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A CRITERION FOR REFLECTIVENESS OF NORMAL EXTENSIONS WITH AN APPLICATION TO MONOIDS

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Dedicated to Manuela Sobral on the occasion of her seventieth birthday

ABSTRACT: We prove that the so-called *special homogeneous surjections* are reflective amongst surjective homomorphisms of monoids. To do so, we use the recent result that these special homogeneous surjections are the normal (= central) extensions with respect to the admissible Galois structure Γ_{Mon} determined by the *Grothendieck group* adjunction together with the classes of surjective homomorphisms. It is well known that such a reflection exists when the left adjoint functor of an admissible Galois structure preserves all pullbacks of fibrations along split epimorphic fibrations, a property which we show to fail for Γ_{Mon}. We give a new sufficient condition for the normal extensions in an admissible Galois structure to be reflective, and we then show that this condition is indeed fulfilled by Γ_{Mon}.

KEYWORDS: categorical Galois theory; admissible Galois structure; central, normal, trivial extension; Grothendieck group; group completion; homogeneous split epimorphism, special homogeneous surjection of monoids.
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1. Introduction

The original aim of our present work was to answer the following question: *Is the category of* special homogeneous surjections *of monoids* [3, 4] *a reflective subcategory of the category of surjective monoid homomorphisms?* Since we recently showed [17] that these special homogeneous surjections are the *normal* extensions in an admissible Galois structure [10, 11], we were at first convinced that this would be an immediate consequence of some known abstract Galois-theoretical result such as the ones in [13, 12]. At a second glance, we could not find any result to apply in the given situation.

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The case of *commutative* monoids is after all not too difficult. Here we can just use Theorem 7.1 of [13], since the reflector from commutative monoids to abelian groups preserves pullbacks of surjective homomorphisms along split epimorphisms (Proposition 4.2). The problem is that, for general (non-commutative) monoids, the Grothendieck group construction need not preserve such pullbacks—see Example 5.1.

It turns out, however, that the special homogeneous surjections of monoids do indeed form a reflective subcategory. The process of proving this result leads naturally to a new criterion for reflectiveness of normal extensions amongst the fibrations of an admissible Galois structure (our Theorem 2.12, closely related to the subject of the recent article [7]): if the categories under consideration are Barr-exact and all fibrations in the Galois structure are regular epimorphisms, then it suffices that *the category of normal extensions is closed under coequalisers of reflexive graphs in the category of fibrations.* This sufficient condition holds for monoids (Theorem 5.5).

Thus a problem in the category of monoids lead to a general result in categorical Galois theory. Conversely, Galois theory helped us to better understand a non-obvious result on monoids, which we believe is interesting for its own sake: it allowed us to extended Proposition 2.3.5 in [4] twice, first to Proposition 5.4, then to Theorem 5.5 which says that *the category of special homogeneous surjections is closed under coequalisers of reflexive graphs of surjective homomorphisms of monoids.*

Before specialising to monoids, we first focus on reflectiveness results in a general Galois-theoretic setting (Section 2). In Subsection 2.8 we use the construction of the normalisation functor proposed in [8] to prove our main result (Theorem 2.12). We then, in Section 3, recall the definitions and main results concerning special homogenous surjections from [3, 4] as well as the main results from [17], giving the link between special homogeneous surjections and normal extensions, which are needed throughout the subsequent sections. In Section 4 we treat the commutative case, which easily leads to Theorem 4.3, the reflectiveness of special homogeneous surjections of commutative monoids. The non-commutative case is treated in the final Section 5, where we first give a counterexample against the preservation of pullbacks of split epimorphisms along spit epimorphisms by the Grothendieck group functor (Example 5.1), to prove then that the technique of Section 2 is applicable (Theorem 5.5) and to obtain Theorem 5.6—thus answering our original question.

2. Reflectiveness of normal extensions

In this section we work towards a general result on reflectiveness of normal extensions in an admissible Galois structure: Theorem 2.12 which says that, if the fibrations in the Galois structure are regular epimorphisms, and normal extensions are closed under coequalisers of reflexive graphs, then the normal extensions are reflective amongst the fibrations.

2.1. Galois structures. We start by recalling the definition of *(admissible) Galois structure* as well as the concepts of *trivial*, *normal* and *central extension* arising from it [10, 11, 12]. We consider the context of Barr-exact categories [1] which is general enough for our purposes and allows us to avoid some technical difficulties.

Definition 2.2. A **Galois structure** $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ consists of an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift left, "H"] \end{tikzcd}$$

with unit $\eta\colon 1_{\mathscr{C}} \Rightarrow HI$ and counit $\epsilon\colon IH \Rightarrow 1_{\mathscr{X}}$ between Barr-exact categories $\mathscr{C}$ and $\mathscr{X}$, as well as classes of morphisms $\mathscr{E}$ in $\mathscr{C}$ and $\mathscr{F}$ in $\mathscr{X}$ such that:

(G1) $\mathscr{E}$ and $\mathscr{F}$ contain all isomorphisms;
(G2) $\mathscr{E}$ and $\mathscr{F}$ are pullback-stable;
(G3) $\mathscr{E}$ and $\mathscr{F}$ are closed under composition;
(G4) $H(\mathscr{F}) \subseteq \mathscr{E}$;
(G5) $I(\mathscr{E}) \subseteq \mathscr{F}$.

We call the morphisms in $\mathscr{E}$ and $\mathscr{F}$ **fibrations** [11]. The following definitions are given with respect to a Galois structure Γ.

