Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 14–09

KNOT COMPLEMENTS WITH MERIDIONAL ESSENTIAL SURFACES OF ARBITRARILY HIGH GENUS

JOÃO MIGUEL NOGUEIRA

ABSTRACT: We show the existence of infinitely many prime knots each of which having in its complement meridional essential surfaces with two boundary components and arbitrarily high genus.

KEYWORDS: Essential surface, meridional surface.
AMS SUBJECT CLASSIFICATION (2010): 57M25, 57N10.

1. Introduction

Let K be a knot in S^3 with complement $E(K) = S^3 - N(K)$, where $N(K)$ is a regular neighborhood of K, and let F be a properly embedded compact surface in $E(K)$. We say that F is *meridional* if the boundary components of F are meridians of the torus $\partial N(K)$. The surface F is said *incompressible* if for each disk D embedded in $E(K)$ with $\partial D = F \cap D$ then ∂D bounds a disk in F. We also say that F is *boundary incompressible* if for each disk D embedded in $E(K)$ with $D \cap F$ an arc a, $D \cap E(K)$ an arc a and $\partial D = a \cup b$, then a co-bounds a disk in F. If F is incompressible and boundary incompressible, and not boundary-parallel, we say that F is *essential*.
Essential surfaces have a very important role in knot theory (and 3-manifold topology). In particular, it has been a subject of active research studying the existence of closed essential surfaces or meridional essential surfaces in knot complements. This existence question has been approached for several classes of knots, for instance: 2-bridge knots [5] and [7]; Montesinos knots [14]; fibered knots [11]; links with a certain $2n$-plat projection [4] and [10]; tunnel number one knots [6], [1] and [2]; closed 3-braids [3] and [9].
A particularly interesting phenomena is the existence of knots with the property that their complements have closed essential surfaces of arbitrarily high genus. The first examples of knots with this property were given by Lyon

Received February 27, 2014.
This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.

[11]* using fibered knots. Later Oertel [14] and recently Li [8] also give examples of knots having closed essential surfaces of arbitrarily high genus in their complements.

Considering meridional surfaces instead, we show in this paper that there are also no general bounds for the genus of meridional essential surfaces in the complements of (prime) knots by proving the following theorem and its corollary.

Theorem 1. *There are infinitely many prime knots each of which having the property that its complement has a meridional essential surface of genus g and two boundary components for all positive g.*

Corollary 1.1. *There are infinitely many knots each of which having the property that its complement has a meridional essential surface of genus g and two boundary components for all $g \geq 0$.*

A composite knot is a knot with a meridional essential annulus in its complement. Then, in particular, Theorem 1 states that some prime knots have the property that they can be decomposed by surfaces of all positive genus as composite knots are decomposed by spheres.

The proof of Theorem 1 follows a similar approach as for the examples given by Lion in [11] and by Li in [8]. However, instead of using composite knots we use a decomposition of prime knots along certain essential tori separating the knot into two arcs. The main techniques for the proof are classical in 3-manifold topology and the reference used for standard definitions and notation in knot theory is Rolfsen's book [15]. Throughout this paper we work in the piecewise linear category.

2. Construction of the knots

Let H be a solid torus and γ an embedded graph in H, as in Figure 1.

The graph γ is topologically a circle connected to two segments, a_1 and a_2, at a boundary point of each. The other two boundary points of $a_1 \cup a_2$ are in ∂H. There is a separating disk D_H in H intersecting γ transversely at a point of each segment a_1 and a_2, and decomposing H into a solid torus and a 3-ball B_H where $(B_H, B_H \cap \gamma)$ is a 2-string essential free tangle† with $B_H \cap \gamma$

*In this paper [11] Lyon also presents an example of a knot with essential spanning surfaces of arbitrarily high genus.

†See the Appendix, section 4, for a definition of n-string essential free tangle and an example of a 2-string essential free tangle with both strings knotted.

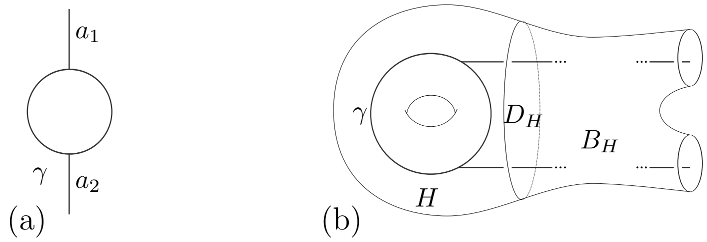

FIGURE 1: The graph γ, in (a), and its embedding into the solid torus H, in (b).

two knotted arcs in B_H. (See Figure 1(b).)

