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HYPERSTRUCTURES ON COURANT ALGEBROIDS

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Abstract: We introduce the notion of hypersymplectic structure on a Courant algebroid and we prove the existence of a $1-1$ correspondence between hypersymplectic and hyperkähler structures. We show that hypersymplectic structures on Courant algebroids encompass hyperkähler and hyperkähler structures with torsion on Lie algebroids. Cases of hypersymplectic structures on Courant algebroids which are doubles of Lie, quasi-Lie and proto-Lie bialgebroids are investigated.

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1. Introduction

In the past years, hyperstructures on Courant algebroids deserved the attention of several authors. Namely, we mention Bursztyn *et al.* [6] who discussed hyperkähler structures and Stiénon [11] for the case of hypercomplex structures. In the present article we introduce and study hypersymplectic structures and, more generally, $\boldsymbol{\varepsilon}$-hypersymplectic structures on Courant algebroids. A very interesting feature of hypersymplectic structures on Courant algebroids is that they are in $1-1$ correspondence with hyperkähler structures.

Inspired by hypersymplectic structures on manifolds, defined by Xu in [13], we introduced in [4] the notion of hypersymplectic structure in the setting of Lie algebroids (see also [1]). A hypersymplectic structure on a Lie algebroid A is a triplet $(\omega_1, \omega_2, \omega_3)$ of symplectic forms on A, such that the square of the transition morphisms, endomorphisms of A constructed out of the 2-forms ω_i and their inverses, is equal to $\pm \mathrm{id}_A$. The idea of extending the theory of hypersymplectic structures to Courant algebroids is not simply an exercise of generalization but it has a strong motivation that we explain next.

Hyperkähler structures with torsion on manifolds, also known as HKT structures, first appear in [7] in relation with sigma models in string theory

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and, since then, HKT and other geometries with torsion caught the interest of many physicists and mathematicians. In a separate article [5], we study hypersymplectic and hyperkähler structures with torsion on Lie algebroids. When these geometrical structures carrying a nonzero torsion are considered on manifolds or on Lie algebroids, they are substantially different from those with vanishing torsion (the hypersymplectic and the hyperkähler cases). In the current article we show that, although hypersymplectic and hypersymplectic structures with torsion on Lie algebroids are different in nature, when we look at them in the Courant algebroid setting, they become of the same type. The same happens with hyperkähler and hyperkähler structures with torsion. This explains the interest of going from Lie algebroids to Courant algebroids. The idea which is behind our results, is to associate to each triplet $(\omega_1, \omega_2, \omega_3)$ of non-degenerate 2-forms on a Lie algebroid A, with inverse $(\pi_1, \pi_2, \pi_3) \in (\Gamma(\wedge^2 A))^3$, a triplet of endomorphisms $\mathcal{S}_i = \begin{bmatrix} 0 & \varepsilon_i \pi_i^\sharp \\ \omega_i^\flat & 0 \end{bmatrix}$, $\varepsilon_i = \pm 1, i = 1,2,3$, of the vector bundle $A \oplus A^*$ equipped with a Courant structure which is successively considered as being the double of a Lie bialgebroid, of a quasi-Lie bialgebroid and of a proto-Lie bialgebroid. Then, choosing a suitable Courant structure on $A \oplus A^*$, we prove that having a hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on A, with or without torsion, is equivalent to having a hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on $A \oplus A^*$. More involved situations are those where, besides the structures considered on A, the vector bundle A^* itself is endowed with a hypersymplectic structure, with or without torsion, determined by (π_1, π_2, π_3). We also prove that, under some conditions, this is equivalent to $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ being a hypersymplectic structure on $A \oplus A^*$.

Besides the Introduction, the article contains seven sections. Since many of the computations are done using the big bracket, Section 2 contains a brief review of the supergeometric setting as well as the main notions around the Courant algebroid definition. In Section 3 we introduce the notion of $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid, which is the more general case that we consider, and we explore the properties of the morphisms induced by this structure. Section 4 and 5 treat the case where $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. The main result of Section 4 is that the transition morphisms $\mathcal{T}_i$ are Nijenhuis. We also show that if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on a Courant algebroid (E, Θ), it is also hypersymplectic for the Courant structure on E deformed by $\mathcal{T}_i$ or by $\mathcal{S}_i$. In Section 5 we prove a $1-1$ correspondence theorem between hypersymplectic and hyperkähler structures on a Courant algebroid. Moreover, we show how the transition morphisms $\mathcal{T}_i$ can take the role of the morphisms $\mathcal{S}_i$ to define a new hypersymplectic structure on the Courant algebroid. Sections 6, 7 and 8 are devoted to examples of hypersymplectic structures on $A \oplus A^*$, equipped with several Courant structures. We start with the simplest case in Section 6. We prove that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure on a Lie algebroid (A, μ) if and only if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu)$. In Section 7 the Courant structure on $A \oplus A^*$ is the double of a Lie bialgebroid $((A, A^*), \mu, \gamma)$ and we prove that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on $(A \oplus A^*, \mu + \gamma)$ if and only if $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure on (A, μ) and (π_1, π_2, π_3) is a hypersymplectic structure on (A^*, γ). The particular case of a triangular Lie bialgebroid is also considered. The class of examples we give in Section 8, deal with the notion of hypersymplectic structure with torsion on a Lie algebroid. This is a structure that generalizes the hypersymplectic case, where the non-degenerate 2-forms ω_i are not closed but satisfy the condition $N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3$, with N_i the transition morphisms. We show that having a hypersymplectic structure with torsion $(\omega_1, \omega_2, \omega_3)$ on a Lie algebroid (A, μ), with π_i weak-Poisson, is equivalent to $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ being a hypersymplectic structure on $A \oplus A^*$ equipped with a Courant structure determined by a quasi-Lie bialgebroid structure on (A, A^*). The next case that we treat is when both (A, μ) and (A^*, γ) are equipped with hypersymplectic structures with torsion. We show that, under some conditions, this is equivalent to having a hypersymplectic structure on $A \oplus A^*$ equipped with a Courant structure determined by a proto-Lie bialgebroid structure on (A, A^*).

2. Preliminaries on Courant algebroids

We begin this section by introducing the supergeometric setting, following the same approach as in [12, 10] (see also [1]). Given a vector bundle $A \to M$, we denote by $A[n]$ the graded manifold obtained by shifting the fibre degree by n. The graded manifold $T^*[2]A[1]$ is equipped with a canonical symplectic structure which induces a Poisson bracket on its algebra of functions $\mathcal{F} := C^\infty(T^*[2]A[1])$. This Poisson bracket is sometimes called the *big bracket* (see [9]).

Let us describe locally this Poisson algebra. Fix local coordinates $x_i, p^i, \xi_a, \theta^a$, $i \in \{1, \dots, n\}, a \in \{1, \dots, d\}$, in $T^*[2]A[1]$, where x_i, ξ_a are local coordinates on $A[1]$ and p^i, θ^a are their associated moment coordinates. In these local coordinates, the Poisson bracket is given by

$$\{p^i, x_i\} = \{\theta^a, \xi_a\} = 1, \quad i = 1, \dots, n, \ a = 1, \dots, d,$$

while all the remaining brackets vanish.

The Poisson algebra of functions $\mathcal{F}$ is endowed with an $(\mathbb{N} \times \mathbb{N})$-valued bidegree. We define this bidegree (locally but it is well defined globally, see [12, 10]) as follows: the coordinates on the base manifold M, x_i, $i \in \{1, \dots, n\}$, have bidegree $(0, 0)$, while the coordinates on the fibres, ξ_a, $a \in \{1, \dots, d\}$, have bidegree $(0, 1)$ and their associated moment coordinates, p^i and θ^a, have bidegree $(1, 1)$ and $(1, 0)$, respectively. We denote by $\mathcal{F}^{k,l}$ the space of functions of bidegree (k, l) and we verify that the big bracket has bidegree $(-1, -1)$, i.e.,

$$\{\mathcal{F}^{k_1,l_1}, \mathcal{F}^{k_2,l_2}\} \subset \mathcal{F}^{k_1+k_2-1,l_1+l_2-1}.$$

This construction is a particular case of a more general one [10] in which we consider a vector bundle E equipped with a fibrewise non-degenerate symmetric bilinear form $\langle ., . \rangle$. In this more general setting, we consider the graded symplectic manifold $\mathcal{E} := p^*(T^*[2]E[1])$, which is the pull-back of $T^*[2]E[1]$ by the map $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, . \rangle)$. We denote by $\mathcal{F}_E$ the graded algebra of functions on $\mathcal{E}$, i.e., $\mathcal{F}_E := C^\infty(\mathcal{E})$. The algebra $\mathcal{F}_E$ is equipped with the canonical Poisson bracket, denoted by $\{., .\}$, which has degree -2. Notice that $\mathcal{F}^0_E = C^\infty(M)$ and $\mathcal{F}^1_E = \Gamma(E)$. Under these identifications, the Poisson bracket of functions of degrees 0 and 1 is given by

$$\{f, g\} = 0, \ \ \{f, X\} = 0 \quad \text{and} \quad \{X, Y\} = \langle X, Y \rangle,$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

When $E := A \oplus A^*$ (with A a vector bundle over M) and when $\langle ., . \rangle$ is the usual symmetric bilinear form:

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X), \quad \forall X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*), \qquad (1)$$

the algebras $\mathcal{F} = C^\infty(T^*[2]A[1])$ and $\mathcal{F}_{A\oplus A^*}$ are isomorphic Poisson algebras [10] and the two constructions above coincide.

Let us recall that a *Courant* structure on a vector bundle E equipped with a fibrewise non-degenerate symmetric bilinear form $\langle .,. \rangle$ is a pair $(\rho, [.,.])$, where the *anchor* ρ is a bundle map from E to TM and the *Dorfman bracket* $[.,.]$ is a $\mathbb{R}$-bilinear (not necessarily skew-symmetric) map on $\Gamma(E)$ satisfying

i) $\rho(X) \cdot \langle Y, Z \rangle = \langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle$,
ii) $\rho(X) \cdot \langle Y, Z \rangle = \langle X, [Y,Z] + [Z,Y] \rangle$,
iii) $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$,

for all $X, Y, Z \in \Gamma(E)$.