Definition 2.3. A **trivial extension** is a fibration $f\colon A \to B$ in $\mathscr{C}$ such that the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & HI(A) \arrow[d, "HI(f)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

is a pullback. A **central extension** is a fibration f whose pullback $p^*(f)$ along *some* fibration p is a trivial extension. A **normal extension** is a fibration such that its kernel pair projections are trivial extensions.

It is well known and easy to see that trivial extensions are always central extensions and that any normal extension is automatically central.

For any object B in $\mathscr{C}$, there is an induced adjunction

$$(\mathscr{E} \downarrow B) \underset{H^B}{\overset{I^B}{\rightleftarrows}} (\mathscr{F} \downarrow I(B)),$$

where we write $(\mathscr{E} \downarrow B)$ for the full subcategory of the slice category $(\mathscr{C} \downarrow B)$ determined by morphisms in $\mathscr{E}$; similarly for $(\mathscr{F} \downarrow I(B))$. The functor I^B is the restriction of I, and H^B sends a fibration $g\colon X \to I(B)$ to the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d, "H^B(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & H(X) \arrow[d, "H(g)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

of $H(g)$ along η_B.

Definition 2.4. A Galois structure $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ is said to be **admissible** when, for every object B in $\mathscr{C}$, the functor H^B is full and faithful.

The admissibility condition amounts to reflectiveness of trivial extensions amongst fibrations. More precisely, we have that:

(1) by Proposition 2.5 below, the replete image of the functor H^B is precisely the category $\mathsf{Triv}(B)$ of trivial extensions over B;
(2) $\mathsf{Triv}(B)$ is a reflective subcategory of $(\mathscr{E} \downarrow B)$;
(3) $H^B I^B\colon (\mathscr{E} \downarrow B) \to \mathsf{Triv}(B)$ is its reflector.

By Proposition 5.8 in [9], we obtain a left adjoint, called the **trivialisation functor**

$$\mathrm{Triv}\colon \mathsf{Fib}(\mathscr{C}) \to \mathsf{TExt}(\mathscr{C}),$$

to the inclusion of the category $\mathsf{TExt}(\mathscr{C})$ of trivial extensions in $\mathscr{C}$ into the full subcategory $\mathsf{Fib}(\mathscr{C})$ of the category of arrows in $\mathscr{C}$ determined by the fibrations.

Proposition 2.5. [13, Proposition 2.4] *If Γ is an admissible Galois structure, then $I\colon \mathscr{C} \to \mathscr{X}$ preserves pullbacks along trivial extensions. Hence a fibration is a trivial extension if and only if it is a pullback of some fibration in $\mathscr{X}$. In particular, the trivial extensions are pullback-stable, so that every trivial extension is a normal extension.* ∎

2.6. A result on reflectiveness of central extensions. Consider an admissible Galois structure Γ as in Definition 2.4. Given an object B in $\mathscr{C}$, we let $\mathsf{Centr}(B)$ denote the full subcategory of $(\mathscr{E} \downarrow B)$ determined by the central extensions over B. When it exists, the left adjoint to the inclusion functor $\mathsf{Centr}(B) \hookrightarrow (\mathscr{E} \downarrow B)$ is written $\mathrm{Centr}\colon (\mathscr{E} \downarrow B) \to \mathsf{Centr}(B)$ and called the **centralisation functor**. In [8], it is explained how Theorem 7.1 in [13] gives us:

Theorem 2.7. *If $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ is an admissible Galois structure and the functor I preserves pullbacks of fibrations along split epimorphic fibrations, then $\mathsf{Centr}(B)$ is a reflective subcategory of $(\mathscr{E} \downarrow B)$, for any object B in $\mathscr{C}$.* ■

In Section 4 we shall explain how this result may be used to prove that special homogeneous surjections of commutative monoids are reflective amongst surjective commutative monoid homomorphisms (Theorem 4.3) while it is not applicable in the non-commutative case (Example 5.1). Therefore we now work towards an alternative for Theorem 2.7 which allows us to prove essentially the same result under different assumptions—Theorem 2.12, which leads to Theorem 5.6 extending the reflectiveness result from the commutative monoids case to general monoids.

2.8. Reflectiveness of normal extensions. Given an admissible Galois structure Γ as in Definition 2.4 and an object B in $\mathscr{C}$, we denote by $\mathsf{Norm}(B)$ the full subcategory of $(\mathscr{E} \downarrow B)$ determined by the normal extensions over B. When it exists, the left adjoint to the inclusion functor $\mathsf{Norm}(B) \hookrightarrow (\mathscr{E} \downarrow B)$ is denoted by $\mathrm{Norm}\colon (\mathscr{E} \downarrow B) \to \mathsf{Norm}(B)$ and called the **normalisation functor (over B)**. We also write

$$\mathrm{Norm}\colon \mathsf{Fib}(\mathscr{C}) \to \mathsf{Norm}(\mathscr{C})$$

for the left adjoint to the inclusion $\mathsf{Norm}(\mathscr{C}) \hookrightarrow \mathsf{Fib}(\mathscr{C})$ (where $\mathsf{Norm}(\mathscr{C})$ is determined by the normal extensions in $\mathscr{C}$) which exists as soon as the normalisation functors over all objects B exist (again by Proposition 5.8 in [9]).

We use the construction proposed in [8] and prove that it does indeed provide us with a normalisation functor as soon as the Galois structure Γ is admissible and satisfies the following conditions:

(G6) the morphisms in $\mathscr{E}$ and in $\mathscr{F}$ are regular epimorphisms;
(G7) the category of normal extensions in $\mathscr{C}$ is closed under coequalisers of reflexive graphs in $\mathsf{Fib}(\mathscr{C})$.

