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FIBER SURFACES FROM ALTERNATING STATES

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Abstract: In this paper we define alternating Kauffman states of links and we characterize when the induced state surface is a fiber. In addition we give a different proof of a similar theorem of Futer on homogeneous states.

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1. Introduction

Given a diagram D of a link L we can construct a collection of disjoint disks connected by a twisted band at each crossing. We thus obtain a surface whose boundary is the link L. The disks and bands are defined by how we split the crossings in the diagram of L. At each crrossing there are two choices of *resolutions* for the split: an A-resolution or a B-resolution, as presented in Figure 1.

A *Kauffman state* σ of a link diagram D is a choice of resolution for each crossing of D. The resulting surface S_σ is called the *state surface* of σ. The boundaries of the disks induce a decomposition of the plane into connected components that we call *regions*. The well known Seifert surface of an oriented diagram of a link is a particular case of a state surface, where the resolution of each crossing is defined by the orientation. It has been an interest of research to identify fibered knots and their fibers. We are interested in understanding when a state surface is a fiber. In the work of Futer, Kalfagianni and Purcell [1, 2] it was studied for *homogeneous states*, that is when all resolutions of the diagram in each region are the same. (See Theorem 2.)

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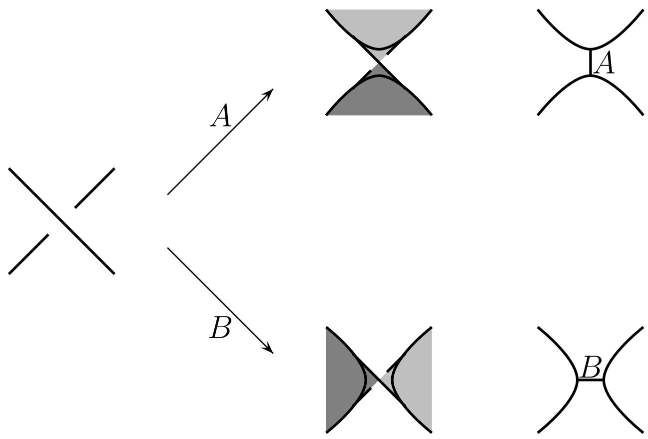

FIGURE 1: Resolutions

In this paper we give a different proof of this theorem and we prove a similar theorem for a different type of Kauffman states, as in the next definition.

Definition 1. A Kauffman state σ is said to be *alternating* when for each circle defined by σ, with a choice of orientation on its boundary, if two consecutive crossings attached to it in the same region have the same resolution then they are adjacent to the same circles defined by σ.

Before we present our main result, we associate two graphs to each state of a link diagram. The *state graph* G_σ has one vertex for each disk and one edge for each band defined by σ. We label the edges by the resolution of the respective crossings. The *reduced graph* G'_σ is obtained from G_σ by eliminating duplicated edges, with the same label, between two vertices. From the state surface S_σ we define also a *reduced surface* S'_σ by cutting duplicated bands with the same label attached to the same pair of disks. We note that the graphs G_σ and G'_σ are not abstract graphs but instead they are embedded in the surfaces S_σ and S'_σ as their spines. An *inner cycle*, of the state graph or a reduced version of it, is an innermost cycle in a certain region. Our main result is the following.

Theorem 1. *Let σ be an alternating state of a link diagram D_L. Then $E(L)$ fibers over the circle with fiber S_σ if and only if the reduced graph G'_σ is a tree.*

The next examples illustrate that the classes of link diagrams in theorems 1 and 2 are distinct. Certain states can be both homogeneous and alternating, as for example the Seifert state of the Figure eight knot as in Figure 2.

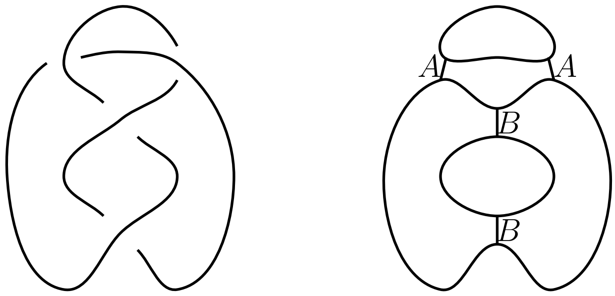

FIGURE 2: The Seifert state of this Figure eight knot diagram is a fiber by theorems 1 and 2.

