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FIBREWISE INJECTIVITY IN ORDER AND TOPOLOGY

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Abstract: This paper studies injectivity for continuous maps between T0-spaces. The new characterizations obtained establish a parallelism between characterizations of injective monotone maps between ordered sets and of injective continuous maps between T0-spaces.

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1. Introduction

Injective T0-spaces were characterized by Dana Scott in 1970 [11] as those T0-spaces which are continuous lattices with respect to their specialization order. The corresponding fibrewise problem, that is to characterize the continuous maps which are injective, has proved to be a difficult task. In [5] it was shown that, as injective T0-spaces are the retracts of powers of the Sierpinski space, injective continuous maps between T0-spaces are the retracts of partial products of the Sierpinski space. Moreover, recently in [2] we characterized them using a fibrewise way-below relation, showing a way of considering a fibrewise notion of continuous lattice. However, a direct topological characterization of these maps was still missing. As in many other problems in topology, an analysis of the corresponding results in the context of ordered sets may give some guidance towards the solution of the problem, since the category of finite orders and monotone maps is isomorphic to the category of finite T0-spaces and continuous maps.

In a category $\mathbf{C}$, an object X is said to be *injective* if for every extremal monomorphism $m : M \to Y$ and every morphism $g : M \to X$ there exists an

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extension $\overline{g}$ of g to Y, so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & X \\ {\scriptstyle m}\big\downarrow & \nearrow_{\overline{g}} & \\ Y & & \end{array}$$

is commutative. A morphism $f : X \to B$ is injective if it is an injective object in the comma category $\mathbf{C} \downarrow B$; that is, for every extremal monomorphism $m : M \to Y$, $h : Y \to B$, $g : M \to X$ making the following diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & X \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[h]{} & B \end{array}$$

commute, there exists an extension $\overline{g}$ of g to Y over B: $\overline{g}\,m = g$ and $f\,\overline{g} = h$.

We recall that, in the category **Ord** of ordered sets and monotone maps, an ordered set is injective if, and only if, it is complete. For monotone maps injectivity can be characterized as follows.

Theorem A. [12] *A monotone map $f : X \to B$ is injective in* **Ord** $\downarrow B$ *if and only if it satisfies the following conditions:*

(I) $X_b = f^{-1}(b)$ *is complete, for every* $b \in B$*;*

(II) f *is a fibration (that is for every* $x \in X$ *and* $b \in B$ *with* $f(x) \leq b$ $\{x' \in X_b \,|\, x \leq x'\}$ *has a minimum element), and a cofibration (=dual of fibration).*

Theorem B. [12] *A monotone map $f : X \to B$ is injective in* **Ord** $\downarrow B$ *if and only if it satisfies the following conditions:*

(0) f *is convex (that is, for all* $x, y \in X$, $b \in B$ *with* $x \leq y$ *and* $f(x) \leq b \leq f(y)$, *there exists* $z \in X_b$ *such that* $x \leq z \leq y$*).*

(I) $X_b = f^{-1}(b)$ *is complete, for every* $b \in B$*;*

(III) f *is homogeneous (that is for all* $b \leq b' \in B$, $(x_i)_{i \in I}$ *and* $(x'_i)_{i \in I}$ *in* X_b *and* $X_{b'}$ *respectively, with* $x_i \leq x'_i$ *for every* i, $\bigvee x_i \leq \bigvee x'_i$, *with these joins calculated in the fibres) and cohomogeneous (=dual of homogeneous).*

In this paper we present characterizations of injective continuous maps that resemble the results of **Ord**. While the characterization of Theorem 3.7 has no direct connection to Theorems A and B, Theorems 4.6 and 5.3 are topological instances of Theorems A and B. Indeed, in topology condition (I) translates into the condition (1) X_b *is continuous lattice for each* $b \in B$, used throughout the paper, while convexity for monotone maps is categorically characterized as exponentiability, used in our Theorem 5.3. For finite spaces, both conditions (2) and (2″) coincide with f being a fibration in the sense of condition (II), while condition (3′) translates into homogeneity of f. (See also [3], where our focus was on the parallelism between these conditions.)

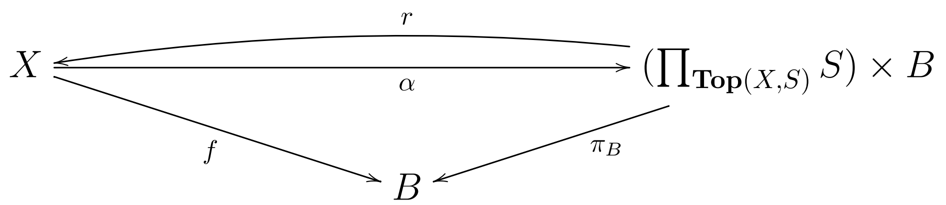
Injective continuous maps are part of a weak factorization system as explained in [1, 2]. They played a crucial role in the development of the theory of lax orthogonal factorization systems presented in [8], and we hope that the novelty of our approach sheds new light to the study of these maps.

2. From injective spaces to injective continuous maps

In this section we review existing results that extend to continuous maps well-known properties of injective T0-spaces.

As it is well-known, a T0-space is injective if and only if it is a retract of a power of the Sierpinski space S. The fibrewise version of this result, which extends it, can be found in [5]. It uses the fact that, for every continuous map $f : X \to B$ between T0-spaces, there is an embedding $\alpha : X \to (\prod_{\mathbf{Top}(X,S)} S) \times B$ defined by $\pi_g \cdot \alpha = g$, for any $g \in \mathbf{Top}(X, S)$, and $\pi_B \cdot \alpha = f$.

Theorem 2.1. *A continuous map* $f : X \to B$ *between T0-spaces is injective if, and only if, it is a (fibrewise) retract of* $\pi_B : (\prod_{\mathbf{Top}(X,S)} S) \times B \to B$*, that is there exists a retraction* r *of* α *making the diagram*



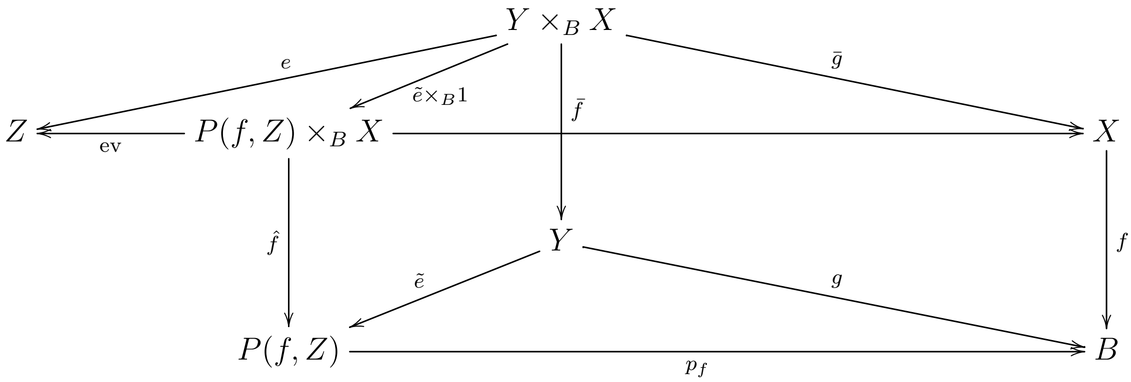
commute.

In [5] there is another characterization of injective continuous maps in $\mathbf{Top}_0$ that will be useful in the sequel, and that focus on exponentiability. In order to present it we first recall the notions of exponentiable morphism and of partial product (see [10] and [9] for details).

Definitions 2.1. In a category $\mathbf{C}$ with finite limits,

(1) a morphism $f : X \to B$ is said to be *exponentiable* if the pullback-functor $f \times_B - : \mathbf{C} \downarrow B \to \mathbf{C} \downarrow B$ has a right adjoint;

(2) given a morphism $f : X \to B$ and an object Z, the *partial product of Z over f* is a pair $(p_f : P(f,Z) \to B, \mathrm{ev} : P(f,Z) \times_B X \to Z)$ such that, given any other pair $(g : Y \to B, e : Y \times_B X \to Z)$, there is a unique morphism $\tilde{e} : Y \to P(f,Z)$ making the following diagram commute:



(2.1)

From results of [10, 9, 5] it follows that:

Theorem 2.2. *Given a continuous map $f : X \to B$ in $\mathbf{Top}_0$, the following conditions are equivalent:*

(i) *f is exponentiable;*
(ii) *the partial product of of the Sierpinski space S over f exists.*

We denote by $\mathcal{O}(X)$ the topology of the topological space X and by X_b the fibre $f^{-1}(b)$ of the map $f : X \to B$, for every $b \in B$. Here, in the Sierpinski space $S = \{0, 1\}$, $\{1\}$ is the non-trivial open subset. Recall, from [10] and [5], that the partial product of S over $f : X \to B$ can be described as:

$$P(f,S) = \{(b,U) \,|\, b \in B,\ U \in \mathcal{O}(X_b)\},\ \ p_f(b,\mathcal{U}) = b,$$

and $\mathcal{U} \subseteq P(f,S)$ is open whenever $\mathcal{U}$ is saturated, binding, with the finite union property, where:

– $\mathcal{U}$ is *saturated* if $(b,U) \in \mathcal{U}$ and $U \subseteq V \in \mathcal{O}(X_b)$ implies $(b,V) \in \mathcal{U}$;
– $\mathcal{U}$ is *binding* if, for every open subset W of X, $\{b \in B \,|\, (b,W_b) \in \mathcal{U}\}$ is open in B;
– $\mathcal{U}$ has the *finite union property* if, for any $b \in B$ and any subset $\mathring{A}$ of $\mathcal{O}(X_b)$, if $(b, \bigcup \mathring{A}) \in \mathcal{U}$ then there is a finite subset $\mathcal{F} \subseteq \mathring{A}$ with $(b, \cup\mathcal{F}) \in \mathcal{U}$.