This is related to the results in [7] where the problem of reflectiveness of normal extensions is studied in a much more general setting. The present construction is essentially a simple version of the one proposed in [5], which strictly speaking cannot be applied in the current context.

2.9. The construction. Given a fibration $f\colon A \to B$, we pull it back along itself, then we take kernel pairs vertically as on the left hand side of the diagram in Figure 1. We apply the trivialisation functor to obtain the upper right part of the diagram, then we take the coequaliser $\underline{f}$ on the right to get the morphism $\mathrm{Norm}(f)$ and the comparison η_f^{Norm}. The normality of $\mathrm{Norm}(f)$ comes from condition (G7) and the fact that all trivial extensions are normal extensions (Proposition 2.5).

$$\begin{tikzcd}
\mathrm{Eq}(\pi_2) \arrow[r, two heads, "\pi_1'"'] \arrow[rr, dashed, bend left, "\eta^{\mathrm{Triv}}_{\pi_1'}"] \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] \arrow[from=d] & \mathrm{Eq}(f) \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] \arrow[from=d] & \mathrm{Eq}(\pi_2)_{\mathrm{Triv}} \arrow[l, two heads, "\mathrm{Triv}(\pi_1')"] \arrow[d, shift left=1.5] \arrow[d, shift right=1.5] \arrow[from=d] \\
\mathrm{Eq}(f) \arrow[r, two heads, "\pi_1"] \arrow[rr, dashed, bend right] \arrow[d, two heads, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"'] & \mathrm{Eq}(f)_{\mathrm{Triv}} \arrow[l, two heads, "\mathrm{Triv}(\pi_1)"'] \arrow[d, two heads, "\underline{f}"] \\
A \arrow[r, two heads, "f"] \arrow[rr, dashed, bend right, "\eta_f^{\mathrm{Norm}}"'] & B & \overline{A}. \arrow[l, two heads, "\mathrm{Norm}(f)"']
\end{tikzcd}$$

FIGURE 1. The construction of $\mathrm{Norm}(f)$

2.10. The universal property. Let us prove that $\mathrm{Norm}(f)$ is universal amongst normal extensions over B. Suppose that $f = g\circ\alpha$, where $g\colon C \to B$ is a normal extension. First note that all steps of the construction are functorial. Next, since g is a normal extension, we have $\mathrm{Norm}(g) = g$, $\overline{C} = C$ and $\eta_g^{\mathrm{Norm}} = 1_C$. So we get an induced morphism $\overline{\alpha}\colon \overline{A} \to C$ such that $g\circ\overline{\alpha} = \mathrm{Norm}(f)$ and $\overline{\alpha}\circ\eta_f^{\mathrm{Norm}} = \alpha$, which proves the existence of a factorisation. Now for the uniqueness, suppose that β, $\gamma\colon \overline{A} \to C$ are such that

$$g\circ\beta = \mathrm{Norm}(f) = g\circ\gamma \quad \text{and} \quad \beta\circ\eta_f^{\mathrm{Norm}} = \alpha = \gamma\circ\eta_f^{\mathrm{Norm}}.$$

We write π_1^f, π_2^f and π_1^g, π_2^g for the kernel pair projections of f and g, respectively. From the fact that g is a normal extension, we have $\mathrm{Triv}(\pi_1^g) = \pi_1^g$

and $\underline{g} = \pi_2^g$. Since $g\circ\alpha\circ\mathrm{Triv}(\pi_1^f) = f\circ\mathrm{Triv}(\pi_1^f) = \mathrm{Norm}(f)\circ\underline{f} = g\circ\beta\circ\underline{f}$ and, likewise, $g\circ\alpha\circ\mathrm{Triv}(\pi_1^f) = g\circ\gamma\circ\underline{f}$, we find morphisms

$$\widetilde{\beta} = \langle\alpha\circ\mathrm{Triv}(\pi_1^f), \beta\circ\underline{f}\rangle, \widetilde{\gamma} = \langle\alpha\circ\mathrm{Triv}(\pi_1^f), \gamma\circ\underline{f}\rangle\colon \mathrm{Eq}(f)_{\mathrm{Triv}} \to \mathrm{Eq}(g)$$

such that $\pi_2^g\circ\widetilde{\beta} = \beta\circ\underline{f}$ and $\pi_2^g\circ\widetilde{\gamma} = \gamma\circ\underline{f}$ while

$$\pi_1^g\circ\widetilde{\beta} = \alpha\circ\mathrm{Triv}(\pi_1^f) \qquad \text{and} \qquad \pi_1^g\circ\widetilde{\gamma} = \alpha\circ\mathrm{Triv}(\pi_1^f).$$

Now $\widetilde{\beta} = \widetilde{\gamma}$ follows from the uniqueness in the universal property of the trivial extension $\mathrm{Triv}(\pi_1^f)$: indeed, $\widetilde{\beta}\circ\eta^{\mathrm{Triv}}_{\pi_1^f} = \alpha \times_{1_B} \alpha = \widetilde{\gamma}\circ\eta^{\mathrm{Triv}}_{\pi_1^f}$. Hence $\beta = \gamma$.

2.11. The result. Thus, keeping Proposition 5.8 in [9] in mind, we obtain:

Theorem 2.12. *Let $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ be an admissible Galois structure such that the conditions* (G6) *and* (G7) *hold. If B is an object of $\mathscr{C}$, then $\mathsf{Norm}(B)$ is a reflective subcategory of $(\mathscr{E} \downarrow B)$. As a consequence, normal extensions are reflective amongst fibrations.* ■

This is what we shall use in Section 5 to prove that special homogeneous monoid surjections are reflective amongst surjective monoid homomorphisms.