Denote by T a regular neighborhood of γ in H and suppose there is a properly embedded arc s in T, as in Figure 2(a), with the boundary of s in $N(\gamma) \cap \partial H$. There is a separating disk D_T in T, intersecting s at two points

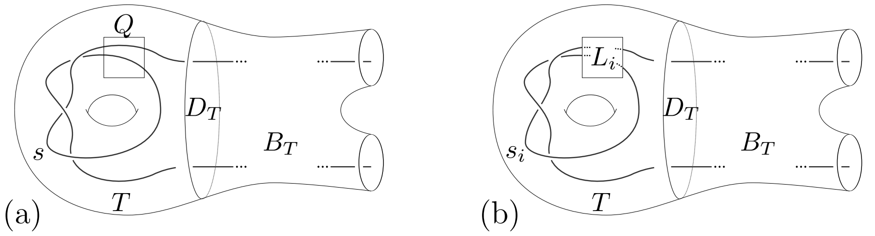

FIGURE 2: The solid torus T with the string s, in (a), and the solid torus T_i with the string s_i in T_i, in (b).

and decomposing T into a 3-ball B_T and a solid torus, where $s \cap \partial T$ is in ∂B_T, the two arcs $B_T \cap s$ in B_T are knotted, and the tangle $(B_T, B_T \cap s)$ is essential and free.

We say that an arc properly embedded in a solid torus is *essential* if it isn't *boundary parallel*, that is the arc doesn't co-bound an embedded disk in the solid torus with a segment in the boundary of the solid torus, and if the boundary of the solid torus is incompressible in the complement of the arc. In Lemma 1 we prove that s is essential in T.

Consider a ball Q in $T - B_T$ intersecting s at two parallel trivial arcs, as in Figure 2(a), and an infinite collection of knots L_i, $i \in \mathbb{N}$. We replace the two parallel trivial arcs by two parallel arcs with the pattern[‡] of a knot L_i, as in Figure 2(b). After this tangle replacement, we denote by s_i the string obtained from s, by T_i the solid torus T containing s_i, by γ_i the graph γ whose regular neighborhood is T_i, and by H_i the solid torus H containing T_i.

[‡]By a properly embedded arc in a ball B having the *pattern* of a knot K we mean that when we cap off the arc with a string in ∂B we get the knot K.

Let $E_H(T)$ be the exterior of T in H, that is the closure of $H - T$, and $E_T(s)$ be the exterior of $N(s)$ in T, that is the closure of $T - N(s)$. The following lemmas are relevant for the next section, and they are also valid if we replace s by s_i, T by T_i and H by H_i in their statements.

Lemma 1.
(a) *The surfaces ∂H and ∂T are incompressible in $E_H(T)$.*
(b) *The arc s is essential in T.*

Proof:
(a) First we prove that ∂H is incompressible in $E_H(T)$. As T is a regular neighborhood of γ this is equivalent to prove that ∂H is incompressible in $H - \gamma$. The graph γ in H is defined by a circle c and two segments a_1 and a_2, each with an end in the circle and the other in ∂H, and H is a regular neighborhood of c. Hence, the boundary of a properly embedded disk D in H disjoint from c bounds a disk O in ∂H. Furthermore, as D is disjoint from γ and each segment a_1 and a_2 intersects ∂H at a single point, the disk O is disjoint from γ. Then, the boundary of every embedded disk in $E_H(T)$ with boundary in ∂H bounds a disk in $\partial H - \partial H \cap \partial T$, which means ∂H is incompressible in $E_H(T)$.
We prove similarly that ∂T is incompressible in $E_H(T)$. Let D be a properly embedded disk in $E_H(T)$ with boundary in ∂T. We have $T = N(c) \cup N(a_1) \cup N(a_2)$. As a_1 and a_2 have each an end in ∂H and in c, we can isotope the boundary of D to $N(c)$. As H is a regular neighborhood of c we have that ∂D bounds a disk O in $\partial N(c)$. As l_1 and l_2 only have an end in c, we have that O is a disk in ∂T. Hence, ∂T is incompressible in $E_H(T)$.

(b) To prove that s is essential T we have to prove that ∂T is incompressible in $E_T(s)$ and that s isn't boundary parallel. We start by showing that ∂T is incompressible in $E_T(s)$. As the tangle $(B_T, B_T \cap s)$ is essential, the boundary of a properly embedded disk in $B_T - B_T \cap s$ bounds a disk in $\partial B_T - (\partial B_T \cap s)$. Let R_T be the solid torus separated from T by D_T. Then, from construction $R_T \cap s$ is a string in the solid torus R_T that when capped off by an arc in D_T we get the $(2, 3)$-torus knot boundary parallel in R_T. Hence, every disk in $R_T - R_T \cap s$ with boundary in ∂T has boundary bounding a disk in ∂T. Also, if a disk in $R_T - R_T \cap s$ has boundary in D_T then its boundary bounds a disk in D_T disjoint from $s \cap D_T$. Suppose D is a disk properly embedded in $E_T(s)$ with boundary in ∂T. From the previous observations, the disk