The next theorem shows how a Courant structure can be defined in the supergeometric setting.

Theorem 2.1. [10] *There is a* $1-1$ *correspondence between Courant structures on* $(E, \langle .,. \rangle)$ *and functions* $\Theta \in \mathcal{F}^3_E$ *such that* $\{\Theta, \Theta\} = 0$.

The anchor and Dorfman bracket associated to a given $\Theta \in \mathcal{F}^3_E$ are defined, for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by the derived bracket expressions

$$\rho(X) \cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [X,Y] = \{\{X, \Theta\}, Y\}.$$

Let $(E, \langle .,. \rangle, \Theta)$ be a Courant algebroid and $I : E \to E$ a vector bundle endomorphism. The transpose morphism $I^* : E^* \simeq E \to E^* \simeq E$ is defined by $\langle I^* u, v \rangle = \langle u, Iv \rangle$ for all $u, v \in E$. If $I = I^*$ (respectively, $I = -I^*$), the morphism I is said to be symmetric (resp. skew-symmetric). The morphism I is orthogonal if $I \circ I^* = \mathrm{id}_E$.

When I is skew-symmetric, we may deform Θ by setting $\Theta_I := \{I, \Theta\} \in \mathcal{F}^3_E$. The deformation of Θ_I by a skew-symmetric morphism J is denoted by $\Theta_{I,J}$, i.e. $\Theta_{I,J} = \{J, \{I, \Theta\}\}$. The *concomitant* $C_\Theta(I,J)$ of two skew-symmetric morphisms I and J, on a Courant algebroid $(E, \langle .,. \rangle, \Theta)$, is given by [2]:

$$C_\Theta(I,J) = \Theta_{I,J} + \Theta_{J,I}. \tag{2}$$

Recall that a vector bundle endomorphism $I : E \to E$ on a Courant algebroid $(E, \langle .,. \rangle, \Theta)$ is a *Nijenhuis morphism* if its Nijenhuis torsion $\mathcal{T}_\Theta I$ vanishes. When $I^2 = \lambda \, \mathrm{id}_E$, for some $\lambda \in \mathbb{R}$, we have [8, 1]

$$\mathcal{T}_\Theta I = \frac{1}{2}(\Theta_{I,I} - \lambda \Theta). \tag{3}$$

If $I^2 = -\mathrm{id}_E$ (resp. $I^2 = \mathrm{id}_E$) then I is said to be an *almost complex* (resp. *almost para-complex*) *structure.* If moreover $\mathcal{T}_\Theta I = 0$, then I a *complex* (resp. *para-complex*) *structure.*

When $E = A \oplus A^*$ and $\langle .,. \rangle$ is the usual symmetric bilinear form (1), a Courant structure $\Theta \in \mathcal{F}^3_E$ can be decomposed using the bidegrees:

$$\Theta = \mu + \gamma + \phi + \psi,$$

with $\mu \in \mathcal{F}^{1,2}_{A\oplus A^*}, \gamma \in \mathcal{F}^{2,1}_{A\oplus A^*}, \phi \in \mathcal{F}^{0,3}_{A\oplus A^*} = \Gamma(\wedge^3 A^*)$ and $\psi \in \mathcal{F}^{3,0}_{A\oplus A^*} = \Gamma(\wedge^3 A)$. We recall from [9] that, when $\gamma = \phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle .,. \rangle)$ *if and only if* (A, μ) is a Lie algebroid; when $\phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle .,. \rangle)$ *if and only if* $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid and when $\phi = 0$ (resp. $\psi = 0$), Θ is a Courant structure on $(A \oplus A^*, \langle .,. \rangle)$ *if and only if* $((A, A^*), \mu, \gamma, \psi)$ (resp. $((A, A^*), \mu, \gamma, \phi)$) is a quasi-Lie bialgebroid. In the more general case, $\Theta = \mu+\gamma+\phi+\psi$ is a Courant structure *if and only if* $((A, A^*), \mu, \gamma, \psi, \phi)$ is a proto-Lie bialgebroid.

Throughout this article, for simplicity, we shall often denote a Courant algebroid $(E, \langle .,. \rangle, \Theta)$ by the pair (E, Θ) and when we write μ, γ, ϕ or ψ, we are assuming that these functions are in $\mathcal{F}^{1,2}_{A\oplus A^*}$, $\mathcal{F}^{2,1}_{A\oplus A^*}$, $\mathcal{F}^{0,3}_{A\oplus A^*}$ or $\mathcal{F}^{3,0}_{A\oplus A^*}$, respectively.

3. Hypersymplectic structures on Courant algebroids

In this section we introduce the notion of an $\boldsymbol{\varepsilon}$-*hypersymplectic structure* on a Courant algebroid (E, Θ) and study the main relations and properties of the induced morphisms. In order to simplify the notation, when I and J are endomorphisms of E, the composition $I \circ J$ will be denoted by IJ.

Definition 3.1. An $\boldsymbol{\varepsilon}$-*hypersymplectic structure* on a Courant algebroid (E, Θ) is a triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ of skew-symmetric endomorphisms $\mathcal{S}_i : E \to E$, $i = 1, 2, 3$, such that

i) ${\mathcal{S}_i}^2 = \varepsilon_i \, \mathrm{id}_E$,
ii) $\mathcal{S}_i\mathcal{S}_j = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_j\mathcal{S}_i, \;\; i \neq j \in \{1, 2, 3\}$
iii) $\Theta_{\mathcal{S}_i,\mathcal{S}_i} = \varepsilon_i\Theta$,

where the parameters $\varepsilon_i = \pm 1$ form the triplet $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

From conditions i) and iii) of Definition 3.1, and using formula (3), we immediately have the following proposition.

Proposition 3.2. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ). Then, $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ are Nijenhuis morphisms.*

Given an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on (E, Θ), let us define the morphisms $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ by setting

$$\mathcal{T}_i := \varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1}, \tag{4}$$

where the indices must be considered as elements of $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$.

Remark 3.3. We consider $1, 2$ and 3 as the representative elements of the equivalence classes of $\mathbb{Z}_3$, i.e., $\mathbb{Z}_3 := \{[1], [2], [3]\}$. In what follows, although we often omit the brackets $[\,\cdot\,]$, and write n instead of $[n]$, all the indices (and the corresponding computations) must be considered in $\mathbb{Z}_3$.

The morphisms $\mathcal{T}_i$, $i = 1, 2, 3$, are seen as transition maps between the morphisms $\mathcal{S}_j$, $j = 1, 2, 3$. In fact we have, for all $i \in \mathbb{Z}_3$,

$$\mathcal{S}_{i-1}\mathcal{T}_i = \mathcal{S}_{i+1}.$$

The picture in Figure 1 is a good way to visualize these relations. For example, considering the bottom triangle, we can verify that $\mathcal{S}_2\mathcal{T}_3 = \mathcal{S}_1$ and $\varepsilon_1\mathcal{T}_3\mathcal{S}_1 = \varepsilon_2\mathcal{S}_2$. For the latter equality we use the fact that the inverse of morphism $\mathcal{S}_i$ is $\varepsilon_i\mathcal{S}_i$.

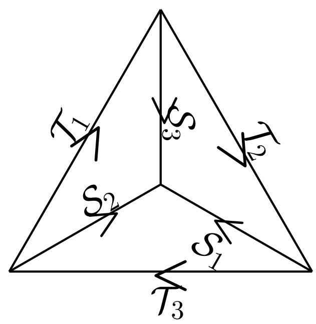

FIGURE 1.

Proposition 3.4. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ). The morphisms $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy the following relations for all $i = 1, 2, 3$:*

i) $\mathcal{T}_i^* = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_i$.
ii) $\mathcal{T}_i^2 = \varepsilon_i \,\mathrm{id}_E$;
iii) $\mathcal{T}_{i-1}\mathcal{T}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_{i+1}\mathcal{T}_{i-1} = \varepsilon_i\mathcal{T}_i$;
iv) $\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1 = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3 = \mathrm{id}_E$.

Proof: Using conditions i) and ii) of Definition 3.1 and also equation (4), we have:

i) $\mathcal{T}_i^* = (\varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1})^* = \varepsilon_{i-1}\mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_i\varepsilon_{i+1}\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_i$, where we also used the fact that the endomorphisms $\mathcal{S}_i$ are skew-symmetric;

ii) $\mathcal{T}_i^2 = \mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i-1}^2\mathcal{S}_{i+1}^2 = \varepsilon_i \,\mathrm{id}_E$;

iii) $\mathcal{T}_{i-1}\mathcal{T}_{i+1} = \varepsilon_{i+1}\varepsilon_i\mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_i\mathcal{S}_{i-1} = \varepsilon_{i+1}\mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_i\varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_i\mathcal{T}_i$. This proves one part of the statement and we use it to prove the second equality of the statement. In fact, from $(\mathcal{T}_{i-1}\mathcal{T}_{i+1})^2 = (\varepsilon_i\mathcal{T}_i)^2 = \varepsilon_i \,\mathrm{id}_E$ and using item ii), we have

$$\mathcal{T}_{i-1}\mathcal{T}_{i+1} = \varepsilon_i(\mathcal{T}_{i-1}\mathcal{T}_{i+1})^{-1} = \varepsilon_i(\mathcal{T}_{i+1})^{-1}(\mathcal{T}_{i-1})^{-1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_{i+1}\mathcal{T}_{i-1}.$$

iv) By item iii), $\mathcal{T}_3\mathcal{T}_2 = \varepsilon_1\mathcal{T}_1$, then, using item ii),

$$\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1 = \varepsilon_1\mathcal{T}_1^2 = \mathrm{id}_E.$$

Furthermore, using item iii) three times we can change the order of $\mathcal{T}_i$'s in the product $\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1$ to get

$$\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1 = (\varepsilon_1\varepsilon_2\varepsilon_3)^3\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3 = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3.$$

■

Remark 3.5. In the particular case where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, the triplet $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ is an *almost hypercomplex structure* on the Courant algebroid (E, Θ) in the terminology of [11].