3. Revision of some known results for monoids

In this section we recall the main results from [17], where it is shown that the group completion of monoids determines an admissible Galois structure with respect to surjective homomorphisms. Moreover, the corresponding central (= normal) extensions are precisely the special homogeneous surjections.

3.1. The Grothendieck group of a monoid. The **Grothendieck group** (or **group completion**) of a monoid $(M, \cdot, 1)$ is given by a group $\mathrm{Gp}(M)$ and a monoid homomorphism $M \to \mathrm{Gp}(M)$ which is universal with respect to monoid homomorphisms from M to a group [14, 15, 16]. Explicitly, we can define $\mathrm{Gp}(M) = \mathrm{GpF}(M)/\mathrm{N}(M)$, where $\mathrm{GpF}(M)$ denotes the free group on M and $\mathrm{N}(M)$ is the normal subgroup generated by elements of the form $[m_1][m_2][m_1 \cdot m_2]^{-1}$. We shall simply write m_1m_2 instead of $m_1 \cdot m_2$ from now on. This gives us an equivalence relation $\equiv$ on $\mathrm{GpF}(M)$ generated by $[m_1][m_2] \equiv [m_1m_2]$ with equivalence classes $\overline{[m_1][m_2]} = \overline{[m_1m_2]}$. An arbitrary element in $\mathrm{Gp}(M)$ is an equivalence class of words, which may be represented by a word of the form

$$[m_1][m_2]^{-1}[m_3][m_4]^{-1}\cdots[m_n]^{\iota(n)} \quad \text{or} \quad [m_1]^{-1}[m_2][m_3]^{-1}[m_4]\cdots[m_n]^{\iota(n)},$$

where $\iota(n) = \pm 1$, $n \in \mathbb{N}$, $m_1, \ldots, m_n \in M$ and no further cancellation is possible.

We write Mon for the category of monoids and Gp for the category of groups. The Grothendieck group construction determines an adjunction

$$\begin{tikzcd} \mathsf{Mon} \arrow[r, shift left, "\mathrm{Gp}"] \arrow[r, phantom, "\perp"] & \mathsf{Gp}, \arrow[l, shift left, "\mathrm{Mon}"] \end{tikzcd} \tag{$*$}$$

where Mon is the forgetful functor. To simplify notation, we write $\mathrm{Gp}(M)$ instead of $\mathrm{MonGp}(M)$ when referring to the monoid structure of $\mathrm{Gp}(M)$. The counit is $\epsilon = 1_{\mathsf{Gp}}$ and the unit is defined, for any monoid M, by

$$\eta_M \colon M \to \mathrm{Gp}(M) \colon m \mapsto \overline{[m]}.$$

By choosing the classes of morphisms $\mathscr{E}$ and $\mathscr{F}$ to be the surjections in Mon and Gp, respectively, we obtain a Galois structure

$$\Gamma_{\mathrm{Mon}} = (\mathsf{Mon}, \mathsf{Gp}, \mathrm{Mon}, \mathrm{Gp}, \eta, \epsilon, \mathscr{E}, \mathscr{F}).$$

This Galois structure was studied in the article [17], with as main result its Theorem 2.2:

Theorem 3.2. *The Galois structure Γ_{Mon} is admissible.* ∎

3.3. Special homogeneous surjections. We recall the definition and some results concerning special homogenous surjections from [3, 4] which are needed in the sequel.

Definition 3.4. Let f be a split epimorphism of monoids, with a chosen splitting s, and N its (canonical) kernel

$$\begin{tikzcd} N \arrow[r, tail, "k"'] & X \arrow[r, two heads, shift left, "f"] & Y. \arrow[l, shift left, "s"] \end{tikzcd} \tag{$\dagger$}$$

The split epimorphism (f, s) is said to be **right homogeneous** when, for every element $y \in Y$, the function $\mu_y \colon N \to f^{-1}(y)$ defined through multiplication on the right by $s(y)$, so $\mu_y(n) = n\, s(y)$, is bijective. Similarly, we can define a **left homogeneous** split epimorphism: the function $N \to f^{-1}(y) \colon n \mapsto s(y)\, n$ is a bijection for all $y \in Y$. A split epimorphism is said to be **homogeneous** when it is both right and left homogeneous.

Definition 3.5. Given a surjective homomorphism g of monoids and its kernel pair

$$\begin{tikzcd} \mathrm{Eq}(g) \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta" description] \arrow[r, two heads, "g"] & Y, \end{tikzcd}$$

the morphism g is called a **special homogeneous surjection** when (π_1, Δ) (or, equivalently, (π_2, Δ)) is a homogeneous split epimorphism.

The next two results illustrate the connection between special homogeneous surjections and the notions of trivial, central and normal extensions arising from the Galois structure Γ_{Mon}. The admissibility of Γ_{Mon} is essential to the coincidence of central and normal extensions in this context.

Proposition 3.6. [17, Proposition 4.2] *For a split epimorphism f of monoids, the following statements are equivalent:*

(i) *f is a trivial extension;*
(ii) *f is a special homogeneous surjection.* ■

Theorem 3.7. [17, Theorem 4.4] *For a surjective homomorphism g of monoids, the following statements are equivalent:*

(i) *g is a central extension;*
(ii) *g is a normal extension;*
(iii) *g is a special homogeneous surjection.* ■

4. The case of commutative monoids

In this section we focus on the commutative case, proving that special homogeneous surjections are reflective amongst surjective homomorphisms of commutative monoids.