D isn't disjoint from D_T. We assume that D intersects D_T transversely in a collection of arcs and simple closed curves, with $|D \cap D_T|$ minimal. If D intersects D_T in simple closed curves then consider an innermost one in D and the respective innermost disk O. From the previous observations, we have that ∂O bounds a disk in D_T. Therefore, by an isotopy of D along the ball bounded by $D \cup D_T$ we can reduce $|D \cap D_T|$, which contradicts its minimality. Hence, $D \cap D_T$ is a collection of arcs. Consider an outermost arc α between the arcs $D \cap D_T$ in D and the respective outermost disk, that we also denote by O. If O is in B_T then ∂O bounds a disk O' in ∂B_T intersecting $D_T - s$ at a disk. Suppose O is in R_T. If O is essential in R_T then O intersects at least twice the $(2,3)$-torus knot obtained from $R_T \cap s$ by capping off the ends of this string in D_T. However, ∂O intersects at most once this knot, wether α separates the components of $D_T \cap s$ in D_T or not. This implies that O intersects $R_T \cap s$, which is contradiction with O being disjoint from s. Therefore, O is inessential in R_T and ∂O bounds a disk O' in ∂R_T intersecting $D_T - s$ at a disk. In both cases, ∂O bounds a disk O' intersecting $D_T - s$ at a disk. If we isotope D along the ball bounded by $O \cup O'$ we reduce $|D \cap D_T|$ and contradict its minimality. Hence, ∂T is incompressible in $E_T(s)$.
Now we prove that s isn't boundary parallel in T. Suppose now that D is embedded in T co-bounded by s and an arc b in ∂T. Following a similar argument as before we can prove that D doesn't intersect D_T at simple closed curves and arcs with both ends in b. Hence, $D \cap D_T$ is a collection of two arcs, each with an end in s and the other end in a. However, the disk components these arcs separate from D imply that the strings of the tangle $(B_T, B_T \cap s)$ are trivial, which contradicts this tangle being essential. Hence, s is not boundary parallel in T and, together with ∂T being incompressible in the exterior of s in T, we have that s is essential in T. ■

Lemma 2. *There is no properly embedded disk in $E_H(T)$*

(a) *intersecting one of the disks of $T \cap \partial H$ at a single point; or*
(b) *with boundary the union of an arc in ∂T and an arc in ∂H, and not bounding a disk in $\partial E_H(T)$.*

Proof: Let D be a properly embedded disk in $E_H(T)$. Following an argument as in Lemma 1 we can assume that $|D \cap D_H|$ is minimal and that $D \cap D_H$ is a collection of essential arcs in $D_H - D_H \cap T$ with ends in $D_H \cup T$ and ∂D_H. Consider also $B_H \cap T$, which is a collection of two cylinders C_1 and C_2, and assume that $|D \cap (B_H \cap T)|$ is minimal. If some arcs of $\partial D \cap C_i$ have ends in the same boundary component of the annlus $\partial C_i - C_i \cap \partial B_H$, $i = 1, 2$, then by using an innermost curve argument we can reduce $|D \cap (B_H \cap T)|$ and contradict its minimality. Therefore, $D \cap C_i$ is a collection of essential arcs in the annulus $\partial C_i - C_i \cap \partial B_H$, $i = 1, 2$.

(a) Assume $C_1 \cap \partial H$ is the disk of $T \cap \partial H$ that D intersects exactly once. The components of $D \cap B_H$ are a collection of disks. Consequently, one component of $D \cap B_H$ is a disk in $B_H - B_H \cap T$ intersecting $C_1 \cap \partial H$ once. This means that $C_1 \cap \partial H$ is primitive with respect to the complement of $B_H \cap T$ in B_H. Hence, as the complement of $C_1 \cup C_2$ is a handlebody (because $(B_H, B_H \cap \gamma)$ is a free tangle), the complement of C_2 in B_H is a solid torus. Then the core of C_2 is unknotted, which is a contradiction to the assumprion that $B_H \cap \gamma$ is a collection of two knotted arcs in B_H.

(b) Suppose there is a disk D as in the statement with $\partial D = a \cup b$, where a is an arc in ∂T and b and arc in ∂H.
The intersection of D with $\partial T \cap \partial H$ is the boundary of α (and β), and notice that $C_1 \cup C_2$ intersect ∂H at $\partial T \cap \partial H$. If a is disjoint from D_H then a co-bounds a disk in the boundary of $B_H \cap T$ with an arc in $\partial T \cap \partial H$. Using this disk, we can isotope a to ∂H to get the resulting disk D with ∂D in ∂H. Consequently, from Lemma 1(a), ∂D bounds a disk in $\partial E_H(T)$. Therefore, a intersects D_H in at least two points. As observed at the beggining o the proof, the arcs $\partial D \cap C_i$ have ends in the distinct boundary component of the annlus $\partial C_i - C_i \cap \partial B_H$, $i = 1, 2$. As $\partial D \cap C_i \subset a$ and are essential arcs of the annlus $\partial C_i - C_i \cap \partial B_H$, $i = 1, 2$, we have that a intersects D_H in at most two points. Therefore, a intersects D_H at two points.