Given an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on a Courant algebroid (E, Θ), we may define an endomorphism $\mathcal{G} : E \to E$ by setting, for all $i = 1, 2, 3$,

$$\mathcal{G} := \mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_{i-1}. \tag{5}$$

Notice that $\mathcal{G}$ is well defined by (5). In fact, since $\mathcal{S}_i\mathcal{S}_j = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_j\mathcal{S}_i$, for $i \neq j$, we obviously have $\mathcal{G} = \mathcal{S}_3\mathcal{S}_2\mathcal{S}_1 = \mathcal{S}_1\mathcal{S}_3\mathcal{S}_2 = \mathcal{S}_2\mathcal{S}_1\mathcal{S}_3$.

Proposition 3.6. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ). Then, the morphism $\mathcal{G}$, given by (5), satisfies the following properties:*

i) $\mathcal{G}^* = -\varepsilon_1\varepsilon_2\varepsilon_3\mathcal{G}$;

ii) $\mathcal{G}^2 = \mathrm{id}_E$.

Proof: i) An easy computation using the skew-symmetry of each $\mathcal{S}_i$ and condition ii) in Definition 3.1, gives

$$\mathcal{G}^* = (\mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_{i-1})^* = -\mathcal{S}_{i-1}\mathcal{S}_i\mathcal{S}_{i+1} = -\varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_i = -\varepsilon_1\varepsilon_2\varepsilon_3\mathcal{G}.$$

ii) The proof is immediate using properties of $\mathcal{S}_i$ from Definition 3.1:

$$\mathcal{G}^2 = (\mathcal{S}_3\mathcal{S}_2\mathcal{S}_1)^2 = \varepsilon_1\varepsilon_2\varepsilon_3\,\mathcal{S}_3^2\mathcal{S}_2^2\mathcal{S}_1^2 = \mathrm{id}_E.$$

■

The next proposition shows that, for each i, the morphisms $\mathcal{G}$, $\mathcal{S}_i$ and $\mathcal{T}_i$ commute pairwise and each one is obtained out of the other two.

Proposition 3.7. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ). The morphisms $\mathcal{S}_i, \mathcal{T}_i$ and $\mathcal{G}, i = 1, 2, 3,$ satisfy the following relations:*

i) $\mathcal{T}_i\mathcal{S}_i = \mathcal{S}_i\mathcal{T}_i = \varepsilon_{i-1}\mathcal{G}$;
ii) $\mathcal{G}\mathcal{S}_i = \mathcal{S}_i\mathcal{G} = \varepsilon_{i-1}\varepsilon_i\mathcal{T}_i$;
iii) $\mathcal{G}\mathcal{T}_i = \mathcal{T}_i\mathcal{G} = \varepsilon_{i-1}\varepsilon_i\mathcal{S}_i$.

Moreover, for all $i \neq j \in \{1, 2, 3\}$,

iv) $\mathcal{S}_j\mathcal{T}_i = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_i\mathcal{S}_j = \begin{cases} \mathcal{S}_{i+1}, & j = i-1 \\ \varepsilon_i\mathcal{S}_{i-1}, & j = i+1. \end{cases}$

Proof: i) Using (4) and the condition ii) of Definition 3.1 twice, we get

$$\mathcal{T}_i\mathcal{S}_i = \varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_i = \varepsilon_{i-1}\mathcal{S}_i\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \mathcal{S}_i\mathcal{T}_i.$$

On the other hand, from (4) and (5) we have

$$\mathcal{T}_i\mathcal{S}_i = \varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_i = \varepsilon_{i-1}\mathcal{G}.$$

ii) From item i) we have $\mathcal{T}_i\mathcal{S}_i = \varepsilon_{i-1}\mathcal{G}$ and composing with $\mathcal{S}_i$, on the right, we get $\mathcal{T}_i(\mathcal{S}_i)^2 = \varepsilon_{i-1}\mathcal{G}\mathcal{S}_i$ or, equivalently, $\varepsilon_{i-1}\varepsilon_i\mathcal{T}_i = \mathcal{G}\mathcal{S}_i$. For the other equality, we start with $\mathcal{S}_i\mathcal{T}_i = \varepsilon_{i-1}\mathcal{G}$ and compose with $\mathcal{S}_i$, on the left, to obtain $(\mathcal{S}_i)^2\mathcal{T}_i = \varepsilon_{i-1}\mathcal{S}_i\mathcal{G}$; so that $\varepsilon_{i-1}\varepsilon_i\mathcal{T}_i = \mathcal{S}_i\mathcal{G}$.

iii) Analogous to the proof of item ii), but composing with $\mathcal{T}_i$ instead of $\mathcal{S}_i$.

iv) Let us prove the case $j = i - 1$. Using (4) and the condition i) of Definition 3.1, we have

$$\mathcal{S}_{i-1}\mathcal{T}_i = \varepsilon_{i-1}\mathcal{S}_{i-1}^2\mathcal{S}_{i+1} = \mathcal{S}_{i+1}.$$

Moreover, by (4) and conditions i) and ii) of Definition 3.1, we get

$$\mathcal{T}_i\mathcal{S}_{i-1} = \varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_i\varepsilon_{i+1}\mathcal{S}_{i-1}^2\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i+1},$$

which completes the proof of the statement. The case $j = i + 1$ is analogous.

■

When $\mathcal{G} = \mathrm{id}_E$, then $\mathcal{S}_i = \varepsilon_{i-1}\varepsilon_i\mathcal{T}_i$ and, if* $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an *hypercomplex structure*, in the sense of [11], on the Courant algebroid (E, Θ).

The relations between $\mathcal{S}_i, \mathcal{T}_j$ and $\mathcal{G}$, for all $i, j = 1, 2, 3$, may be visualized in Figure 2.

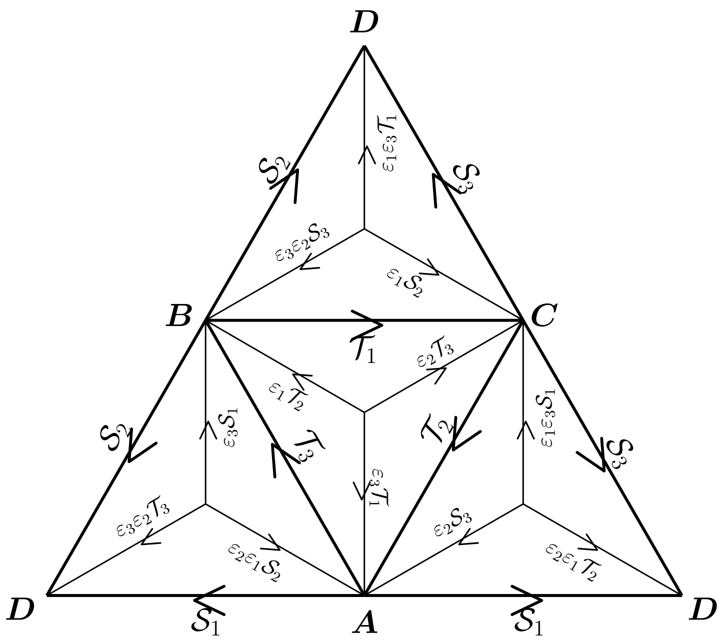

FIGURE 2.

This is to be understood as the pattern for a tetrahedron $ABCD$. The metric $\mathcal{G}$ does not appear in Figure 2 but in Figure 3, after building the tetrahedron, $\mathcal{G}$ appears as the altitude of the tetrahedron $ABCD$.

*Notice that, because of Proposition 3.6 i), $\mathcal{G} = \mathrm{id}_E$ implies $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.

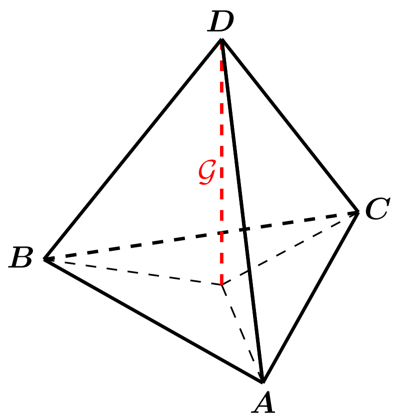

FIGURE 3.

When $\mathcal{G} = \mathrm{id}_E$, there is an identification given by $\varepsilon_i \mathcal{S}_i = \varepsilon_{i-1}\mathcal{T}_i$ between upper edges of the tetrahedron $ABCD$ and their projections onto the face ABC (see Figure 3). In other words, in this case the tetrahedron degenerates into a (flat) triangle.

The next proposition shows the behaviour of $\mathcal{G}$ and $\mathcal{T}_i$, $i = 1, 2, 3$, under the bilinear form $\langle ., . \rangle$.

Proposition 3.8. *Let* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *be an* $\boldsymbol{\varepsilon}$*-hypersymplectic structure on a Courant algebroid* $(E, \langle ., . \rangle, \Theta)$*. The maps* $\mathcal{G}$ *and* $\mathcal{T}_i$*,* $i = 1, 2, 3$*, satisfy*

$$\langle \mathcal{G}\mathcal{T}_i(X), \mathcal{T}_i(Y) \rangle = \varepsilon_{i-1}\varepsilon_{i+1} \langle \mathcal{G}(X), Y \rangle,$$

for all $X, Y \in \Gamma(E)$*.*

Proof: Using Proposition 3.4 i) and ii) and Proposition 3.7 iii) we have:

$$\begin{aligned} \langle \mathcal{G}\mathcal{T}_i(X), \mathcal{T}_i(Y) \rangle &= \varepsilon_1\varepsilon_2\varepsilon_3 \langle \mathcal{T}_i\mathcal{G}\mathcal{T}_i(X), Y \rangle = \varepsilon_1\varepsilon_2\varepsilon_3 \langle \mathcal{G}\mathcal{T}_i^2(X), Y \rangle \\ &= \varepsilon_{i-1}\varepsilon_{i+1} \langle \mathcal{G}(X), Y \rangle. \end{aligned}$$

■

From Definition 3.1 and Propositions 3.4, 3.6 and 3.7, we realize that the parameter $\varepsilon_1\varepsilon_2\varepsilon_3 = \pm 1$ is determinant for some basic properties of the morphisms $\mathcal{T}_i$, $\mathcal{S}_j$ and $\mathcal{G}$, $i, j = 1, 2, 3$, and for the relations between them. We shall see, in the remaining sections of this paper, that the case $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ is the more interesting one.

