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NOTES ON POINT-FREE REAL FUNCTIONS AND SUBLOCALES

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Dedicated to Manuela Sobral

Abstract: Using the technique of sublocales we present a survey of some known facts (with a few new ones added) on point-free real functions. The subjects treated are, e.g., images and preimages, semicontinuity, algebraic structure (point-free real arithmetics), zero and cozero parts, z-embeddings, z-open and z-closed maps, disconnectivity, small sublocales and supports.

Keywords: Frame, locale, sublocale lattice, localic map, frame of reals, real function, upper semicontinuous, lower semicontinuous, ring of continuous functions, completely separated sublocales, z-embedding, z-open map, z-closed map, perfectly normal frame, small sublocale, support.

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Introduction

The frame of reals, $\mathfrak{L}(\mathbb{R})$ ("point-free real numbers"), was originally introduced by Joyal in an unpublished manuscript [23] and thoroughly studied by Banaschewski in [3] (see also Johnstone [22]). As one might expect, it was not defined as the lattice $\Omega(\mathbb{R})$ of open sets in the standard real line $\mathbb{R}$ but as a primarily algebraic entity, the free frame generated by pairs of rational numbers (which one can intuitively view as rational intervals) factorized by natural relations (see 2.3 below). Under the Axiom of Choice, $\mathfrak{L}(\mathbb{R})$ is indeed isomorphic with $\Omega(\mathbb{R})$, but the point is to have the frame of point-free reals as a frame in its own right and to be able to avoid choice whenever possible (it should be noted that one can prove in a choice-free way for instance that

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$\mathfrak{L}(\mathbb{R})$ is the completion of the frame of rationals or that it is continuous, that is, locally compact, see [3]).

Once one has the frame of real numbers, one can also represent continuous real functions on a general frame L, namely as frame homomorphisms $h\colon \mathfrak{L}(\mathbb{R}) \to L$. This was originally done by Banaschewski ([3] – see [26] for further references). However, the classical theory of real functions, not necessarily continuous, calls for a point-free counterpart as well. An appropriate definition was presented in [15] and developed in subsequent papers (e.g. [17, 6]). A classical (general) real function on a space $(X, \Omega(X))$ is a continuous real function on the discrete space $(X, \mathfrak{P}(X))$. The lattice $\mathfrak{P}(X)$ of all subsets of X has a natural counterpart in $\mathcal{S}(L)^{\mathrm{op}}$ where $\mathcal{S}(L)$ is the co-frame of all sublocales of L. Hence, a (general) real function on L can be represented as a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{\mathrm{op}}$.

The present paper is inspired by [25]. Using extensively the technique of sublocales, we present a survey of some facts on point-free real functions. Most of the results are not new; the originality is essentially in the presentation. Our main goal is to show how zero sets may be considered in the localic setting (as *zero sublocales*) and then how several important notions and results about real functions may be rewritten and directly proved using this tool.

After some necessary preliminaries we introduce the point-free real functions and prove a few facts, in particular some results concerning images and preimages of sublocales are discussed. Then, semicontinuous functions and their relation with the continuous ones are mentioned. In the following section, point-free algebraic operations on $\mathfrak{L}(\mathbb{R})$ are studied, with special attention paid to the addition, multiplication, maximum and minimum. Next we turn to cozero and zero sublocales. The concept of *cozero element* is a well-known standard topic and its sublocale counterpart is straightforward, but there are no reasonable *zero elements* while in the context of sublocales we obtain a sensible notion. This approach allows to formulate the basics of the theory in a way very much parallel to the classical book of Gillman and Jerison [12]. We illustrate this in a miscellany of topics.

Regarding general background, we refer to Picado and Pultr [26] for frames and locales and to Banaschewski [3] and Ball and Walters-Wayland [1] for specific information on continuous functions on frames.

1. Preliminaries I. Free constructions

We will work with point-free real numbers as they are usually described in literature, that is, by generators subject to relations. Since the free generators come from a set that is in fact a meet-semilattice (while its elements are used in the free construction simply as elements of a set) we think that it may be useful for the reader to confront the free frames over sets with free frames over semilattices.

1.1. Free semilattice with 1. For a set X define

$$F(X) = \{A \subseteq X \mid A \text{ finite}\}$$

ordered by $\leq\, = \,\supseteq$ so that we have the meet $A \wedge B = A \cup B$. Denote by β_X the mapping

$$\beta_X = (x \mapsto \{x\})\colon X \to F(X).$$

Then we have for each meet-semilattice S with 1 and each mapping $f\colon X \to S$ precisely one meet-semilattice homomorphism $\overline{f}\colon F(X) \to S$ such that $\overline{f}\beta_X = f$ and $\overline{f}(\emptyset) = 1$, namely the homomorphism defined by $\overline{f}(A) = \bigwedge_{x\in A} f(x)$.

1.2. Free frame generated by a semilattice with 1. For a meet-semilattice S with 1 set

$$\mathfrak{D}(S) = \{U \subseteq S \mid {\downarrow}U = U \neq \emptyset\}.$$

$\mathfrak{D}(S)$ is a frame with unions for joins and intersections for meets and if we denote by α_S the mapping

$$\alpha_S = (s \mapsto {\downarrow}s)\colon S \to \mathfrak{D}(S)$$

we have a meet-semilattice homomorphism such that for each frame L and each meet-semilattice homomorphism $h\colon S \to L$ there is precisely one frame homomorphism $\widetilde{h}\colon \mathfrak{D}(S) \to L$ such that $\widetilde{h}\alpha_S = h$, namely that defined by $\widetilde{h}(U) = \bigvee_{s\in U} h(s)$.

1.2.1. The free frame over a set can be now obtained combining F and $\mathfrak{D}$, that is, as $\mathfrak{D}F(X)$.

1.3. Free frames over a set and over a meet-semilattice confronted. Now suppose we have a construction of a frame based on a set which is endowed by a meet-semilattice structure. We will compare the free constructions as over the carrier $|S|$ and the one based directly on the semilattice S.

We will use the standard factorization procedure as e.g. in [26, III.11]. On $\mathfrak{D}F(|S|)$ define a relation

$$M = \{(\downarrow A, \downarrow B) \mid \bigwedge A = \bigwedge B \text{ in } S\}$$

and write $\kappa\colon \mathfrak{D}F(|S|) \to \mathfrak{D}F(|S|)/M$ for the quotient map. Consider the following diagram:

$$\begin{tikzcd}
{|S|} \arrow[r, "\beta_{|S|}"] \arrow[d, "\mathrm{id}"'] & F(|S|) \arrow[r, "\alpha_{F(|S|)}"] \arrow[d, "h=\overline{\mathrm{id}}"] & \mathfrak{D}F(|S|) \arrow[dl, "\widetilde{h}"'] \arrow[d, "\kappa"] \\
S \arrow[r, "\alpha_S"] \arrow[rr, bend right, "f"'] & \mathfrak{D}(S) \arrow[r, shift right, "\psi"'] & \mathfrak{D}F(|S|)/M \arrow[l, shift right, "\phi"']
\end{tikzcd}$$

Since $\widetilde{h}$ obviously respects the relation M we have a frame homomorphism ϕ such that $\phi\kappa = \widetilde{h}$. Further, define a mapping

$$f\colon S \to \mathfrak{D}F(|S|)/M$$

by setting $f(s) = \kappa(\downarrow\{s\})$. By the definition of M, f is a meet-semilattice homomorphism and hence there is a frame homomorphism ψ such that $\psi\alpha_S = f$. Now we have

$$\begin{aligned}
\phi\psi\,(\downarrow s) &= \phi\psi\alpha_S(s) = \phi f(s) = \phi\kappa\,(\downarrow\{s\}) = \widetilde{h}\,(\downarrow\{s\}) = \\
&= \widetilde{h}\alpha_{F(|S|)}(\{s\}) = h(\{s\}) = h\beta_{|S|}(s) = \alpha_S(s) = \downarrow s \quad \text{and} \\
\psi\phi\,(\kappa\,(\downarrow\{s\})) &= \psi\widetilde{h}\,(\downarrow\{s\}) = \psi\widetilde{h}\alpha_{F(|S|)}(\{s\}) = \psi\alpha_S(s) = f(s) = \kappa\,(\downarrow\{s\})
\end{aligned}$$

so that $\phi\psi$ and $\psi\phi$ are identical on systems of generators and hence ϕ and ψ are mutually inverse homomorphisms.

Thus, if we represent a construction based on factorizing $\mathfrak{D}(S)$ identifying pairs from a relation R as a free construction on $|S|$ we only have to consider the relation $R \cup M$ with the M as above instead of R.

2. Preliminaries II

2.1. Localic maps. Since frames can be viewed as a natural generalization of spaces while the natural functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ from the category of topological spaces into that of frames ($\Omega(f)$ sending an open set $U \subseteq Y$ to $f^{-1}[U]$ for any morphism $f\colon X \to Y$ in $\mathbf{Top}$) is contravariant, one introduces the *category of locales* $\mathbf{Loc}$ as the dual of the category of frames (then the natural open set functor becomes even a full embedding on the important subcategory of sober spaces). Often one just considers the formal $\mathbf{Frm}^{\mathrm{op}}$ but it is of advantage to represent it as a concrete category with morphisms as well-defined maps. For this purpose one defines a *localic map* $f\colon L \to M$ as the right Galois adjoint of a frame homomorphism $h = f^*\colon M \to L$. This can be done since frame homomorphisms preserve suprema; but of course not every mapping preserving infima is a localic one. Here is a characterization (see [26] or [25]).

Let $f\colon L \to M$ have a left adjoint $f^*\colon M \to L$. Then it is a localic map iff

(1) $f[L \smallsetminus \{1\}] \subseteq M \smallsetminus \{1\}$, *and*
(2) $f(f^*(a) \to b) = a \to f(b)$

($\to$ is the Heyting operation in the frames L resp. M).

2.2. The frame of sublocales. A *sublocale* of a frame L is a subset $S \subseteq L$ such that

(1) $M \subseteq S$ implies $\bigwedge M \in S$, and
(2) if $a \in L$ and $s \in S$ then $a \to s \in S$.

This concept expresses the intuition of a natural subobject of L understood as a generalized space; in the category of locales and localic maps the inclusion $j\colon S \subseteq L$ is a localic extremal monomorphism, hence indeed a sub-locale (in the frame perspective, it is the image of a nucleus). The set of all sublocales ordered by inclusion, denoted by

$$\mathcal{S}(L),$$

is a co-frame, with the lattice operations

$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i \quad \text{and} \quad \bigvee_{i\in J} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in J} S_i\}.$$

We have the closed resp. open sublocales

$$\mathfrak{c}(a) = {\uparrow} a \quad \text{resp.} \quad \mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\}$$

modelling closed resp. open subspaces. They are complements of each other, and the $\mathfrak{o}(a)$ are in a natural one-one correspondence with the elements of L, preserving joins and finite meets.

We will need, rather, the opposite of $\mathcal{S}(L)$, the *frame of sublocales*, denoted

$$\mathfrak{S}(L),$$

with $S \leq T$ iff $S \supseteq T$ and

$$\bigvee_{i\in J} S_i = \bigcap_{i\in J} S_i \quad \text{and} \quad \bigwedge_{i\in J} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in J} S_i\}.$$

Note that $\mathfrak{S}(L)$ is isomorphic with the frame of congruences on L and we have a natural frame embedding

$$\mathfrak{c}_L\colon L \to \mathfrak{S}(L) \quad (a \mapsto \mathfrak{c}(a)). \tag{2.2.1}$$

This means that there is a one-one correspondence between the elements of L and the closed sublocales of L, agreeing in arbitrary joins and finite meets in L and $\mathfrak{S}(L)$.

For our purposes it is particularly important that the operation $\mathfrak{S}$ can be iterated.

2.2.1. Note. Introducing the frame $\mathfrak{S}(L)$ may seem to be just an ad hoc inversion of the order for technical purposes, but it is not so. The co-frame and frame of sublocales are in fact two entities with different roles.

The co-frame $\mathcal{S}(L)$ corresponds to the collection $\mathrm{Sub}(X)$ of subspaces of a space in classical topology which is naturally worked with as with a co-frame. It is, of course, a Boolean algebra, that is, both a co-Heyting and Heyting one, but note that when computing with subspaces the co-Heyting operation of difference $B \smallsetminus A$ is ubiquitous while the Heyting $B \to A = A \cup (X \smallsetminus B)$ is hardly ever employed.

On the other hand, $\mathfrak{S}(L)$ represents a (generalized) *space* of sublocales, loosely analogous to the idea of a space of subspaces. Now our "generalized space of generalized subspaces" is universal, while the classical spaces of subspaces vary according to their purpose and usually concern special subspaces only. But anyway we have here a generalized space $\mathfrak{S}(L)$ naturally extending the original L (see the frame embedding above; from the covariant point of view we have a natural localic quotient $(S \mapsto \bigwedge S)\colon \mathfrak{S}(L) \to L$).