We can restrict the group completion to commutative monoids: it is easily seen that then Γ_{Mon} restricts to an admissible Galois structure

$$\Gamma_{\mathrm{CMon}} = (\mathsf{CMon}, \mathsf{Ab}, \mathrm{CMon}, \mathrm{Gp}|_{\mathsf{CMon}}, \eta', \epsilon', \mathscr{E}', \mathscr{F}')$$

induced by the (co)restriction

$$\begin{tikzcd} \mathsf{CMon} \arrow[r, shift left=1.5, "\mathrm{Gp}|_{\mathsf{CMon}}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, shift left=1.5, "\mathrm{CMon}"] \end{tikzcd}$$

of the adjunction $(*)$ to the category of commutative monoids CMon and the category of abelian groups Ab. To simplify notation, we write $\mathrm{Gp}\colon \mathsf{CMon} \to \mathsf{Ab}$

instead of $\mathrm{Gp}|_{\mathsf{CMon}}$ for the Grothendieck group functor restricted to commutative monoids.

If M is a commutative monoid, $\mathrm{Gp}(M)$ can be described in the following way: $\mathrm{Gp}(M) = (M \times M)/\sim$, where $(m_1, m_2) \sim (n_1, n_2)$ if and only if there exists $z \in M$ such that $m_1 + n_2 + z = m_2 + n_1 + z$, using the additive notation in M. Comparing this description of $\mathrm{Gp}(M)$ with the previous one for general monoids, we have that the equivalence class of $(m_1, m_2) \in M \times M$ corresponds to the class $\overline{[m_1]} - \overline{[m_2]}$ in $\mathrm{Gp}(M)$.

It is easy to check that Proposition 3.6 and Theorem 3.7 still hold for the admissible Galois structure Γ_{CMon}.

The following results show that the conditions of Theorem 2.7 hold for Γ_{CMon}. Together with Theorem 3.7 applied to the commutative case, we see that special homogeneous surjections are reflective amongst surjections in CMon.

Proposition 4.1. *The Grothendieck group functor* $\mathrm{Gp}\colon \mathsf{CMon} \to \mathsf{Ab}$ *preserves pullbacks of split epimorphisms along split epimorphisms.*

Proof: In the category CMon, consider the pullback

$$\begin{tikzcd}
X \times_Z Y \arrow[r, shift left, "\pi_Y"] \arrow[d, shift left, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left] \arrow[d, shift left, "g"] \\
X \arrow[u, shift left] \arrow[r, shift left, "f"] & Z \arrow[l, shift left, "s"] \arrow[u, shift left, "t"]
\end{tikzcd}$$

of the split epimorphisms (f, s) and (g, t). Applying the Grothendieck group functor gives us the diagram

$$\begin{tikzcd}
& & \mathrm{Gp}(Y) \arrow[dl, shift left] \arrow[dr, shift left, "\mathrm{Gp}(g)"] & \\
\mathrm{Gp}(X \times_Z Y) \arrow[r, dotted, two heads, "h"] \arrow[urr, bend left, "\mathrm{Gp}(\pi_Y)"] \arrow[drr, bend right, "\mathrm{Gp}(\pi_X)"'] & P \arrow[ur, shift left] \arrow[dr, shift left] \arrow[rr, phantom, "\lrcorner", very near start] & & \mathrm{Gp}(Z) \arrow[ul, shift left, "\mathrm{Gp}(t)"] \arrow[dl, shift left, "\mathrm{Gp}(s)"] \\
& & \mathrm{Gp}(X) \arrow[ul, shift left] \arrow[ur, shift left, "\mathrm{Gp}(f)"] &
\end{tikzcd}$$

in which the comparison morphism h is a surjective group homomorphism by Lemma 3.1 in [2]. We now show that h is a monomorphism. An element of $\mathrm{Gp}(X \times_Z Y)$ is of the form $\overline{[(x_1, y_1)]} - \overline{[(x_2, y_2)]}$, with $x_i \in X$ and $y_i \in Y$.

Suppose that

$$h(\overline{[(x_1, y_1)]} - \overline{[(x_2, y_2)]}) = (\overline{[x_1]} - \overline{[x_2]}, \overline{[y_1]} - \overline{[y_2]}) = 0 \in P.$$

Then $\overline{[x_1]} - \overline{[x_2]} = 0 \in \mathrm{Gp}(X)$ and $\overline{[y_1]} - \overline{[y_2]} = 0 \in \mathrm{Gp}(Y)$. This means that there exist $x \in X$ and $y \in Y$ such that $x_1 + 0 + x = 0 + x_2 + x$ and $y_1 + 0 + y = 0 + y_2 + y$. We consider the pair $(x + sg(y),\, y + tf(x)) \in X \times_Z Y$ which gives

$$\begin{aligned}(x_1, y_1) + (x + sg(y),\, y + tf(x)) &= (x_1 + x + sg(y),\, y_1 + y + tf(x))\\ &= (x_2 + x + sg(y),\, y_2 + y + tf(x))\\ &= (x_2, y_2) + (x + sg(y),\, y + tf(x)).\end{aligned}$$

We conclude that $\overline{[(x_1, y_1)]} - \overline{[(x_2, y_2)]} = 0$. ■

Proposition 4.2. *The Grothendieck group functor* $\mathrm{Gp}\colon \mathsf{CMon} \to \mathsf{Ab}$ *preserves pullbacks of surjective homomorphisms along split epimorphisms.*