4. Hypersymplectic on deformed Courant structures

In this section we consider an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on a Courant algebroid (E, Θ) such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. As we have already remarked, the condition $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ determines some properties of the morphisms $\mathcal{S}_i$, $\mathcal{T}_i$, $i = 1, 2, 3$, and $\mathcal{G}$. Namely, we prove that the $\mathcal{T}_i$'s are Nijenhuis morphisms and we show that we may deform the Courant structure Θ by $\mathcal{S}_i$ or by $\mathcal{T}_i$, without loosing the property of $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ being hypersymplectic.

Let us recall a result from [2].

Proposition 4.1. *Let I and J be two anti-commuting endomorphisms on a Courant algebroid (E, Θ). Then, for all sections X and Y of E,*

$$2\,\mathcal{T}_\Theta(IJ)(X,Y) = \Big(\mathcal{T}_\Theta I(JX, JY) - J\left(\mathcal{T}_\Theta I(JX,Y) + \mathcal{T}_\Theta I(X, JY)\right) - \\ - J^2(\mathcal{T}_\Theta I(X,Y))\Big) + \underset{I,J}{\circlearrowleft},$$

where $\underset{I,J}{\circlearrowleft}$ stands for permutation of I and J. In particular, if I and J have vanishing Nijenhuis torsion then so has IJ.

As a direct consequence of Propositions 3.2, 4.1 and 3.4 ii), we get the following:

Theorem 4.2. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ) such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. Then, for each $i = 1, 2, 3$,*

i) the transition morphism $\mathcal{T}_i$ is a Nijenhuis morphism;
ii) if $\varepsilon_i = -1$, $\mathcal{T}_i$ is a complex structure;
iii) if $\varepsilon_i = 1$, $\mathcal{T}_i$ is a para-complex structure.

In the case $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, the triplet $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a *hypercomplex structure* on the Courant algebroid (E, Θ) in the sense of [11], see Remark 3.5.

In [2] we defined a *Nijenhuis pair* on a Courant algebroid (E, Θ) as a pair (I, J) of anti-commuting Nijenhuis morphisms such that $C_\Theta(I, J) = 0$ (see (2)). We shall see that an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid with $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ induces several Nijenhuis pairs. First, we need the next result which can be directly obtained from Proposition 3.13 in [2].

Proposition 4.3. *Let I and J be two anti-commuting endomorphisms on a Courant algebroid (E, Θ), with vanishing Nijenhuis torsion. Then, we have*

$$C_\Theta(I, IJ)(X, Y) = I(C_\Theta(I, J)(X, Y))$$

for all sections $X, Y \in \Gamma(E)$.

Furthermore, in the case where the square of the endomorphisms I and J is a multiple of the identity, we shall prove, in the next proposition, that $C_\Theta(I, J)$ vanishes.

Proposition 4.4. *Let I and J be two anti-commuting endomorphisms on a Courant algebroid (E, Θ), with vanishing Nijenhuis torsion and such that $I^2 = \lambda_I \operatorname{id}_E$ and $J^2 = \lambda_J \operatorname{id}_E$, for some $\lambda_I, \lambda_J \in \mathbb{R}\backslash\{0\}$. Then,*

$$C_\Theta(I, J) = 0.$$

Proof: From $I^2 = \lambda_I \operatorname{id}_E$ we have $J = \lambda_I^{-1} I(IJ)$ and, for all $X, Y \in \Gamma(E)$,

$$C_\Theta(I, J)(X, Y) = \lambda_I^{-1} C_\Theta(I, I(IJ))(X, Y) = \lambda_I^{-1} I(C_\Theta(IJ, I)(X, Y)), \tag{6}$$

where, in the last equality, we used Proposition 4.3 and the fact that $C_\Theta(\cdot, \cdot)$ is symmetric. Using $J^2 = \lambda_J \operatorname{id}_E$ and Proposition 4.3, Equation (6) becomes

$$\begin{aligned} C_\Theta(I, J)(X, Y) &= \lambda_I^{-1}\lambda_J^{-1} I(C_\Theta(IJ, IJ(J))(X, Y)) \\ &= \lambda_I^{-1}\lambda_J^{-1} I(IJ)(C_\Theta(IJ, J)(X, Y)) = -\lambda_J^{-1} J(C_\Theta(JI, J)(X, Y)), \end{aligned}$$

where we used the fact that $IJ = -JI$ in the last equality. Finally, applying once more Proposition 4.3, we get

$$C_\Theta(I, J)(X, Y) = -\lambda_J^{-1} J^2(C_\Theta(I, J)(X, Y)) = -C_\Theta(I, J)(X, Y).$$

Therefore,

$$C_\Theta(I, J)(X, Y) = 0,$$

for all $X, Y \in \Gamma(E)$. ■

From Proposition 4.4 we see that when two Nijenhuis morphisms are (para-)complex structures, i.e., $\lambda_I = \pm 1$ and $\lambda_J = \pm 1$, it is sufficient that they anti-commute to form a Nijenhuis pair. This is the case when we have a Courant algebroid equipped with an $\boldsymbol{\varepsilon}$-hypersymplectic structure such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, as it is stated in the next proposition.

Proposition 4.5. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ) such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. Then, we have $C_\Theta(\mathcal{S}_i, \mathcal{S}_j) = C_\Theta(\mathcal{T}_i, \mathcal{T}_j) = C_\Theta(\mathcal{S}_i, \mathcal{T}_j) = 0$, for all $i \neq j \in \{1, 2, 3\}$. In this case, for all $i \neq j \in \{1, 2, 3\}$, the pairs $(\mathcal{S}_i, \mathcal{S}_j), (\mathcal{T}_i, \mathcal{T}_j)$ and $(\mathcal{S}_i, \mathcal{T}_j)$ are Nijenhuis pairs.*

Next we show that when a triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ), it is also an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Courant algebroid deformed by $\mathcal{T}_i$ or by $\mathcal{S}_i$.

Theorem 4.6. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid (E, Θ), with $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. Then, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Courant algebroids $(E, \Theta_{\mathcal{T}_i})$ and $(E, \Theta_{\mathcal{S}_i})$, $i = 1, 2, 3$.*

Proof: Notice that both $\mathcal{T}_i$ and $\mathcal{S}_i$ are Nijenhuis with respect to Θ, so that $\Theta_{\mathcal{T}_i}$ and $\Theta_{\mathcal{S}_i}$ are Courant structures on E [8]. We only show that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on $(E, \Theta_{\mathcal{T}_i})$; the other case is similar. We compute, using the Jacobi identity,

$$\begin{aligned}(\Theta_{\mathcal{T}_i})_{\mathcal{S}_i,\mathcal{S}_i} &= \{\mathcal{S}_i, \{\mathcal{S}_i, \{\mathcal{T}_i, \Theta\}\}\} = \{\mathcal{S}_i, \{\mathcal{T}_i, \{\mathcal{S}_i, \Theta\}\}\} \\ &= \{\mathcal{T}_i, \Theta_{\mathcal{S}_i,\mathcal{S}_i}\} = \varepsilon_i \Theta_{\mathcal{T}_i}.\end{aligned}$$

For $i \neq j$ we have,

$$\begin{aligned}(\Theta_{\mathcal{T}_i})_{\mathcal{S}_j,\mathcal{S}_j} &= \Theta_{\mathcal{S}_j,\mathcal{S}_j\mathcal{T}_i} - 2C_\Theta(\mathcal{S}_j, \mathcal{S}_j\mathcal{T}_i) \\ &= \varepsilon_j\Theta_{\mathcal{T}_i} - 2C_\Theta(\mathcal{S}_j, \mathcal{S}_{i+1}) \\ &= \varepsilon_j\Theta_{\mathcal{T}_i}\end{aligned}$$

where we have used the formula $C_\Theta(I, IJ) = \frac{1}{2}(\Theta_{I,I,J} - \Theta_{J,I,I})$, that holds for all skew-symmetric anti-commuting endomorphisms I and J of E [2], Proposition 3.7 and Proposition 4.5. So, for all $i, j = 1, 2, 3$, we have $(\Theta_{\mathcal{T}_i})_{\mathcal{S}_j,\mathcal{S}_j} = \varepsilon_j\Theta_{\mathcal{T}_i}$ which completes the proof. ∎

5. 1-1 correspondence

In this section we keep considering an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on a Courant algebroid such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. We define hyperkähler structures on Courant algebroids and prove a $1-1$ correspondence between hypersymplectic and hyperkähler structures. We show how we may switch the roles of the morphisms $\mathcal{S}_i$ and $\mathcal{T}_i$ to get a set of equivalent structures; these structures are summarized in a diagram at the end of the section.

First, we have to distinguish two different cases amongst $\boldsymbol{\varepsilon}$-hypersymplectic structures $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$:

- when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is said to be a *hypersymplectic structure*;
- otherwise, the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is said to be a *para-hypersymplectic structure*.

Note that all para-hypersymplectic structures satisfy, eventually after a cyclic permutation of the indices, $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$. In the sequel, every para-hypersymplectic structure will be considered in such form.

Given a Courant algebroid, we may define, in a natural way, a notion of (pseudo-)metric.