2.2.2. Interior and closure. Originally, the closure resp. interior of a sublocale S in $\mathcal{S}(L)$ is the smallest closed sublocale containing S, that is,

$$\overline{S} = \bigcap\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\} = \uparrow \bigwedge S \ (= \mathfrak{c}(\bigwedge S))$$

resp. the largest open sublocale contained in S, that is,

$$S^\circ = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\}.$$

We work, however, in $\mathfrak{S}(L)$ and hence we should not forget that here

$$\overline{S} \leq S \leq S^\circ.$$

Note that $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$ and $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ (see [26, III.6 and III.8]).

Further, we recall that the *rather below* relation $\prec$ in a frame is defined by $a \prec b \equiv a^* \vee b = 1$. Note that for any $a, b \in L$,

$$a \prec b \quad \text{iff} \quad \mathfrak{c}(a)^\circ \leq \mathfrak{c}(b) \quad \text{iff} \quad \overline{\mathfrak{o}(a)} \geq \mathfrak{o}(b). \tag{2.2.2}$$

Indeed: $a \prec b$ iff $a^* \vee b = 1$ iff $\mathfrak{o}(a^*) \wedge \mathfrak{o}(b) = 0$ iff $\mathfrak{c}(a)^\circ \wedge \mathfrak{o}(b) = 0$ iff $\mathfrak{c}(a)^\circ \leq \mathfrak{c}(b)$; furthermore, $\mathfrak{c}(a)^\circ \leq \mathfrak{c}(b)$ iff $\mathfrak{o}(a^*) \leq \mathfrak{c}(b)$ iff $\mathfrak{c}(a^*) \geq \mathfrak{o}(b)$ iff $\overline{\mathfrak{o}(a)} \geq \mathfrak{o}(b)$.

The completely below relation $\prec\!\!\prec$ is the interpolative modification of the rather below relation ([8]). Elements $a, b \in L$ satisfy $a \prec\!\!\prec b$ if and only if there exists a subset $\{a_q \mid q \in [0,1] \cap \mathbb{Q}\} \subseteq L$ with $a_0 = a$ and $a_1 = b$ such that $a_p \prec a_q$ whenever $p < q$ in $[0,1] \cap \mathbb{Q}$.

2.2.3. Images and preimages. Let $f\colon L \to M$ be a localic map. If $S \subseteq L$ is a sublocale then the standard set-theoretical image $f[S]$ is a sublocale of M. The set-theoretical preimage $f^{-1}[S]$ of a sublocale is not necessarily a sublocale, but there is the largest sublocale contained in $f^{-1}[S]$ which we denote by $f_{-1}[S]$ and refer to it as the (sublocale) preimage (see [25]). In $\mathcal{S}(L)$ we have the Galois adjunction

$$f[S] \subseteq T \quad \text{iff} \quad S \subseteq f_{-1}[T]$$

which in $\mathfrak{S}(L)$ becomes

$$f_{-1}[T] \leq S \quad \text{iff} \quad T \leq f[S]. \tag{$\mathfrak{S}$-image-preimage}$$

$f_{-1}[-]\colon \mathfrak{S}(M) \to \mathfrak{S}(L)$ is a frame homomorphism that preserves complements and assigns closed resp. open sublocales to closed resp. open sublocales. In more detail, $f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$ and $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$. By the adjunction ($\mathfrak{S}$-image-preimage), $f[-]\colon \mathfrak{S}(L) \to \mathfrak{S}(M)$ is the associated localic map. (See, e.g., [25, Ch. 7] or [26, III.6,9].)

We will use the symbol $g[A]$ also for set-theoretic image of any function and any set; since our $f[S]$ coincides with the set-theoretic image, there is no danger of confusion. It is, however, necessary to be careful with $f_{-1}[-]$ and $f^{-1}[-]$.

2.3. The frame of reals. Considering the standard order in $\mathbb{Q}$ we will define the *frame of reals*, denoted

$$\mathfrak{L}(\mathbb{R}),$$

as $\mathfrak{D}(\mathbb{Q}^{\mathrm{op}} \times \mathbb{Q})/R$ where R consists of all pairs

- $({\downarrow}(p, q) \cup {\downarrow}(r, s), {\downarrow}(p, s))$ with $p \leq r < q \leq s$,
- $({\downarrow}(p, q), \bigcup\{(r, s) \mid p < r < s < q\})$ and
- $(\bigcup_{p,q\in\mathbb{Q}}(p, q), 1)$.

All ${\downarrow}(r, s)$ are saturated; encoding them simply as (r, s) we can think of the *frame of reals* as in [3], that is, as of a frame generated by all ordered pairs $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ satisfying the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.
(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

(Note that (R1) plays the role of the relation M from 1.3.)

Equivalently, $\mathfrak{L}(\mathbb{R})$ may be defined as the frame with generators of the form $(p, –)$ and $(–, q)$, $p, q \in \mathbb{Q}$, subject to the relations

(r1) $(p, –) \wedge (–, q) = 0$ whenever $p \geq q$.
(r2) $(p, –) \vee (–, q) = 1$ whenever $p < q$.
(r3) $(p, –) = \bigvee_{r>p}(r, –)$, for every $p \in \mathbb{Q}$.
(r4) $(–, q) = \bigvee_{s<q}(–, s)$, for every $q \in \mathbb{Q}$.
(r5) $\bigvee_{p\in\mathbb{Q}}(p, –) = 1$.
(r6) $\bigvee_{q\in\mathbb{Q}}(–, q) = 1$.

With $(p, q) = (p, –) \wedge (–, q)$ one goes back to (R1)-(R4).

2.4. Coproducts of frames. For details about coproducts of frames see, e.g., [26] (or [22]). For us it is important that such a coproduct $L \oplus M$ with the coproduct injections $\iota_L : L \to L \oplus M$, $\iota_M : M \to L \oplus M$ is generated by the elements $a \oplus b = \iota_L(a) \wedge \iota_M(b)$ such that

- $\bigvee(a_i \oplus b) = (\bigvee a_i) \oplus b$ and $\bigvee(a \oplus b_i) = a \oplus (\bigvee b_i)$,

- $(a_1 \oplus b_1) \wedge (a_2 \oplus b_2) = (a_1 \wedge a_2) \oplus (b_1 \wedge b_2)$, and
- for $a, b, c, d \neq 0$, $a \oplus b \leq c \oplus d$ iff $a \leq c$ and $b \leq d$.

2.5. Algebraic operations. An n-ary algebraic operation on a locale L is, in frame language, a frame homomorphism

$$\omega\colon L \to \underbrace{L \oplus \cdots \oplus L}_{n \text{ times}}$$

where $\oplus$ designates the coproduct in **Frm**. Thus in particular we have operations on point-free reals

$$\omega\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R}). \tag{$*$}$$

It should be explained how thus defined operations are related to the operations on classical real numbers.

The functor $\Omega\colon$ **Top** $\to$ **Loc** does not generally preserve products (that is, does not necessarily send products in **Top** to coproducts in **Frm**). If we take a product of topological spaces $p_i\colon X \times \cdots \times X \to X$, $i = 1, \ldots, n$, we have in the diagram

$$\begin{tikzcd}
\Omega(X) \oplus \cdots \oplus \Omega(X) \arrow[rr, "\pi"] & & \Omega(X \times \cdots \times X) \\
& \Omega(X) \arrow[ul, "\iota_i"] \arrow[ur, "\Omega(p_i)"'] &
\end{tikzcd}$$

a unique frame homomorphism π satisfying $\pi\iota_i = \Omega(p_i)$. It is quite a nice homomorphism being onto and dense. Hence, in the localic language, π embeds $\Omega(X \times \cdots \times X)$ into $\Omega(X) \oplus \cdots \oplus \Omega(X)$ as a dense sublocale. But it is not an isomorphism and hence an algebraic operation $\alpha\colon X^n \to X$ does not necessarily translate into an algebraic operation

$$\Omega(\alpha)\colon \Omega(X) \to \Omega(X) \oplus \cdots \oplus \Omega(X).$$

Luckily enough, the space of reals $\mathbb{R}$ is locally compact and finite products of locally compact spatial locales coincide with the topological products. This holds under the Axiom of Choice, which also implies an isomorphism $\mathfrak{L}(\mathbb{R}) \cong \Omega(\mathbb{R})$. Without any choice principle it has been proved in [3] that $\mathfrak{L}(\mathbb{R})$ is continuous (locally compact). This justifies thinking of the operations in $\mathfrak{L}(\mathbb{R})$ defined in $(*)$ above as of counterparts of the classical ones.

2.6. Cozero elements. Of central importance in the theory of continuous real functions in a frame L are the cozero elements of L. A *cozero element* of L is an element of the form $h((-,0) \vee (0,-))$ for some frame homomorphism $h\colon \mathfrak{L}(\mathbb{R}) \to L$, and for any such map we refer to $h((-,0) \vee (0,-))$ as $\operatorname{coz} h$.

It is good to know that cozero elements can be alternatively described without reference to the frame of reals as follows [5]: $a \in L$ is a cozero element if and only if $a = \bigvee_{n\in\mathbb{N}} a_n$ for some $a_n \prec\!\prec a$, $n = 1, 2, \ldots$ (equivalently, $a_n \prec\!\prec a_{n+1}$; note that here the plain $\prec$ suffices).

3. Real functions

3.1. The adjoint situation between **Top** and **Loc** yields in particular a natural isomorphism

$$\mathbf{Loc}(\Omega(X), \mathfrak{L}(\mathbb{R})) \xrightarrow{\sim} \mathbf{Top}(X, \Sigma\mathfrak{L}(\mathbb{R}))$$

for each space X. Since $\Sigma\mathfrak{L}(\mathbb{R})$ is homeomorphic with the usual space of reals (that we denote by $\mathbb{R}$) we obtain

$$\mathbf{Loc}(\Omega(X), \mathfrak{L}(\mathbb{R})) \xrightarrow{\sim} \mathbf{Top}(X, \mathbb{R}).$$

This means that the continuous real-valued functions on X are completely described by localic maps $\Omega(X) \to \mathfrak{L}(\mathbb{R})$ and motivates (and justifies) introducing a continuous real function on a general frame L as a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$ (see [3]).

What about arbitrary members of $\mathbb{R}^X$, that is, *arbitrary*, not necessarily continuous, real functions on X?

First, we should notice that each element of $\mathbb{R}^X$ is automatically a continuous real function if we regard X with the discrete topology and therefore that there is a bijection between $\mathbb{R}^X$ and $\mathbf{Top}((X, \mathfrak{P}(X)), \mathbb{R})$, and thus with

$$\mathbf{Loc}(\mathfrak{P}(X), \mathfrak{L}(\mathbb{R})).$$

Now, for a general frame L, the role of the lattice $\mathfrak{P}(X)$ of all subspaces of X should be taken by the frame $\mathfrak{S}(L)$ of all sublocales of L. This justifies thinking of localic maps $\mathfrak{S}(L) \to \mathfrak{L}(\mathbb{R})$ as of *general real functions* on L (introduced in [15] as frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \mathfrak{S}(L)$).

Given a real function $f\colon \mathfrak{S}(L) \to \mathfrak{L}(\mathbb{R})$, we will denote by

$$f^*\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{S}(L)$$

the associated frame homomorphism (recall 2.1). The set of all real functions on L will be denoted by $\mathrm{F}(L)$.

3.1.1. Recall 2.1: any $f \in \mathrm{F}(L)$ preserves arbitrary meets including 1, we have $f(S) = 1$ only if $S = \{1\}$, and $f(f^*(p,q) \to S) = (p,q) \to f(S)$.

3.2. In accordance with the classical notation used for real functions ([12]) we also denote $f^*(p,\text{—})$ and $f^*(\text{—},q)$ by

$$[f > p] \quad \text{and} \quad [f < q]$$

respectively. We immediately see that for each $p, q \in \mathbb{Q}$,

$$[f > p] = \bigwedge\{S \in \mathfrak{S}(L) \mid f(S) \geq (p,\text{—})\}$$

and

$$[f < q] = \bigwedge\{S \in \mathfrak{S}(L) \mid f(S) \geq (\text{—},q)\}.$$

3.2.1. The following facts follow immediately from relations (r1)-(r6).

Facts. For every $f \in \mathrm{F}(L)$ and $p, q \in \mathbb{Q}$ we have that

(1) $p \geq q \Rightarrow [f < q] \wedge [f > p] = 0$.
(2) $p < q \Rightarrow [f > p] \vee [f < q] = 1$.
(3) $\bigvee_{r>p}[f > r] = [f > p]$.
(4) $\bigvee_{s<q}[f < s] = [f < q]$.
(5) $\bigvee_{p\in\mathbb{Q}}[f > p] = 1$.
(6) $\bigvee_{q\in\mathbb{Q}}[f < q] = 1$.