Proof: We extend Proposition 4.1 to the case of pullbacks of surjective homomorphisms along split epimorphisms. Consider the pullback

$$\begin{tikzcd}
X \times_Z Y \arrow[r, shift left, "\pi_Y"] \arrow[d, two heads, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left] \arrow[d, two heads, "g"] \\
X \arrow[r, shift left, "f"] & Z \arrow[l, shift left, "s"]
\end{tikzcd}$$

of a split epimorphism (f, s) along a surjective homomorphism g. Taking kernel pairs vertically gives a pullback of split epimorphisms, which is preserved by the Grothendieck group functor. By right exactness of this functor, we get a diagram of reflexive graphs—in fact, internal groupoids, see for instance [6]—with their coequalisers as on the left:

$$\begin{tikzcd}
\mathrm{Gp}(\mathrm{Eq}(\pi_X)) \arrow[r, shift left] \arrow[d, shift right=2] \arrow[d, shift left=2] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Gp}(\mathrm{Eq}(g)) \arrow[l, shift left] \arrow[d, shift right=2] \arrow[d, shift left=2] \\
\mathrm{Gp}(X \times_Z Y) \arrow[u] \arrow[r, shift left, "\mathrm{Gp}(\pi_Y)"] \arrow[d, "\mathrm{Gp}(\pi_X)"'] & \mathrm{Gp}(Y) \arrow[u] \arrow[l, shift left] \arrow[d, "\mathrm{Gp}(g)"] \\
\mathrm{Gp}(X) \arrow[r, shift left, "\mathrm{Gp}(f)"] & \mathrm{Gp}(Z) \arrow[l, shift left, "\mathrm{Gp}(s)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Gp}(\mathrm{Eq}(\pi_X)) \arrow[r, shift left, "\rho_2"] \arrow[d, "\rho_1"'] & \mathrm{Gp}(\mathrm{Eq}(g)) \arrow[l, shift left] \arrow[d] \\
\mathrm{Eq}(\mathrm{Gp}(\pi_X)) \arrow[r, shift left] \arrow[d, shift right=2] \arrow[d, shift left=2] & \mathrm{Eq}(\mathrm{Gp}(g)) \arrow[l, shift left] \arrow[d, shift right=2] \arrow[d, shift left=2] \\
\mathrm{Gp}(X \times_Z Y) \arrow[u] \arrow[r, shift left, "\mathrm{Gp}(\pi_Y)"] & \mathrm{Gp}(Y). \arrow[u] \arrow[l, shift left]
\end{tikzcd}$$

We have to show that its bottom square is a pullback. Thanks to a well known result for regular categories (see 6.10 in [1]), it suffices to prove that the diagram consisting of the regular images of the given internal groupoids, so the bottom diagram of kernel pairs of $\mathrm{Gp}(\pi_X)$ and $\mathrm{Gp}(g)$ on the right above, is still a pullback. Note that its upper square is a pullback since it is a regular pushout [2, Proposition 3.2], while the pair (ρ_1, ρ_2) is jointly monomorphic. So the outer rectangle and the upper square are pullbacks, thus the bottom square is also a pullback by Proposition 2.7 in [12]. ∎

The proposition above, together with Theorem 2.7, gives then the following:

Theorem 4.3. *Special homogeneous surjections are reflective amongst surjections in* CMon. ∎

5. Arbitrary monoids

In this section we prove that special homogeneous surjections are reflective amongst surjective homomorphisms of monoids. For this purpose we can no longer apply Theorem 2.7 like in the commutative monoid case, since Proposition 4.1 and Proposition 4.2 are not true for the Grothendieck group functor $\mathrm{Gp}\colon \mathsf{Mon} \to \mathsf{Gp}$, as the following counterexample shows. For the case of arbitrary monoids, we show that condition (G7) holds, so that we can apply Theorem 2.12 to obtain the result.

Example 5.1. Consider the pullback

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{F}\{x,z\} \arrow[d, "\beta"] \\ \mathrm{F}\{x,y\} \arrow[r, "\alpha"'] & \mathrm{F}\{x\} \end{tikzcd}$$

in Mon, where $\mathrm{F}\{x,y\}$, $\mathrm{F}\{x,z\}$ and $\mathrm{F}\{x\}$ are the free monoids with generators x and y, x and z, and x, respectively, and α and β are the morphisms deleting y and z, respectively. Observe that both α and β are split epimorphisms. Then the elements of R are pairs (σ_1, σ_2), where σ_1 is a word in x and y, σ_2 is a word in x and z, and σ_1 and σ_2 contain the same number of letters x. As a monoid, R is then generated by the pairs $(y, 1)$, $(1, z)$ and (x, x). Since the pair $(y, 1)$ commutes with $(1, z)$ in R, we have that R is isomorphic to the sum $\mathrm{FC}\{(y,1),(1,z)\} + \mathrm{F}\{(x,x)\}$, where $\mathrm{FC}\{(y,1),(1,z)\}$ is the free commutative monoid on two generators $(y,1)$ and $(1,z)$. In other terms, R is isomorphic to $(\mathbb{N}\times\mathbb{N})+\mathbb{N}$. Applying the Grothendieck group functor we obtain the diagram

$$\begin{tikzcd}
 & & \mathrm{Gp}(\mathrm{F}\{x,z\}) \arrow[dr, "\mathrm{Gp}(\beta)"] & \\
\mathrm{Gp}(R) \arrow[rr, dotted, "h"] \arrow[urr, bend left, "\mathrm{Gp}(\delta)"] \arrow[drr, bend right, "\mathrm{Gp}(\gamma)"'] & & P \arrow[u] \arrow[d] & \mathrm{Gp}(\mathrm{F}\{x\}) \\
 & & \mathrm{Gp}(\mathrm{F}\{x,y\}) \arrow[ur, "\mathrm{Gp}(\alpha)"'] &
\end{tikzcd}$$

in which P is the pullback of $\mathrm{Gp}(\alpha)$ along $\mathrm{Gp}(\beta)$ and h is the induced morphism. It is immediate to see that $\mathrm{Gp}(R)$ is isomorphic to $(\mathbb{Z}\times\mathbb{Z})+\mathbb{Z}$, where the three copies of $\mathbb{Z}$ are generated by $(y,1)$, $(1,z)$ and (x,x), respectively. The map h is not a monomorphism: for example, the equivalence class represented by the word

$$[(y,1)][(x,x)]^{-1}[(1,z)][(x,x)][(y,1)]^{-1}[(x,x)]^{-1}[(1,z)]^{-1}[(x,x)]$$

clearly belongs to the kernel of h, but it is not equivalent to the empty word in $\mathrm{Gp}(R)$.