In this case , given any map $g : Y \to B$ and any $e : Y \times_B X \to S$, it is easy to give a description of the map $\tilde{e}$ which corresponds to e by the universal property of the partial product. Indeed, we have that (for details, see Remark 1.4 in [4]):

$$\tilde{e}(y) = \Big(g(y), \overline{g}(\overline{f}^{-1}(y) \cap e^{-1}(1))\Big)$$

In particular, if $g = 1_B$ and $e : X \to S$ is the characteristic map of an open set V of X, then

$$\tilde{e}(b) = (b, X_b \cap V)$$

Proposition 2.3 ([5]). *Let $f : X \to B$ be a continuous map in $\mathbf{Top}_0$.*

1. *If f is injective, then it is exponentiable.*
2. *If f is exponentiable, then:*
   a. *the continuous map $p_f : P(f, S) \to B$ is injective;*
   b. *there is an embedding $\varepsilon : X \to P(p_f, S)$ making the diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\ \varepsilon\ } & P(p_f, S) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p_{p_f}} & \\ & B & \end{array} \tag{2.2}$$

In the diagram, $\varepsilon(x) = (f(x), \mathcal{O}_x)$, where $\mathcal{O}_x = \{U \in \mathcal{O}(X_{f(x)}) \mid x \in U\}$, and $p_{p_f}(b, \mathcal{U}) = b$.

These results lead to the characterization of injective continuous maps of [5]:

Theorem 2.4. *A continuous map between T0-spaces is injective if and only if it is a (fibrewise) retract of a partial product of S.*

3. Continuous lattices suffice

In this section we will use results of [6] to obtain new characterizations of injective continuous maps. We start by presenting a theorem that subsumes the arguments used in the proof of Theorem 1.2 of [6].

Proposition 3.1. *Given an adjunction $F \dashv G : \mathbf{B} \to \mathbf{A}$, with unit η and counit ε, and classes $\mathcal{M}$ and $\mathcal{N}$ of embeddings in $\mathbf{A}$ and $\mathbf{B}$ respectively, with $F(\mathcal{M}) \subseteq \mathcal{N}$, then G(B) is $\mathcal{M}$-injective provided that B is $\mathcal{N}$-injective.*

Proof: Assume that B is $\mathcal{N}$-injective and let $m : A \to A'$ belong to $\mathcal{M}$ and $f : A \to G(B)$ be any $\mathbf{A}$-morphism. Since $F(m) : F(A) \to F(A')$ belongs to $\mathcal{N}$, there exists $\overline{f} : F(A') \to B$ such that $\overline{f} \cdot F(m) = \varepsilon_B \cdot F(f)$. Therefore $G(\overline{f}) \cdot \eta_{A'} \cdot m = f$ as claimed. ■

Theorem 3.2. *Given a pair of adjunctions* $< F, G, \eta, \varepsilon >$ *and* $< G, H, \sigma, \mu >$

$$\begin{array}{ccc} & \xrightarrow{\quad F \quad} & \\ & \perp & \\ \mathbf{A} & \xleftarrow{\quad G \quad} & \mathbf{B} \\ & \perp & \\ & \xrightarrow[\quad H \quad]{} & \end{array}$$

and classes $\mathcal{M}$ *and* $\mathcal{N}$ *of embeddings in* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively, with* $F(\mathcal{M}) \subseteq \mathcal{N}$ *and* $G(\mathcal{N}) \subseteq \mathcal{M}$*, the following conditions are equivalent, for an* $\mathbf{A}$*-object* A *with* $\eta_A \in \mathcal{M}$*:*

(i) A *is* $\mathcal{M}$*-injective;*
(ii) $H(A)$ *is* $\mathcal{N}$*-injective and* η_A *is a section.*

Proof: (i) ⇒ (ii): From Proposition 3.1 it follows that $H(A)$ is $\mathcal{N}$-injective; moreover, $\mathcal{M}$-injectivity of A, together with $\eta_A \in \mathcal{M}$, gives the desired retraction for η_A.

(ii) ⇒ (i): Proposition 3.1 assures that $GH(A)$ is $\mathcal{M}$-injective. To show that A is $\mathcal{M}$-injective, we show that $\mu_A : GH(A) \to A$ is a retraction, using the retraction $\rho : GF(A) \to A$ of η_A: consider the mate $\hat{\rho} : F(A) \to H(A)$ via the second adjunction and the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & GF(A) & \xrightarrow{\quad\rho\quad} & A \\ & & {\scriptstyle G(\hat{\rho})}\searrow & & \nearrow{\scriptstyle \mu_A} \\ & & & GH(A) & \end{array} \tag{3.3}$$

■

The category **PsTop** of pseudotopological spaces is cartesian closed (in fact, a quasi-topos), hence for every pseudotopological space B the functor $\Pi_B : \mathbf{PsTop} \to \mathbf{PsTop} \downarrow B$, with $\Pi_B(X) = (X \times B \xrightarrow{\pi_B^X} B)$ and $\Pi_B(f) = f \times 1_B$, has a right adjoint $\mathrm{S} : \mathbf{PsTop} \downarrow B \to \mathbf{PsTop}$. We recall that, for every $f : X \to B$,

$$\mathrm{S}(f) = \{s : B \to X \text{ continuous} \mid f \cdot s = 1_B\}.$$

(For details on this adjunction we refer to [10], while a description of the pseudotopology of $\mathrm{S}(f)$ can be found in [7].) Since the functor Π_B is the right adjoint to the forgetful functor $\mathrm{Dom} : \mathbf{PsTop} \downarrow B \to \mathbf{PsTop}$, with $\mathrm{Dom}(f : X \to B) = X$ and $\mathrm{Dom}(h) = h$, we can apply Theorem 3.2, considering in both **PsTop** and **PsTop** $\downarrow B$ the class $\mathcal{H}$ of topological embeddings (that is, embeddings between topological spaces):

$$\mathbf{PsTop}\downarrow B \;\underset{\xrightarrow[S]{}}{\overset{\xrightarrow{\text{Dom}}}{\underset{\perp}{\overset{\perp}{\xleftarrow{\;\Pi_B\;}}}}}\; \mathbf{PsTop} \tag{3.4}$$

Theorem 3.3. *Let X and B be topological spaces. A continuous map $f : X \to B$ is $\mathcal{H}$-injective in* **PsTop** *if and only if:*
(a) *the map $\langle 1_X, f\rangle : (X, f) \to (X \times B, \pi_B^X)$ is a section in* **PsTop** $\downarrow B$*;*
(b) $\mathrm{S}(f)$ *is $\mathcal{H}$-injective in* **PsTop**.

Proof: We apply Theorem 3.2 to the adjunctions of (3.4) observing that $\eta_{(X,f)} = \langle 1_X, f\rangle$. ■

Remarks 3.1. *1.* If X and B are topological $T0$-spaces and the continuous map $f : X \to B$ is injective in $\mathbf{Top}_0 \downarrow B$, then the Theorem says that $\langle 1_X, f\rangle$ is a section in $\mathbf{Top}_0 \downarrow B$. Moreover, if $\mathrm{S}(f)$ is a topological $T0$-space, then f is injective in $\mathbf{Top}_0 \downarrow B$ provided that $\langle 1_X, f\rangle$ is a section and $\mathrm{S}(f)$ is a continuous lattice.
2. When $f : X \to B$ is injective in **PsTop** $\downarrow B$, diagram (3.3) provides a section of the evaluation map $\mathrm{ev} : \mathrm{S}(f) \times B \to X$ over B. Indeed, if $\rho : X \times B \to X$ is a retraction of $\langle 1_X, f\rangle$ in **PsTop** $\downarrow B$, it defines $\hat{\rho} : X \to \mathrm{S}(f)$ so that $\mathrm{ev} \cdot \langle \hat{\rho}, f\rangle = 1_X$.

In parallel with the pseudotopological space $\mathrm{S}(f)$, for a continuous map $f : X \to B$ between T0-spaces we consider the topological *space of continuous sections of f*

$$\mathrm{Sec}(f) = \{s : B \to X \text{ continuous} \mid f \cdot s = 1_X\}$$

endowed with the subspace topology induced by the embedding

$$\mathrm{Sec}(f) \xrightarrow{\ \sigma\ } \prod_{b\in B} X_b \ , \ \text{ with } \sigma(s) = (s(b))_{b\in B}.$$

Lemma 3.4. *The identity map $\iota : \mathrm{S}(f) \to \mathrm{Sec}(f)$ is continuous.*

Proof: For any $b \in B$, the evaluation map $\mathrm{ev} : \mathrm{S}(f)\times B \to X$ maps $\mathrm{S}(f)\times\{b\}$ into X_b, so that its restriction and corestriction

$$\mathrm{S}(f) \xrightarrow{\ \mathrm{ev}_b\ } X_b \ , \ \text{ with } \mathrm{ev}_b(s) = s(b),$$

is continuous. Then, from the equality

$$(\mathrm{S}(f) \xrightarrow{\mathrm{ev}_b} X_b) = (\mathrm{S}(f) \xrightarrow{\iota} \mathrm{Sec}(f) \xrightarrow{\sigma} \prod_{b\in B} X_b \xrightarrow{\pi_b} X_b),\ b \in B,$$

and the fact that σ is an embedding and π_b are product projections, it follows that ι is continuous. ■

Proposition 3.5. *If $f : X \to B$ is injective in $\mathbf{Top}_0 \downarrow B$, then:*

1. *For each $b \in B$, X_b is a continuous lattice.*
2. *For any set of continuous sections of f, its pointwise supremum is continuous.*

Proof: 1. Injective maps are pullback-stable, hence $X_b \to 1$ is injective in $\mathbf{Top}_0 \downarrow 1$, which is equivalent to injectivity of X_b in $\mathbf{Top}_0$, that is X_b is a continuous lattice.