Remarks. (1) It follows immediately from (2) that

$$[f < q]^* \leq [f > p] \quad \text{and} \quad [f > p]^* \leq [f < q]$$

for any $p < q$.
(2) Note that the frame $\mathfrak{S}(L)$ does not necessarily satisfy the De Morgan law for meets. Nevertheless we have

$$[f > p]^* = \bigvee\{S^* \in \mathfrak{S}(L) \mid f(S) \geq (p,\text{—})\}.$$

Indeed:
"$\leq$": Let $T = [f > p]$. Since $f(T) = ff^*(p,\text{—}) \geq (p,\text{—})$, we have $T^* \in \{S^* \mid f(S) \geq (p,\text{—})\}$.
"$\geq$": Since $f(S) \geq (p,\text{—})$ if and only if $S \geq f^*(p,\text{—}) = [f > p]$, the inequality $f(S) \geq (p,\text{—})$ implies $S^* \leq [f > p]^*$. ■

Similarly,

$$[f < q]^* = \bigvee\{S^* \in \mathfrak{S}(L) \mid f(S) \geq (\text{—},q)\}.$$

We will denote $[f > p]^*$ and $[f < q]^*$ by

$$[f \le p] \quad \text{resp.} \quad [f \ge q].$$

It follows from Facts (1) and (2) that

$$[f < p] \le [f \le p] \le [f < q] \quad \text{and} \quad [f > q] \le [f \ge q] \le [f > p]$$

for any $p < q$ in $\mathbb{Q}$. Also, we will write $[f = p]$ to denote $[f \le p] \wedge [f \ge p]$ and $[p < f < q]$ to denote $[f > p] \wedge [f < q]$.

3.3. Recall 2.2.2. We have now the image map $f[-]\colon \mathfrak{S}(\mathfrak{S}(L)) \to \mathfrak{S}(\mathfrak{L}(\mathbb{R}))$ (a localic map) and the preimage map $f_{-1}[-]\colon \mathfrak{S}(\mathfrak{L}(\mathbb{R})) \to \mathfrak{S}(\mathfrak{S}(L))$ (a frame homomorphism that preserves complements).

3.3.1. It follows from 2.2.3 that, for every $f \in \mathrm{F}(L)$ and every $p, q \in \mathbb{Q}$,

(P1) $f_{-1}[\mathfrak{c}(p,q)] = f^{-1}[\mathfrak{c}(p,q)] = \mathfrak{c}([p < f < q])$,
(P2) $f_{-1}[\mathfrak{c}(p,—)] = \mathfrak{c}([f > p])$ and $f_{-1}[\mathfrak{c}(—,q)] = \mathfrak{c}([f < q])$,
(P3) $f_{-1}[\mathfrak{o}(p,q)] = \mathfrak{o}([p < f < q])$, and
(P4) $f_{-1}[\mathfrak{o}(p,—)] = \mathfrak{o}([f > p])$ and $f_{-1}[\mathfrak{o}(—,q)] = \mathfrak{o}([f < q])$.

3.4. $\mathrm{F}(L)$ is partially ordered by

$$f \le g \equiv \forall\, q \in \mathbb{Q},\ \forall\, S \in \mathfrak{S}(L)\ [(—,q) \le f(S) \Rightarrow (—,q) \le g(S)].$$

Proposition. *Let $f, g \in \mathrm{F}(L)$. The following are equivalent:*

(1) $f \le g$.
(2) $\forall\, p \in \mathbb{Q}$, $\forall\, S \in \mathfrak{S}(L)$, $[(p,—) \le g(S) \Rightarrow (p,—) \le f(S)]$.
(3) $\forall q \in \mathbb{Q}$, $(—,q) \le g([f < q])$.
(4) $\forall p \in \mathbb{Q}$, $(p,—) \le f([g > p])$.
(5) $\forall q \in \mathbb{Q}$, $g_{-1}[\mathfrak{c}(—,q)] \le f_{-1}[\mathfrak{c}(—,q)]$.
(6) $\forall p \in \mathbb{Q}$, $f_{-1}[\mathfrak{c}(p,—)] \le g_{-1}[\mathfrak{c}(p,—)]$.
(7) $\forall q \in \mathbb{Q}$, $f_{-1}[\mathfrak{o}(—,q)] \le g_{-1}[\mathfrak{o}(—,q)]$.
(8) $\forall p \in \mathbb{Q}$, $g_{-1}[\mathfrak{o}(p,—)] \le f_{-1}[\mathfrak{o}(p,—)]$.
(9) $\forall q \in \mathbb{Q}$, $[g < q] \le [f < q]$.
(10) $\forall p \in \mathbb{Q}$, $[f > p] \le [g > p]$.

Proof: (1)⇒(3) is obvious since $(—,q) \le ff^*(—,q) = f([f < q])$.

(3)⇒(1): Let $(—,q) \le f(S)$, that is, $[f < q] \le S$. Then $g([f < q]) \le g(S)$ and consequently $(—,q) \le g(S)$.

(2)⇔(4) is similar.

(3)⇔(5) and (4)⇔(6) follow from (P2).

(5)⇔(7) and (6)⇔(8) follow from the fact that $f_{-1}[-]$ preserves complements.

(9)⇒(10): By Facts 3.2.1 (3) and (1) and Remark 3.2.1 (1) we have that

$$[f > p] = \bigvee_{r>p} [f > r] \leq \bigvee_{r>p} [f < r]^* \leq \bigvee_{r>p} [g < r]^* \leq [g > p].$$

(10)⇒(9) is similar. Finally, (3)⇔(9) and (4)⇔(10) are obvious. ■

3.5. An f in $\mathrm{F}(L)$ is *lower semicontinuous* if

$$[f > p] \in \mathfrak{c}_L[L] \quad \text{for each } p \in \mathbb{Q},$$

i.e. $f_{-1}[\mathfrak{c}(p, \text{—})] \in \mathfrak{c}_{\mathcal{S}(L)}[\mathfrak{c}_L[L]]$. It is *upper semicontinuous* if

$$[f < q] \in \mathfrak{c}_L[L] \quad \text{for each } q \in \mathbb{Q}.$$

Then, f is *continuous* if it is both upper and lower semicontinuous. We denote by

$$\mathrm{LSC}(L), \mathrm{USC}(L) \text{ and } \mathrm{C}(L), \text{ respectively,}$$

the sets of all lower semicontinuous, upper semicontinuous, and continuous real functions on L.

The fact that f is continuous means that $f_{-1}[\mathfrak{c}(p, q)] \in \mathfrak{c}_{\mathcal{S}(L)}[\mathfrak{c}_L[L]]$ for each $p, q \in \mathbb{Q}$. According to the isomorphism $\mathfrak{c}_L[L] \cong L$ we can then identify $f_{-1}[\mathfrak{c}(p, q)]$ with an element of $\mathfrak{c}_L[L]$ and by complementation we may also view each $f_{-1}[\mathfrak{o}(p, q)]$ as an open sublocale of L. Hence $f_{-1}[S] \in \mathcal{S}(L)$ for any $S \in \mathcal{S}(\mathfrak{L}(\mathbb{R}))$ (recall that any element in $\mathcal{S}(L(R))$ is a join of finite meets of open and closed elements). This explains why any $f \in \mathrm{C}(L)$ may be regarded as a localic map $L \to \mathfrak{L}(\mathbb{R})$ via the 1-1 correspondence

$$f \mapsto f \circ \mathfrak{c}_L \tag{3.5.1}$$

Under this identification we have

$$[f > p] = \bigwedge\{a \in L \mid f(a) \geq (p, \text{—})\} \text{ and } [f < q] = \bigwedge\{a \in L \mid f(a) \geq (\text{—}, q)\}.$$

Note that then $\mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$.

4. Algebraic operations

4.1. Consider an n-ary operation ω on $\mathbb{Q}$ (that is, a continuous function $\omega\colon \mathbb{Q}^n \to \mathbb{Q}$). We then have a frame homomorphism $\Omega(\omega)\colon \Omega(\mathbb{Q}) \longrightarrow \Omega(\mathbb{Q}^n)$. Note that this is not an algebraic operation on $\Omega(\mathbb{Q})$ (Ω does not preserve powers of $\mathbb{Q}$; in fact, by the closed subgroup theorem there is in particular no localic addition on $\Omega(\mathbb{Q})$, see [21]).

The frame $\mathfrak{L}(\mathbb{R})$ is the completion of $\Omega(\mathbb{Q})$ (both taken with the uniformity derived from the respective metric uniformities – [3]), where the completion homomorphism $\gamma\colon \mathfrak{L}(\mathbb{R}) \to \Omega(\mathbb{Q})$ is given by $(p,q) \mapsto]p,q[= \{x \in \mathbb{Q} \mid p < x < q\}$. Then $\mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R})$ (n summands) is the completion of $\Omega(\mathbb{Q}^n)$ ([3]) with the completion map $\overline{\gamma}$ given by the coproduct diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[r, "\iota_i"] \arrow[d, "\gamma"'] & \mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R}) \arrow[d, dotted, "\overline{\gamma}"] \\
\Omega(\mathbb{Q}) \arrow[r, "\Omega(p_i)"'] & \Omega(\mathbb{Q}^n)
\end{tikzcd}$$

(where the p_i, $i = 1, \ldots, n$, are the projections $\mathbb{Q}^n \to \mathbb{Q}$). The general theory of completion (in particular, the general criterion in [7] for the liftability of a frame homomorphism between uniform frames to their completions) guarantees the existence of an operation $\widetilde{\omega}$ on $\mathfrak{L}(\mathbb{R})$ that completes the diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[r, dotted, "\widetilde{\omega}"] \arrow[d, "\gamma"'] & \mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R}) \arrow[d, "\overline{\gamma}"] \\
\Omega(\mathbb{Q}) \arrow[r, "\Omega(\omega)"'] & \Omega(\mathbb{Q}^n)
\end{tikzcd}$$

to a commuting square. It is easy to check that the operation $\widetilde{\omega}$ is given by

$$\widetilde{\omega}(p,q) = \bigvee \Big\{ \bigoplus_{i=1}^{n} (r_i, s_i) \mid \prod_{i=1}^{n}]r_i, s_i[\subseteq \omega^{-1}\left(]p,q[\right) \Big\}. \tag{4.1.1}$$

Hence, $\widetilde{\omega}(p,\text{—}) = \bigvee_{r>p} \widetilde{\omega}(p,r) = \bigvee \Big\{ \bigoplus_{i=1}^{n} (r_i, s_i) \mid \prod_{i=1}^{n}]r_i, s_i[\subseteq \omega^{-1}\left(]p,+\infty[\right) \Big\}$ and $\widetilde{\omega}(\text{—},q) = \bigvee_{s<q} \widetilde{\omega}(s,q) = \bigvee \Big\{ \bigoplus_{i=1}^{n} (r_i, s_i) \mid \prod_{i=1}^{n}]r_i, s_i[\subseteq \omega^{-1}\left(]-\infty,q[\right) \Big\}$.

4.2. Examples. (1) For each $r \in \mathbb{Q}$, the nullary operation ω_r in $\mathbb{Q}$ that picks the rational r yields the operation $\widetilde{\omega}_r \colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2} = \{0,1\}$ given by

$$\widetilde{\omega}_r(p,q) = 1 \text{ if and only if } r \in \,]p,q[.$$

(2) For each $\lambda > 0$ in $\mathbb{Q}$, the unary operation ω_λ representing the scalar multiplication by λ yields $\widetilde{\omega}_\lambda \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ defined by

$$\widetilde{\omega}_\lambda(p,q) = \left(\tfrac{p}{\lambda}, \tfrac{q}{\lambda}\right).$$

Similarly, for each $\lambda < 0$ in $\mathbb{Q}$, ω_λ lifts to $\widetilde{\omega}_\lambda$ given by $\widetilde{\omega}_\lambda(p,q) = \left(\tfrac{q}{\lambda}, \tfrac{p}{\lambda}\right)$.