Definition 5.1. A *pseudo-metric* on a Courant algebroid $(E, \langle .,. \rangle, \Theta)$ is a symmetric and orthogonal bundle automorphism $G : E \to E$. If moreover G is *positive definite*, that is, $\langle G(e), e \rangle > 0$, for all non vanishing sections $e \in \Gamma(E)$, then the prefix "pseudo" is removed and G is said to be a *metric* on $(E, \langle .,. \rangle, \Theta)$.

In the sequel, we do not require the metric to be positive definite. However, in order to simplify the terminology we shall omit the prefix "pseudo", although we deal with pseudo-metrics.

Remark 5.2. On the above definition of a metric G, assuming that G is symmetric, the orthogonality condition ($GG^* = \mathrm{id}_E$) can be replaced by an almost para-complex condition ($G^2 = \mathrm{id}_E$).

The next proposition follows directly from the Proposition 3.6 and Remark 5.2.

Proposition 5.3. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a (para-)hypersymplectic structure on a Courant algebroid $(E, \langle .,. \rangle, \Theta)$. Then, the morphism $\mathcal{G}$ given by equation (5) is a metric on E.*

Next, we define the notions of hermitian and para-hermitian pair on a Courant algebroid.

Definition 5.4. A *hermitian* (resp., *para-hermitian*) *pair*[†] on a Courant algebroid $(E, \langle .,. \rangle, \Theta)$ is a pair (J, G) where J is a complex (resp., para-complex) structure and G is a metric such that, for all $X, Y \in \Gamma(E)$,

$$\langle G(JX), JY \rangle = \langle G(X), Y \rangle, \qquad (\text{resp., } \langle G(JX), JY \rangle = -\langle G(X), Y \rangle) \, .$$

[†]Rigourously, we should say *pseudo-hermitian* and *para-pseudo-hermitian* but, as it was already mentioned, we omit the prefix "pseudo".

Remark 5.5. In the above definition, if J is skew-symmetric the (para-)hermiticity condition, $\langle G(JX), JY\rangle = \pm\langle G(X), Y\rangle$, is equivalent to $GJ = JG$.

As a direct consequence of Proposition 3.8, we have the following:

Proposition 5.6. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a (para-)hypersymplectic structure on a Courant algebroid $(E, \langle .,.\rangle, \Theta)$.*

i) If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure, then $(\mathcal{T}_i, \mathcal{G})$ is a hermitian pair, for all $i = 1, 2, 3$.

ii) If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a para-hypersymplectic structure, then $(\mathcal{T}_1, \mathcal{G})$ and $(\mathcal{T}_2, \mathcal{G})$ are para-hermitian pairs while $(\mathcal{T}_3, \mathcal{G})$ is a hermitian pair.

Let us define (para-)hyperkähler structures on a Courant algebroid[‡] and see how they are related to (para-)hypersymplectic structures.

Definition 5.7. A quadruple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a *hyperkähler* (resp., *para-hyperkähler*) *structure*[§] on a Courant algebroid $(E, \langle .,.\rangle, \Theta)$ if the following is satisfied:

i) $\mathcal{G}$ is a metric;

ii) $\mathcal{T}_1$ and $\mathcal{T}_2$ are anti-commuting complex (resp., para-complex) endomorphisms and $\mathcal{T}_3 = \mathcal{T}_1\mathcal{T}_2$;

iii) $(\mathcal{G}, \mathcal{T}_j)_{j=1,2}$ are hermitian (resp., para-hermitian) pairs;

iv) $\mathcal{T}_\Theta(\mathcal{G}\mathcal{T}_j) = 0, \ j = 1, 2, 3.$

Notice that, when $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a (para-)hyperkähler structure, $(\mathcal{G}, \mathcal{T}_3)$ is a hermitian pair and the morphisms $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ pairwise anti-commute.

Theorem 5.8. *The triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic (resp., para-hypersymplectic) structure on a Courant algebroid (E, Θ) if and only if $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a hyperkähler (resp., para-hyperkähler) structure on (E, Θ).*

Proof: If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on (E, Θ) then, using previous results, we easily conclude that $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a (para-)hyperkähler structure on (E, Θ).

On the other hand, if $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a (para-)hyperkähler structure on (E, Θ), we define

$$\mathcal{S}_i := \varepsilon_i\varepsilon_{i-1}\mathcal{G}\mathcal{T}_i.$$

‡In [6], hyperkähler structures on Courant algebroids are called generalized hyper-Kähler structures.

§Again, we omit the prefix "pseudo".

Then, we have

$$\mathcal{S}_i^2 = \mathcal{G}\mathcal{T}_i\mathcal{G}\mathcal{T}_i = \mathcal{G}^2\mathcal{T}_i^2 = \mathcal{T}_i^2 = \varepsilon_i \mathrm{id}_E,$$

where we used the fact that $\mathcal{G}$ and $\mathcal{T}_i$ commute and $\mathcal{G}^2 = \mathrm{id}_E$ (see Remarks 5.5 and 5.2). Moreover,

$$\mathcal{S}_i\mathcal{S}_{i+1} = \varepsilon_i\varepsilon_{i-1}\varepsilon_{i+1}\varepsilon_i\mathcal{G}\mathcal{T}_i\mathcal{G}\mathcal{T}_{i+1} = \varepsilon_{i-1}\varepsilon_{i+1}\mathcal{T}_i\mathcal{T}_{i+1} = -\varepsilon_{i-1}\varepsilon_{i+1}\mathcal{T}_{i+1}\mathcal{T}_i = -\mathcal{S}_{i+1}\mathcal{S}_i,$$

where we used the fact that the morphisms $\mathcal{T}_i, i = 1, 2, 3$, pairwise anticommute. Finally, because $\mathcal{S}_i^2 = \varepsilon_i \mathrm{id}_E$ and $\mathcal{T}_\Theta \mathcal{S}_i = 0$ (see item iv) of Definition 5.7) we conclude that $\Theta_{\mathcal{S}_i,\mathcal{S}_i} = \varepsilon_i\Theta$. Therefore, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on (E, Θ). ■

Next, we see that the tetrahedron model (see Figure 3), besides being an efficient way to summarize all the algebraic relations between the morphisms of a (para-)hypersymplectic structure, is an accurate representation that enables us to discover new relations. In fact, the next theorem shows that the symmetries of the tetrahedron are symmetries of the (para-)hypersymplectic structures on Courant algebroids. These symmetries can not exist for (para-)hypersymplectic structures on Lie algebroids (see definition in [3]).

Theorem 5.9. *The triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic (resp., para-hypersymplectic) structure on a Courant algebroid (E, Θ) if and only if $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a hypersymplectic (resp., para-hypersymplectic) structure on (E, Θ). Furthermore, both (para-)hypersymplectic structures determine equal or opposite metrics.*

Proof: If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure then, definitions and previous results yield,

$$\begin{cases} \mathcal{S}_i^2 = \mathcal{T}_i^2 = \varepsilon_i \mathrm{id}_E; \\ \mathcal{S}_i\mathcal{S}_j + \mathcal{S}_j\mathcal{S}_i = \mathcal{T}_i\mathcal{T}_j + \mathcal{T}_j\mathcal{T}_i = \mathcal{S}_i\mathcal{T}_j + \mathcal{T}_j\mathcal{S}_i = 0; \\ \mathcal{T}_\Theta(\mathcal{S}_i) = \mathcal{T}_\Theta(\mathcal{T}_i) = 0. \end{cases}$$

Thus, $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a (para-)hypersymplectic structure.

Now, let us assume that $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a (para-)hypersymplectic structure. In this case, the transition morphisms are $\varepsilon_1\varepsilon_3\mathcal{T}_1, \varepsilon_1\varepsilon_3\mathcal{S}_2$ and $\varepsilon_1\varepsilon_3\mathcal{S}_3$ and, using the first part of the proof, we conclude that $(\mathcal{S}_1, \varepsilon_1\varepsilon_3\mathcal{S}_2, \varepsilon_1\varepsilon_3\mathcal{S}_3)$ is a (para-)hypersymplectic structure. Therefore, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure.

Finally, because $\mathcal{S}_3\mathcal{S}_2\mathcal{S}_1 = \varepsilon_1\varepsilon_3\mathcal{T}_3\mathcal{T}_2\mathcal{S}_1$, the metrics induced by both (para-)hypersymplectic structures are equal or opposite. ■

Applying successively Theorems 5.8 and 5.9, we conclude that a (para-)hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on (E, Θ) induces several (para-)hypersymplectic and (para-)hyperkähler structures on (E, Θ), as we see in the next diagram.

$$\begin{tikzcd}
(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) \text{ (para-)hypersymplectic} \arrow[r, leftrightarrow, "Thm\ 5.8"] \arrow[d, leftrightarrow, "Thm\ 5.9"'] & (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G}) \text{ (para-)hyperkähler} \arrow[d, leftrightarrow] \\
(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3) \text{ (para-)hypersymplectic} \arrow[r, leftrightarrow, "Thm\ 5.8"] \arrow[d, leftrightarrow, "Thm\ 5.9"'] & (\mathcal{T}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{G}) \text{ (para-)hyperkähler} \arrow[d, leftrightarrow] \\
(\mathcal{T}_1, \mathcal{S}_2, \mathcal{T}_3) \text{ (para-)hypersymplectic} \arrow[r, leftrightarrow, "Thm\ 5.8"] \arrow[d, leftrightarrow, "Thm\ 5.9"'] & (\mathcal{S}_1, \mathcal{T}_2, \mathcal{S}_3, \mathcal{G}) \text{ (para-)hyperkähler} \arrow[d, leftrightarrow] \\
(\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_3) \text{ (para-)hypersymplectic} \arrow[r, leftrightarrow, "Thm\ 5.8"] & (\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_3, \mathcal{G}) \text{ (para-)hyperkähler}
\end{tikzcd}$$

6. Hypersymplectic structures on Lie algebroids

The purpose of this section is to present a first example of an ε-hypersymplectic structure on a Courant algebroid, which is constructed out of an ε-hypersymplectic structure on a Lie algebroid. First, we recall the definition and some properties of the latter [1, 3].