2. Since each fibre X_b is a continuous lattice, the product $\prod\limits_{b\in B} X_b$ is a continuous lattice. Given a set $S \subseteq \mathrm{Sec}(f)$, the map $\check{s} : B \to X$ which assigns to each $b \in B$ the join $\bigvee\{s(b) \,|\, s \in S\}$ in X_b is a section of f. To show that $\check{s}$ is continuous, we consider the spaces $S_0 = S \uplus \{0\}$, with the topology generated by $\{\{s, 0\},\ s \in S\}$, and $S_1 = S \uplus \{0, 1\}$, with the topology generated by $\{\{s, 0, 1\},\ \{0\},\ s \in S\}$. In $Z = B \times S_0$ and $Y = B \times S_1$ we consider the topologies generated by $U \times W$, $U \subseteq B$, $W \subseteq S_n$, $n = 0, 1$, such that either both are open or $W = \{0\}$. The inclusion $\zeta : Z \to Y$ is a topological embedding. We consider now maps $h : Z \to X$ and $k : Y \to B$, with $h(b, s) = s(b)$ for every $s \in S$, $h(b, 0) = \bigvee s(b)$, and k the projection over B. The following diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[k]{} & B \end{array}$$

is commutative, k is continuous since the topology on Y is finer than the product topology, and h is also continuous as we show next. For any open subset U of X,

$$\begin{aligned} h^{-1}(U) &= \{(b,s)\,|\,s(b)\in U\}\cup\{(b,0)\,|\,\bigvee s(b)\in U\}\\ &= \bigcup_{s\in S}(s^{-1}(U)\times\{s,0\})\cup(\{b\,|\,\bigvee s(b)\in U\}\times\{0\})\end{aligned}$$

(because U is upwards-closed), hence $h^{-1}(U)$ is open in Z. Injectivity of f guarantees the existence of a continuous map $d:Y\to X$ such that $f\cdot d=k$ and $d\cdot\zeta=h$. Now it is easy to check that $\check{s}$ is the composition of the continuous maps

$$B\cong B\times\{1\}\longrightarrow Y\xrightarrow{d}X$$

hence it is continuous as claimed. In fact, for any $b\in B$ and $s\in S$, in Y $(b,s)\le(b,1)\le(b,0)$ for all $s\in S$, and so, from

$$s(b)=d(b,s)\le d(b,1)\le d(b,0)=\bigvee s(b)=\check{s}(b)$$

we conclude that $d(b,1)=\bigvee s(b)$.

This way we obtain that $\mathrm{Sec}(f)$ is closed under suprema in $\prod_{b\in B}X_b$. ■

If the fibres of $f:X\to B$ are non-empty continuous lattices, we can define the *maximum section* $\overline{s}$ and the *minimum section* $\underline{s}$ by $\overline{s}(b)=\bigvee X_b$ and $\underline{s}(b)=\bigwedge X_b$, with join and meet calculated in the fibre X_b. (When necessary to stress that they are sections of f, we will use $\overline{s}_f$ and $\underline{s}_f$.)

Proposition 3.6. *Let $f:X\to B$ be a continuous surjective map in $\mathbf{Top}_0$ such that its fibres are continuous lattices. Then:*

1. *f is open if and only if the maximum section of f is continuous;*
2. *f is closed if and only if the minimum section of f is continuous.*

Proof: 1. If $U\subseteq X$ is open, hence upwards-closed, then it is easy to check that $\overline{s}^{-1}(U)=f(U)$, and the result follows.

The proof of 2. is analogous, arguing with closed subsets instead of open ones. ■

From the previous proposition we can conclude immediately that every injective continuous map is closed. That it is also open is a consequence of condition (2) of next theorem.

Theorem 3.7. *A continuous map $f:X\to B$ is injective in $\mathbf{Top}_0\downarrow B$ if and only if it satisfies the following conditions.*

(1) *For any* $b \in B$, X_b *is a continuous lattice.*
(2) *The morphism* $\langle 1_X, f\rangle : (X, f) \to (X \times B, \pi_B)$ *is a section over* B.
(3) $\mathrm{Sec}(f)$ *is a continuous lattice.*

Proof: We have already seen that conditions (1)-(3) are necessary for the injectivity of f. Using Theorem 3.2, to conclude that f is injective when such conditions hold, it is enough to prove that $\mathrm{Sec}(f)$ is the object $\mathrm{S}(f)$ of sections of f in **PsTop**, which follows from Lemma 3.4 once we have shown that the identity map $\iota' : \mathrm{Sec}(f) \to \mathrm{S}(f)$ is continuous, or, equivalently, that the evaluation map $e : \mathrm{Sec}(f)\times B \to X$ is continuous. Let $(t, y) \in \mathrm{Sec}(f)\times B$ and U be an open subset of X with $x = e(t, y) = t(y) \in U$. Since $t = \bigvee\{s \mid s \ll t\}$ in $\mathrm{Sec}(f)$, and then also $t(y) = \bigvee\{s(y) \mid s(y) \ll t(y)\}$ in X_y, there exists $s \ll t$ such that $s(y) \in U$. Therefore $\emptyset \neq s(B) \cap U$ is an open subset of $s(B)$, and then its image under f is an open subset of B, since f is an open map (in fact a homeomorphism when restricted to $s(B)\cap U$). Now, $V := \Uparrow s \times f(s(B)\cap U)$ is open in $\mathrm{Sec}(f) \times B$, $(t, y) \in V$ and $e(V) \subseteq U$ since, for any $s' \ll s$ and any $b = f(x')$ with $x' \in U$ and $x' = s(b')$ for some $b' \in B$,

$$e(s', b) = s'(b) = s'(f(x')) \leq s(f(x')) = s(f(s(b'))) = s(b') = x' \in U$$

and U is upwards-closed. ■

The arguments used prove in fact the following

Corollary 3.8. *If* $f : X \to B$ *is injective in* $\mathbf{Top}_0 \downarrow B$, *then* $\mathrm{S}(f)$ *and* $\mathrm{Sec}(f)$ *are isomorphic, yielding a (co)restriction of the functor* $\mathrm{S} : \mathbf{PsTop} \downarrow B \to \mathbf{PsTop}$ *to* $\mathrm{S} : \mathrm{Inj}(\mathbf{Top}_0 \downarrow B) \to \mathrm{Inj}(\mathbf{Top}_0) \cong \mathbf{ContLat}$.

4. From injective monotone maps to injective continuous maps

Our next goal is to obtain a characterization that avoids Condition (2) of Theorem 3.7, so that it uses only the 'internal' properties of $f : X \to B$. First we present some auxiliary results.

Proposition 4.1. *Let* $f : X \to B$ *be a continuous map between T0-spaces. Then, for the following conditions,* (2) $\Rightarrow$ (2′) $\Rightarrow$ (2″).

(2) *The morphism* $\langle 1_X, f\rangle : (X, f) \to (X \times B, \pi_B)$ *is a section over* B.
(2′) *for each* $x \in X$ *and* $b \in B$ *with* $f(x) \leq b$, *there exists* $x_b \in X_b$ *such that* $x \leq x_b$ *and, for any net* $(x_\lambda)_\lambda$ *in* X_b, *if* $(x_\lambda)_\lambda$ *converges to* x *then* $(x_\lambda)_\lambda$ *also converges to* x_b.

(2″) *for* $x \in X$ *and* $b \in B$ *with* $f(x) \leq b$, $\{x' \in X_b | x \leq x'\}$ *has minimum element* x_b.

Proof: (2) ⇒ (2′): Let $\rho : X \times B \to X$ be a retraction of $\langle 1_X, f\rangle$ over B. For $x \in X$ and $b \in B$ with $f(x) \leq b$, let $x_b := \rho(x, b)$. Then $x = \rho(x, f(x)) \leq \rho(x, b) = x_b$. Given a net $(x_\lambda)_\lambda$ in X_b with $x_\lambda \to x$, $\rho(x_\lambda, b) \to \rho(x, b) = x_b$.

(2′) ⇒ (2″): Let x' be in X_b with $x' \geq x$. Applying condition (2′) to the constant net x', we get that such a net x' converges also to x_b, which means $x' \geq x_b$. ∎

Proposition 4.2. *If* $f : X \to B$ *is injective in* $\mathbf{Top}_0 \downarrow B$, *then:*

(3′) *for any directed set* Λ *and any* I*-indexed family of nets* $((x^i_\lambda)_\lambda \in \Lambda)_{i\in I}$, *if* $(x^i_\lambda)_\lambda$ *converges to* x^i, $f(x^i_\lambda) = b_\lambda$ *and* $f(x^i) = b$, *then* $(\bigvee_i x^i_\lambda)_\lambda$ *converges to* $\bigvee_i x^i$.