(3) For the binary operations $\wedge$ and $\vee$ we have $\widetilde{\wedge}, \widetilde{\vee} \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R})$ defined respectively by

$$\widetilde{\wedge}(p,q) = (p,q) \oplus (p,\text{—}) \vee (p,\text{—}) \oplus (p,q) \quad \text{and}$$

$$\widetilde{\vee}(p,q) = (p,q) \oplus (\text{—},q) \vee (\text{—},q) \oplus (p,q).$$

Equivalently,

$$\widetilde{\wedge}(p,\text{—}) = (p,\text{—}) \oplus (p,\text{—}) \quad \text{and} \quad \widetilde{\wedge}(\text{—},q) = ((\text{—},q) \oplus 1) \vee (1 \oplus (\text{—},q)),$$

$$\widetilde{\vee}(p,\text{—}) = ((p,\text{—}) \oplus 1) \vee (1 \oplus ((p,\text{—})) \quad \text{and} \quad \widetilde{\vee}(\text{—},q) = (\text{—},q) \oplus (\text{—},q).$$

(4) For the binary operation $+$ we have $\widetilde{+} \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R})$ defined by

$$\widetilde{+}(p,q) = \bigvee_{r \in \mathbb{Q}} \left(\left(r, r + \tfrac{q-p}{2}\right) \oplus \left(p - r, \tfrac{p+q}{2} - r\right)\right).$$

Indeed, for a fixed $(p,q) \in \mathfrak{L}(\mathbb{R})$ we have that (see Figure 1)

$$\begin{aligned} +^{-1}(]p,q[) &= \{(x,y) \in \mathbb{Q} \times \mathbb{Q} \mid p < x + y < q\} \\ &= \{(x,y) \mid \exists r \in \mathbb{Q} : r < x < r + \tfrac{q-p}{2} \text{ and } p - r < y < \tfrac{p+q}{2} - r\} \\ &= \bigcup_{r \in \mathbb{Q}} \left(\left]r, r + \tfrac{q-p}{2}\right[\times \left]p - r, \tfrac{p+q}{2} - r\right[\right) \cap (\mathbb{Q} \times \mathbb{Q}). \end{aligned}$$

Hence (see Figure 1)

$$\widetilde{+}(p,\text{—}) = \bigvee_{r \in \mathbb{Q}} ((r,\text{—}) \oplus (p - r,\text{—})) \quad \text{and} \quad \widetilde{+}(\text{—},q) = \bigvee_{s \in \mathbb{Q}} ((\text{—},s) \oplus (\text{—},q - s)).$$

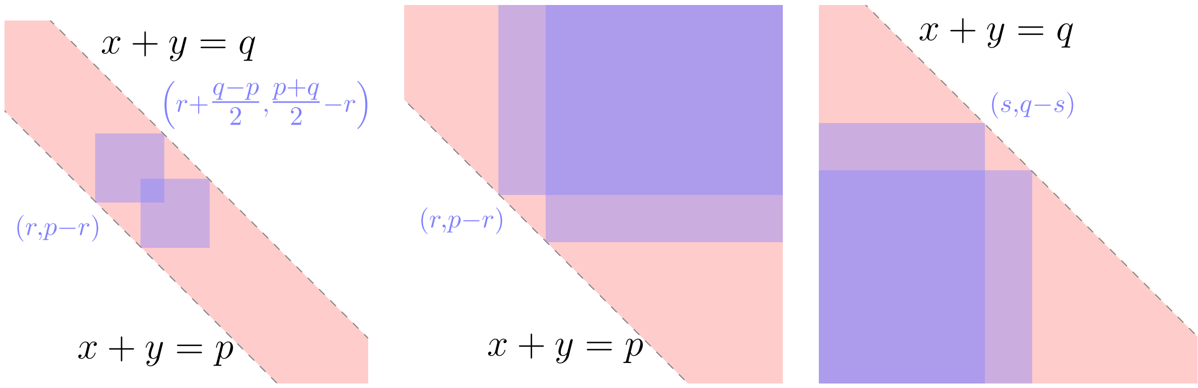

FIGURE 1. Sum

(5) For the binary operation $\cdot$ we have $\widetilde{\cdot}\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R})$ defined by (for simplicity we only describe it on the generators $(p,\text{—})$ and $(\text{—},q)$)

$$\widetilde{\cdot}(p,\text{—}) = \begin{cases} \bigvee\limits_{r>0} \Big(\big((r,\text{—}) \oplus \big(\tfrac{p}{r},\text{—}\big)\big) \vee \big((\text{—},-r) \oplus \big(\text{—},-\tfrac{p}{r}\big)\big)\Big) & \text{if } p>0,\\ (0,\text{—}) \oplus (0,\text{—}) \vee (\text{—},0) \oplus (\text{—},0) & \text{if } p=0,\\ \widetilde{\cdot}(0,\text{—}) \vee \bigvee_{r>0} \big((-r,r) \oplus \big(\tfrac{p}{r},-\tfrac{p}{r}\big)\big) & \text{if } p<0, \end{cases}$$

$$\widetilde{\cdot}(\text{—},q) = \begin{cases} \bigvee\limits_{s>0} \Big(\big((\text{—},-s) \oplus \big(-\tfrac{q}{s},\text{—}\big)\big) \vee \big((s,\text{—}) \oplus \big(\text{—},\tfrac{q}{s}\big)\big)\Big) & \text{if } q<0,\\ (\text{—},0) \oplus (0,\text{—}) \vee (0,\text{—}) \oplus (\text{—},0) & \text{if } q=0,\\ \widetilde{\cdot}(\text{—},0) \vee \bigvee_{s>0} \big((-s,s) \oplus \big(-\tfrac{q}{s},\tfrac{q}{s}\big)\big) & \text{if } q>0. \end{cases}$$

In fact, for a fixed $(p,\text{—}) \in \mathfrak{L}(\mathbb{R})$ (with $p>0$) we have (see Figure 2)

$$\begin{aligned}(\,\cdot\,)^{-1}\left(]p,+\infty[\right) &= \{(x,y) \in \mathbb{Q}\times\mathbb{Q} \mid p < x\cdot y\}\\ &= \bigcup_{r>0} \Big(\big(]r,+\infty[\times \big]\tfrac{p}{r},+\infty\big[\big) \cup \big(]-\infty,-r[\times \big]-\infty,-\tfrac{p}{r}\big[\big)\Big) \cap (\mathbb{Q}\times\mathbb{Q})\,.\end{aligned}$$

Hence $\widetilde{\cdot}(p,\text{—}) = \bigvee_{r>0} \big((r,\text{—}) \oplus \big(\tfrac{p}{r},\text{—}\big) \vee (\text{—},-r) \oplus \big(\text{—},-\tfrac{p}{r}\big)\big)$

In the case $p=0$ we have (see Figure 2)

$$\begin{aligned}(\,\cdot\,)^{-1}\left(]0,+\infty[\right) &= \{(x,y) \in \mathbb{Q}\times\mathbb{Q} \mid 0 < x\cdot y\}\\ &= \left(\left(]0,+\infty[\times]0,+\infty[\right) \cup \left(]-\infty,0[\times]-\infty,0[\right)\right) \cap (\mathbb{Q}\times\mathbb{Q})\end{aligned}$$

and hence $\widetilde{\cdot}(0,\text{—}) = (0,\text{—}) \oplus (0,\text{—}) \vee (\text{—},0) \oplus (\text{—},0)$. Of course, this case could be included in the previous case since $\bigvee_{r>0} ((r,\text{—}) \oplus (0,\text{—}) \vee (\text{—},-r) \oplus (\text{—},0)) = (0,\text{—}) \oplus (0,\text{—}) \vee (\text{—},0) \oplus (\text{—},0)$.

Finally, when $p < 0$ we have that (see Figure 2)

$$\begin{aligned}(\,\cdot\,)^{-1}\,(]p,+\infty[) &= \{(x,y) \in \mathbb{Q}\times\mathbb{Q} \mid p < x\cdot y\}\\ &= (((]0,+\infty[\times]0,+\infty[) \cup (]-\infty,0[\times]-\infty,0[)) \cap (\mathbb{Q}\times\mathbb{Q})) \cup \\ &\quad \cup \left(\bigcup_{r>0} ((]-r,r[\times]\tfrac{p}{r},-\tfrac{p}{r}[)) \cap (\mathbb{Q}\times\mathbb{Q})\right).\end{aligned}$$

Hence $\widetilde{\cdot}(p,\text{—}) = \widetilde{\cdot}(0,\text{—}) \vee \bigvee_{r>0} \left((-r,r) \oplus \left(\frac{p}{r},-\frac{p}{r}\right)\right)$.

A similar situation holds for the generators of the form $(\text{—},q)$.

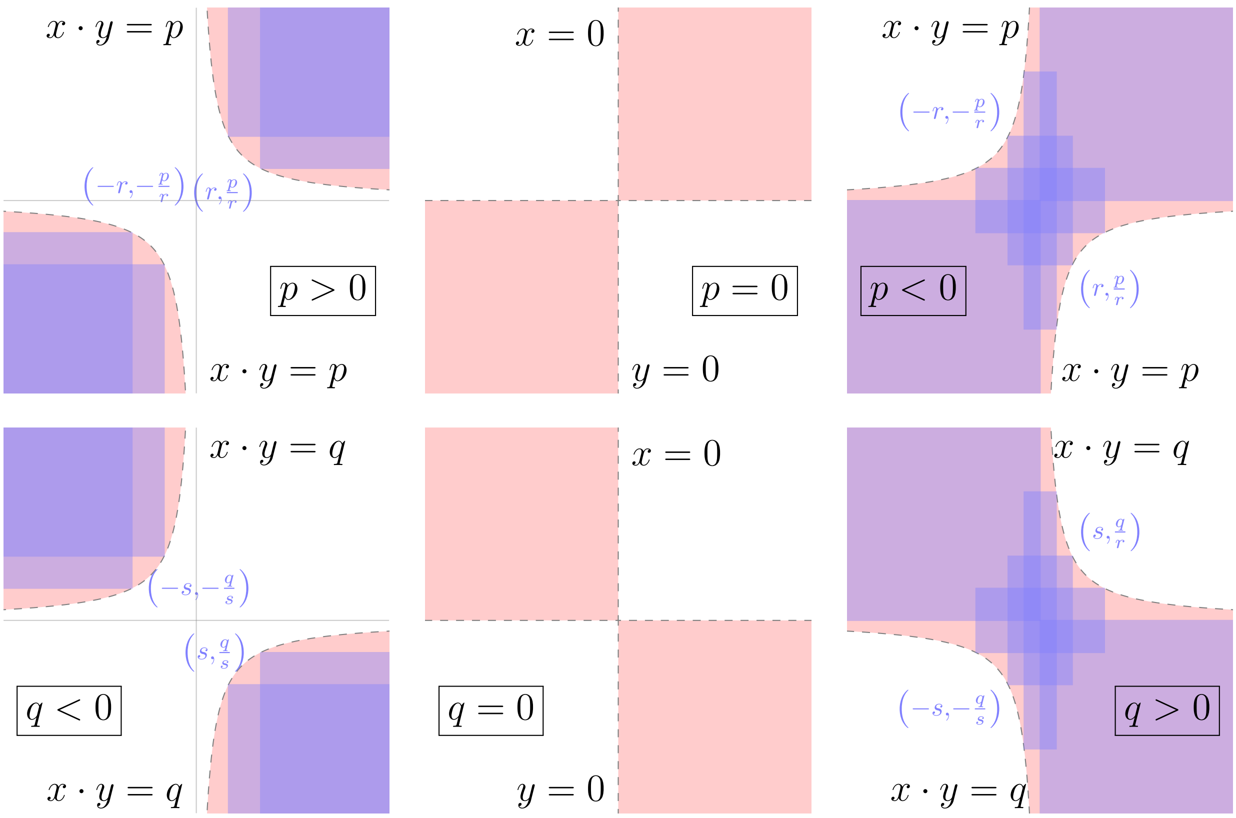

FIGURE 2. Product

4.3. Now there is a standard canonical process of lifting for any frame L the operations on $\mathfrak{L}(\mathbb{R})$ to operations on $\mathrm{C}(L)$. Indeed, consider an n-ary operation on $\mathfrak{L}(\mathbb{R})$,

$$\widetilde{\omega}\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R}).$$

Given $f_i \in \mathrm{C}(L)$, $i = 1, \dots, n$, let $f^* = (f_1^*, \dots, f_n^*)$ be the frame homomorphism given by the coproduct diagram

$$\begin{tikzcd} \mathfrak{L}(\mathbb{R}) \arrow[r, "\widetilde{\omega}"] & \mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R}) \arrow[d, dotted, "f^*"'] & \mathfrak{L}(\mathbb{R}) \arrow[l, "\iota_i"'] \arrow[dl, "{f_i}^*"] \\ & L & \end{tikzcd}$$

Combining it with $\widetilde{\omega}$, one gets an $\widetilde{\omega}(f_1, \dots, f_n) \in \mathrm{C}(L)$ associated with the frame homomorphism $f^* \cdot \widetilde{\omega}$. Since f^* is given by $f^*(x_1 \oplus \cdots \oplus x_n) = x_1 \wedge \cdots \wedge x_n$, the formula (4.1.1) yields

$$\widetilde{\omega}(f_1, \dots, f_n)^*(p, q) = \bigvee \Big\{ \bigwedge_{i=1}^{n} {f_i}^*(r_i, s_i) \mid \prod_{i=1}^{n}]r_i, s_i[\subseteq \omega^{-1} (]p, q[) \Big\}. \quad (4.3.1)$$

Then

$$\widetilde{\omega}(f_1, \dots, f_n)^*(p, \text{—}) = \bigvee \Big\{ \bigwedge_{i=1}^{n} {f_i}^*(r_i, s_i) \mid \prod_{i=1}^{n}]r_i, s_i[\subseteq \omega^{-1} (]p, +\infty[) \Big\}$$

and

$$\widetilde{\omega}(f_1, \dots, f_n)^*(\text{—}, q) = \bigvee \Big\{ \bigwedge_{i=1}^{n} {f_i}^*(r_i, s_i) \mid \prod_{i=1}^{n}]r_i, s_i[\subseteq \omega^{-1} (]-\infty, q[) \Big\}.$$

4.4. Examples. (1) For each $r \in \mathbb{Q}$, the nullary operation $\widetilde{\omega}_r \colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}$ induces the *constant* function $\boldsymbol{r} \in \mathrm{C}(L)$ defined by

$$[p < \boldsymbol{r} < q] = 1 \text{ if and only if } r \in]p, q[.$$

(2) For each $\lambda > 0$ in $\mathbb{Q}$ and any $f \in \mathrm{C}(L)$, the unary operation $\widetilde{\omega}_\lambda \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ induces the *multiplication by a scalar* function λf defined by

$$[p < \lambda f < q] = [\tfrac{p}{\lambda} < f < \tfrac{q}{\lambda}].$$

For $\lambda < 0$, λf is defined by

$$[p < \lambda f < q] = [\tfrac{q}{\lambda} < f < \tfrac{p}{\lambda}].$$

In particular, we write $-f$ for $(-1)f$ and so $[p < -f < q] = [-q < f < -p]$.