An *ε-hypersymplectic structure* on a Lie algebroid (A, μ) is a triplet $(\omega_1, \omega_2, \omega_3)$ of symplectic forms with inverse Poisson bivectors (π_1, π_2, π_3) such that the transition endomorphisms N_1, N_2 and N_3 on A, defined by

$$N_i := \pi_{i-1}^{\#} \circ \omega_{i+1}^{\flat}, \quad i \in \mathbb{Z}_3, \tag{7}$$

satisfy

$$N_i^2 = \varepsilon_i \mathrm{id}_A, \quad i = 1, 2, 3. \tag{8}$$

An important property of the transitions morphisms N_i, $i = 1, 2, 3$, is that

$$\mathcal{T}_\mu N_i = 0,$$

i.e., they are Nijenhuis morphisms.

Having an ε-hypersymplectic structure on a Lie algebroid (A, μ), we define $g \in \bigotimes^2 A^*$ by setting, for all $X, Y \in \Gamma(A)$,

$$g(X, Y) := \langle g^\flat X, Y \rangle,$$

where $g^\flat : A \longrightarrow A^*$ is given by

$$g^\flat := \varepsilon_3\varepsilon_2\ {\omega_3}^\flat \circ {\pi_1}^\sharp \circ {\omega_2}^\flat. \tag{9}$$

The definition of $g^\flat$ is not affected by a circular permutation of the indices in equation (9), that is,

$$g^\flat = \varepsilon_{i-1}\varepsilon_{i+1}\ {\omega_{i-1}}^\flat \circ {\pi_i}^\sharp \circ {\omega_{i+1}}^\flat, \tag{10}$$

for all $i \in \mathbb{Z}_3$. Moreover, we have

$$(g^\flat)^* = -\varepsilon_1\varepsilon_2\varepsilon_3\ g^\flat,$$

which means that g is symmetric or skew-symmetric, depending on the sign of the product $\varepsilon_1\varepsilon_2\varepsilon_3$. When $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, the morphism $g^\flat$ defined by (9) determines a *pseudo-metric* on A.

Let (A, μ) be a Lie algebroid and consider the Courant algebroid $(A \oplus A^*, \mu)$. If we take a triplet $(\omega_1, \omega_2, \omega_3)$ of 2-forms and a triplet (π_1, π_2, π_3) of bivectors on A, we may define the skew-symmetric bundle endomorphisms $\mathcal{S}_i : A \oplus A^* \to A \oplus A^*$, $i = 1, 2, 3$,

$$\mathcal{S}_i := \begin{bmatrix} 0 & \varepsilon_i\, \pi_i^\sharp \\ \omega_i^\flat & 0 \end{bmatrix}. \tag{11}$$

In order to simplify the writing and if there is no risk of confusion, we shall omit the symbols $\sharp$ and $\flat$ and denote the morphisms $\omega_i^\flat$ and $\pi_i^\sharp$ by ω_i and π_i, respectively. Moreover, in the supergeometric setting, we have

$$\mathcal{S}_i(X + \alpha) = \{X + \alpha, \omega_i + \varepsilon_i\pi_i\},$$

for all $X + \alpha \in A \oplus A^*$.

Lemma 6.1. *Let ω_1, ω_2 and ω_3 be 2-forms on a Lie algebroid (A, μ) and π_1, π_2 and π_3 bivectors on A. Consider the vector bundle morphisms N_1, N_2 and N_3 on A, given by (7), and the bundle endomorphisms $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ on $A \oplus A^*$, given by (11). Then, for all $i = 1, 2, 3$,*

i) $\mathcal{S}_i^2 = \varepsilon_i \mathrm{id}_{\mathrm{A}\oplus\mathrm{A}^*} \Leftrightarrow \pi_{\mathrm{i}} \circ \omega_{\mathrm{i}} = \mathrm{id}_{\mathrm{A}}$,
ii) $\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\, \mathcal{S}_{i+1}\mathcal{S}_{i-1} \Leftrightarrow N_i^2 = \varepsilon_i \mathrm{id}_{\mathrm{A}}$.

Proof: A simple computation gives i). To prove ii), we notice that $N_i^2 = \varepsilon_i\, \mathrm{id}_{\mathrm{A}}$ is equivalent to [3]

$$\omega_{i+1} \circ \pi_{i-1} = \varepsilon_i\, \omega_{i-1} \circ \pi_{i+1},$$

for all $i \in \mathbb{Z}_3$. On the other hand, we have

$$\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \begin{bmatrix} \varepsilon_{i-1}\, \pi_{i-1} \circ \omega_{i+1} & 0 \\ 0 & \varepsilon_{i+1}\, \omega_{i-1} \circ \pi_{i+1} \end{bmatrix}$$

and

$$\begin{aligned} \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_1\varepsilon_2\varepsilon_3 & \begin{bmatrix} \varepsilon_{i+1}\, \pi_{i+1} \circ \omega_{i-1} & 0 \\ 0 & \varepsilon_{i-1}\, \omega_{i+1} \circ \pi_{i-1} \end{bmatrix} \\ & = \begin{bmatrix} \varepsilon_{i-1}\varepsilon_i\, \pi_{i+1} \circ \omega_{i-1} & 0 \\ 0 & \varepsilon_{i+1}\varepsilon_i\, \omega_{i+1} \circ \pi_{i-1} \end{bmatrix}. \end{aligned}$$

So, $\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i+1}\mathcal{S}_{i-1}$ if and only if $\omega_{i+1} \circ \pi_{i-1} = \varepsilon_i\, \omega_{i-1} \circ \pi_{i+1}$ and this completes the proof. ∎

Proposition 6.2. *A triplet $(\omega_1, \omega_2, \omega_3)$, with inverse (π_1, π_2, π_3), is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid (A, μ) if and only if the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu)$, with $\mathcal{S}_i$, $i = 1, 2, 3$, given by (11).*

Proof: Suppose that $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid (A, μ) and π_i is the inverse of ω_i, $i = 1, 2, 3$. According to Definition 3.1 and Lemma 6.1, we only have to check that $\mu_{\mathcal{S}_i, \mathcal{S}_i} = \varepsilon_i \mu$, for $i = 1, 2, 3$. A simple computation, using the fact that π_i is a Poisson bivector, gives:

$$\begin{aligned} \{\mathcal{S}_i, \{\mathcal{S}_i, \mu\}\} &= \{\omega_i + \varepsilon_i\pi_i, \{\omega_i + \varepsilon_i\pi_i, \mu\}\} = \varepsilon_i\{\omega_i, \{\pi_i, \mu\}\} \\ &= \varepsilon_i\, \mu, \end{aligned}$$

where we used, in the last equality, the formula

$$\{\mathrm{id}_A, \chi\} = (q-p)\chi, \quad \chi \in \mathcal{F}^{(p,q)}_{A\oplus A^*}. \tag{12}$$

Conversely, assume that the endomorphisms $\mathcal{S}_i = \begin{bmatrix} 0 & \varepsilon_i\, \pi_i \\ \omega_i & 0 \end{bmatrix}$, $i = 1, 2, 3$, form an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu)$. Using again Lemma 6.1, we only have to prove that the non-degenerate 2-forms ω_i are symplectic. From

$$\{\omega_i + \varepsilon_i\pi_i, \{\omega_i + \varepsilon_i\pi_i, \mu\}\} = \varepsilon_i\mu,$$

we get $\{\pi_i, \{\pi_i, \mu\}\} = 0$, which means that π_i is a Poisson bivector on (A, μ). But π_i being a Poisson bivector on (A, μ) is equivalent to ω_i being a symplectic form on (A, μ). ∎

Under the conditions of Proposition 6.2, the transition morphisms of the $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on $(A \oplus A^*, \mu)$, defined by (4), are given by

$$\mathcal{T}_i = \begin{bmatrix} N_i & 0 \\ 0 & \varepsilon_1\varepsilon_2\varepsilon_3\, N_i{}^* \end{bmatrix}, \quad i = 1, 2, 3,$$

where N_i is the transition morphism of the $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on the Lie algebroid (A, μ), see (7). The endomorphism $\mathcal{G} : A \oplus A^* \to A \oplus A^*$ defined by (5) is given by

$$\mathcal{G} = \begin{bmatrix} 0 & (g^\flat)^{-1} \\ g^\flat & 0 \end{bmatrix},$$

where $g^\flat : A \to A^*$ is defined by (9).

7. Hypersymplectic structures on Lie bialgebroids

In this section we present a class of examples of $\boldsymbol{\varepsilon}$-hypersymplectic structures on a Courant algebroid $(A \oplus A^*, \mu + \gamma)$, which is the double of a Lie bialgebroid $((A, A^*), \mu, \gamma)$.

Having in mind that a bivector π on A can be seen as a 2-form on A^*, through the identification $A = (A^*)^*$, we have the following result.

Proposition 7.1. *Let $((A, A^*), \mu, \gamma)$ be a Lie bialgebroid and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a triplet of bundle endomorphisms of $A \oplus A^*$, with $\mathcal{S}_i$ given by (11).*

The triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \gamma)$ if and only if $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Lie algebroid (A, μ), (π_1, π_2, π_3) is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Lie algebroid (A^*, γ) and π_i is the inverse of ω_i, $i = 1, 2, 3$.