But in general a state isn't both homogenous and alternating. For instance, in the Figure 3 the Seifert state is alternating and not homogeous.

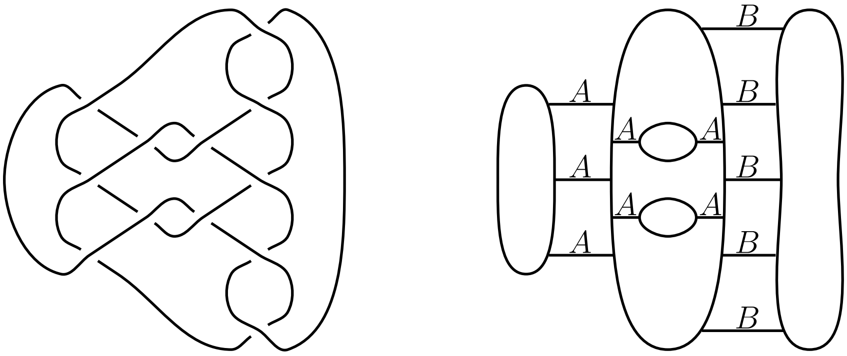

FIGURE 3: The knot $12n0328$ is prime, the Seifert state of this diagram is alternating and not homogenous, and the corresponding state surface is a fiber by Theorem 1.

Furthermore, in the following example the Seifert state is homogeneous but not alternating.

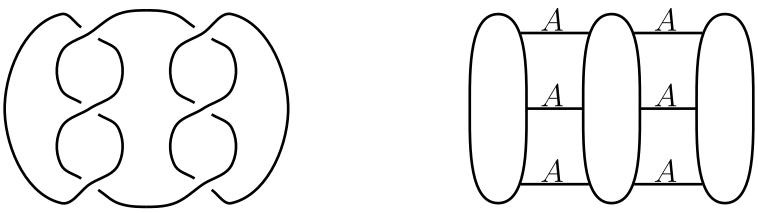

FIGURE 4: The Seifert state of this granny knot diagram is homogenous and not alternating, and the corresponding state surface is a fiber by Theorem 2.

The reduced graph of the state in the examples of Figure 2 is a tree, so in this particular case the state surface is a fiber. We notice that if G_σ has edges with different labels between the same pair of vertices then G'_σ is not a tree and, by Theorem 1, S_σ is not a fiber.

In section 2 we prove this theorem using Murasugi sums and results of Gabai on knot fibration. In section 3 we give a different, homological proof, of the following theorem of Futer [1] on homogeneous states. The techniques used are similar to the ones in the paper [4] by the first author, where he studies the fibration of augmented link complements.

Theorem 2. *Let σ be a homogeneous state of a link diagram D_L. Then $E(L)$ fibers over the circle with fiber S_σ if and only if the reduced graph G'_σ is a tree.*

2. Fibers from alternating states

For this section we use a specific concept of graph decomposition: We say that two vertices, v and w, *decompose* a graph G into components $G_1, \dots, G_k$ if
$G = G_1 \cup \dots \cup G_k$ and $G_i \cap G_j \subseteq \{v, w\}$, for $i \neq j$. We also make use of the the following theorem by Gabai [3] on Murasugi sum and knot fibration.

Theorem 3 (Gabai)**.** *Let $T \subset S^3$, with $\partial T = L$, be a Murasugi sum of oriented surfaces $T_i \subset S^3$, with $\partial T_i = L_i$, for $i = 1, 2$. Then $S^3 - L$ is fibered with fiber T if and only if $S^3 - L_i$ is fibered with fiber T_i for $i = 1, 2$.*

With the following lemma we are able to prove that we neither lose fibration information by working with the reduced state graph nor with graph decomposition.