(3) For the binary operations $\widetilde{\wedge}$ and $\widetilde{\vee}$ and any $f, g \in \mathrm{C}(L)$, we have the *meet* and the *join* of f and g given by

$$[f \wedge g > p] = [f > p] \wedge [g > p] \quad \text{and} \quad [f \wedge g < q] = [f < q] \vee [g < q],$$

$$[f \vee g > p] = [f > p] \vee [g > p] \quad \text{and} \quad [f \vee g < q] = [f < q] \wedge [g < q].$$

Of course, $f \leq g$ iff $f \wedge g = f$ iff $f \vee g = g$.

(4) For the binary operation $\widetilde{+}$ and any $f, g \in \mathrm{C}(L)$, we have the *sum* of f and g given by

$$[f+g>p] = \bigvee_{r\in\mathbb{Q}} ([f>r] \wedge [g>p-r]) \quad \text{and}$$
$$[f+g<q] = \bigvee_{s\in\mathbb{Q}} ([f<q] \wedge [g<q-s]).$$

Of course, if $L = \mathfrak{S}(M)$ we get the above results for any $f, g \in \mathrm{F}(M)$. It follows immediately that

- if $f, g \in \mathrm{LSC}(M)$ then $f+g \in \mathrm{LSC}(M)$,
- if $f, g \in \mathrm{USC}(M)$ then $f+g \in \mathrm{USC}(M)$.

Defining $f-g = f+((-1)g)$ we also have that

- if $f \in \mathrm{LSC}(M)$ and $g \in \mathrm{USC}(M)$ then $f-g \in \mathrm{LSC}(M)$,
- if $f \in \mathrm{USC}(M)$ and $g \in \mathrm{LSC}(M)$ then $f-g \in \mathrm{USC}(M)$.

Note also that $f \leq g$ iff $g - f \geq \mathbf{0}$.

(5) For the binary product $\widetilde{\cdot}$ and $f, g \in \mathrm{F}(M)$ we have the *product* of f and g:

$$[f\cdot g>p] = \begin{cases} \bigvee_{r>0} \big(([f>r] \wedge [g>\frac{p}{r}]) \vee ([f<-r] \wedge [g<-\frac{p}{r}])\big) & \text{if } p>0, \\ ([f>0] \wedge [g>0]) \vee ([f<0] \wedge [g<0]) & \text{if } p=0, \\ [f\cdot g>0] \vee \bigvee_{r>0} \big([-r<f<r] \wedge [\frac{p}{r}<g<-\frac{p}{r}]\big) & \text{if } p<0, \end{cases}$$

and

$$[f\cdot g<q] = \begin{cases} \bigvee_{s>0} \big(([f<-s] \wedge [g>-\frac{q}{s}]) \vee ([f>s] \wedge [g<\frac{q}{s}])\big) & \text{if } q<0, \\ ([f<0] \wedge [g>0]) \vee ([f>0] \wedge [g<0]) & \text{if } q=0, \\ [f\cdot g<0] \vee \bigvee_{s>0} \big([-s<f<s] \wedge [-\frac{q}{s}<g<\frac{q}{s}]\big) & \text{if } q>0. \end{cases}$$

In particular, if $\mathbf{0} \leq f, g \in \mathrm{F}(M)$ then $[f<0]=[g<0]=0$ and $[f>-r]=[g>-r]=1$ for any $r>0$ and so for each $p<0$ we have that

$$
\begin{aligned}
{[f \cdot g>p]} &= ([f>0] \wedge [g>0]) \vee \bigvee_{r>0}\big([-r<f<r] \wedge [\tfrac{p}{r}<g<-\tfrac{p}{r}]\big) \\
&= \Big([f>0] \vee \bigvee_{r>0}\big([-r<f<r] \wedge [\tfrac{p}{r}<g<-\tfrac{p}{r}]\big)\Big) \wedge \\
&\qquad \wedge \Big([g>0] \vee \bigvee_{r>0}\big([-r<f<r] \wedge [\tfrac{p}{r}<g<-\tfrac{p}{r}]\big)\Big) \\
&= \Big(\bigvee_{r>0}\big([f>-r] \wedge \big([f>0] \vee [\tfrac{p}{r}<g<-\tfrac{p}{r}]\big)\big)\Big) \wedge \\
&\qquad \wedge \Big(\bigvee_{r>0}\big(([g>0] \vee [-r<f<r]) \wedge [g>\tfrac{p}{r}]\big)\Big) \\
&= \Big(\bigvee_{r>0}\big([f>0] \vee [\tfrac{p}{r}<g<-\tfrac{p}{r}]\big)\Big) \wedge \Big(\bigvee_{r>0}([g>0] \vee [-r<f<r])\Big) \\
&= \Big([f>0] \vee \bigvee_{r>0}[\tfrac{p}{r}<g<-\tfrac{p}{r}]\Big) \wedge \Big([g>0] \vee \bigvee_{r>0}[-r<f<r]\Big)=1.
\end{aligned}
$$

On the other hand, $[-r<f<r]=[f<r]$ and $[-r<g<r]=[g<r]=0$ for any $r>0$ and so in this case the product of f and g reduces to

$$
[f \cdot g>p]= \begin{cases}\bigvee_{r>0}\big([f>r] \wedge [g>\frac{p}{r}]\big) & \text{if } p \geq 0, \\ 1 & \text{if } p<0,\end{cases}
$$

and

$$
[f \cdot g<q]= \begin{cases}0 & \text{if } q \leq 0, \\ \bigvee_{s>0}\big([f<s] \wedge [g<\frac{q}{s}]\big) & \text{if } q>0.\end{cases}
$$

Consequently

- if $\mathbf{0} \leq f, g \in \mathrm{LSC}(M)$ then $f \cdot g \in \mathrm{LSC}(M)$, and
- if $\mathbf{0} \leq f, g \in \mathrm{USC}(M)$ then $f \cdot g \in \mathrm{USC}(M)$.

If $f, g \leq \mathbf{0}$ then $f \cdot g=(-f) \cdot(-g)$ and hence

- if $\mathbf{0} \geq f, g \in \mathrm{LSC}(M)$ then $f \cdot g \in \mathrm{USC}(M)$, and
- if $\mathbf{0} \geq f, g \in \mathrm{USC}(M)$ then $f \cdot g \in \mathrm{LSC}(M)$.

5. Cozero and zero sublocales

5.1. We say that a sublocale S of a frame L is a *cozero sublocale* if it is of the form

$$[f < 0] \vee [f > 0]$$

for some $f \in \mathrm{C}(L)$. We will use the notation

$$S = \mathrm{Coz}_L(f) \text{ or simply } S = \mathrm{Coz}(f).$$

Dually, S is a *zero sublocale* if it is of the form

$$[f = 0] = [f \geq 0] \wedge [f \leq 0] = [f < 0]^* \wedge [f < 0]^*$$

for some $f \in \mathrm{C}(L)$. We will use the notation

$$S = \mathrm{Z}_L(f) \text{ or simply } S = \mathrm{Z}(f).$$

5.2. Of course, each cozero sublocale is a closed sublocale $\mathfrak{c}(a)$ for some $a \in L$. By (3.5.1) and 2.6 the a is a cozero element. 2.6 also allows to characterize cozero and zero sublocales without reference to the frame of reals.

Proposition. *Let $S \in \mathcal{S}(L)$. Then S is a cozero sublocale if and only if $S = \mathfrak{c}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ for some $a_n \prec\!\prec a$, $n = 1, 2, \ldots$.* ■

Each zero sublocale, being a complement of a cozero sublocale, is an open sublocale $\mathfrak{o}(a)$ for some cozero element a. Note that since complemented elements are always cozero elements, both cozero sublocales and zero sublocales of L are cozero elements of the frame $\mathcal{S}(L)$ (but not conversely).

5.3. Properties of cozero and zero sublocales. Evidently $\mathrm{Coz}(\mathbf{1}) = \mathrm{Z}(\mathbf{0}) = 1$, $\mathrm{Coz}(\mathbf{0}) = \mathrm{Coz}(\mathbf{1}) = 0$, and

$$\mathrm{Coz}(f) = \mathrm{Coz}(f^n), \quad \mathrm{Z}(f) = \mathrm{Z}(f^n) \quad \text{for all } n \in \mathbb{N}.$$

Furthermore

$$\mathrm{Coz}(f) = \mathrm{Coz}(|f|) \quad \text{and} \quad \mathrm{Z}(f) = \mathrm{Z}(|f|)$$

(recall that $|f|$ is defined as $f \vee (-f)$). Indeed, from the formulas in 4.4 we get

$$[f \vee (-f) < 0] = [f < 0] \wedge [-f < 0] = [f < 0] \wedge [f < 0] = 0$$

and $[f \vee (-f) > 0] = [f < 0] \vee [f < 0]$, hence $\mathrm{Coz}(|f|) = [f < 0] \vee [f < 0]$.

Further we have:

(Z1) $\mathrm{Coz}(f) = 0$ iff $f = \mathbf{0}$ iff $\mathrm{Z}(f) = 1$.

(Z2) $\mathrm{Coz}(f - \boldsymbol{r}) = [f > r] \vee [f < r]$ and $\mathrm{Z}(f - \boldsymbol{r}) = [f \geq r] \wedge [f \leq r]$.

(Z3) $\mathrm{Coz}(f+g) \le \mathrm{Coz}(f) \vee \mathrm{Coz}(g)$ and $\mathrm{Z}(f+g) \ge \mathrm{Z}(f) \wedge \mathrm{Z}(g)$.

(Z4) $\mathrm{Coz}(|f| \wedge |g|) = \mathrm{Coz}(f) \wedge \mathrm{Coz}(g) = \mathrm{Coz}(f \cdot g)$ and $\mathrm{Z}(|f| \wedge |g|) = \mathrm{Z}(f) \vee \mathrm{Z}(g) = \mathrm{Z}(f \cdot g)$.

Proof: (Z1) and (Z2) are straightforward.

(Z3) By 4.4,

$$[f+g<0] = \bigvee_{s\in\mathbb{Q}} ([f<s] \wedge [g<-s]) \le \bigvee_{s\le 0} [f<s] \vee \bigvee_{s>0} [g<-s] = [f<0] \vee [g<0].$$

Similarly, $[f+g>0] \le [f>0] \vee [g>0]$ and so $\mathrm{Coz}(f+g) \le \mathrm{Coz}(f) \vee \mathrm{Coz}(g)$.

(Z4) We have

$$[f\cdot g<0] = ([f<0] \wedge [g>0]) \vee ([f>0] \wedge [g<0])$$

and $[f\cdot g>0] = ([f>0] \wedge [g>0]) \vee ([f<0] \wedge [g<0])$. Hence

$$\mathrm{Coz}(f\cdot g) = [f\cdot g<0] \vee [f\cdot g>0] = \mathrm{Coz}(f) \wedge \mathrm{Coz}(g). \quad \blacksquare$$

5.3.1. Remarks. (1) Every closed sublocale of the form $[f<0]$ is a cozero sublocale: $[f<0] = \mathrm{Coz}(f \wedge \mathbf{0}) = \mathrm{Coz}(f - |f|)$. Indeed,

$$\mathrm{Coz}(f \wedge \mathbf{0}) = [f \wedge \mathbf{0} < 0] \vee [f \wedge \mathbf{0} > 0] = [f<0].$$

Likewise, $[f>0] = \mathrm{Coz}(f \vee \mathbf{0}) = \mathrm{Coz}(f + |f|)$. Hence $[f \ge 0] = \mathrm{Z}(f \wedge \mathbf{0}) = \mathrm{Z}(f - |f|)$ and $[f \le 0] = \mathrm{Z}(f \vee \mathbf{0}) = \mathrm{Z}(f + |f|)$.

More generally, for any rational r, $[f<r] = \mathrm{Coz}(f \wedge \boldsymbol{r})$, $[f>r] = \mathrm{Coz}(f \vee \boldsymbol{r})$, $[f \ge r] = \mathrm{Z}(f \wedge \boldsymbol{r})$ and $[f \le r] = \mathrm{Z}(f \vee \boldsymbol{r})$.

(2) By (Z4), $Z(|f| \wedge \mathbf{1}) = Z(f)$ for any $f \in \mathrm{C}(L)$. Let $\mathrm{C}^*(L)$ denote the set of all bounded elements of $\mathrm{C}(L)$. Since $|f| \wedge \mathbf{1} \in \mathrm{C}^*(L)$, this shows that $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ yield the same zero sublocales in L.

(3) Given $f \in \mathrm{C}(L)$, we have the formula

$$[f<0] = \bigvee_{n\in\mathbb{N}} [f \le -\tfrac{1}{n}].$$

Indeed: $[f<0] = \bigvee_{n\in\mathbb{N}} [f < -\frac{1}{n}] \le \bigvee_{n\in\mathbb{N}} [f \le -\frac{1}{n}] \le [f<0]$.