Proof: We use Lemma 6.1 noticing that $\pi_i \circ \omega_i = \mathrm{id}_A \Leftrightarrow \omega_i \circ \pi_i = \mathrm{id}_{A^*}$ and $N_i^2 = \varepsilon_i \mathrm{id}_A \Leftrightarrow (N_i^*)^2 = \varepsilon_i \mathrm{id}_{A^*}$, $i = 1, 2, 3$, so that conditions i) and ii) of Definition 3.1 are satisfied if and only if π_i and ω_i are inverses of each other and (8) holds. Moreover, using the bidegrees of $\mathcal{F}^3_{A \oplus A^*}$, we have

$$\{\mathcal{S}_i, \{\mathcal{S}_i, \mu + \gamma\}\} = \varepsilon_i(\mu + \gamma) \Leftrightarrow \begin{cases} \{\omega_i, \{\pi_i, \mu\}\} + \{\pi_i, \{\omega_i, \mu\}\} = \mu \\ \{\omega_i, \{\pi_i, \gamma\}\} + \{\pi_i, \{\omega_i, \gamma\}\} = \gamma \\ \{\omega_i, \{\omega_i, \gamma\}\} = 0 \\ \{\pi_i, \{\pi_i, \mu\}\} = 0. \end{cases} \tag{13}$$

The fourth equation on the right-hand side of (13) means that π_i is a Poisson bivector on (A, μ), which is equivalent to ω_i being a symplectic form on (A, μ).

The third equation on the right-hand side of (13) means that ω_i, seen as a bivector on A^*, is Poisson on the Lie algebroid (A^*, γ), which is equivalent to saying that π_i is symplectic on (A^*, γ). Concerning the first and second equations on the right-hand side of (13), when π_i and ω_i are inverse of each other, they are equivalent to

$$\{\pi_i, \{\omega_i, \mu\}\} = 0 \quad \text{and} \quad \{\omega_{\mathrm{i}}, \{\pi_{\mathrm{i}}, \gamma\}\} = 0, \tag{14}$$

respectively. Contracting, on the left, the first equation of (14) with ω_i and the second equation of (14) with π_i, (14) becomes equivalent to

$$\{\omega_i, \mu\} = 0 \quad \text{and} \quad \{\pi_{\mathrm{i}}, \gamma\} = 0,$$

respectively. This completes the proof. ■

It is well known that a Poisson bivector π_i on (A, μ) determines a Lie algebroid structure on A^*; we denote by μ_{π_i} this induced structure. In [4] we proved that if $(\omega_1, \omega_2, \omega_3)$ is an ε-hypersymplectic structure on a Lie algebroid (A, μ) and π_i is the inverse of ω_i, $i = 1, 2, 3$, then the triplet (π_1, π_2, π_3) is an ε-hypersymplectic structure on the Lie algebroid (A^*, μ_{π_i}). So, given an ε-hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on a Lie algebroid (A, μ), Proposition 7.1 yields that the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \mu_{\pi_i})$. Conversely, if $((A, A^*), \mu, \mu_{\pi_i})$, $i = 1, 2, 3$, is a Lie bialgebroid and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \mu_{\pi_i})$, then, by Proposition 7.1, $(\omega_1, \omega_2, \omega_3)$ is an ε-hypersymplectic structure on the Lie algebroid (A, μ).

Thus, we have proved:

Corollary 7.2. *The triplet $(\omega_1, \omega_2, \omega_3)$, with inverse (π_1, π_2, π_3), is an ε-hypersymplectic structure on the Lie algebroid (A, μ) if and only if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \mu_{\pi_i})$, with $\mathcal{S}_i$ given by (11), $i = 1, 2, 3$.*

8. Hypersymplectic structures with torsion on Lie algebroids

In this section we pretend to study a class of examples of hypersymplectic structures on Courant algebroids determined by some structures on Lie algebroids which are called *hypersymplectic with torsion*. These are introduced and discussed in [5] and may be considered as being equivalent to hyperkähler structures with torsion, also known as HKT structures [7]. The hypersymplectic structures with torsion on Lie algebroids provide examples of hypersymplectic structures (without torsion) on Courant algebroids which are doubles of quasi-Lie bialgebroids and even in the more general case where the Courant structure is the double of a proto-Lie bialgebroid.

We give the definition of a hypersymplectic structure with torsion on a Lie algebroid (A, μ) and postpone the study of this structure to [5].

Let ω_1, ω_2 and ω_3 be nondegenerate 2-forms on a Lie algebroid (A, μ), with inverses π_1, π_2 and $\pi_3 \in \Gamma(\wedge^2 A)$, respectively, and consider the transition morphisms $N_1, N_2, N_3 : A \to A$ given by (7).

Definition 8.1. The triplet $(\omega_1, \omega_2, \omega_3)$ is a *hypersymplectic structure with torsion* on the Lie algebroid (A, μ) if

$$N_i^{\,2} = -\mathrm{id}_A, \quad i = 1, 2, 3, \quad \text{and} \quad N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3, \tag{15}$$

where $N_i \mathrm{d}\omega_i(X, Y, Z) = \mathrm{d}\omega_i(N_i X, N_i Y, N_i Z)$, for all $X, Y, Z \in \Gamma(A)$ and d stands for the differential of the Lie algebroid (A, μ).

When the non-degenerate 2-forms ω_1, ω_2 and ω_3 are closed, then they are symplectic forms and the right hand side of (15) is trivially satisfied. In this case, the triplet $(\omega_1, \omega_2, \omega_3)$ is a *hypersymplectic structure* on (A, μ), that is, an $\boldsymbol{\varepsilon}$-hypersymplectic structure with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ (see Section 6).

The next lemma will be useful in what follows.

Lemma 8.2. *Let $((A, A^*), \mu, \gamma, \psi, \phi)$ be a proto-Lie bialgebroid, $\pi \in \Gamma(\wedge^2 A)$ and $\omega \in \Gamma(\wedge^2 A^*)$ inverse of each other and $\varepsilon = \pm 1$. Then,*

i) $\{\pi, \{\pi, \mu\}\} = -2\,\varepsilon\,\psi \Leftrightarrow 2\,\varepsilon\,\{\pi, \{\omega, \mu\}\} = -\{\omega, \{\omega, \psi\}\}$;

ii) $\{\omega, \{\omega, \gamma\}\} = -2\,\varepsilon\,\phi \Leftrightarrow 2\,\varepsilon\,\{\omega, \{\pi, \gamma\}\} = -\{\pi, \{\pi, \phi\}\}$;

Proof: i) Let us assume that $\{\pi, \{\pi, \mu\}\} = -2\,\varepsilon\,\psi$. Then,

$$\{\omega, \{\pi, \{\pi, \mu\}\}\} = -2\,\varepsilon\,\{\omega, \psi\}$$

and the Jacobi identity together with (12) gives

$$\{\pi, \{\pi, \{\omega, \mu\}\}\} = -2\,\varepsilon\,\{\omega, \psi\}.$$

Thus,

$$\{\omega, \{\pi, \{\pi, \{\omega, \mu\}\}\}\} = -2\,\varepsilon\,\{\omega, \{\omega, \psi\}\}$$

or, equivalently,

$$\{\pi, \{\omega, \mu\}\} + \{\pi, \{\omega, \{\pi, \{\omega, \mu\}\}\}\} = -2\,\varepsilon\,\{\omega, \{\omega, \psi\}\}. \tag{16}$$

Finally, (16) gives

$$2\{\pi,\{\omega,\mu\}\} = -\varepsilon\,\{\omega,\{\omega,\psi\}\}.$$

Now, we assume that $-\{\omega,\{\omega,\psi\}\} = 2\,\varepsilon\,\{\pi,\{\omega,\mu\}\}$. Then,

$$-\{\pi,\{\omega,\{\omega,\psi\}\}\} = 2\,\varepsilon\,\{\pi,\{\pi,\{\omega,\mu\}\}\}$$

which is equivalent to

$$-\{\omega,\psi\} - \{\omega,\{\pi,\{\omega,\psi\}\}\} = 2\,\varepsilon\,\{\pi,\{\pi,\{\omega,\mu\}\}\}.$$

Thus,

$$-\,\{\pi,\{\omega,\psi\}\} - \{\pi,\{\omega,\{\pi,\{\omega,\psi\}\}\}\} = 2\,\varepsilon\,\{\pi,\{\pi,\{\pi,\{\omega,\mu\}\}\}\}. \tag{17}$$

From (17) we get, applying the Jacobi identity and (12) several times,

$$\begin{aligned} -3\psi - 3\{\pi,\{\omega,\psi\}\} &= -2\,\varepsilon\,\{\pi,\{\pi,\mu\}\} + 2\,\varepsilon\,\{\pi,\{\pi,\{\omega,\{\pi,\mu\}\}\}\} \\ \Leftrightarrow\ -6\psi = \varepsilon\,\{\pi,\{\omega,\{\pi,\{\pi,\mu\}\}\}\} &\Leftrightarrow\ -2\psi = \ \varepsilon\,\{\pi,\{\pi,\mu\}\}. \end{aligned}$$

ii) The proof is similar to case i).

■

Now, we have to mention that the definition of hypersymplectic structure with torsion on a Lie algebroid can be given using the inverses of the non-degenerate 2-forms ω_i. More precisely,

$(\omega_1,\omega_2,\omega_3)$ is a hypersymplectic structure with torsion on (A,μ) if and only if

$$N_i^2 = -\mathrm{id_A} \quad \textit{and} \quad [\pi_1,\pi_1] = [\pi_2,\pi_2] = [\pi_3,\pi_3], \tag{18}$$

where $[.,.]$ is the bracket of multivectors on A. The equivalence of the two definitions is given in [5].

The next proposition gives a first example of a hypersymplectic structure on a Courant algebroid, which is constructed out of a hypersymplectic structure with torsion on a Lie algebroid.

Proposition 8.3. *Let $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ be a triplet of bundle endomorphisms of $A\oplus A^*$, with $\mathcal{S}_i$ given by (11).*

If $(\omega_1,\omega_2,\omega_3)$ is a hypersymplectic structure with torsion on a Lie algebroid (A,μ) such that the inverses π_1, π_2 and π_3 are weak-Poisson [¶] *and ψ =*

[¶] A bivector π on a Lie algebroid (A,μ) is *weak-Poisson* if $\{\mu,\{\{\pi,\mu\},\pi\}\} = 0$ or, equivalently, $\{\mu,[\pi,\pi]\} = 0$.