Lemma 1. *Let G_σ be a state graph and suppose there are two vertices, v and w, adjacent by the edge X, that decompose G_σ into connected components X, H_1, H_2, …, H_k. (See Figure 5.) Consider also the state surface S_i induced by σ and the subgraph $X \cup H_i$ of G_σ, $i = 1, \dots, k$. Then, S_σ is a fiber if and only if each surface $S_1, \dots, S_k$ is a fiber with respect to its boundary.*

Proof: We start by proving that S_σ is a Murasugi sum of the surfaces $S_1, \dots, S_k$. Consider one of the connected components H_l. If H_l contains only one of the vertices v or w, then using the disk associated to this vertex and X we can decompose S_l from S_σ by a Murasugi sum. (See Figure 6.) Notice that S_l is also the state surface of H_l, since X contains a terminal vertex in $X \cup H_l$.

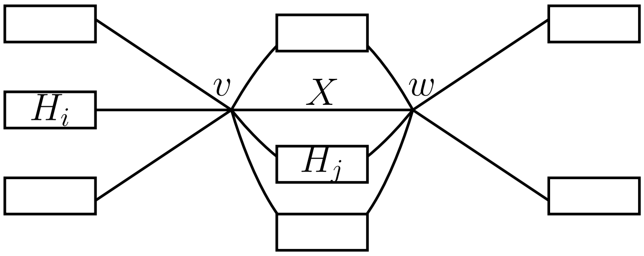

FIGURE 5: Representation of the decomposition of G_σ by $v \cup w$.

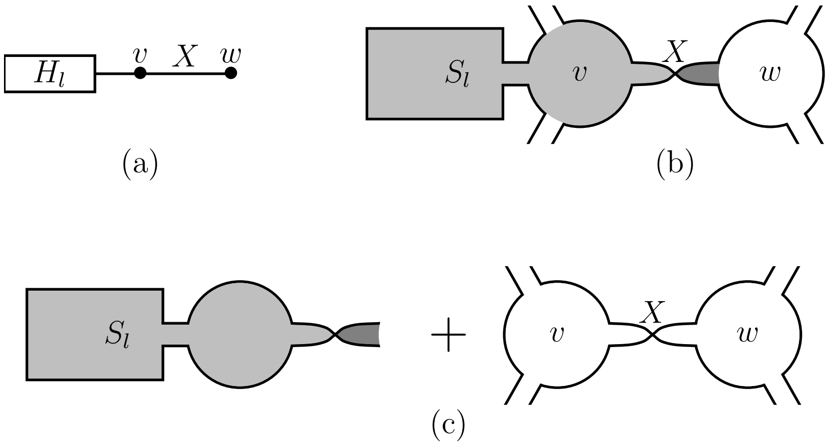

FIGURE 6: When H_l is adjacent to only v, (a), there is a decomposition of S_σ, (b), by $v \cup w$ as a Murasugi sum, (c).

Let us assume now that H_l contains $v \cup w$. Suppose, without loss of generality, that H_l is innermost with respect to X, i.e., there is no other component between H_l and X in the state graph. We can decompose S_l from S_σ by a Murasugi sum as depicted in Figure 7.
Repeating this procedure with subsequent innermost components we obtain the claimed Murasugi sum decomposition. Therefore, by Theorem 3, S_σ is a fiber if and only if each surface $S_l, \ldots, S_k$ is a fiber with respect to its boundary. ■

Corollary 1. *Suppose that there are two edges with end points v and w. If the edges have different labels then the surface S_σ is not a fiber. If the edges have the same label then S_σ is a fiber if and only if the surface obtained by cutting the band corresponding to one of the edges is a fiber.*

Proof: Under the statement of this Corollary, in Lemma 1 one of the surfaces S_i is either an annulus, when the edges have different labels, or a Hopf band,