We use the admissibility of Γ_{Mon} to obtain a variation on Proposition 2.3.5 in [4]. We denote by $\mathsf{Pt}(\mathsf{Mon})$ the **category of points** in Mon, whose objects are the split epimorphisms of monoids with a chosen section, and whose morphisms are pairs of monoid homomorphisms which then form commutative squares with both the split epimorphisms and their sections.

Lemma 5.2. *In a regular category, pulling back along a morphism of regular epimorphisms preserves regular pushout squares.*

Proof: A square is a regular pushout square if and only if it decomposes as a composite of two squares of regular epimorphisms

$$\begin{tikzcd}
A' \arrow[r, two heads] \arrow[d, two heads] & B'\times_B A \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads] \\
B' \arrow[r, equal] & B' \arrow[r, two heads, "h"'] & B
\end{tikzcd}$$

where the square on the right is a pullback. Given a regular epimorphism $r\colon C'\to C$ and a morphism $(f',f)\colon r\to h$, pulling back the given regular pushout square along it now yields a regular pushout square over r. ■

Lemma 5.3. *The functor* $\mathrm{Triv}|_{\mathsf{Pt}(\mathsf{Mon})}\colon \mathsf{Pt}(\mathsf{Mon}) \to \mathsf{TExt}(\mathsf{Mon})$ *preserves coequalisers of equivalence relations.*

Proof: Consider a reflexive graph in $\mathsf{Pt}(\mathsf{Mon})$ with its coequaliser

$$\begin{tikzcd}
R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] \arrow[d, shift left, "f''"] & A' \arrow[l] \arrow[r, two heads, "g"] \arrow[d, shift left, "f'"] & A \arrow[d, shift left, "f"] \\
S \arrow[r, shift left=2, "s_2"] \arrow[r, shift right=2, "s_1"'] \arrow[u, shift left, "s''"] & B' \arrow[l] \arrow[r, two heads, "h"'] \arrow[u, shift left, "s'"] & B. \arrow[u, shift left, "s"]
\end{tikzcd} \tag{‡}$$

For the sake of simplicity we shall assume that R and S are equivalence relations. Since the Grothendieck group functor Gp preserves all coequalisers, we obtain a reflexive graph in $\mathsf{Pt}(\mathsf{Gp})$ with its coequaliser

$$\begin{tikzcd}
\mathrm{Gp}(R) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left] & \mathrm{Gp}(A') \arrow[l] \arrow[r, two heads, "\mathrm{Gp}(g)"] \arrow[d, shift left] & \mathrm{Gp}(A) \arrow[d, shift left] \\
\mathrm{Gp}(S) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[u, shift left] & \mathrm{Gp}(B') \arrow[l] \arrow[r, two heads, "\mathrm{Gp}(h)"'] \arrow[u, shift left] & \mathrm{Gp}(B). \arrow[u, shift left]
\end{tikzcd}$$

Since the inclusion $\mathsf{Gp} \to \mathsf{Mon}$ preserves regular epimorphisms and kernel pairs, this diagram is still a reflexive graph with its coequaliser when considered in the category $\mathsf{Pt}(\mathsf{Mon})$: indeed, the comparison $\mathrm{Gp}(R) \to \mathrm{Eq}(\mathrm{Gp}(g))$ is a regular epimorphism, and similarly for $\mathrm{Gp}(S) \to \mathrm{Eq}(\mathrm{Gp}(h))$.

Now we pull back along η_B, $\eta_{B'}$ and η_S to obtain the diagram

$$\begin{tikzcd}
R_{\mathrm{Triv}} \arrow[r, dotted] \arrow[d, shift left, "\mathrm{Triv}(f'')"] & \eta_S^*(\mathrm{Eq}(\mathrm{Gp}(g))) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left] & A'_{\mathrm{Triv}} \arrow[l] \arrow[r, dotted] \arrow[d, shift left, "\mathrm{Triv}(f')"] & A_{\mathrm{Triv}} \arrow[d, shift left, "\mathrm{Triv}(f)"] \\
S \arrow[r, equal] \arrow[u, shift left] & S \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[u, shift left] & B' \arrow[l] \arrow[r, two heads, "h"'] \arrow[u, shift left] & B. \arrow[u, shift left]
\end{tikzcd}$$

Using that, in the category of groups, any square of regular epimorphisms between split epimorphisms is a regular pushout, via Lemma 5.2 it is not hard to see that the dotted arrows in this diagram are regular epimorphisms. Since, moreover, pullbacks preserve kernel pairs, we see that $\mathrm{Triv}(f)$, being the coequaliser of its kernel pair, is also the coequaliser of the reflexive graph $\mathrm{Triv}(f'') \rightrightarrows \mathrm{Triv}(f')$. ■