$-\frac{1}{2}[\pi_i,\pi_i]$, $i=1,2,3$, then $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ is a hypersymplectic structure on the Courant algebroid $(A\oplus A^*,\mu+\psi)$.

Conversely, if $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ is a hypersymplectic structure on a Courant algebroid $(A\oplus A^*,\mu+\psi)$, then $\psi=-\frac{1}{2}[\pi_i,\pi_i]$ and $(\omega_1,\omega_2,\omega_3)$ is a hypersymplectic structure with torsion on the Lie algebroid (A,μ), where π_i is the inverse of ω_i and is weak-Poisson, $i=1,2,3$.

Proof: First, notice that given $\psi\in\Gamma(\wedge^3 A)$, $\mu+\psi$ is a Courant algebroid structure on $A\oplus A^*$ if and only if

$$\{\mu+\psi,\mu+\psi\}=0 \Leftrightarrow \begin{cases}\{\mu,\mu\}=0\\ \{\mu,\psi\}=0\end{cases} \Leftrightarrow \begin{cases}(A,\mu)\,\text{is a Lie algebroid}\\ \{\mu,\psi\}=0.\end{cases} \tag{19}$$

Let us assume that $(\omega_1,\omega_2,\omega_3)$ is a hypersymplectic structure with torsion on a Lie algebroid (A,μ) such that $\{\mu,\{\{\pi_i,\mu\},\pi_i\}\}=0$ and $\psi=-\frac{1}{2}[\pi_i,\pi_i]$, $i=1,2,3$. From (19), $\mu+\psi$ is a Courant structure on $A\oplus A^*$ while, from Lemma 6.1, conditions i) and ii) of Definition 3.1 are satisfied. For condition iii) of Definition 3.1, we have

$$\{\mathcal{S}_i,\{\mathcal{S}_i,\mu+\psi\}\}=-\mu-\psi \Leftrightarrow \begin{cases}\{\omega_i,\{\pi_i,\mu\}\}+\{\pi_i,\{\omega_i,\mu\}\}-\{\omega_i,\{\omega_i,\psi\}\}=\mu\\ -\{\pi_i,\{\pi_i,\mu\}\}+\{\pi_i,\{\omega_i,\psi\}\}=\psi\end{cases}$$

$$\Leftrightarrow \begin{cases}2\{\pi_i,\{\omega_i,\mu\}\}=\{\omega_i,\{\omega_i,\psi\}\}\\ \{\pi_i,\{\pi_i,\mu\}\}=2\psi.\end{cases} \tag{20}$$

According to Lemma 8.2, the two conditions of (20) are equivalent and they hold from the very definition of ψ.

Let us now assume that $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ is a hypersymplectic structure on a Courant algebroid $(A\oplus A^*,\mu+\psi)$. Then, the pair (A,μ) is a Lie algebroid and, from (20), the 3-vector ψ is given by $\psi=-\frac{1}{2}[\pi_i,\pi_i]$, $i=1,2,3$. Thus, $[\pi_1,\pi_1]=[\pi_2,\pi_2]=[\pi_3,\pi_3]$ and, from Lemma 6.1 and (18), we get that $(\omega_1,\omega_2,\omega_3)$ is a hypersymplectic structure with torsion on (A,μ). ■

In the last theorem of this article we show that having a Lie bialgebroid (A,A^*) equipped with a hypersymplectic structure with torsion on A and a hypersymplectic structure with torsion on A^* is equivalent to having a hypersymplectic structure (without torsion) on $A\oplus A^*$ equipped with a Courant structure which is the double of a proto-Lie bialgebroid.

Let us consider the following conditions that will be needed in the next theorem, where $\psi \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$,

$$\begin{cases} \{\mu, \psi\} = 0 \\ \{\gamma, \phi\} = 0 \\ \{\psi, \phi\} = 0. \end{cases} \tag{21}$$

Theorem 8.4. *Let $((A, A^*), \mu, \gamma)$ be a Lie bialgebroid and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a triplet of bundle endomorphisms of $A \oplus A^*$, with $\mathcal{S}_i$ given by (11).*

If $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on the Lie algebroid (A, μ) with inverses π_i such that (π_1, π_2, π_3) is a hypersymplectic structure with torsion on the Lie algebroid (A^*, γ), and if $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$ and $\phi = -\frac{1}{2}\{\omega_i, \{\gamma, \omega_i\}\}$, $i = 1, 2, 3$, satisfy (21), then $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on $(A \oplus A^*, \mu + \gamma + \psi + \phi)$.

Conversely, if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on a Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi + \phi)$, then $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on (A, μ), (π_1, π_2, π_3) is a hypersymplectic structure with torsion on (A^*, γ), ψ and ϕ are given by $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$ and $\phi = -\frac{1}{2}\{\omega_i, \{\gamma, \omega_i\}\}$, $i = 1, 2, 3$, and satisfy (21).

Proof: Let us assume that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on (A, μ), (π_1, π_2, π_3) is a hypersymplectic structure with torsion on (A^*, γ), $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$, $\phi = -\frac{1}{2}\{\omega_i, \{\gamma, \omega_i\}\}$, $\{\mu, \psi\} = 0$, $\{\gamma, \phi\} = 0$, $i = 1, 2, 3$, and $\{\psi, \phi\} = 0$. We start by proving that $\mu + \gamma + \psi + \phi$ is a Courant structure on $A \oplus A^*$. We have

$$\{\mu + \gamma + \psi + \phi, \, \mu + \gamma + \psi + \phi\} = 0 \Leftrightarrow \begin{cases} \{\mu, \mu\} = -2\{\gamma, \phi\} \\ \{\gamma, \gamma\} = -2\{\mu, \psi\} \\ \{\mu, \gamma\} = -\{\psi, \phi\} \\ \{\gamma, \psi\} = 0 \\ \{\mu, \phi\} = 0. \end{cases} \tag{22}$$

The first equation in the right hand side of (22) holds as a consequence of the fact that (A, μ) is a Lie algebroid, so that $\{\mu, \mu\} = 0$, and $\{\gamma, \phi\} = 0$. The second equation of (22) holds for analogous reasons. We have $\{\mu, \gamma\} = 0$ because $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid and $\{\psi, \phi\} = 0$, by assumption. Thus, the third equation of (22) also holds. Let us check that $\{\gamma, \psi\} = 0$

and $\{\mu, \phi\} = 0$. We only prove the first one (the second is similar):

$$\begin{aligned}
\{\gamma, \psi\} &= -\frac{1}{2}\{\gamma, \{\pi_i, \{\mu, \pi_i\}\}\} \\
&= -\frac{1}{2}\{\{\gamma, \pi_i\}, \{\mu, \pi_i\}\} - \frac{1}{2}\{\pi_i, \{\gamma, \{\mu, \pi_i\}\}\} \\
&= -\frac{1}{2}\{\{\gamma, \pi_i\}, \{\mu, \pi_i\}\} - \frac{1}{2}\{\pi_i, \{\mu, \{\gamma, \pi_i\}\}\} = 0,
\end{aligned}$$

where we used $\{\mu, \gamma\} = 0$. Now, let us prove that

$$\{\mathcal{S}_i, \{\mathcal{S}_i, \mu + \gamma + \psi + \phi\}\} = -\mu - \gamma - \psi - \phi. \tag{23}$$

Equation (23) is equivalent to

$$\begin{cases}
\{\omega_i, \{\omega_i, \psi\}\} - \{\omega_i, \{\pi_i, \mu\}\} - \{\pi_i, \{\omega_i, \mu\}\} = -\mu \\
-\{\omega_i, \{\{\pi_i, \gamma\}\} - \{\pi_i, \{\omega_i, \gamma\}\} + \{\pi_i, \{\pi_i, \phi\}\} = -\gamma \\
\{\omega_i, \{\omega_i, \gamma\}\} - \{\omega_i, \{\pi_i, \phi\}\} = -\phi \\
\{\pi_i, \{\pi_i, \mu\}\} - \{\pi_i, \{\omega_i, \psi\}\} = -\psi
\end{cases}$$

or to

$$\begin{cases}
\{\omega_i, \{\omega_i, \psi\}\} = 2\{\pi_i, \{\omega_i, \mu\}\} \\
\{\pi_i, \{\pi_i, \phi\}\} = 2\{\omega_i, \{\pi_i, \gamma\}\} \\
\{\omega_i, \{\omega_i, \gamma\}\} = 2\phi \\
\{\pi_i, \{\pi_i, \mu\}\} = 2\psi.
\end{cases} \tag{24}$$

The third and fourth equations of (24) are simply the definitions of ϕ and ψ and, according to Lemma 8.2, they are equivalent to the second and the first equations, respectively. Applying Lemma 6.1, the first part of the proof is complete.

Now, we assume that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi + \phi)$. From (24), we get $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$ and $\phi = -\frac{1}{2}\{\omega_i, \{\gamma, \omega_i\}\}$, for $i = 1, 2, 3$. Thus, $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on the Lie algebroid (A, μ) and (π_1, π_2, π_3) is a hypersymplectic structure with torsion on the Lie algebroid (A^*, γ), (see Lemma 6.1 and (18)). Moreover, $\mu + \gamma + \psi + \phi$ being a Courant structure with (μ, γ) a Lie bialgebroid structure, (22) implies $\{\mu, \psi\} = 0$, $\{\gamma, \phi\} = 0$ and $\{\psi, \phi\} = 0$. ∎

If, in Theorem 8.4, we take $\phi = 0$, then the Lie algebroid (A^*, γ) is equipped with a hypersymplectic structure (without torsion) determined by (π_1, π_2, π_3).

So, Theorem 8.4 shows that having a Lie bialgebroid (A, A^*) equipped with a hypersymplectic structure with torsion on A and a hypersymplectic structure on A^* is equivalent to having a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi)$, which is the double of the quasi-Lie bialgebroid $((A, A^*), \mu, \gamma, \psi)$.

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