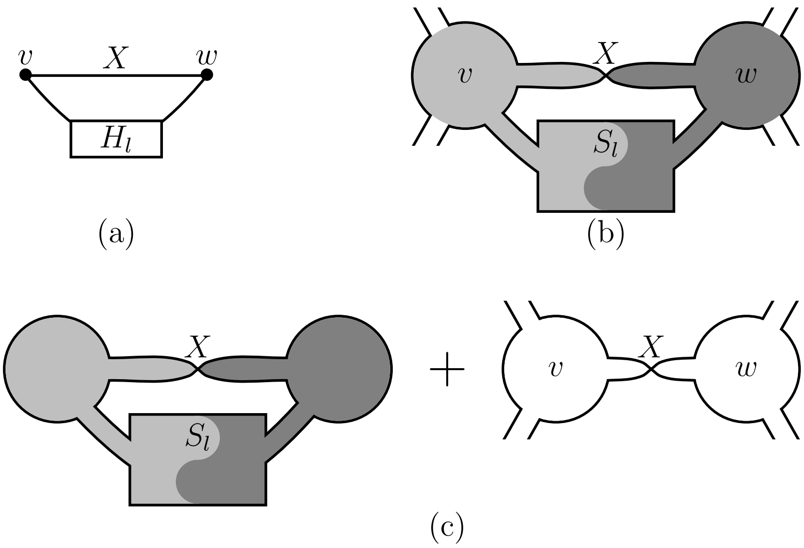

FIGURE 7: When H_l is adjacent to both v and w, (a), there is a decomposition of S_σ, (b), by $v \cup w$ as a Murasugi sum, (c).

when the edges have the same label. In the former case, as an annulus is not a fiber, the surface S_σ is not a fiber; in the latter case, since an Hopf band is a fiber, then S_σ is a fiber if and only if the remaining Murasugi summands are fibers, that is the surface obtained by cutting the band corresponding to one of the edges is a fiber. ■

Remark 1. In light of Corollary 1 we assume from now on that the state graph G_σ has no edges with different labels adjacent to the same pair of vertices.

Let L be a link and σ a state for a diagram D_L of L. We denote by L' the boundary of the reduced surface S'_σ, and we observe that the reduced graph G'_σ is the state graph associated to S'_σ.

Corollary 2. *The link L' is fibered by S'_σ if and only if L is fibered by S_σ.*

Proof: This is a immediate consequence of Corollary 1. ■

Lemma 2. *Let G_σ be a state graph and suppose there are two vertices, v and w, that decompose G_σ into two connected components X and Y, and there is an alternating path α from v to w, in Y, that together with X define an inner cycle. (See Figure 8.) Consider also the state surface S_y induced by σ and Y, and the state surface S_x induced by σ and $X \cup \alpha$. Then, S_σ is a fiber if and only if each surface, S_x and S_y, is a fiber with respect to its boundary.*

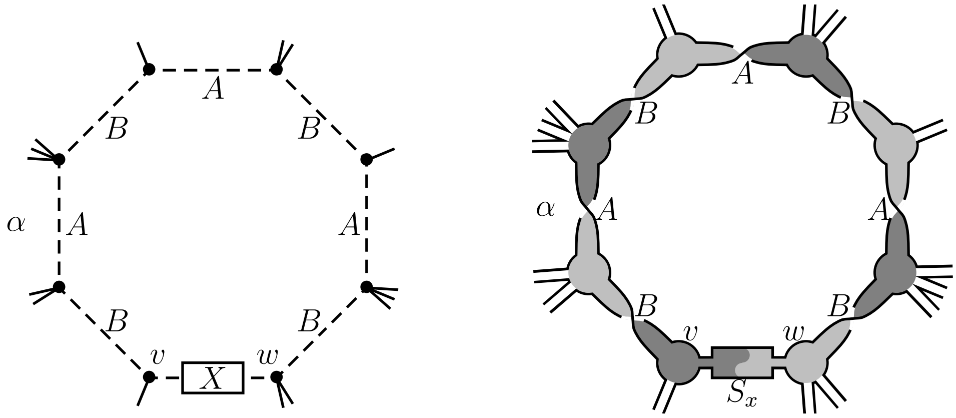

FIGURE 8: An alternating path α in that together with X define an inner cycle.

Proof: Since $X \cup \alpha$ defines an inner cycle and α is alternating, with respect to the labels, then there is a ball Q intersecting S_σ at S_x with the band associated with α in ∂Q. In this way, we can decompose S_σ as a Murasugi sum of S_x and S_y, as depicted in Figure 9.