In terms of the structure of sublocales in $\mathcal{S}(L)$ this formula shows that *every cozero sublocale is a G_δ-sublocale*, i.e. a countable intersection of open sublocales, while *every zero sublocale is an F_σ-sublocale* (i.e. a countable join of closed sublocales). Note that by [14, Proposition 3.5] the converse implications hold under normality: in a normal frame, *every closed G_δ-sublocale is a cozero sublocale* and *every open F_σ-sublocale is a zero sublocale.*

5.3.2. Lemma. *The following are equivalent for $f, g \in \mathrm{C}(L)$:*

(1) $f \leq g$.
(2) $[g - f \leq r] = 0$ *for every* $r < 0$.
(3) $[g - f \geq 0] = 1$.
(4) $Z(g - f) = [g - f \leq 0]$.

Proof: (1)⇒(2): If $f \leq g$ then $g - f \geq \mathbf{0}$ and thus $[g - f \leq r] = [g - f > r]^* \leq [\mathbf{0} > r]^* = 1^* = 0$ for every $r < 0$.

(2)⇒(3): By Fact 3.2.1(4), $[g-f \geq 0] = [g-f < 0]^* = \bigwedge_{r<0}[g-f < r]^* = 1$.

(3)⇒(4) is obvious.

(4)⇒(1): By the hypothesis, $[g - f \leq 0] = Z(g - f) \leq [g - f \geq 0]$ and so $[g - f \geq 0] = [g - f \geq 0] \vee [g - f \leq 0] = 1$. Hence $g - f \geq \mathbf{0}$, i.e., $f \leq g$. ■

5.3.3. Corollary. *The following are equivalent for $g \in \mathrm{C}(L)$:*

(1) $g \geq \mathbf{0}$.
(2) $[g \leq r] = 0$ *for every* $r < 0$.
(3) $[g \geq 0] = 1$.
(4) $Z(g) = [g \leq 0]$. ■

5.3.4. Proposition. *For any $f, g \in \mathrm{C}(L)$,*

$$Z(f) \wedge Z(g) = Z(|f| \vee |g|) = Z(|f| + |g|).$$

Proof: Since $Z(f) = Z(|f|)$ we may take $f, g \geq \mathbf{0}$. Evidently, $[f > 0] \vee [g > 0] = [f \vee g > 0] \leq [f + g > 0]$. Then, by complementation and using the preceding corollary, $Z(f) \wedge Z(g) = Z(f \vee g) \geq Z(f + g)$. The conclusion follows from (Z3). ■

5.3.5. Corollary. *If $\mathbf{0} \leq f \leq g$ in $\mathrm{C}(L)$ then $Z(f) \geq Z(g) = Z(f \vee g) = Z(f + g)$.* ■

5.3.6. Remark. The set $\mathrm{Coz}(\mathfrak{S}(L))$ of all cozero sublocales is immediately seen to be a sublattice of $\mathfrak{S}(L)$ by rules (Z4) and Proposition 5.3.4. Furthermore, it is closed under countable joins (and thus a sub-σ-frame of $\mathfrak{S}(L)$). Indeed, consider any cozero sublocale $\mathfrak{c}(a_n)$ and using countable dependent choice take $b_{n_k} \prec\!\!\prec a_n$, $k = 1, 2, \ldots$, such that $a_n = \bigvee_{k\in\mathbb{N}} b_{n_k}$ for each n. Then

$a = \bigvee_{n,k\in\mathbb{N}} b_{n_k}$ where $b_{n_k} \prec\!\prec a$ for $n, k \in \mathbb{N}$, and these b_{n_k} form a countable set. The additional fact that the set of all cozero sublocales is a normal σ-frame can either be deduced directly or from the well-known fact that the σ-frame of all cozero elements of L is normal.

Evidently, the set $\mathrm{Z}(\mathfrak{S}(L))$ of all zero sublocales of L is a sublattice of $\mathfrak{S}(L)$ closed under countable meets.

5.4. Completely separated sublocales. The notion of complete separation in pointfree topology was first introduced in [1] in terms of quotient maps and cozero elements and equivalently reformulated in [13] in terms of sublocales and continuous real functions.

Two sublocales S and T of L are said to be *completely separated* if there exists an $f \in \mathrm{C}(L)$ satisfying $\mathbf{0} \leq f \leq \mathbf{1}$ such that

$$f(S) \geq (0, \text{—}) \quad \text{and} \quad f(T) \geq (\text{—}, 1)$$

(equivalently, $S \geq [f > 0]$ and $T \geq [f < 1]$). This implies, in particular, that $S^* \leq [f \leq 0]$ and $T^* \leq [f \geq 1]$. Hence:

5.4.1. Proposition. *If S and T are completely separated, then there exist a cozero sublocale C and a zero sublocale Z such that $S \geq C \geq Z \geq T^*$.* ■

5.4.2. Lemma. *The following are equivalent for $a, b \in L$:*

(1) $b \prec\!\prec a$.

(2) $\mathfrak{o}(b)$ *and* $\mathfrak{c}(a)$ *are completely separated.*

(3) *There exist a cozero sublocale C and a zero sublocale Z such that* $\mathfrak{c}(a) \geq C^\circ \geq C \geq Z \geq \mathfrak{c}(b)$.

(4) *There exist a cozero sublocale C and a zero sublocale Z such that* $\mathfrak{o}(b) \geq C \geq Z \geq \overline{Z} \geq \mathfrak{o}(a)$.

Proof: (1)⇒(2): Let $\{x_r \mid r \in [0,1] \cap \mathbb{Q}\} \subseteq L$ with $x_0 = b$ and $x_1 = a$ such that $x_p \prec x_q$ whenever $p < q$ in $[0,1] \cap \mathbb{Q}$. First we extend the index set to $\mathbb{Q}$ by setting $x_r = 0$ for $r < 0$ and $x_r = 1$ for $r > 1$. Then it is a straightforward exercise to check that the formulas

$$[f > p] = f^*(p, \text{—}) = \bigvee_{r>p} \overline{\mathfrak{o}(x_r)} \quad \text{and} \quad [f < q] = f^*(\text{—}, q) = \bigvee_{s<q} \mathfrak{o}(x_s^*)$$

define an $f \in \mathrm{C}(L)$, i.e., that f^* turns the defining relations of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathfrak{S}(L)$. Finally,

- $[f > 0] = \bigvee_{r>0} \overline{\mathfrak{o}(x_r)} \leq \overline{\mathfrak{o}(x_0)} \leq \mathfrak{o}(x_0) = \mathfrak{o}(b)$.

- $[f<1]=\bigvee_{s<1}\mathfrak{o}(x_s^*)\le\mathfrak{c}(x_1)=\mathfrak{c}(a)$ (since $x_s^*\vee x_1=1$ for each $s<1$).

(2)⇒(3): Let $f\in\mathrm{C}(L)$ such that $\mathbf{0}\le f\le\mathbf{1}$, $\mathfrak{o}(b)\ge[f>0]$ and $\mathfrak{c}(a)\ge[f<1]$. Then $Z=[f\le\frac{1}{3}]\in\mathrm{Z}(\mathfrak{S}(L))$, $C=[f<\frac{2}{3}]\in\mathrm{Coz}(\mathfrak{S}(L))$ and

$$\mathfrak{c}(b)\le[f\le 0]\le Z\le C\le C^\circ=[f<\tfrac{2}{3}]^\circ\le[f\le\tfrac{2}{3}]^\circ=[f\le\tfrac{2}{3}]\le[f<1]^\circ\le\mathfrak{c}(a).$$

The equivalence of (3) and (4) is obvious, by complementation.

(4)⇒(1): Let C and Z be respectively a cozero sublocale and a zero sublocale satisfying $\mathfrak{o}(b)\ge C\ge Z\ge\overline{Z}\ge\mathfrak{o}(a)$. Furthermore, let $g,h\in\mathrm{C}(L)$ such that $\mathbf{0}\le g,h\le\mathbf{1}$, $C=[g>0]$ and $Z=[h\le 0]$. It is easy to check that the formulas

$$[f>p]=\begin{cases}1 & \text{if } p<0,\\ C=[g>0] & \text{if } p=0,\\ \bigvee\limits_{r\in(0,1)\cap\mathbb{Q}}[g>\frac{r}{1-p}]\wedge[h<\frac{r}{p}] & \text{if } 0<p<1,\\ 0 & \text{if } p\ge 1,\end{cases}$$

and

$$[f<q]=\begin{cases}0 & \text{if } q\le 0,\\ \bigvee\limits_{r\in(0,1)\cap\mathbb{Q}}[g<\frac{r}{1-q}]\wedge[h>\frac{r}{q}] & \text{if } 0<q<1,\\ Z^*=[h>0] & \text{if } q=1,\\ 1 & \text{if } q>1,\end{cases}$$

define an $f\in\mathrm{C}(L)$, i.e., that f^* turns the defining relations of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathfrak{S}(L)$. Let $x_0=b$, $x_1=a$ and $x_r=\mathfrak{c}_L^{-1}([f<r])$ for each $r\in(0,1)\cap\mathbb{Q}$. Then

$$\overline{\mathfrak{o}(b)}\ge C=[g>0]=[f>0]\ge[f\ge q]=[f<q]^*=\mathfrak{o}(x_q)$$

for every $q>0$, hence $x_0=b\prec x_q$. On the other hand,

$$\overline{\mathfrak{o}(x_q)}=\overline{[f\ge q]}\ge\overline{[f\ge 1]}=\overline{[h\le 0]}=\overline{Z}\ge\mathfrak{o}(a)$$

for every $q<1$, hence $x_q\prec a=x_1$. Finally, $x_p\prec x_q$ for $p<q$ in $(0,1)\cap\mathbb{Q}$ because $\overline{\mathfrak{o}(x_p)}=\overline{[f\ge p]}\ge[f>p]\ge[f\ge q]=\mathfrak{o}(x_q)$. ∎

5.4.3. Corollary. (Cf. [1, Prop. 2.1.4]) *The following are equivalent for $a,b\in L$:*

(1) $b\prec\!\prec a$.

(2) *There is a frame homomorphism* $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ *with* $\mathbf{0} \le \varphi \le \mathbf{1}$ *and such that* $\varphi(0,\text{—}) \wedge b = 0$ *and* $\varphi(\text{—},1) \le a$.

(3) *There are cozero elements* c *and* d *in* L *such that* $c^* \vee a = 1$, $c \vee d = 1$ *and* $b \wedge d = 0$. ■

5.4.4. Remark. In Proposition 2.1.4 in [1] the authors include the equivalence

$$b \prec\!\!\prec a \iff \text{there is a cozero element } c \text{ in } L \text{ such that } b \prec c \prec a.$$

The necessity follows easily from the equivalence (1)⇔(3) above: if c and d are cozero elements in L such that $c^* \vee a = 1$, $c \vee d = 1$ and $b \wedge d = 0$ then $c \prec a$ and $b^* \vee c \ge d \vee c = 1$, hence $b \prec c$. However, the converse implication does not hold as some easy counterexamples show. In [1] the authors refer for a proof of the equivalence to [2]. This is mistaken, but their fundamental results hold true.

5.5. Recall that a frame L is *completely regular* if $a = \bigvee\{b \in L \mid b \prec\!\!\prec a\}$ for every $a \in L$. It follows easily from the characterization (3) in the above lemma that

Corollary. *A frame is completely regular iff every closed sublocale is the join of all cozero sublocales containing it iff every open sublocale is the meet of all zero sublocales it contains.*

Proof: If L is completely regular then $\mathfrak{c}(a) = \bigvee\{\mathfrak{c}(b) \in \mathfrak{Z}(L) \mid b \prec\!\!\prec a\}$ for every $a \in L$. It follows from Lemma 5.4.2 that for each $b \in L$ satisfying $b \prec\!\!\prec a$ there exists some cozero sublocale C_b such that $\mathfrak{c}(b) \le C_b \le \mathfrak{c}(a)$. Then $\mathfrak{c}(a) \le \bigvee\{C_b \in \mathfrak{Z}(L) \mid b \prec\!\!\prec a\} \le \mathfrak{c}(a)$.

Conversely, assume that $\mathfrak{c}(a) = \bigvee\{C \in \mathrm{Coz}\,(\mathfrak{Z}(L)) \mid C \le \mathfrak{c}(a)\}$ for every $a \in L$. Let $C = \mathfrak{c}(x_C)$ be a cozero sublocale such that $C \le \mathfrak{c}(a)$. It follows from Proposition 5.2 that there exist $(a_n^C)_{n\in\mathbb{N}}$ such that $a_n^C \prec\!\!\prec x_C$ for each $n \in \mathbb{N}$ and $C = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n^C)$. Then $a_n^C \prec\!\!\prec a$ for every such C and every n and $\mathfrak{c}(a) = \bigvee_{C,n} \mathfrak{c}(a_n^C)$. In conclusion $a = \bigvee_{C,n} a_n^C \le \bigvee\{b \mid b \prec\!\!\prec a\} \le a$.

The second part follows immediately by complementation. ■

This is the pointfree counterpart of the classical well-known fact that a topological space is completely regular iff every open set is the union of the cozero sets it contains.