FIGURE 3: The arcs $D \cap D_H$ in D when a single arc intersects a, in (a), and when two arcs intersect a, in (b).

Hence, $D \cap D_H$ is a collection of arcs with at least an end in $a \cap D_H$. If $a \cap D_H$ is the boundary of a single arc component of $D \cap D_H$, as in Figure 3(a), then the strings of the tangle $(B_H, B_H \cap \gamma)$ are parallel, which contradicts this tangle being a 2-string essential free tangle (see Lemma 2.1 in [13]). If each point of $a \cap D_H$ cobounds an arc of $D \cap D_H$ with the other end in b, as in Figure 3(b), then the two strings of $(B_H, B_H \cap \gamma)$ are trivial, which also contradicts this tangle being essential. ■

To construct the knots that are the main study of this paper we identify the solid tori H and H_i along their boundaries, defining an Heegaard decomposition $H \cup H_i$ of S^3, such that ∂s is identified with ∂s_i. From construction, $K_i = s \cup s_i$ is a knot in S^3, for $i \in \mathbb{N}$, and in the next proposition we prove these knots are prime.

Proposition 1. *The knots K_i, $i \in \mathbb{N}$, are an infinite collection of distinct prime knots.*

To prove that the knots K_i are prime we use the following technical result. Let K and L be non-trivial knots. Take a ball B intersecting K in two parallel trivial arcs with the tangle $(B^c, B^c \cap K)$ being locally unknotted. Replace the arcs of $B \cap K$ in B by two parallel arcs with the pattern of L, and denote this new knot by K_L.

Lemma 3. *The knot K_L is prime.*

Proof: If the knot K_L is trivial then it bounds a disk D in S^3. Then, ∂D intersects ∂B at four points. Suppose that $|D \cap \partial B|$ is minimal. By an innermost curve argument, as used before, we can show that $D \cap \partial B$ is a collection of two arcs. The strings of $B \cap K_L$ are knotted and each can't co-bound an outermost disk of $D - D \cap \partial B$ with an arc in ∂B. Hence, the arcs of $D \cap \partial B$ have an end on each string of $B \cap K_L$ and co-bound together with the strings a disk in B. Each arc of $D \cap \partial B$ also co-bounds a disk with a string of $K_L \cap B^c$. Therefore, if we replace the tangle $(B, B \cap K_L)$ with the tangle $(B, B \cap K)$ we obtain a disk in S^3 bounded by K, which is a contradiction because K is knotted. Hence, the knot K_L is also non-trivial. Now we prove that K_L is prime. Suppose there is a decomposing sphere S for K_L. As $(B, B \cap K_L)$ is defined by two parallel strings in B, using the disk co-bounded by the two strings $B \cap K_L$ in B we can show that S can be assumed disjoint from B. However, this means that S is in B^c, which contradicts $(B^c, B^c \cap K_L)$ being locally unknotted. ■

As for the construction of the knots K_i, we construct a knot K by identifying two copies of H, say H and H', along their boundaries, defining an

Heegaard decomposition of S^3, and such that the two copies of s, say s and s' resp., are also identified along their boundaries. As s is essential in H we have that ∂H defines a meridional incompressible surface in the exterior of K, which means that K is not trivial. We also denote the copy of the solid torus T of H in H' by T'.
We will use this knot K, the knots L_i and the construction of Lemma 3 to characterize the knots K_i, but first we need the following lemma. Let Q be the ball as in Figure 2 and Q^c its complement.