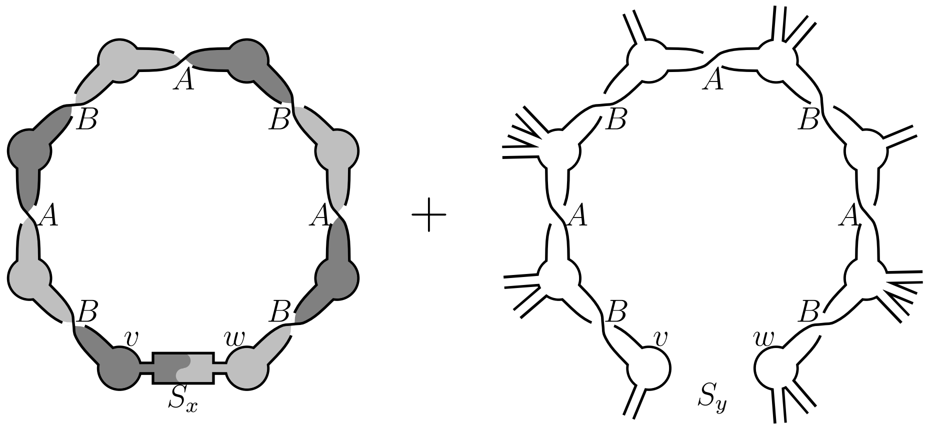

FIGURE 9: Decomposition of S_σ by α as a Murasugi sum of S_x and S_y.

From the result of Gabai and this Murasugi sum we have the statement of the lemma. ■

Lemma 3. *If the state graph G_σ has an inner cycle that is alternating with respect to the labels A or B then S_σ is not a fiber of L.*

Proof: Consider an inner cycle γ of G_σ. In Lemma 2, let X be one edge of γ and α the remaining edges. Then, S_σ is a fiber if and only if S_x and S_y are fibers. Since γ is alternating then S_x is an annulus, which is not a fiber of its boundary. Hence, S_σ is not a fiber of L. ■

Proof of Theorem 1: We start by observing that if G'_σ is a tree then S'_σ is a disk, and hence a fiber of L'. Therefore, by Corollary 2 L is fibered by S_σ. Suppose now that G'_σ has a cycle. Then G_σ has also a cycle. Consider an inner cycle α of G_σ. Two consecutive edges of α are also consecutive in the common vertex. Since σ is alternating then these two edges have different resolutions. Hence, α is alternating. Consequently, by Lemma 3 the state surface S_σ is not a fiber of L. ■

3. A new proof of Theorem 2

In this section we present a different proof of Theorem 2. This is the main theorem in [1], where it is proved inductively via Murasugi sums together with Theorem 3 to deduce fibering information. Some of these ideas were also independently used in the work of the first author [4] and in the previous section. The proof we present is a consequence of Stallings' fibration criteria [5].

Theorem 4 (Stallings). *Let $T \subset S^3$ be a compact, connected, oriented surface with nonempty boundary ∂T. Let $T \times [-1, 1]$ be a regular neigborhood of T and let $T^+ = T \times \{1\} \subset S^3 - T$. Let $f = \varphi|_T$, where $\varphi : T \times [-1, 1] \longrightarrow T^+$ is the projection map. Then T is a fiber for the link ∂T if and only if the induced map $f_* : \pi_1(T) \longrightarrow \pi_1(S^3 - T)$ is an isomorphism.*

We describe the induced map in the case T is the state surface associated to the reduced graph of a homogeneous link. We will see that when G'_σ is a tree, the reduced surface S'_σ is a disk and the map f_* is trivial, as desired. When G'_σ has cycles, we show that the map f_* cannot be an isomorphism by showing that the corresponding map on first homology is not an isomorphism. By decomposing the homogeneous link along cut vertices in the graph, we only need to prove this result for all-A or all-B states. We provide the proof for the case of an all-A state, the other case being similar.