5.6. We can now characterize cozero and zero sublocales without reference to the frame of reals.

5.6.1. Corollary. *Let S be a sublocale of L.*

(1) *S is a cozero sublocale if and only if $S = \bigvee_{n\in\mathbb{N}} S_n$ where for each $n\in\mathbb{N}$, S_n is a closed sublocale for which there exists a cozero sublocale C_n and a zero sublocale Z_n satisfying $S_n \leq Z_n \leq C_n \leq C_n^\circ \leq S$.*

(2) *S is a zero sublocale if and only if $S = \bigwedge_{n\in\mathbb{N}} S_n$ where for each $n\in\mathbb{N}$, S_n is an open sublocale for which there exists a zero sublocale Z_n and a cozero sublocale C_n satisfying $S \leq \overline{Z_n} \leq Z_n \leq C_n \leq S_n$.*

(3) *S is both a cozero and zero sublocale if and only if it is both closed and open.*

Proof: (1) Let S be a cozero sublocale. By Proposition 5.2, we may write $S = \mathfrak{c}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ where each $a_n \prec\!\prec a$. Then, using Lemma 5.4.2 and 2.2.2, we have $\mathfrak{c}(a_n) \leq Z_n \leq C_n \leq C_n^\circ \leq \mathfrak{c}(a)$ for some cozero sublocale C_n and some zero sublocale Z_n. The converse follows from the fact that countable joins of cozero sublocales are cozero sublocales.
(2) can be be proved similarly by complementation.
(3) For sufficiency: let $S = \mathfrak{c}(a) = \mathfrak{o}(b)$, then a and b are complements of each other and hence cozero elements. ∎

5.6.2. Corollary. *Every zero sublocale of a zero sublocale of L is a zero sublocale of L.*

Proof: Let $S = \mathfrak{o}(a)$ be a zero sublocale of L and let T be a zero sublocale of S. Then a is a cozero element, that is, $a = \bigvee_{n\in\mathbb{N}} a_n$ where $a_n \prec a_{n+1}$ for each $n \in \mathbb{N}$. The fact that $a_n \prec a_{n+1}$ in L means that there is some $c_n \in L$ such that $a_n \wedge c_n = 0$ and $a_{n+1} \vee c_n = 1$. On the other hand, T being an open sublocale of S means that $T = \mathfrak{o}(b) \cap S = \mathfrak{o}(b) \cap \mathfrak{o}(a) = \mathfrak{o}(a \wedge b)$ for some $b \in S$ and being a zero sublocale means that b is a cozero element of S. Therefore $b = \bigvee_m b_m$ with $b_m \prec b_{m+1}$ in S and, consequently, there is some $d_m \in S$ such that $b_m \wedge d_m = a \to 0 = a^*$ and $b_{m+1} \vee d_m = 1$. But then each $c_n \vee d_n$ satisfies

$$(a_n \wedge b_n) \wedge (c_n \vee d_n) = a_n \wedge b_n \wedge d_n = a_n \wedge a^* \leq a \wedge a^* = 0$$

and

$$(a_{n+1} \wedge b_{n+1}) \vee (c_n \vee d_n) = (a_{n+1} \vee c_n \vee d_n) \wedge (b_{n+1} \vee c_n \vee d_n) = 1.$$

Hence $a_n \wedge b_n \prec a_{n+1} \wedge b_{n+1}$ in L. Since $a \wedge b = \bigvee_{n\in\mathbb{N}}(a_n \wedge b_n)$, this makes sure that $a \wedge b$ is a cozero element of L and thus T is a zero sublocale of L. ■

5.7. Remark. Consider the characteristic function χ_S defined for a complemented sublocale S of L by

$$[\chi_S > p] = \begin{cases} 1 & \text{if } p < 0, \\ S^* & \text{if } 0 \le p < 1, \\ 0 & \text{if } p \ge 1, \end{cases} \quad \text{and} \quad [\chi_S < q] = \begin{cases} 0 & \text{if } q \le 0, \\ S & \text{if } 0 < q \le 1, \\ 1 & \text{if } q > 1. \end{cases}$$

These functions are precisely the idempotents of the ring $\mathrm{F}(L)$ (see [17]). In any ring A, the set $\mathrm{Idp}(A)$ of idempotents forms a Boolean algebra with meet, join and complement given by

$$a \wedge b = ab, \quad a \vee b = a + b - ab, \quad \neg a = 1 - a.$$

This shows immediately that the map $\mathrm{F}(L) \to \mathfrak{S}(L)$ given by $f \mapsto \mathrm{Coz}(f)$ restricts to a Boolean isomorphism between $\mathrm{Idp}(\mathrm{F}(L))$ and the set of all complemented sublocales of L (in particular, there is a Boolean isomorphism between $\mathrm{Idp}(\mathrm{C}(L))$ and the set of all complemented elements of L). This isomorphism might be helpful for a better understanding of the class of complemented sublocales of a locale L and problem 7.3 raised at the end of [24].

6. z-embeddings

6.1. Proposition. *Let $\varphi\colon L \to M$ be a localic map. Then:*

(1) $\varphi_{-1}[-]$ *preserves cozero sublocales.*
(2) $\varphi_{-1}[-]$ *preserves zero sublocales.*

Proof: Since $\varphi_{-1}[-]$ preserves complements, it is enough to show one of the statements. Let $C = \mathrm{Coz}_M(g)$ for some $\mathbf{0} \le g \in \mathrm{C}(M)$. Since $\varphi_{-1}[-]$ assigns closed sublocales to closed sublocales, the composite $f \equiv g \circ \varphi[-]$ belongs to $\mathrm{C}(L)$.

$$\begin{tikzcd} \mathfrak{S}(L) \arrow[d, shift right, "\varphi[-]"'] \arrow[drr, bend left, "f"] & & \\ \mathfrak{S}(M) \arrow[u, shift right, "\varphi_{-1}[-]"'] \arrow[rr, "g"'] & & \mathfrak{L}(\mathbb{R}) \end{tikzcd}$$

Finally, $\varphi_{-1}[C] = \varphi_{-1}[g^*(0,\text{—})] = (g \circ \varphi[-])^*(0,\text{—}) = \mathrm{Coz}_L(f)$. ■

6.2. Therefore each localic map $\varphi\colon L \to M$ induces frame homomorphisms
$$\varphi_{-1}[-]\colon \mathrm{Coz}\,(\mathfrak{S}(M)) \to \mathrm{Coz}\,(\mathfrak{S}(L))$$
and
$$\varphi_{-1}[-]\colon \mathrm{Z}\,(\mathfrak{S}(M)) \to \mathrm{Z}\,(\mathfrak{S}(L)).$$
The former will be called *the cozero map* and the latter will be called *the zero map.*

Of course, the zero map is onto (and one says that φ is a *z-embedding*) iff the cozero map is onto (when one says that φ is *coz-onto*); on the other hand, the zero map is dense (that is, $\varphi_{-1}[\mathfrak{o}(a)] = 0$ implies $\mathfrak{o}(a) = 0$; one refers to φ as a *z-dense* map) iff the cozero map is codense (that is, $\varphi_{-1}[\mathfrak{c}(a)] = 1$ implies $\mathfrak{c}(a) = 1$; in this case one says that φ is *coz-codense*).

In particular, for each sublocale S of L, the cozero map $j_{-1}\colon \mathfrak{S}(L) \to \mathfrak{S}(S)$ induced by the embedding $j\colon S \hookrightarrow L$ is given by $j_{-1}[T] = T \vee S = T \cap S$. When j is coz-onto, the sublocale S is said to be *z-embedded* in L. We have, immediately:

6.2.1. Proposition. *A sublocale S of L is z-embedded if and only if for every zero sublocale Z of S there is a zero sublocale W of L such that $W \cap S = Z$.* ■

This property captures the corresponding notion of being z-embedded for a subspace of a topological space.

6.2.2. Remarks. (1) Let S be a sublocale of L. Recall that an $f \in \mathrm{C}^*(S)$ is said to have a *continuous bounded extension* to L if there exists a $\overline{f} \in \mathrm{C}^*(L)$ such that $\overline{f} j_S = f$ for the embedding $j_S\colon \mathfrak{S}(S) \to \mathfrak{S}(L)$. The sublocale S is then said to be *C^*-embedded in L* if every $f \in \mathrm{C}^*(S)$ has a continuous bounded extension to L.

Each C^*-embedded sublocale of L is z-embedded in L since every zero sublocale of S is the zero sublocale of a bounded continuous real function (by Remark (2) in 5.3.1).

(2) It also follows immediately that an open sublocale $\mathfrak{o}(a)$ is z-embedded in L iff the surjection $a \wedge -\colon L \to {\downarrow}a$ is a coz-surjection, that is, iff every cozero element of ${\downarrow}a$ is the image under $(a \wedge -)$ of some cozero element in L. Thus an open sublocale $\mathfrak{o}(a)$ is z-embedded iff the element a is what is usually called a *coz-embedded* element [4].

(3) Recall from [4] that a frame L is an *Oz frame* if every $a \in L$ is coz-embedded. Then, immediately:

A locale L is Oz if and only if every open sublocale is z-embedded in L.

6.3. z-Open and z-closed maps. Inspired by the corresponding classical notions in [9, 27], we say that a localic map $\varphi\colon L \to M$ is *z-open* if for every $Z \in \mathrm{Z}(\mathfrak{S}(L))$, $\varphi[Z]$ is an open sublocale of M. Similarly, φ is *coz-closed* if $\varphi[C]$ is a closed sublocale of M for every cozero sublocale C of L. On the other hand, φ is *z-closed* (or *coz-open*) if $\varphi[C]^\circ \le \overline{\varphi[Z]}$ for every zero sublocale Z of L and every cozero sublocale C of L such that $C \le Z$.

6.3.1. Remark. Clearly, open maps and zero-preserving maps (i.e., localic maps that preserve zero sublocales) are z-open. Moreover, "closed+z-open" implies "z-closed".

6.3.2. Proposition. *Let $\varphi\colon L \to M$ be a localic map. If L is completely regular and φ is z-open, then φ is open.*

Proof: Let S be an open sublocale of L. By Corollary 5.5 we may write $S = \bigwedge Z_i$ for some zero sublocales Z_i. Then $\varphi[S] = \bigwedge \varphi[Z_i]$. By the hypothesis, each $\varphi[Z_i]$ is open in M and hence $\varphi[S]$ is also open in M. ■

7. Disconnectivity and perfect normality

7.1. Now recall from [1] that a frame L is *basically disconnected* if $a^* \vee a^{**} = 1$ for every cozero element a. Further, L is a *P-frame* (resp. *almost P-frame*) if $a^* \vee a = 1$ (resp. $a = a^{**}$) for every cozero element a.

7.1.1. Proposition. *The following are equivalent for a frame L:*

(1) *L is basically disconnected.*
(2) *For every zero sublocale $\mathrm{Z}(f)$ of L, $\overline{\mathrm{Z}(f)} \vee \overline{\mathrm{Coz}(f)^\circ} = 1$.*
(3) *For every zero sublocale $\mathrm{Z}(f)$ of L and every $a \in L$,*

$$\mathrm{Z}(f) \vee \mathfrak{o}(a) = 1 \implies \overline{\mathrm{Z}(f)} \vee \overline{\mathfrak{o}(a)} = 1.$$

(4) *For every zero sublocale $\mathrm{Z}(f)$ of L, $\overline{\mathrm{Z}(f)}$ is an open sublocale.*

Proof: (1)⇒(2): Let $\mathrm{Z}(f) = \mathfrak{o}(b)$ with b a cozero element of L. Then $\overline{\mathrm{Z}(f)} \vee \overline{\mathrm{Coz}(f)^\circ} = \mathfrak{c}(b^*) \vee \mathfrak{c}(b^{**}) = 1$.

(2)⇒(3): Let $\mathrm{Z}(f)$ be a zero sublocale such that $\mathrm{Z}(f) \vee \mathfrak{o}(a) = 1$. Then $\mathrm{Coz}(f) \le \mathfrak{o}(a)$ and hence by the hypothesis we may conclude that $1 = \overline{\mathrm{Z}(f)} \vee \overline{\mathrm{Coz}(f)} \le \overline{\mathrm{Z}(f)} \vee \overline{\mathfrak{o}(a)}$.

(3)⇒(4): Let $\mathrm{Z}(f) = \mathfrak{o}(b)$ with $b \in \mathrm{Coz}\, L$. Since $\mathrm{Z}(f) \vee \mathfrak{o}(b^*) = 1$, it follows that $1 = \overline{\mathrm{Z}(f)} \vee \overline{\mathfrak{o}(b^*)}$. Hence $\overline{\mathrm{Z}(f)} \ge \overline{\mathfrak{o}(b^*)}^{\,*} = \overline{\mathrm{Z}(f)}^{\,\circ}$.

(4)⇒(1): Let a be a cozero element of L. Since $\mathfrak{o}(a)$ is a zero sublocale it follows that $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ is an open sublocale and so there exists $b \in L$ such that $\mathfrak{o}(a^*) = \mathfrak{c}(b)$. Hence $1 = a^* \vee b \leq a^* \vee a^{**}$. ■

7.1.2. Proposition. *Let L be a frame.*

(1) *L is a P-frame if and only if every zero sublocale is a closed sublocale.*
(2) *L is an almost P-frame if and only if every zero sublocale is the interior of its closure.*

Proof: (1) Let $\mathrm{Z}(f) = \mathfrak{o}(a)$ for some cozero element $a \in L$. Then $a^* \vee a = 1$ and so $\mathrm{Z}(f) = \mathfrak{c}(a^*)$. Conversely, let $a \in L$ be a cozero element. Since $\mathfrak{o}(a)$ is a zero sublocale it follows that there exists $b \in L$ such that $\mathfrak{o}(a) = \mathfrak{c}(b)$. Hence $1 = a \vee b \leq a \vee a^*$.