Proof: Given $((x^i_\lambda)_\lambda)_{i\in I}$ as in (3′), define the topological spaces $\Lambda^\infty = \Lambda \uplus \{\infty\}$, with basic open sets $\{\lambda\}$ and $\uparrow \lambda \cup \{\infty\}$, for $\lambda \in \Lambda$, and $I_0 = I \uplus \{0\}$, with the topology generated by $\{i, 0\}$, $i \in I$, as in the proof of Proposition 3.5. Let $Y = \Lambda^\infty \times I_0$ and $Z = Y \setminus \{(\infty, 0)\}$. The maps $h : Z \to X$, with $h(\lambda, i) = x^i_\lambda$, $h(\infty, i) = x^i$ and $h(\lambda, 0) = \bigvee_i x^i_\lambda$, and $k : Y \to B$, with $k(\lambda, i) = k(\lambda, 0) = b_\lambda$ and $k(\infty, i) = k(\infty, 0) = b$, are continuous and make the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{k} & B \end{array}$$

commute, where ζ is the inclusion. Hence, there exists a diagonal $d : Y \to X$ with $f \cdot d = k$ and $d \cdot \zeta = h$. Since d is continuous and $(\lambda, 0) \xrightarrow[\lambda]{} (\infty, 0) \geq (\infty, i)$, we have

$$d(\lambda, 0) = \bigvee_i x^i_\lambda \xrightarrow[\lambda]{} d(\infty, 0) = x \geq d(\infty, i) = x^i.$$

Hence $x \geq \bigvee_i x^i$ and then $\bigvee_i x^i_\lambda \xrightarrow[\lambda]{} \bigvee_i x^i$ as claimed. ∎

Proposition 4.3. *Let* $f : X \to B$ *be a continuous map between T0-spaces. Then condition* (3′) *of Proposition 4.2 implies:*

(3″) *Given a net* $(x_\lambda)_\lambda$ *in* X, *if* $(f(x_\lambda))_\lambda$ *converges to* $b \in B$, *then* $\lim x_\lambda \cap X_b$ *has a top element* x_Λ.

Proof: We consider Λ^∞ as in the proof of Proposition 4.2 and the inclusion β of its (discrete) subspace Λ. We define $h : \Lambda \to X$ by $h(\lambda) = x_\lambda$, and $k : \Lambda^\infty \to B$ by $k(\lambda) = f(x_\lambda)$ and $k(\infty) = b$. The maps h and k are continuous, $f \cdot h = k \cdot \beta$ and so there is a diagonal $d : \Lambda^\infty \to X$ which gives a limit point $d(\infty)$ of $(x_\lambda)_\lambda$ in the fibre X_b. Now, to see that $\lim x_\lambda \cap X_b$ has a top element, we apply condition (3′) of Proposition 4.2 to $I := \lim x_\lambda \cap X_b$, $x^i_\lambda := x_\lambda$ and $x^i := i$ for every $i \in I$. ■

Proposition 4.4. *If $\zeta : (X, f) \to (Z, g)$ is an embedding in $\mathbf{Top}_0 \downarrow B$, with $f : X \to B$ injective in $\mathbf{Top}_0 \downarrow B$, then ζ has a largest continuous retraction $\overline{r} : Z \to X$ over B.*

Proof: By injectivity of f, the set

$$R := \{r \mid r \text{ is a continuous retraction of } \zeta \text{ in } \mathbf{Top}_0 \downarrow B\}$$

is non-empty. We consider the topological space $R_0 = R \uplus \{0\}$ as before (that is, with basic open sets $\{r, 0\}$, $r \in R$, so that $r \leq 0$ in the specialization order), and the embedding $\zeta_0 : \tilde{Z} = (Z \times R) \cup (X \times \{0\}) \to Z \times R_0$. We define $h : \tilde{Z} \to X$ by $h(z, r) = r(z)$ and $h(x, 0) = x$, for $x \in X$, $z \in Z$ and $r \in R$, which is easily checked to be continuous. Since the diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{h} & X \\ {\scriptstyle \zeta_0}\downarrow & & \downarrow{\scriptstyle f} \\ Z \times R_0 & \xrightarrow{g \cdot \pi_{R_0}} & B \end{array}$$

is commutative, injectivity of $f : X \to B$ guarantees the existence of a continuous map $\overline{h} : Z \times R_0 \to X$ such that $\overline{h} \cdot \zeta_0 = h$ and $f \cdot \overline{h} = g \cdot \pi_{R_0}$. The continuous map $\overline{r} : Z \to X$ with $\overline{r}(z) = \overline{h}(z, 0)$ is the required retraction: $\overline{r}(\zeta(x)) = h(x, 0) = x$, and, moreover, since $r \leq 0$ for each $r \in R$, one has $r(z) = \overline{h}(z, r) \leq \overline{h}(z, 0) = \overline{r}(z)$. ■

Remark 4.1. If $f : X \to B$ is injective in $\mathbf{Top}_0 \downarrow B$, Proposition 4.4 says, in particular, that $\langle 1_X, f\rangle : X \to X \times B$ has a largest continuous retraction $\overline{r} : X \times B \to B$ over B. Note that, according to Proposition 4.1 and its proof, for all $x \in X$ and $b \in B$ with $f(x) \leq b$, $\overline{r}(x, b) = x_b$ for x_b as in condition (2′).

Proposition 4.5. *Let $f : X \to B$ be a continuous map between T0-spaces satisfying*

(1) *X_b is a continuous lattice.*

(2″) *For $x \in X$ and $b \in B$ with $f(x) \leq b$, $\{x' \in X_b | x \leq x'\}$ has minimum element x_b.*

Then the largest continuous retraction $r_b : X \to X_b$ of the inclusion of X_b in X given by Proposition 4.4 is defined by

$$r_b(x) = \inf\{x' \in X_b \,|\, x \leq x'\} = \begin{cases} x_b & \text{if } f(x) \leq b, \\ \max X_b & \text{otherwise.} \end{cases}$$

Proof: Let $x \in X$ with $f(x) \leq b$: then $x \leq x_b$ and $r_b(x) \leq r_b(x_b) = x_b$. The map $r : \{x\} \cup X_b \to X_b$ with $r(x) = x_b$ and the identity otherwise is continuous. By the injectivity of X_b, we can extend r to X, obtaining a retraction $\tilde{r} : X \to X_b$, with $x_b = \tilde{r}(x) \leq r_b(x)$. Consequently, $x_b = r_b(x)$. If $f(x) \nleq b$, there exists in B an open neighbourhood of $f(x)$ not containing b and then x is open in the induced topology on $\{x\} \cup X_b$. As a consequence, the map $r : \{x\} \cup X_b \to X_b$ with $r(x) = x_b = \max X_b$ and the identity otherwise is continuous (any open set in X_b contains x_b). The result follows as in the previous case. ∎

Theorem 4.6. *A continuous map $f : X \to B$ is injective in $\mathbf{Top}_0 \downarrow B$ if and only if it satisfies the following conditions:*

(1) *For each $b \in B$, X_b is a continuous lattice;*

(2′) *For each $x \in X$ and $b \in B$ with $f(x) \leq b$, there exists $x_b \in X_b$ such that $x \leq x_b$ and, for any net $(x_\lambda)_\lambda$ in X_b, if $(x_\lambda)_\lambda$ converges to x then $(x_\lambda)_\lambda$ also converges to x_b.*

(3″) *Given a net $(x_\lambda)_\lambda$ in X, if $(f(x_\lambda))_\lambda$ converges to $b \in B$, then $\lim x_\lambda \cap X_b$ has a top element x_Λ.*

(4) *For each $x \in X$ and U open neighbourhood of x, there exists a continuous section s of f and an open neighbourhood W of $f(x)$ such that $\dagger s(W) = \{x' | f(x') \in W, x' \geq s(f(x'))\}$ is a neighbourhood of x contained in U.*

Proof: Let $f : X \to B$ be injective in $\mathbf{Top}_0 \downarrow B$. Theorem 3.7 and Propositions 4.1, 4.3 and 4.4 guarantee that (1), (2′) and (3″) are valid. It remains to show (4). Let $x_0 \in X$ and U be an open neighbourhood of x_0. Since $U \cap X_{f(x_0)}$ is open in $X_{f(x_0)}$ and $x_0 = \bigvee\{x \in X_{f(x_0)} \,|\, x \ll x_0\}$, there exists $\tilde{x} \in U \cap X_{f(x_0)}$ such that $\tilde{x} \ll x_0$. Let $\rho : X \times B \to X$ be a retraction of

$\langle 1_X, f\rangle$, let $\hat{\rho} : X \to \mathrm{S}(f)$ be its mate, and denote the section $\hat{\rho}(\tilde{x}) : B \to X$ of f by $\tilde{s}$. By definition, $\tilde{s}(f(\tilde{x})) = \tilde{x}$. We will show that $W := f(\tilde{s}(B) \cap U)$ is the required open neighbourhood of $f(x_0)$:

- W is an open subset of B (as shown in the proof of Theorem 3.7).
- $\dagger\tilde{s}(W) \subset U$, since U is upwards-closed on each fibre.
- To show that $\dagger\tilde{s}(W)$ is a neighbourhood of x_0, we recall from Remark 3.1.2 that ρ induces an embedding $\langle\hat{\rho}, f\rangle : X \to \mathrm{S}(f) \times B$. Since $\mathrm{S}(f)$ is a continuous lattice, $\Uparrow\tilde{s}$ is an open subset of $\mathrm{S}(f)$. Then $V := \Uparrow\tilde{s} \times W$ is an open subset of $\mathrm{S}(f) \times B$. Moreover:
  - $(\hat{\rho}(x_0), f(x_0)) \in V$, since $\tilde{x} \ll x_0$ implies $\hat{\rho}(\tilde{x}) = \tilde{s} \ll \hat{\rho}(x_0)$ and $f(x_0) \in W$, and
  - $\langle\hat{\rho}, f\rangle^{-1}(V) \subseteq \dagger\tilde{s}(W)$, since if, for $x \in X$, $\hat{\rho}(x) \gg \tilde{s}$ and $f(x) \in f(\tilde{s}(B) \cap U)$, then $f(x) = f(x')$ with $x' = \tilde{s}(f(x')) \in U$; then $x = \hat{\rho}(x)(f(x)) \geq \hat{\rho}(x')(f(x)) = x' = \tilde{s}(f(x'))$ and $f(x') \in f(\tilde{s}(B) \cap U) = W$.