First note that in the absence of cut vertices in the graph G'_σ, the surface S'_σ is a checkerboard surface. If the graph G'_σ is a tree, then the surface S'_σ is a disk. Hence S'_σ is a fiber, and by Corollary 2 the surface S_σ is also a fiber.

Suppose now that S_σ is a fiber but G'_σ has cycles. We will prove that this contradicts Stalling's theorem. First note that the fundamental group of the surface S'_σ is free. Consider the inner cycles $\alpha_1, ..., \alpha_n$ in G'_σ oriented in the counter-clockwise direction. Since S_σ is a fiber, it is orientable, hence S'_σ is also orientable and we choose a base point a of $\pi_1(S'_\sigma)$ such that, when seen from above the projecting plane, we see the base point a in the "+" side of S'_σ. Finally, add arcs $h_1, ..., h_n$ from a to each of the inner cycles above. This gives loops $\beta_i = h_i \alpha_i h_i^{-1}$, based at a. This set of based loops corresponds to a generating set for $\pi_1(S'_\sigma)$. These generators will be denoted by $u_1, ..., u_n$.

Since the surface S'_σ is a checkerboard surface, its complement $S^3 - S'_\sigma$ also has a free fundamental group. We now describe a generating set for this group. There are two types of white regions in the projecting plane: one unbounded region and n bounded ones, which correspond to the inner cycles of G'_σ. Let C_0 denote the unbounded white region determined by S'_σ and let A_i denote a white region determined by the inner cycle α_i. Let $\gamma_i \subset S^3 - S'_\sigma$ be a semi-circle with one endpoint in C_0 and the other in A_i, lying under the projecting plane. Let $f : S'_\sigma \longrightarrow S^3 - S'_\sigma$ be the function described in theorem 4. Associated to each region A_i we construct a simple closed curve by connecting the endpoints of the arc γ_i to the point $f(a)$ by straight line segments. Each of these curves is oriented so that, starting at $f(a)$, we move along the line segment connecting $f(a)$ to the endpoint of γ_i in A_i, then move along γ_i to the second endpoint and then back to $f(a)$ through the second line segment. We have built loops with base point $f(a)$ corresponding to a set of generators for $\pi_1(S^3 - S'_\sigma)$. These generators are denoted by $x_1, ..., x_n$, according to the label of region they cross.

Let S'^{+}_σ be the copy of S'_σ in $S^3 - S'_\sigma$ parallel to S'_σ, obtained from S'_σ by pushing it in the "+" direction. This is formally defined by the map $f : S'_\sigma \longrightarrow S^3 - S'_\sigma$ described in Theorem 4. The induced map f_* can be described by determining the image of each generator $u_i \in \pi_1(S'_\sigma)$. We write $f_*(u_i)$ as a word on the generators $x_1, ..., x_n$, given by the image the loop $\beta_i = h_i \alpha_i h_i^{-1}$:

$$f_*(u_i) = w_{h_i} w_{\alpha_i} w_{h_i}^{-1}$$

where w_{h_i} is the word on the letters $x_1, ..., x_n$ given by the image of the arcs h_i under the map f. The word w_{α_i} is obtained by the image of the cycle α_i as follows. Suppose that α_i and α_j have a common edge. Vertices are labeled "+" or "−", depending on the side of the surface they lie. We have two possibilities:

Case 1. The orientation induced on the edge by α_i is from a "+" vertex to a "−" vertex. In this case we write the letter x_i. (See Figure 10 left.)

Case 2. The orientation induced on the edge by α_i is from a "$-$" vertex to a "$+$" vertex. In this case we write the letter x_j^{-1}. (See Figure 10 right.)

FIGURE 10: Case 1 (left); case 2 (right).

Remark 2. It is important to notice the inner cycle α_i may share an edge with the unbounded region C_0. If this is the case, in 2 above, we write no letters corresponding to this edge.

Remark 3. Observe that the loops α_i and α_j induce reverse orientations on the edges they share. Therefore, when we write the letters corresponding to the loop α_j, the letter corresponding to this edge is the same letter as α_i, with opposite sign, i.e., either x_i^{-1} or x_j. This is illustrated in figure 10.

