $h_n(x) = a(x - x_1)\cdots(x - x_n)$ and $g_n(x) = b(x-\zeta_1)\cdots(x-\zeta_n)$ be polynomials with real and simple zeros, where a and b are real positive constants.*

(*i*) *If*

$$\zeta_1 < x_1 < \cdots < \zeta_n < x_n,$$

then, for any real constant $c > 0$, the polynomial

$$f(x) = h_n(x) + cg_n(x)$$

has n real zeros $\eta_1 < \cdots < \eta_n$ which interlace with the zeros of $h_n(x)$ and $g_n(x)$ in the following way

$$\zeta_1 < \eta_1 < x_1 < \cdots < \zeta_n < \eta_n < x_n.$$

Moreover, each $\eta_k = \eta_k(c)$ is a decreasing function of c and, for each $k = 1, \ldots, n$,

$$\lim_{c\to\infty} \eta_k = \zeta_k \quad \text{and} \quad \lim_{c\to\infty} c[\eta_k - \zeta_k] = \frac{-h_n(\zeta_k)}{g_n'(\zeta_k)}.$$

(*ii*) *If*

$$x_1 < \zeta_1 < \cdots < x_n < \zeta_n,$$

then, for any positive real constant $c > 0$, the polynomial

$$f(x) = h_n(x) + cg_n(x)$$

has n real zeros $\eta_1 < \cdots < \eta_n$ which interlace with the zeros of $h_n(x)$ and $g_n(x)$ as follows

$$x_1 < \eta_1 < \zeta_1 < \cdots < x_n < \eta_n < \zeta_n.$$

Moreover, each $\eta_k = \eta_k(c)$ is an increasing function of c and, for each $k = 1, \ldots, n$,

$$\lim_{c\to\infty} \eta_k = \zeta_k \quad \text{and} \quad \lim_{c\to\infty} c[\zeta_k - \eta_k] = \frac{h_n(\zeta_k)}{g_n'(\zeta_k)}.$$

References

[1] D. Barrios and A. Branquinho, *Complex high order Toda and Volterra lattices*, J. Difference Equ. Appl. **15** (2) (2009), 197–213.

[2] C. F. Bracciali, D. K. Dimitrov, and A. Sri Ranga, *Chain sequences and symmetric generalized orthogonal polynomials*, J. Comput. Appl. Math. **143** (2002), 95–106.

[3] A. Branquinho and F. Marcellán, *Generating new classes of orthogonal polynomials*, Int. J. Math. Math. Sci. **19** (4) (1996), 643–656.

[4] M. I. Bueno, A. Deaño, and E. Tavernetti, *A new algorithm for computing the Geronimus transformation with large shifts*, Numer. Alg. **54** (2010), 101–139.

[5] M. I. Bueno and F. Marcellán, *Darboux tranformations and perturbation of linear functionals*, Linear Algebra Appl. **384** (2004), 215–242.

[6] T. S. Chihara, *An Introduction to Orthogonal Polynomials*. Mathematics and its Applications Series, Gordon and Breach, New York, 1978.

[7] M. Derevyagin and F. Marcellán, *A note on the Geronimus transformation and Sobolev orthogonal polynomials*, Numer. Algorithms, DOI 10.1007/s11075-013-9788-6, (2013), 1-17.

[8] D. K. Dimitrov, M. V. Mello, and F. R. Rafaeli, *Monotonicity of zeros of Jacobi-Sobolev type orthogonal polynomials*, Appl. Numer. Math. **60** (2010), 263–276.

[9] J. Dini, P. Maroni, *La multiplication d'une forme linéaire par une forme rationnelle. Application aux polynômes de Laguerre-Hahn*, Ann. Polon. Math. **52** (1990), 175–185.

[10] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, in Numerical Mathematics and Scientific Computation Series, Oxford University Press. New York. 2004.

[11] Y. L. Geronimus, *On the polynomials orthogonal with respect to a given number sequence and a theorem*, in: W. Hahn and I. A. Nauk, (eds.), vol. **4** (1940), 215–228 (in Russian).

[12] Y. L. Geronimus, *On the polynomials orthogonal with respect to a given number sequence*, Zap. Mat. Otdel. Khar'kov. Univers. i NII Mat. i Mehan. **17** (1940), 3–18.

[13] F. A. Grünbaum, *Variations on a theme of Heine and Stieltjes: An electrotatic interpretation of the zeros of certain polynomials*, J. Comp. Appl. Math. **99** (1998), 189–194.

[14] E. J. Huertas, F. Marcellán and F. R. Rafaeli, *Zeros of orthogonal polynomials generated by canonical perturbations of measures*, Appl. Math. Comput. **218** (13) (2012), 7109-7127.

[15] E. Hendriksen and H. Van Rossum, *Semiclassical orthogonal polynomials*, Lect. Notes in Math., Springer Verlag (1985) Vol. **1171**, Cl. Brezinski et coll. ed., p. 354–361.

[16] M. E. H. Ismail, *More on electrostatic models for zeros of orthogonal polynomials*, Numer. Funct. Anal. Optimiz. **21** (2000), 191–204.

[17] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, Vol. **98**. Cambridge University Press, Cambridge UK. 2005.

[18] F. Marcellán and P. Maroni, *Sur l'adjonction d' une masse de Dirac à une forme réguliére et semi-classique*, Annal. Mat. Pura ed Appl. **162** (1992), 1–22.

[19] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x-c)^{-1}L$*, Period. Math. Hungar. **21** (3) (1990), 223–248.

[20] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, in *Orthogonal Polynomials and Their Applications*, C. Brezinski et al. Editors. Annals. Comput. Appl. Math. **9**. Baltzer, Basel. 1991, 95–130.

[21] A. Ronveaux and F. Marcellán, *Differential equation for classical-type orthogonal polynomials*, Canad. Math. Bull. **32** (4), (1989), 404–411.

[22] J. Shohat, *On mechanical quadratures, in particular, with positive coefficients*, Trans. Amer. Math. Soc. **42** (3) (1937), 461–496.

[23] V. Spiridonov, A. Zhedanov, *Discrete Darboux transformations, the discrete-time Toda lattice, and the Askey-Wilson polynomials*, Methods Appl. Anal. **2** (4) (1995), 369–398.

[24] V. Spiridonov, A. Zhedanov, *Discrete-time Volterra chain and classical orthogonal polynomials*, J. Phys. A: Math. Gen. **30** (1997), 8727–8737.

[25] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ, Vol. **23**, 4th ed., Amer. Math. Soc., Providence, RI, 1975.

[26] V. B. Uvarov, *The connection between systems of polynomials orthogonal with respect to different distribution functions*, USSR Compt. Math. Phys. **9** (6) (1969), 25–36.

[27] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85** (1997), 67–83.

[28] G. J. Yoon, *Darboux transforms and orthogonal polynomials*, Bull. Korean Math. Soc. **39** (2002), 359–376.

Amílcar Branquinho
CMUC, Department of Mathematics, Universidade de Coimbra, FCTUC, Largo D. Dinis, 3001-454 Coimbra, Portugal

E-mail address: ajplb@mat.uc.pt

Edmundo J. Huertas
CMUC, Department of Mathematics, Universidade de Coimbra, FCTUC, Largo D. Dinis, 3001-454 Coimbra, Portugal

E-mail address: ehuertas@mat.uc.pt

Fernando R. Rafaeli
Departamento de Matemática Aplicada, IBILCE, Universidade Estadual Paulista, Brazil

E-mail address: rafaeli@ibilce.unesp.br