To prove the converse, we assume that the conditions (1)-(4) are satisfied, and show that $f : X \to B$ is a retraction of $\pi_B : \prod_{b\in B} X_b \times B \to B$; since the latter map is injective, according to Corollary 1.6 of [5] because $\prod_{b\in B} X_b$ is a continuous lattice by (1), we can conclude that f is injective. For any $x \in X$ and $b \in B$, let $\overline{x}_b = r_b(x)$ as defined in Proposition 4.5. Consider the map $\nu : X \to \prod_{b\in B} X_b \times B$ defined by $\nu(x) = ((\overline{x}_b)_{b\in B}, f(x))$.

- ν is continuous: $\pi_B \cdot \nu = f$ is continuous and, for $\pi'_b : \prod_{b\in B} X_b \times B \to X_b$, $\pi'_b \cdot \nu$ is the largest retraction r_b of the embedding $X_b \to X$ as constructed in Proposition 4.5, for every $b \in B$, hence continuous.
- ν has a retraction $r : \prod_{b\in B} X_b \times B \to X$: for each $(s, b) \in \prod_{b\in B} X_b \times B$, define
$$r(s, b) = \bigwedge\{x_\Lambda \mid (x_\lambda)_{\lambda\in\Lambda} \text{ in } s(B) \text{ with } f(x_\lambda) \to b\}$$
(where x_Λ is as in condition (3′′)). First we observe that $r(s, b) \leq s(b)$ since among the $(x_\lambda)_\lambda$ we may consider the constant net $s(b)$. Secondly, r is continuous: given $(s_0, b_0) \in \prod_{b\in B} X_b \times B$ and an open neighbourhood U of $x_0 := r(s_0, b_0)$, let s and W be as in (4), and $V := int(\dagger s(W)) \subseteq U$ with $x_0 \in V$. Consider the subspace $A = \{b | b \in f(V),\ s_0(b) \notin V\}$ of B. We want to show that b_0 is not in the closure $cl(A)$ of A. Suppose $b_0 \in cl(A)$: then there exists a net $(b_\lambda)_\lambda$ in A with $b_\lambda \to b_0$. From (3′′), it follows that $\lim x_\lambda := s_0(b_\lambda) \cap X_{b_0}$ has a top element $x_\Lambda \geq r(s_0, b_0) = x_0$, by definition of r. This means that $x_\lambda \to x_0$, hence there is a queue contained in V, neighbourhood of x_0. But this is impossible since any $x_\lambda \notin V$. Hence,

$b_0 \notin cl(A)$ and we can take in B an open neighbourhood O of b_0 with $O \cap A = \emptyset$.

Now V_b, defined by $V \cap X_b$ in case $b \in f(V) \cap O$ and X_b otherwise, is an open subset of X_b. Since f is open by (4), $\widetilde{V} := \prod_{b\in B} V_b \times (f(V) \cap O)$ is an open subset of $\prod_{b\in B} X_b \times B$ such that:

(a) $(s_0, b_0) \in \widetilde{V}$, since for any b, $s_0(b) \in V_b$ and $b_0 = f(x_0) \in f(V) \cap O$.

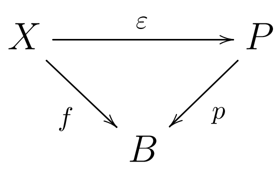
(b) $r(\widetilde{V}) \subseteq U$: if $(t, c) \in \widetilde{V}$, then $t(b) \in V$ and $c \in f(V) \cap O$. So, given any net $(b_\lambda)_\lambda$ in $f(V) \cap O$ with $b_\lambda \to c$, the net $x_\lambda := t(b_\lambda) \in V_{b_\lambda} \subseteq \dagger s(W)$, hence $x_\lambda \geq s(b_\lambda)$ and $s(b_\lambda) \to s(c)$, thanks to the continuity of s. Therefore also $x_\lambda \to s(c)$. By definition of r, $r(t, c) \geq s(c)$, then, since $c \in W$, $r(t, c) \in \dagger s(W) \subset U$.

Finally, $r \cdot \nu = 1_X$: Indeed, if we denote by $s_x : B \to X$ the (not necessarily continuous) section of f with $s_x(b) = \overline{x}_b$, then $r \cdot \nu(x) = s_x(f(x)) = x$. To check the equality we consider any net $(x_\lambda)_\lambda$ in $s_x(B)$, that is, $x_\lambda = \overline{x}_{f(x_\lambda)}$, with $f(x_\lambda) \to f(x)$, and we need to show that $x_\lambda \to x$. Let $\Lambda' = \{\lambda \in \Lambda \,|\, f(x_\lambda) \geq f(x)\}$ and $\Lambda'' = \Lambda \setminus \Lambda'$. If $\lambda \in \Lambda'$, then $x_\lambda = x_{f(x_\lambda)}$, and if $\lambda \in \Lambda''$ then $x_\lambda = \max X_{f(x_\lambda)}$ by definition of $\overline{x}_b$. To show the convergence of $(x_\lambda)_{\lambda\in\Lambda}$ it is enough to show that, in case Λ' and/or Λ'' define subnets, $(x_\lambda)_{\lambda\in\Lambda'}$ and/or $(x_\lambda)_{\lambda\in\Lambda''}$ converge to x. For $\lambda \in \Lambda'$, $x_\lambda = x_{f(x_\lambda)} \geq x$ by definition of $x_{f(x_\lambda)}$, hence $x_\lambda \to x$. Now, let $\lambda \in \Lambda''$. Since f is open by (4), its largest section $\overline{s}$ is continuous. Hence, for $\lambda \in \Lambda''$, $x_\lambda = \overline{s}(f(x_\lambda)) \to \overline{s}(f(x)) \geq x$, and so $x_\lambda \to x$.

■

5. Injectivity via exponentiability

Here we characterize injective continuous maps using exponentiability. Recall from Proposition 2.3 that, if $f : X \to B$ is exponentiable in $\mathbf{Top}_0 \downarrow B$, then f is embeddable in a partial product over S:



with $P = P(p_f, S)$, and $p = p_{p_f}$ injective.

This embedding $\varepsilon : X \to P$ is given by: $\varepsilon(x) = (f(x), \{U \text{ open in } f^{-1}(f(x)) \,|\, x \in U\})$.

Lemma 5.1. *Let $f : X \to B$ be an exponentiable map and $b \in B$ such that*

(1) *X_b is a continuous lattice;*
(2″) *for $x \in X$ and $b \in B$ with $f(x) \leq b$, $\{x' \in X_b | x \leq x'\}$ has minimum element x_b.*

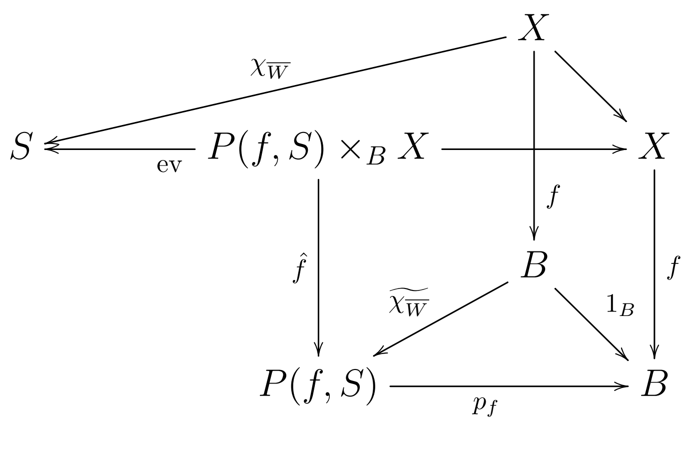
Then $\varepsilon(x_b) = \varepsilon(x)_b = \min\{t \in P_b | \varepsilon(x_b) \leq t\}$.

Proof: Consider $f(x) \leq b$. We have to prove that $\varepsilon(x_b)$ is the minimum element $\varepsilon(x)_b$ of $\{t \in P_b | \varepsilon(x_b) \leq t\}$ in P_b, which exists by Theorem 3.7 and Proposition 4.1, since p is injective.

By definition $x_b \geq x$, so that by continuity $\varepsilon(x_b) \geq \varepsilon(x)_b$. Suppose now $\varepsilon(x_b) \neq \varepsilon(x)_b := (b, \mathcal{U})$. As a consequence there exists a W open in X_b s.t. $x_b \in W$ and $W \notin \mathcal{U}$.