Now we consider the map $\bar{f}_* : H_1(S'_\sigma) \longrightarrow H_1(S^3 - S'_\sigma)$ induced on homology by f_*. Denote by $\bar{u}_1, ..., \bar{u}_n$ the generators of $H_1(S'_\sigma)$, corresponding to the generators of $\pi_1(S'_\sigma)$. The generators of $H_1(S^3 - S'_\sigma)$ are defined similarly and denoted $\bar{x}_1, ..., \bar{x}_n$.

The map $\bar{f}_*$ is given by a $n \times n$ matrix $\mathcal{A} = [a_{ij}]$, where the i-th column is the vector $\bar{f}(\bar{u}_i) \in H_1(S^3 - S'_\sigma)$. By the description of the map f_* and the remarks above, the matrix $\mathcal{A}$ has the following properties:

(i) $a_{ii} \geq 2$;

(ii) $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$

(iii) $a_{ii} \geq \sum_{j \neq i} |a_{ji}|$

(i)follows from the fact that every inner cycle in G'_σ has at least 4 edges; (ii) and (iii) follow from the fact that, when we go through the cycle α_i, at every other edge we write the letter x_i and at the remaining edges we write one of the other letters x_j or write no letters (as in remark 2).

To prove that the map f_* is not an isomorphism if G'_σ is not a tree (i.e., has cycles), it suffices to prove the matrix $\mathcal{A}$ is not invertible over $\mathbb{Z}$. This is straightforward:

Theorem 5. *Let* $\mathcal{A} = [a_{ij}]$ *be such that* $a_{ii} \geq \max(2, \sum\limits_{j \neq i} |a_{ij}|), \forall i \in \{1, \ldots, n\}$. *If* $\det(\mathcal{A}) \neq 0$*, then* $\det(\mathcal{A}) \geq 2$ *and this inequality is sharp.*

Proof: We will prove the theorem by induction on n.

For $n = 1$, $\det(\mathcal{A}) = \det[a_{11}] = a_{11} \geq 2$.

Consider now any $n \in \mathbb{N}$ and suppose that the result is true for $n - 1$.

Suppose $\det(\mathcal{A}) \neq 0$ and let $\mathcal{B} = [b_{ij}] \in M_n(\mathbb{Z})$ be the adjugate matrix of $\mathcal{A}$. Then $\mathcal{A}\mathcal{B} = (\det \mathcal{A})\mathcal{I}_n$.

If all elements of the column j of $\mathcal{B}$ have the same absolute value b_{jj}, then $\det(\mathcal{A}) = \sum_k a_{ij} b_{ji}$ is a multiple of $b_{jj} \geq 2$.

If not, suppose $|b_{ij}| \geq |b_{kj}|, \forall k \in \{1, \ldots, n\}$ and $|b_{ij}| > |b_{kj}|$ for some k. Then

$$\left|\sum_{k=1}^{n} a_{ik} b_{kj}\right| = \left|a_{ii} b_{ij} + \sum_{k \neq i} a_{ik} b_{kj}\right| \geq |a_{ii} b_{ij}| - \sum_{k \neq i} |a_{ik} b_{kj}| > |a_{ii} b_{ij}| - |a_{ii} b_{ij}| = 0.$$

Since $\mathcal{A}\mathcal{B}$ is a diagonal matrix, then $i = j$. Therefore, $|b_{ii}| > |b_{ki}|, \forall k \neq i$. Furthermore, by the induction hypothesis, $b_{ii} \geq 2$. Hence

$$\det(\mathcal{A}) = \left|\sum_{k=1}^{n} a_{ik} b_{ki}\right| = \left|a_{ii} b_{ii} + \sum_{k \neq i} a_{ik} b_{ki}\right| \geq |a_{ii} b_{ii}| - \sum_{k \neq i} |a_{ik} b_{ki}| \geq$$

$$\geq a_{ii} b_{ii} - a_{ii}(b_{ii} - 1) = a_{ii} \geq 2.$$

To see that the inequality is sharp, observe that the determinant of the $n \times n$ tridiagonal matrix

$$\mathcal{A} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$$

is 2, for every $n \in \mathbb{N}$. ∎

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