Lemma 4. *The tangle $(Q^c, Q^c \cap K)$ is locally unknotted.*

Proof: Suppose $(Q^c, Q^c \cap K)$ is locally knotted. Then there is a sphere S bounding a ball P intersecting $Q^c \cap K$ at a single knotted arc. We have that K is in $T \cup T'$ then S intersects T or T'.
Consider the intersection of S with T and T', and suppose it has a minimal number of components. From the construction of the knot K the cores of the solid tori T and T' define a two component link with each component being unknotted.
As the tangle $(B_T, B_T \cap s)$ is free and essential we can assume that S is disjoint from B_T (and similarly, S is disjoint from $B_{T'}$).
The intersections of S with the boudaries of T and T' is a collection of simple closed curves. As S is disjoint from B_T and $B_{T'}$ the curves of intersection are either in $\partial T - B_T$ or in $\partial T' - B_{T'}$. Consider E a disk component of S separated by ∂T and $\partial T'$ from S. If E is not in $T \cup T'$ and its boundary is in ∂T (or similarly $\partial T'$) we have that ∂E is a longitude of T, as S^3 doesn't have a $S^2 \times S^1$ or a Lens space summand. Therefore, the core of T bounds a disk disjoint from T', which is a contradiction as the core of T and T' are linked from construction. Hence, E is in T or T'. If E is in T (or similarly in T') and is disjoint from s then as s is essential in T we have that ∂E bounds a disk in $\partial T - s$. In this case we can reduce the number of components of S intersection with $\partial T \cup \partial T'$, which is a contradiction to its minimality. Then, we can assume that all disks E intersect s or s'. If some disk E intersects either s or s' at two points then some other disk component of S separated by $\partial T \cup \partial T'$ is disjoint from s and s', which is a contradiction to all disks E intersecting s or s'. Then, there is an essential disk E in T (or similarly, in T') that intersects s at a single point. As before, let R_T be the solid torus separated by D_T from T. From the construction of s in T, if we cap off $R_T \cap s$ with an arc in D_T we get a torus knot. Then any essential disk in

R_T intersects the knot in more than one point. As E is disjoint from B_T it is a non-separating disk in R_T intersecting the torus knot at a single point, which is a contradiction. Hence, $(Q^c, Q^c \cap K)$ is locally unknotted. ∎

Proof of Proposition 1: The knots K_i are the knots K_{L_i} obtained from the knots K and L_i with a construction as in Lemma 3. From Lemmas 3 and 4 the knots K_i are prime.
Each knot K_i is also sattelite with companion knot L_i and pattern knot K. Then, from the JSJ-decomposition for compact 3-manifolds we have that the knots K_i, $i \in \mathbb{N}$, are an infinite collection of prime knots. ∎

3. Knots with meridional essential surfaces for all genus

In this section we prove Theorem 1, and its corollary, by showing the knots K_i, $i \in \mathbb{N}$, have meridional essential surfaces of all positive genus and two boundary components. We start by constructing these surfaces, denoted by $F_1, \dots F_g, \dots$ where F_g has genus g, in the complement of an arbitrary knot K_i, and afterwards we prove they are essential in $E(K_i)$. In this construction we denote the boundaries of s and s_i by $\partial_1 s$ $(= \partial_1 s_i)$ and $\partial_2 s$ $(= \partial_2 s_i)$. Denote by X (resp., Y) the punctured torus ∂T (resp., ∂T_i). We also denote by $\partial_i X$ (resp., $\partial_i Y$) the boundary component of X (resp., Y) related to $\partial_i s$, $i = 1, 2$.

The surface F_1 is defined as X. The surface F_2 is obtained from X and Y by identifying the boundary components $\partial_2 X$ and $\partial_2 Y$. In Figure 4 we have a schematic representation of F_1 and F_2.

FIGURE 4: A schematic diagram of surface F_1, in (a), and surface F_2, in (b).

To construct the surfaces F_g, for $g \geq 3$, we follow a general procedure as explained next. In H_i consider a copy of Y and an annulus A, around s_i, defined by $\partial N(s_i) - (\partial N(s_i) \cap \partial H_i)$. We denote by Z the surface obtained by identifying Y and A along the boundaries $\partial_1 Y$ and $\partial_1 A$. Let $n = g - 1$ and $A_1, \dots, A_{n-2}$ be disjoint copies of A disjoint from Z. Consider also n disjoint copies of X in H, denoted by $X_1, \dots, X_n$. Denote $\partial_1 X_j$ (resp., $\partial_2 X_j$)

the boundary component of X_j around $\partial_1 s$ (resp., $\partial_2 s$). Similarly, we label the boundary components of A_j by $\partial_1 A_j$ and $\partial_2 A_j$. To construct F_g we start by attaching ∂X_n and ∂X_{n-1} to the two boundary components of Z respecting the order from $\partial_2 s$. If $g \geq 4$ we also attach $\partial_2 X_{n-2}, \dots, \partial_2 X_1$ to $\partial_2 A_{n-2}, \dots, \partial_2 A_1$, respectively, and $\partial_1 X_n, \dots, \partial_1 X_3$ to $\partial_1 A_{n-2}, \dots, \partial_1 A_1$, respectively. The surface F_g has two boundary components ($\partial_1 X_1$ and $\partial_1 X_2$) and Euler characteristic $-2g$, which means the genus of F_g is g. In Figure 5 we have a schematic representation of F_3 and F_4, and in Figure 6 a representation of the general construction of F_g.