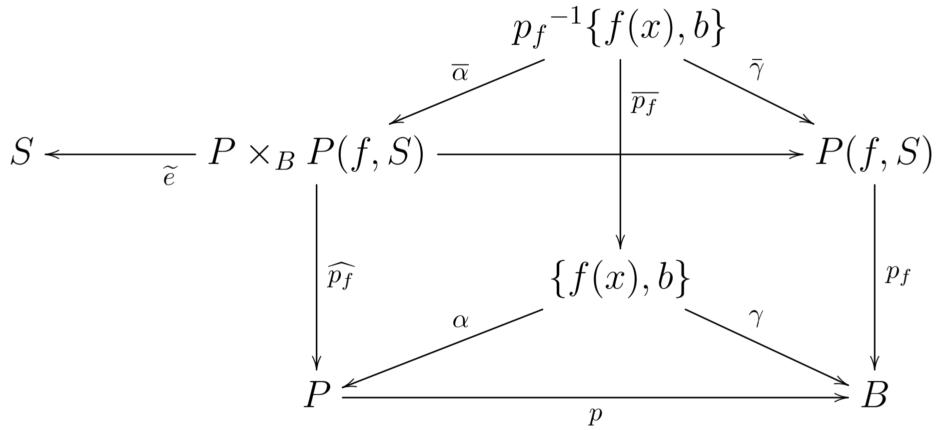
Consider the maximal retraction $r_b : X \to X_b$ of Proposition 4.5 and the open set $r^{-1}(W) = \overline{W}$. Since $r_b(x) = x_b$, we get $x \in \overline{W}$.

Let $\chi_{\overline{W}} : X \to S$ be the characteristic map of $\overline{W}$. In correspondence to $\chi_{\overline{W}}$ there exists, by the universal property of the partial product $P(f, S)$, a continuous map $\widetilde{\chi_{\overline{W}}} : B \to P(f, S)$



with $\widetilde{\chi_{\overline{W}}}(b') = (b', X_{b'} \cap \overline{W})$, for any $b' \in B$.

Consider now the subspace $\{f(x), b\}$ of B, homeomorphic to the Sierpinski space S, and its inclusion γ in B. Let $\alpha : \{f(x), b\} \to P$ be the continuous map given by $\alpha(f(x)) = \varepsilon(x)$ and $\alpha(b) = \varepsilon(x)_b$. Corresponding to α and the pullback of $\gamma = p\alpha$ along p_f:



there exists a continuous map $\overline{\alpha} : p_f{}^{-1}\{f(x), b\} \to P \times_B P(f, S)$ with

$$\overline{\alpha}(f(x), A) = (\varepsilon(x), A) \quad \text{and} \quad \overline{\alpha}(b, A) = (\varepsilon(x)_b, A).$$

Consequently the map $\beta := \mathrm{ev}\overline{\alpha} : p_f{}^{-1}\{f(x), b\} \to S$ is such that

$$\beta(f(x), A) = 1 \quad \Leftrightarrow \quad x \in A \qquad \text{and} \qquad \beta(b, A) = 1 \quad \Leftrightarrow \quad A \in \mathcal{U}.$$

$\beta^{-1}(1)$ is an open subset $D \subseteq p_f{}^{-1}\{f(x), b\}$, and then there is an open set D' of $P(f, S)$ with $D' \cap p_f{}^{-1}\{f(x), b\} = D$.

Now, since $x \in X_{f(x)} \cap \overline{W}$, this means that $\beta(f(x), X_{f(x)} \cap \overline{W}) = 1$, i.e. $(f(x), X_{f(x)} \cap \overline{W}) \in D$, so that $\widetilde{\chi_{\overline{W}}}(f(x)) = (f(x), X_{f(x)} \cap \overline{W}) \in D'$ and $f(x) \in \widetilde{\chi_{\overline{W}}}^{-1}(D')$.

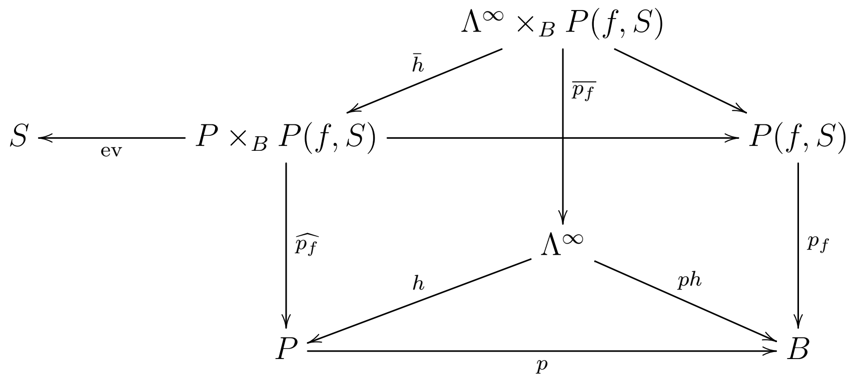
On the contrary, since $W \notin \mathcal{U}$, $\beta(b, W) = 0$ and then $\widetilde{\chi_{\overline{W}}}(b) = (b, X_b \cap \overline{W} = W) \notin D'$. This means that $b \notin \widetilde{\chi_{\overline{W}}}^{-1}(D')$ and this is impossible, being $f(x) \leq b$. ■

Lemma 5.2. *Let $f : X \to B$ be an exponentiable map and $b \in B$ such that*

(1) *X_b is a continuous lattice.*
(3″) *Given a net $(x_\lambda)_{\lambda \in \Lambda}$ in X, if $f(x_\lambda) \neq b$ converges to $b \in B$, then x_λ has a maximum limit point x_Λ in X_b.*

Then $\varepsilon(x_\Lambda)$ is the maximum limit point in P_b of the net $\varepsilon(x_\lambda)$.

Proof: As a consequence of the injectivity of p, we know that the net $\varepsilon(x_\lambda)$ has a maximum limit point $\varepsilon(x_\lambda)_\Lambda = (b, \mathcal{U})$ in P_b. By continuity of ε, we get $\varepsilon(x_\Lambda) \leq \varepsilon(x_\lambda)_\Lambda$. Suppose they are different. This means there is a $W \in \mathcal{U}$, with $x_\Lambda \notin W$. Consider the topological space $\Lambda^\infty = \Lambda \uplus \{\infty\}$, with basic open sets $\{\lambda\}$ and $\uparrow \lambda \cup \{\infty\}$. Define the continuous map $h : \Lambda^\infty \to P$ as $h(\lambda) = \varepsilon(x_\lambda)$ and $h(\infty) = (b, \mathcal{U})$. By the universal property of the pullback, corresponding to h there exists a continuous map $\bar{h} : \Lambda^\infty \times_B P(f,S) \to P \times_B P(f,S)$



with

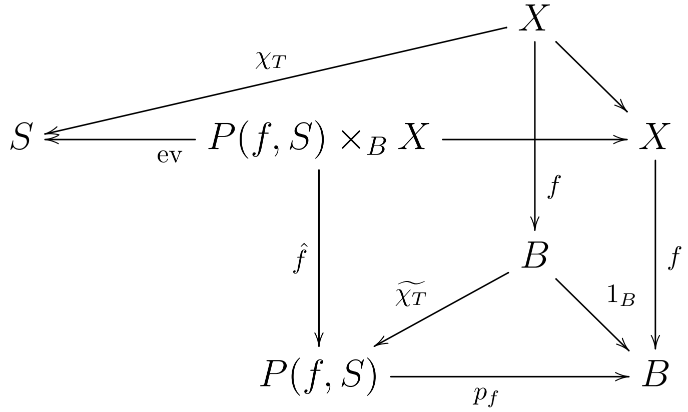
$$\overline{h}(\lambda, f(x_\lambda), A) = (\varepsilon(x_\lambda), A) \quad \text{and} \quad \overline{h}(\infty, b, A) = (\varepsilon(x_\lambda)_\Lambda, A).$$

If we define $e := \mathrm{ev}\overline{h}$, we get

$$e(\lambda, f(x_\lambda), A) = 1 \quad \Leftrightarrow \quad x_\lambda \in A \qquad \text{and} \qquad e(\infty, b, A) = 1 \quad \Leftrightarrow \quad A \in \mathcal{U}.$$

Since $W \in \mathcal{U}$, then $(\infty, b, W) \in e^{-1}(1)$, so that there exists a base open neighborhood $(\uparrow \bar{\lambda} \cap \{\infty\} \times O) \cup (\Lambda^\infty \times_B P(f,S)) \subset e^{-1}(1)$, with $\bar{\lambda} \in \Lambda$ and O an open set of $P(f,S)$. Hence, $(b, W) \in O$ and for any $\lambda \geq \bar{\lambda}$ and any $U_\lambda \in O_\lambda = O \cap p_f^{-1}(f(x_\lambda))$, $e(\lambda, f(x_\lambda), U_\lambda) = 1$, which means $x_\lambda \in U_\lambda$.