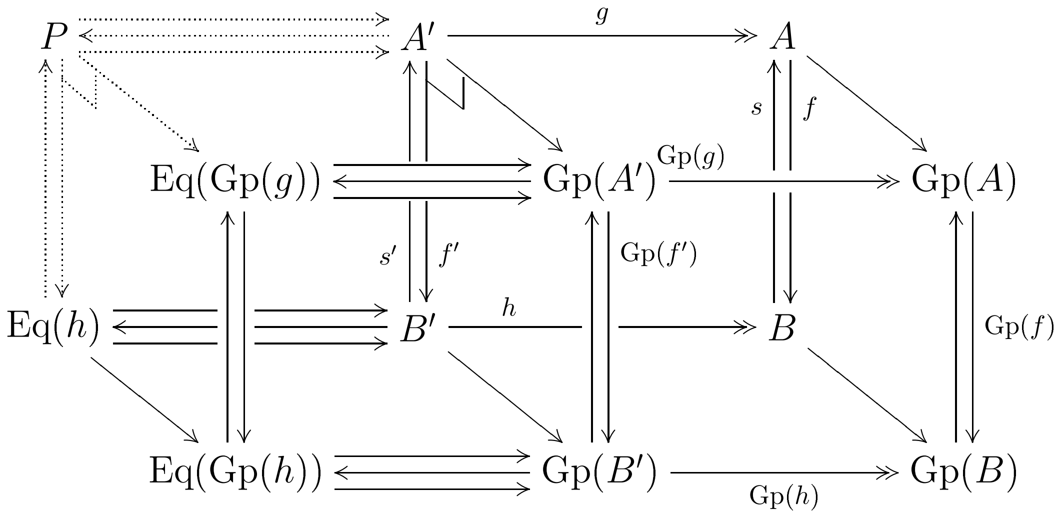

FIGURE 2. Reduction to equivalence relations

Proposition 5.4. *Given a reflexive graph and its coequaliser in* $\mathsf{Pt}(\mathsf{Mon})$ *such as in Diagram* (‡) *where* f'' *and* f' *are special homogeneous surjections, also* f *is a special homogeneous surjection.*

Proof: We first reduce the problem to the situation where R and S are the kernel pairs of g and h, respectively. To do so, it suffices to note—see Figure 2—that the pullback P of $\mathrm{Eq}(h)$, $\mathrm{Eq}(\mathrm{Gp}(h))$ and $\mathrm{Eq}(\mathrm{Gp}(g))$ is precisely the kernel pair of g, so that the induced split epimorphism $\mathrm{Eq}(g) \to \mathrm{Eq}(h)$ is a trivial extension, being a pullback of a fibration in Gp.

We now find the result as a consequence of the fact that special homogeneous split epimorphisms are precisely split epimorphic trivial extensions and Lemma 5.3, which tells us that

$$f = \mathrm{coeq}((r_1, s_1), (r_2, s_2)) = \mathrm{coeq}(((r_1)_{\mathrm{Triv}}, s_1), ((r_2)_{\mathrm{Triv}}, s_2)) = \mathrm{Triv}(f),$$

so f is a trivial extension. ■

The following result shows that condition (G7) holds for monoids.

Theorem 5.5. *The category of special homogeneous surjections is closed under coequalisers of reflexive graphs of surjective homomorphisms of monoids.*

Proof: Consider a reflexive graph of surjections of monoids and its coequaliser in Mon as in the solid part of the diagram in Figure 3. We wish to prove that, if f'' and f' are special homogeneous surjections, then also f is a special homogeneous surjection.

$$\begin{tikzcd}
\mathrm{Eq}(f'') \arrow[r, dotted, shift left=2] \arrow[r, dotted, shift right=2] \arrow[d, dotted, shift left=2] \arrow[d, dotted, shift right=2] & R \arrow[l, dotted] \arrow[r, two heads, "f''"] \arrow[d, shift left=2] \arrow[d, shift right=2] & S \arrow[d, shift left=2] \arrow[d, shift right=2] \\
\mathrm{Eq}(f') \arrow[u, dotted] \arrow[r, dotted, shift left=2] \arrow[r, dotted, shift right=2] \arrow[d, dotted, "\overline{g}"'] & A' \arrow[u] \arrow[l, dotted] \arrow[r, two heads, "f'"] \arrow[d, two heads, "g"'] & B' \arrow[u] \arrow[d, two heads, "h"] \\
\mathrm{Eq}(f) \arrow[r, dotted, shift left=2] \arrow[r, dotted, shift right=2] & A \arrow[l, dotted] \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

FIGURE 3. Closedness of special homogeneous surjections under coequalisers of reflexive graphs

Taking kernel pairs to the left, we want to use Proposition 5.4 together with the fact that special homogeneous surjections are precisely normal (= central) extensions to show that the kernel pair projections of f are trivial extensions. For this argument to be valid, we only need to show that $\overline{g}$ is a surjective homomorphism. This follows from the fact that the coequaliser of $\mathrm{Eq}(f'') \rightrightarrows \mathrm{Eq}(f')$ is an internal groupoid on A. Indeed, by Proposition 5.4, it is a special homogeneous reflexive graph. Thanks to Proposition 4.3.7 in [5], it suffices then to show that the kernels of the projections commute. The kernels of the projections of $\mathrm{Eq}(f')$ commute; thanks to Proposition 2.3.10 in [5], the kernels of the projections of the coequalizer of $\mathrm{Eq}(f'') \rightrightarrows \mathrm{Eq}(f')$ are the regular images of the ones of $\mathrm{Eq}(f')$, and then they also commute, thanks to Proposition 1.6.4 in [2]. Hence the regular image of this internal groupoid is an equivalence relation, so a kernel pair, with coequalizer f, which makes it isomorphic to $\mathrm{Eq}(f)$. ∎

Theorem 2.12 now implies our last result.

Theorem 5.6. *Special homogeneous surjections are reflective amongst surjective monoid homomorphisms.* ∎

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