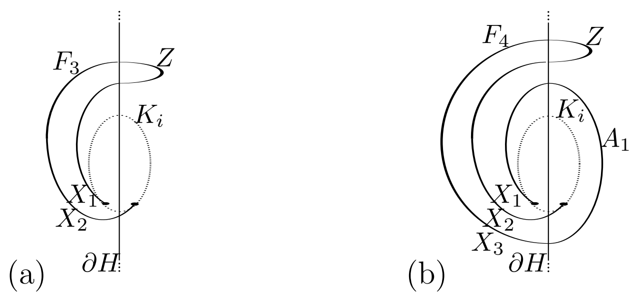

FIGURE 5: A schematic diagram of surface F_3, in (a), and surface F_4, in (b).

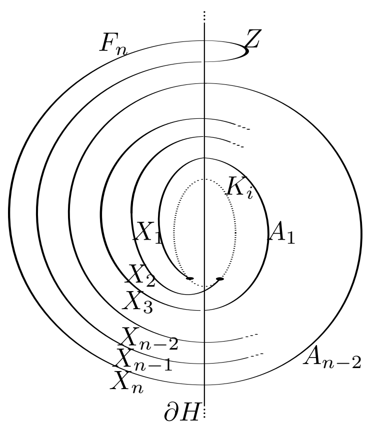

FIGURE 6: A schematic general representation of the surface F_n, for $n \geq 3$.

Proof of Theorem 1. To prove Theorem 1 we consider the knots K_i, $i \in \mathbb{N}$, and show the surfaces F_g, $g \in \mathbb{N}$, to be essential in their complements.
We assume that F_g is not essential in $E(K_i)$ and prove this leads to a contradiction to the knots K_i properties. Let D be a compressing or a boundary compressing disk for F_g in $E(K_i)$. In case D is a boundary compressing disk then $\partial D = a \cup b$ where a is an arc in $\partial E(K_i) - F_g$, with one end in each

compoenent of ∂F_g, and b is an arc in F_g. We also assume $|D \cap \partial H|$ to be minimal. Consequently, using an innermost curve argument, as in Lemma 1, we have that D doesn't intersect ∂H in simple closed curves.

Suppose $g = 1$. By a small isotopy of a neighborhood of ∂F_1 into H if necessary, we can assume that F_1 is in H. If D is a compressing disk for F_1 in $E(K_i)$ then $D \subset H$, as D cannot intersect ∂H in simple closed curves and ∂D is disjoint from ∂H. This is a contradiction to Lemma 1, which says ∂T is incompressible in $E_T(s)$ and in $E_H(T)$. Assume now D is a boundary compressing disk of F_1 in $E(K_i)$. If D is in T then we have a contradiction to Lemma 1(b) for s being essential in T. If D is not in T then, by using an innermost curve argument, we can assume that a intersects ∂H at two points and that $D \cap \partial H$ is an arc separating from D a disk O in H with boundary an arc in ∂H and an arc that we can assume in ∂T having ends in $\partial H \cap \partial T$. Hence, O contradicts Lemma 2(b). Therefore, we have that F_1 is essential in $E(K_i)$.

Suppose $g = 2$. By a small isotopy of a neighborhood of ∂F_2 we can assume that the compoenent of $\partial F_2 \cap X$ is in H and that $\partial F_2 \cap Y$ is in H_i. Suppose D is a compressing disk of F_2 in $E(K_i)$. If D is disjoint from ∂H then D is a compressing disk for X or Y in $E(K_i)$, which is a contradiction to Lemma 1. Then D intersects ∂H at a minimal collection of arcs. Consider an outermost arc α of $D \cap \partial H$ in D and let O be the respective outermost disk, with $O \cap F_2 = \beta$ an arc in X or in Y. Without loss of generality, suppose β is in X. If α or β doesn't co-bound a disk in ∂H or X, respectively, with $\partial_2 X$ we have a contradiction to Lemma 1(b) Lemma 2(b). Otherwise, ∂O bounds a disk O' in $\partial H \cup \partial T$ and using the ball bounded by $O \cup O'$ we can isotope D reducing $|D \cap \partial H|$ which is a contradiction to its minimality.
Suppose now that D is a boundary compressing disk for F_2 in $E(K_i)$. As the two components of ∂F_2 are in opposite sides of ∂H by an innermost curve argument we can prove that a intersects ∂H at a single point. Hence, $D \cap \partial H$ is an arc with one end in a and one end in b and, possibly, arcs with both ends in b, as in Figure 7.

If D is in $T \cup T_i$ then the arcs of $D \cap \partial H$ with both ends in b are in the annulus E separated by $F_2 \cap \partial H$ (that is $\partial_2 X$) from $\partial H - N(K_i) \cap \partial H$. Hence, each arc of $D \cap \partial H$ with ends in b co-bounds a disk in E with $\partial_2 X$.