Since W is open in X_b, then there exists a W' open in X with $W' \cap f^{-1}(b) = W$. Consider now the closure C in X of the set $\{x_\lambda, x_\Lambda\}$ and $T = W' \setminus C$. Let us observe that $X_b \cap C = \downarrow x_\Lambda$, since any $f(x_\lambda) \neq b$. Consequently $X \cap T = W$. Take now the characteristic map $\chi_T : X \to S$. By the exponentiability of f we get the following diagram:



with $\widetilde{\chi_T}(b') = (b', X_{b'} \cap T)$, for any $b' \in B$. In particular $\widetilde{\chi_T}(b) = (b, X_b \cap T = W) \in O$, so that $b \in \widetilde{\chi_T}^{-1}(O)$, which is an open set of B. But for any $\lambda \geq \bar{\lambda}$, if $\widetilde{\chi_T}(f(x_\lambda)) = (f(x_\lambda), X_{f(x_\lambda)} \cap T) \in O_\lambda$, x_λ should be in T and this is not, since in Tb there are no x_λ. As a conclusion, there is a queue $f(x_\lambda) \notin \widetilde{\chi}_T^{-1}(O)$ and this is impossible, since $f(x_\lambda)$ converges to b. ■

Theorem 5.3. *$f : X \to B$ is injective in* $\mathbf{Top_0}/B$ *if and only if*

(0) *f is exponentiable and open.*
(1) *For each $b \in B$, X_b is a continuous lattice.*
(2″) *For $x \in X$ and $b \in B$ with $f(x) \leq b$, $\{x' \in X_b | x \leq x'\}$ has minimum element x_b.*
(3″) *Given a net $(x_\lambda)_{\lambda\in\Lambda}$ in X, if $f(x_\lambda)$ converges to $b \in B$, then x_λ has a maximum limit point x_Λ in X_b.*

Proof: All the above conditions are satisfied if f is injective.

Suppose now that all the conditions are satisfied. Since condition (1) implies that $\Pi_{b\in B}X_b$ is a continuous lattice, it is enough to show that f can be obtained as a retraction of the injective map $\pi_B : \Pi_{b\in B}X_b \times B \to B$.

We split the proof in several steps, showing that:

(a) $\varepsilon : X \to P$ can be corestricted to $\check{P} = P \setminus \overline{s}_p(B)$.
(b) The restriction $\check{p} : \check{P} \to B$ of p is injective.
(c) The retraction $r_{\check{P}} : \Pi_{b\in B}\check{P}_b \times B \to \check{P}$ given in Theorem 4.6, which exists since $\check{p}$ is injective, can be restricted via ε_b to $\Pi_{b\in B}X_b \times B$, yielding a continuous retraction $r_X : \Pi_{b\in B}X_b \times B \to X$ of π_B over f.

(a) *$\varepsilon : X \to P$ can be corestricted to $\check{P} = P \setminus \overline{s}_p(B)$*: The maximum section $\overline{s}_p : B \to P$ of p is defined by $\overline{s}_p(b) = (b, \bigvee P_b)$, and $\bigvee P_b = \mathcal{O}(X_b)$. Therefore $\overline{s}_p(B)$ does not meet $\varepsilon(X) = \{(f(x), \{U \in \mathcal{O}(f(x)) \,|\, x \in U\})\}$.

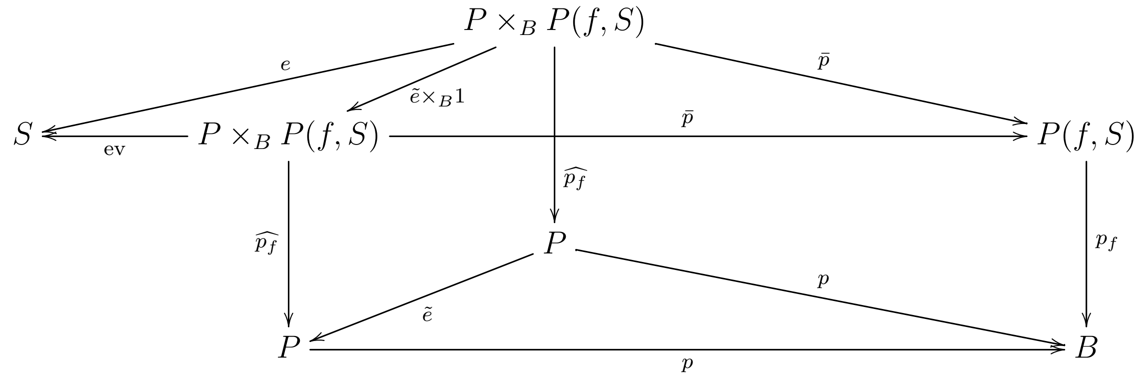
(b) *The restriction $\check{p} : \check{P} \to B$ of p is injective*:

$P_0 = \{(b, \emptyset) \,|\, b \in B\}$ *is a closed subset of* $P(f, S)$: Let $A = P(f, S) \setminus P_0 = \{(b, U) \,|\, b \in B,\, U \in \mathcal{O}(X) \setminus \emptyset\}$. Then A is open since, for each $b \in B$, A_b is clearly saturated and has the finite union property; A is binding because f is open: for each $U \in \mathcal{O}(X)$, $\{b \,|\, U \cap X_b \neq \emptyset\} = f(U) \in \mathcal{O}(B)$.

Now we consider the map

$$\begin{array}{rcl} e : P \times_B P(f, S) & \longrightarrow & S \\ (b, \mathcal{U}, U) & \longmapsto & \begin{cases} 0 & \text{if } U = \emptyset \\ \text{ev}(b, \mathcal{U}, U) & \text{if } U \neq \emptyset \end{cases} \end{array}$$

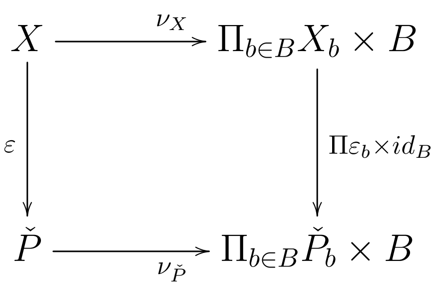
which is continuous, because $e^{-1}(0) = \bar{p}^{-1}(P_0) \cup \mathrm{ev}^{-1}(0)$ is closed. By the universal property of the partial product, there exists a unique continuous map $\tilde{e} : P \to P$ such that $p \cdot \tilde{e} = p$ and $\mathrm{ev} \cdot (\tilde{e} \times_B 1) = e$:



] By construction, if $(b,\mathcal{U}) \in \check{P}$, then $\emptyset \notin \mathcal{U}$, so that e coincides with ev when restricted to $\check{P} \times_B P(f,S)$. Therefore the restriction of $\tilde{e}$ to $\check{P}$ is the identity. Moreover, for any $(b,\mathcal{U}) \in P$, $\tilde{e}(b,\mathcal{U}) = (b,\tilde{\mathcal{U}})$ must belong to $\check{P}$, otherwise $\mathrm{ev}(b,\tilde{\mathcal{U}},\emptyset) = 1 \neq e(b,\mathcal{U},\emptyset) = 0$. Hence the corestriction of $\tilde{e}$ to $\check{P}$ gives a retraction of P into $\check{P}$, and therefore $\check{p}$ is injective. Notice that ε can be considered as an embedding of X into $\check{P}$ and that ε sends the maximal section of f into the maximal section of $\check{p}$.

(c) $\Pi_{b\in B} X_b$ is a continuous lattice which can be embedded via the restrictions ε_b of ε to each fiber of b, in $\Pi_{b\in B}\check{P}_b$. We observe that every ε_b preserves the maximum element on each fiber.

So we can consider the following diagram:



where ν_X and $\nu_{\check{P}}$ are defined as in Theorem 4.6 respectively for f and $\check{p}$:

$\nu_X(x) = ((\overline{x}_b)_{b\in B}, f(x))$, with $\overline{x}_b = x_b$ if $b \geq f(x)$ and $\overline{x}_b = \max X_b$, otherwise

$\nu_{\check{P}}(t) = ((\bar{t}_b)_{b\in B}, \check{p}(t))$, with $\bar{t}_b = t_b$ if $b \geq \check{p}(t)$ and $\bar{t}_b = \max \check{P}_b$ otherwise

Since ε, $\nu_{\check{P}}$ are embeddings, ν_X (which is continuous by the proof of Theorem 4.6, since therein just conditions (1) and (2″) are used) is an embedding itself. The diagram is commutative thanks to Lemma 5.1.

Consider the retraction $r_{\check{P}} : \Pi_{b\in B}\check{P}_b \times B \to \check{P}$ of $\nu_{\check{P}}$ given in Theorem 4.6, which is continuous since $\check{p}$ is injective:

$$r_{\check{P}}(\sigma, b) = \bigwedge\{t_\Lambda \,|\, (t_\lambda)_{\lambda\in\Lambda} \text{ in } \sigma(B) \text{ with } \check{p}(t_\lambda) \to b\}$$

(where t_Λ is as defined in Proposition 4.3).

We now consider the corresponding retraction $r_X : \Pi_{b\in B}X_b \times B \to X$ for f, defined as:

$$r_X(s, b) = \bigwedge\{x_\Lambda \,|\, (x_\lambda)_{\lambda\in\Lambda} \text{ in } s(B) \text{ with } f(x_\lambda) \to b\}$$

(where x_Λ is as in condition (3″)). We want to prove that such r_X is continuous. To this aim it is sufficient to prove that $r_{\check{P}}(\Pi\varepsilon_b \times id_B) = \varepsilon r_X$. We will do it in two steps.

(c1) First, given $(s, b) \in \Pi_{b\in B}X_b\times B$, where s is a (not necessarily continuous) section $s : B \to X$ of f and $b \in B$, we will prove that the set $A = \{x_\Lambda | x_\lambda \in\ s(B),\ f(x_\lambda) \to b\}$ has a minimum $\tilde{x}$ in X_b, so that we will have that $r_X(s, b) = \bigwedge A =$min $A = \tilde{x}$.

(c2) Since $\check{p}$ is injective, $\check{p}$ shares the same properties with f and then by (c1) also $r_{\check{P}}(\varepsilon s, b)$=min $\{\varepsilon(x)_\Lambda \,|\, \varepsilon(x_\lambda)_{\lambda\in\Lambda}$ in $\varepsilon s(B)$ with $f(x_\lambda) \to b\}$. By Lemma 5.2, $\varepsilon(x)_\Lambda = \varepsilon(x_\Lambda)$ and this implies that

$$r_{\check{P}}(\Pi\varepsilon_b\times id_B)(s, b) = r_{\check{P}}(\varepsilon s, b) = \min\{\varepsilon(x_\Lambda) \,|\, \varepsilon(x_\lambda)_{\lambda\in\Lambda} \text{ in } \varepsilon s(B) \text{ with } f(x_\lambda) \to b\} =: \tilde{t}.$$

Since $\tilde{x} \le x_\Lambda$, $\varepsilon(\tilde{x}) \le \varepsilon(x_\Lambda)$, for any Λ, then $\varepsilon(\tilde{x}) \le \tilde{t}$. But $\tilde{x} = x_{\tilde{\Lambda}}$ for some $\tilde{\Lambda}$, so that

$$r_{\check{P}}(\Pi\varepsilon_b \times \mathrm{id}_B)(s, b) = \tilde{t} = \varepsilon(\tilde{x}) = \varepsilon r_X(s, b)$$

r_X is then a retraction of f over B of the injective map $\pi_B : \Pi_{b\in B}X_b \times B \to B$, and then f is injective.

(c1) We start by proving that A is directed in the inverse order of X_b. Let $x_{\Lambda_1}, x_{\Lambda_2} \in A$, maximum limit points respectively of $(x_{\lambda_1}), (x_{\lambda_2})$.

Consider the directed set $\Lambda_1 \times \Lambda_2 \times \mathbb{N}$ ($\mathbb{N}$ natural numbers with the natural order) and define the net $h : \Lambda_1 \times \Lambda_2 \times \mathbb{N} \to X$ as

$$h(\lambda_1, \lambda_2, 2n) = x_{\lambda_1}, h(\lambda_1, \lambda_2, 2n + 1) = x_{\lambda_2}.$$

We prove that $x_{\Lambda_1} \wedge x_{\Lambda_2}$ is the maximum limit point of h in X_b. Indeed, given any open set W with $x_{\Lambda_1} \wedge x_{\Lambda_2} \in W$, since $W \cap X_b$ is upward closed, $x_{\Lambda_1} \in W$, $x_{\Lambda_2} \in W$. Then there exist $\bar{\lambda}_1$ and $\bar{\lambda}_2$ such that for any $\lambda_1 \geq \bar{\lambda}_1$ and $\lambda_2 \geq \bar{\lambda}_2$, $x_{\lambda_1}, x_{\lambda_2} \in W$ and so the queue of h given by $(\bar{\lambda}_1, \bar{\lambda}_2, 1)$ is in W. This means that $h \to x_{\Lambda_1} \wedge x_{\Lambda_2}$. Let now y be another limit point of h in X_b; if V_y is open with $y \in V_y$, there is a queue of h starting from $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{n})$ in V_y. For any $\lambda_1 \geq \tilde{\lambda}_1$, $h(\lambda_1, \tilde{\lambda}_2, 2\tilde{n}) = x_{\lambda_1} \in V_y$ and for any $\lambda_2 \geq \tilde{\lambda}_2$, $h(\tilde{\lambda}_1, \lambda_2, 2\tilde{n}+1) = x_{\lambda_2} \in V_y$, so that

$$x_{\lambda_1} \to y \qquad x_{\lambda_2} \to y \quad \Rightarrow \quad y \leq x_{\Lambda_1} \wedge x_{\Lambda_2}$$

Now we want to prove the existence of a net in $s(B)$ converging to $\bigwedge A =: \tilde{x}$. First of all, for any $a \in A$, fix a net $(x_{\lambda_a})_{\Lambda_a}$ which has a as maximum limit point.

Consider now the product $D = \Pi_{a \in A} \Lambda_a \times A$ directed by the product order. Let $h : D \to X$ be the net given by

$$h(d) = h((\lambda_a)_{a \in A}, \bar{a})) = x_{\lambda_{\bar{a}}}.$$

▷ the net fh converges to b: for any open neighbourhood U of b, $f^{-1}(U)$ is a neighbourhood of each $a \in A$, so that, for any $a \in A$, there is $(\bar{\lambda}_a)$ s.t. if $\lambda_a \geq (\bar{\lambda}_a)$, $x_{\lambda_a} \in f^{-1}(U)$. Then, for any $\tilde{a} \in A$, the queue of fh given by $((\bar{\lambda}_a)_{a \in A}, \tilde{a})$ is such that for any $\lambda_a \geq (\bar{\lambda}_a)$ and any $\bar{a} \geq \tilde{a}$,

$$fh((\lambda_a)_a, \bar{a}) = f(x_{\lambda_{\bar{a}}}) \in U$$

so that fh converges to b and consequently h has a maximum limit point $y \in A$ in X_b.

We want to prove that $y \leq a$ for any $a \in A$, so that $y = \tilde{x}$. For this, it is sufficient to show that, for any a,

$$x_{\lambda_a} \to y, \quad \text{which is equivalent to proving that} \quad \varepsilon(x_{\lambda_a}) \to \varepsilon(y),$$

being ε an embedding.

Let us take an open neighbourhood W of $\varepsilon(y)$. By Theorem 4.6 (4), there exists a continuous section $\sigma : B \to \check{P}$ of $\check{p} : \check{P} \to B$ and a neighbourhood Z of $\check{p}(\varepsilon(y)) = b$, such that $\dagger\sigma(Z) = \{t | \check{p}(t) \in Z, t \geq \sigma(\check{p}(t))\}$ is a neighbourhood of $\varepsilon(y)$ with $\dagger\sigma(Z) \subseteq W$.

Since εh converges to $\varepsilon(y)$, there exists $\tilde{d} = ((\tilde{\lambda}_a)_{a \in A}, \tilde{a})$ such that for any $d = ((\lambda_a)_{a \in A}, \bar{a}) \geq \tilde{d}$, $\varepsilon(h(d)) = \varepsilon(x_{\lambda_{\bar{a}}}) \in \dagger\sigma(Z)$.

Now, for any $\bar{a} \geq \tilde{a}$ and any $\lambda_{\bar{a}} \geq \tilde{\lambda}_{\bar{a}}$, we can define $\bar{d} = ((\bar{\lambda}_a)_{a\in A}, \bar{a})$ with $\bar{\lambda}_a = \tilde{\lambda}_a$ for any $a \neq \bar{a}$ and $\bar{\lambda}_{\bar{a}} = \lambda_{\bar{a}}$. Then

$$\varepsilon(h(\bar{d})) = \varepsilon(x_{\lambda_{\bar{a}}}) \in \dagger\sigma(Z)$$

Then, for any $\bar{a} \geq \tilde{a}$, this queue of $\varepsilon(x_{\lambda_a})$ is a net, say $(t_{\bar{\lambda}})_{\bar{\lambda}\in\bar{\Lambda}_{\bar{a}}}$ in $\check{P}$ with :

- $\check{P}(t_{\bar{\lambda}}) \to b$
- $t_{\bar{\lambda}} \in \dagger\sigma(Z)$
- $t_{\bar{\lambda}} = \varepsilon(x_{\lambda_{\bar{a}}})$ and then its maximum limit point is $\varepsilon(\bar{a})$

This will be sufficient to prove that

$$t_{\bar{\lambda}} \to \sigma(b)$$

and consequently we will have that $\sigma(b) \leq \varepsilon(\bar{a})$, for any $\bar{a} \geq \tilde{a}$.

Let V be an open neighbourhood of $\sigma(b)$, then $V \cap \sigma(Z) = V'$ is an open neighbourhood of $\sigma(b)$ in $\sigma(B)$, and $\check{p}(V')$ is an open neighbourhood of b. Since $\check{p}(t_{\bar{\lambda}}) \to b$, there exists $\hat{\lambda} \in \bar{\Lambda}_{\bar{a}}$ such that, for any $\bar{\lambda} \geq \hat{\lambda}$, $\check{p}(t_{\bar{\lambda}}) \in \check{p}(V') \subseteq Z$. $t_{\bar{\lambda}} \in \dagger\sigma(Z)$, so $t_{\bar{\lambda}} \geq \sigma(\check{p}(t_{\bar{\lambda}}))$, that is $t_{\bar{\lambda}} \in \dagger V' = \dagger\sigma(\check{p}(V')) \subseteq V$, since V is open (hence upward closed on each fiber). We can conclude that $t_{\bar{\lambda}} \to \sigma(b)$ and then, as already noticed, $\sigma(b) \leq \varepsilon(\bar{a})$, for any $\bar{a} \geq \tilde{a}$.

Consider now $a \in A$, $a \not\geq \tilde{a}$; A is directed, hence there is $\bar{a} \in A$ with $\bar{a} \geq \tilde{a}$ and $\bar{a} \geq a$, which means $\bar{a} \leq a$ in X_b. By the argument above, we have that $\sigma(b) \leq \varepsilon(\bar{a})$ and then $\sigma(b) \leq \varepsilon(a)$ in $\check{P}_b$. It follows that $\sigma(b) \leq \varepsilon(a)$, for any $a \in A$. Then, given any neighbourhood W of $\varepsilon(y)$, $\varepsilon(a) \in W$, for any $a \in A$ and in the specialization of $\check{P}_b$ this means that $\varepsilon(y) \leq \varepsilon(a)$, hence $y \leq a$, for any $a \in A$.

Finally, we can conclude that $y = \min A = \bigwedge A = \tilde{x}$. ■

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