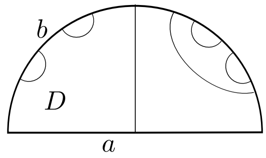

FIGURE 7: Arcs of $D \cap \partial H$ when D is a boundary compressing disk of F_2.

Consider an outermost of such arcs in E, and the respective outermost disk O. By cutting and pasting D along O we contradict Lemma 1(b) or we can reduce $|D \cap \partial H|$ contradicting its minimality. Therefore, in this case, $D \cap \partial H$ is the arc with an end in a. This arc cuts D into two disks, one in T and the other in T_i, contradicting Lemma 1(b). If D is in $E_H(T) \cup E_{H_i}(T_i)$ we consider an outermost arc α between the arcs of $D \cap \partial H$ in D and the respective outermost disk, also denoted by O. If the arc β, that is $\partial O \cap F_2$, co-bounds a disk in F_2 with $F_2 \cap \partial H$, using an argument as before, we can reduce $|D \cap \partial H|$ contradicting its minimality. Otherwise, the disk O is in contradiction to Lemma 2(b). Hence, $D \cap \partial H$ is only the arc with an end in a, and the disk separated by this arc in D are also in contradiction to Lemma 2(b). Consequently, F_2 is essential in $E(K_i)$.

Supose now $g \geq 3$. As in the other cases, we start by isotoping the boundary components of F_g such that $\partial_2 X_1$ is in H and $\partial_2 X_2$ is in H_i. Hence, as in the case for $g = 2$, if D is a boundary compressing disk we can assume that a intersects ∂H at a single point. This means that D intersects ∂H at an arc with one end in a and the other in b and, possibly, in arcs with both ends in b. If D intersects the interior of T_1, we can proceed as in the case $g = 2$ and get a contradiction to Lemma 1(b). Otherwise, the disk D intersects the region of H between X_1 and X_2 in at least a disk component. As the only intersection of D with $\partial E(K_i)$ is a, and b intersects X_1 and X_2, that are disjoint, we have necessarily that D intersects ∂H in arcs with both ends in b. Suppose now that D is a compressing disk. If D is disjoint from ∂H then ∂D is in one surface X_j, in Z or in one annlus A_j. If ∂D is in A_j, as A_j is boundary parallel to $N(K_i)$ we have that ∂D bounds a disk in A_j, which contradicts D being a compressing disk of F_g. If ∂D is in Z, as Z is the union of ∂T_i with an annulus boundary parallel to $N(K_i)$ we can assume that ∂D is in ∂T_i and D is in T_i or $E_{H_i} T_i$, which is a contradiction to Lemma 1. If ∂D is in some surface X_j, as D is disjoint from ∂H, we have the same contradiction. Hence, D intersects ∂H in a collection of arcs. Consequently, if D is a compressing or a boundary compressing disk there are arcs of $D \cap \partial H$ with both ends in F_g. In Figure 8 we have a representation of the arcs of $D \cap \partial H$ in D.

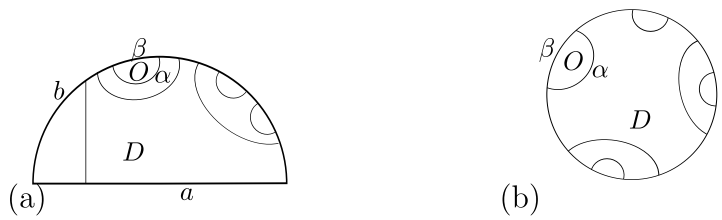

FIGURE 8: The disk D together with the arcs $D \cap \partial H$ when D is a boundary compressing disk, in (a), or a compressing disk, in (b), of F_g.

Consider an outermost arc in D between the arcs of $D \cap \partial H$ with both ends in F_g and denote it by α. In case D is a boundary compressing disk, α is an outermost arc of $D \cap \partial H$ in D between the arcs with both ends in b. Let O be the outermost disk cut from D by α, and let $\partial O = \alpha \cup \beta$ where β is an arc in F_g. The surfaces X_j and X_{j+1} are disjoint, $j = 1, \dots, n-1$, and each two of A_j, A_{j+1} and Z, $j = 1, \dots, n-3$, are also disjoint. Then, there is no outermost arc α with an end in $\partial_1 X_j$ and an end in $\partial_1 X_{j+1}$, or an end in $\partial_2 X_j$ and an end in $\partial_2 X_{j+1}$, $j = 1, \dots, n-1$, unless α is in the annulus separated by $\partial_1 X_{n-1} \cup \partial_1 X_n$ from ∂H and O is in the manifold cut by Z from H_i. In this case, we would be in contradiction to Lemma 1(b). Therefore, α is not essential in an annulus separated by $\partial_1 X_j \cup \partial_1 X_{j+1}$ or by $\partial_2 X_j \cup \partial_2 X_{j+1}$, $j = 1, \dots, n-1$, from ∂H. If α has ends in distinct components of $F_g \cap \partial H$, then it has one end in $\partial_1 X_n$ and the other in $\partial_2 X_n$, with O between H and X_n in H. In this situation, we have a contradiction to Lemma 2(b). Suppose now that α has both ends in the same component of $F \cap \partial H$, say in $\partial_1 X_j$ or in $\partial_2 X_j$. If β is in some annulus A_j then it co-bounds a disk in A_j with $\partial_1 X_j$ (or with $\partial_2 X_j$). By an isotopy of β along this disk, through ∂H, we can reduce $|D \cap \partial H|$ contradicting its minimality. If β is in some X_j or Z and co-bounds a disk with $\partial_1 X_j$ (or with $\partial_2 X_j$), following a similar argument as before, we get a contradiction to the minimality of $|D \cap \partial H|$. Then, β doesn't co-bound a disk in X_j or Z with $\partial_1 X_j$ (or with $\partial_2 X_j$). Suppose, without loss of generality, that α is in the annulus separated from ∂H by $\partial_1 X_j \cup \partial_1 X_{j+1}$, and let O' be the disk that α co-bounds with $\partial_1 X_j$ in the annulus. Then $O \cup O'$ is a compressing disk for X_j (or Y) in H (or H_i, resp.), which contradicts Lemma 1. In a similar way, if β has both ends in $\partial_1 X_1$ $(\partial_2 X_1)$ and α is in the annulus separated from ∂H by $\partial_1 X_1$ (or $\partial_2 X_1$, resp.)

and a component of $\partial H \cap N(K_i)$ we obtain a contradiction to the minimality of $|D \cap \partial H|$ or to the incompressibility of X_1 in H. We are left with the case when α has both ends in $\partial_1 X_n$, $\partial_2 X_n$ or Z and O is between X_n and ∂H in H, or between Z and ∂H_i in H_i, which gives us a contradiction to Lemma 1.
Hence, D cannot be a compressing or a boundary compressing disk.
Altogether we have that the surfaces F_g are essential in $E(K_i)$.

■

The proof Corollary 1.1 now follows naturally.

Proof of Corollary 1.1: In Theorem 1 we proved that the knots K_i, $i \in \mathbb{N}$, are an infinite collection of prime knots with meridional essential surfaces in their complements for each positive genus and two boundary components. Hence, considering the knots K_i connected sum with some other knot, we have infinitely many knots with meridional essential surfaces of genus g and two boundary components for all $g \geq 0$, as in the statement. ■

4. Appendix

In this appendix we define n-string essential free tangles and give an example of a 2-string essential free tangle with both strings knotted.
A n-string tangle is a pair (B, σ) where B is a 3-ball and σ is a collection of n properly embedded disjoint arcs in B. We say that (B, σ) is essential if for every disk D properly embedded in $B - \sigma$ then ∂D bounds a disk in $\partial B - \partial\sigma$. The tangle is said to be free if the fundamental group of $B - \sigma$ is free, or, equivalently, if $B - N(\sigma)$ is a handlebody.
For a string s in a ball B we can consider the knot obtained by capping off s along ∂B, that is by gluing to s an arc in ∂B along the respective boundaries. We denote this knot by $K(s)$.
Let s_1 be an arc in a ball B such that $K(s_1)$ is a trefoil, and consider also an unknotting tunnel t for $K(s_1)$, as in Figure 9.
If we slide ∂t along s_1 into ∂B, as illustrated in Figure 10(a), we get a new string that we denote by s_2, as in Figure 10(b).
The knot $K(s_2)$ is the $(3, -4)$-torus knot, and hence knotted. The tangle $(B, s_1 \cup s_2)$ is free by construction. In fact, as t is an unknotting tunnel of $K(s_1)$, the complement of $N(s_1) \cup N(t)$ in B is a handlebody. Henceforth, by an ambient isotopy, the complement of $N(s_1) \cup N(s_2)$ is also a handlebody. As the tangle $(B, s_1 \cup s_2)$ is free and both strings are knotted then it is necessarily essential. Otherwise, the complement of $N(s_1) \cup N(s_2)$ in B isn't

(a) (b)

FIGURE 9: The string s_1 when capped off along ∂B is a trefoil knot with an unknotting tunnel t.

(a) (b)

FIGURE 10: Construction of a 2-string essential free tangle with both strings knotted from s_1 and the unknotting tunnel t.

a handlebody as it is obtained by gluing two non-trivial knot complements along a disk in their boundaries, which is a contradiction to the tangle $(B, s_1 \cup s_2)$ being free.

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João Miguel Nogueira
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008 EC Santa Cruz, 3001-501 Coimbra, Portugal

E-mail address: nogueira@mat.uc.pt