To conclude (2) just notice that $a = a^{**}$ is equivalent to $\mathfrak{o}(a) = \left(\overline{\mathfrak{o}(a)}\right)^{\circ}$. ■

7.2. The concept of perfectness for topological spaces (due to Heath and Michael [19]) is formulated in frames in the following manner [16]:

A frame is *perfect* if every open sublocale is an F_σ-sublocale, that is, if for each $a \in L$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in L such that $\mathfrak{o}(a) = \bigwedge_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ (equivalently, if every open sublocale is a F_σ-sublocale). By a *perfectly normal frame* we understand a perfect plus normal frame.

Contrarily to the situation in spaces, in the category of frames the dual notion of perfectness (that every closed sublocale is a G_δ-sublocale) is not equivalent (it is in fact stronger). However, under normality, as proved in [14, Proposition 3.5], these conditions are equivalent, and they are also equivalent with the condition

$$\forall a \in L\ \exists (a_n)_{n\in\mathbb{N}} \subseteq L \colon a = \bigvee_{n\in\mathbb{N}} a_n, a_n \prec a. \tag{7.2.1}$$

Thus we have:

7.2.1. Proposition. *A frame is perfectly normal if and only if every open sublocale is a zero sublocale.*

Proof: The necessity follows immediately from the fact (also proved in [14, Proposition 3.5]) that under normality any a satisfying (7.2.1) is a cozero element.

Conversely, if every open sublocale is a zero sublocale then the set of all cozero sublocales is precisely $\mathfrak{c}[L]$. Thus the set $Coz(\mathfrak{S}(L))$ of all cozero sublocales is isomorphic to L and the normality of L follows from the normality of $Coz(\mathfrak{S}(L))$. Perfectness of L is obvious. ■

7.2.2. Corollary. *If a frame L is perfectly normal then every sublocale of L is z-embedded.*

Proof: Let S be a sublocale of L and let Z be a zero sublocale of S. Since Z is an open sublocale of S, it is of the form $\mathfrak{o}(a) \cap S$ for some $a \in S$. By 7.2.1, $\mathfrak{o}(a)$ is a zero sublocale of L. The conclusion follows from Proposition 6.2.1. ■

7.2.3. Remark. This gives numerous examples of z-embedded sublocales that are not C^*-embedded. For example, any non C^*-embedded sublocale of $\mathfrak{L}(\mathbb{R})$ is z-embedded. This means that z-embedding is a considerable weakening of C^*-embedding.

8. The support of a real function

8.1. Small sublocales. We say that a sublocale S of L is *small* if any cozero sublocale contained in S is compact. We set

$$\mathrm{F}_s(L) = \{f \in \mathrm{F}(L) \mid [f = 0] \text{ is small}\}$$

and

$$\mathrm{C}_s(L) = \mathrm{F}_s(L) \cap \mathrm{C}(L) = \{f \in \mathrm{C}(L) \mid \mathrm{Z}(f) \text{ is small}\}.$$

Recall from [10] that an element $a \in L$ is said to be *small* if for any cozero element $x \in L$ such that $a \vee x = 1$ the sublocale $\mathfrak{c}(x)$ is compact.

8.1.1. Proposition. *An element $a \in L$ is small if and only if the sublocale $\mathfrak{o}(a)$ is small.*

Proof: $\Rightarrow$: Let a be small and let S be a cozero sublocale contained in $\mathfrak{o}(a)$. Then $S = \mathfrak{c}(x)$ for some cozero element x so that

$$S = \mathfrak{c}(x) \subseteq \mathfrak{o}(a) \Leftrightarrow \mathfrak{c}(x) \geq \mathfrak{o}(a) \Leftrightarrow a \vee x = 1.$$

Hence $\mathfrak{c}(a)$ is compact.

$\Leftarrow$: Let $\mathfrak{o}(a)$ be small and x a cozero element in L such that $a \vee x = 1$. Arguing as above we obtain that $\mathfrak{c}(x)$ is a zero sublocale contained in $\mathfrak{o}(a)$. Hence $\mathfrak{c}(a)$ is compact. ■

8.1.2. As remarked earlier in 3.5, there is a 1-1 correspondence between $\mathrm{C}(L)$ and the set $\mathcal{R}(L)$ of all frame homomorphisms $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$. We will use the following notation. Given a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$, we denote by f_φ the continuous real function determined by the frame homomorphism

$$(f_\varphi)^* = \mathfrak{c}_L \circ \varphi\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{Z}(L). \tag{8.1.2}$$

Note that $\mathrm{Coz}(f_\varphi) = (f_\varphi)^*((-,0)\vee(0,-)) = \mathfrak{c}(\varphi((-,0)\vee(0,-))) = \mathfrak{c}_L(\mathrm{coz}\,\varphi)$ and $\mathrm{Z}(f_\varphi) = \mathfrak{o}_L(\mathrm{coz}\,\varphi)$. It follows immediately from Proposition 8.1.1 that coz φ is small if and only if $\mathrm{Z}(f_\varphi)$ is small. Moreover, denoting by $\mathcal{R}_s(L)$ the set of all frame homomorphisms $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ whose corresponding cozero elements are small, we have:

8.1.3. Corollary. *There is a 1-1 correspondence between* $\mathcal{R}_s(L)$ *and* $\mathrm{C}_s(L)$.

Proof: Just note that

$$\varphi \in \mathcal{R}_s(L) \Leftrightarrow \text{coz } \varphi \text{ is small} \Leftrightarrow \mathrm{Z}(f_\varphi) \text{ is small} \Leftrightarrow f_\varphi \in \mathrm{C}_s(L),$$

so that the 1-1 correspondence between $\mathcal{R}(L)$ and $\mathrm{C}(L)$ restricts to a 1-1 correspondence between $\mathcal{R}_s(L)$ and $\mathrm{C}_s(L)$. ■

8.2. The support of a real function. Given a real function $f \in \mathrm{F}(L)$, the *support* of f is the closed sublocale $\overline{[f = 0]}$; we denote it by $\mathrm{supp}(f)$. In particular, if $f \in \mathrm{C}(L)$ then $\mathrm{supp}(f) = \overline{\mathrm{Z}(f)}$.

We set

$$\mathrm{F}_K(L) = \{f \in \mathrm{F}(L) \mid \mathrm{supp}(f) \text{ is compact}\}$$

and

$$\mathrm{C}_K(L) = \mathrm{F}_K(L) \cap \mathrm{C}(L).$$

8.2.1. Example. For the characteristic function χ_S (defined for any complemented sublocale S of L), it is clear that $\mathrm{Coz}(\chi_S) = S^*$. Therefore $\mathrm{supp}(\chi_S) = \overline{S}$ and hence $\chi_S \in \mathrm{C}_K(L)$ if and only if $\overline{S}$ is compact.

8.2.2. Observe that $\mathrm{F}_K(L) \subseteq \mathrm{F}_s(L)$ and $\mathrm{C}_K(L) \subseteq \mathrm{C}_s(L)$. Indeed, let $f \in \mathrm{F}(L)$ such that $\mathrm{supp}(f)$ is compact and $\mathrm{Coz}(g)$ a cozero sublocale contained in $[f = 0]$. Then $\mathrm{Coz}(g) \subseteq \overline{[f = 0]}$ and hence $\mathrm{Coz}(g)$ is compact. It follows that $[f = 0]$ is small and so $f \in \mathrm{F}_s(L)$.

Moreover, for any P-frame L, $\mathrm{C}_K(L) = \mathrm{C}_s(L)$. Indeed, let $f \in \mathrm{F}(L)$ such that $\mathrm{Z}(f)$ is small. Then, by Proposition 7.1.2, $\mathrm{Z}(f)$ is a closed sublocale and hence a cozero sublocale, by Proposition 5.6.1 (3). It follows that $\mathrm{supp}(f) =$

$\overline{\mathrm{Z}(f)} = \mathrm{Z}(f)$ is compact (since $\mathrm{supp}(f)$ is a cozero sublocale contained in the small sublocale $\mathrm{Z}(f)$).

8.2.3. Remark. We point out that the support of a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ is implicitly defined in [10, p. 212] as

$$\mathrm{supp}(\varphi) = \uparrow(\mathrm{coz}\ \varphi)^* = \mathfrak{c}_L\left((\mathrm{coz}\ \varphi)^*\right) = \overline{\mathfrak{o}_L\left(\mathrm{coz}\ \varphi\right)}.$$

Let $\mathcal{R}_K(L)$ denote the set of all frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ with compact support. Since $\mathrm{supp}(f_\varphi) = \overline{\mathrm{Z}(f_\varphi)} = \overline{\mathfrak{o}_L(\mathrm{coz}\,\varphi)} = \mathrm{supp}(\varphi)$, we have immediately the following:

8.2.4. Corollary. *There is a 1-1 correspondence between* $\mathcal{R}_K(L)$ *and* $\mathrm{C}_K(L)$.

Proof: Just note that

$$\varphi \in \mathcal{R}_K(L) \Leftrightarrow \mathrm{supp}(\varphi) \text{ is compact} \Leftrightarrow f_\varphi \in \mathrm{C}_K(L),$$

so that the 1-1 correspondence between $\mathcal{R}(L)$ and $\mathrm{C}(L)$ restricts to a 1-1 correspondence between $\mathcal{R}_K(L)$ and $\mathrm{C}_K(L)$. ■

8.2.5. Remark. In [10, Page 212] it is erroneously stated that $\mathcal{R}_K(L) = \mathcal{R}_s(L)$ holds for any basically disconnected frame. Of course, the intended statement is for "P-frames" instead of "basically disconnected frames", that follows from 8.2.2 and corollaries 8.1.3 and 8.2.4 above. The rest of [10], however, is not affected by this slight mistake.

8.3. Real functions which vanish at infinity. We say that a real function $f \in \mathrm{F}(L)$ *vanishes at infinity* if $[-\frac{1}{n} < f < \frac{1}{n}]$ is compact for every $n \in \mathbb{N}$. Let

$$\mathrm{F}_\infty(L) = \{f \in \mathrm{F}(L) \mid f \text{ vanishes at infinity}\}$$

and

$$\mathrm{C}_\infty(L) = \mathrm{F}_\infty(L) \cap \mathrm{C}(L).$$

Note that $\mathrm{F}_s(L) \subseteq \mathrm{F}_\infty(L)$ and $\mathrm{C}_s(L) \subseteq \mathrm{C}_\infty(L)$. Indeed, let $f \in \mathrm{F}(L)$ such that $\mathrm{Z}(f)$ is small. Then $[-\frac{1}{n} < f < \frac{1}{n}]$ is a cozero sublocale contained in $\mathrm{Z}(f)$ and hence $[-\frac{1}{n} < f < \frac{1}{n}]$ is compact for every $n \in \mathbb{N}$.

8.3.1. Remark. Note that this concept was originally considered for frame homomorphisms in [10]: a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ *vanishes at infinity* if $\uparrow\varphi\left(-\frac{1}{n}, \frac{1}{n}\right)$ is compact for each $n \in \mathbb{N}$.

Let $\mathcal{R}_\infty(L)$ denote the set of all frame homomorphisms $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ which vanish at infinity. Note that

$$[-\tfrac{1}{n} < f_\varphi < \tfrac{1}{n}] = \mathfrak{c}_L(\varphi(-\tfrac{1}{n}, \tfrac{1}{n})) = \uparrow\varphi(-\tfrac{1}{n}, \tfrac{1}{n}).$$

This yields immediately the following:

8.3.2. Corollary. *There is a 1-1 correspondence between $\mathcal{R}_\infty(L)$ and $\mathrm{C}_\infty(L)$.*

Proof: Just note that

$$\varphi \in \mathcal{R}_\infty(L) \Leftrightarrow \uparrow\varphi(-\tfrac{1}{n}, \tfrac{1}{n}) \text{ is compact for every } n \in \mathbb{N} \Leftrightarrow f_\varphi \in \mathrm{C}_K(L),$$

so that the 1-1 correspondence between $\mathcal{R}(L)$ and $\mathrm{C}(L)$ restricts to a 1-1 correspondence between $\mathcal{R}_\infty(L)$ and $\mathrm{C}_\infty(L)$. ■

References

[1] R. N. Ball and J. Walters-Wayland, *C- and C*-quotients in Pointfree Topology*, Dissert. Math, Vol. **412**, 2002.

[2] B. Banaschewski, Compactifications of frames, *Math. Nachr.* **149** (1990) 105–116.

[3] B. Banaschewski, *The Real Numbers in Pointfree Topology*, Textos de Matemática, Vol. **12**, University of Coimbra, 1997.

[4] B. Banaschewski, T. Dube, C. Gilmour and J. Walters-Wayland, Oz in pointfree topology, *Quaest. Math.* **32** (2010) 215–227.

[5] B. Banaschewski and C. Gilmour, Pseudocompactness and the cozero part of a frame, *Comment. Math. Univ. Carolinae* **37** (1996) 577–587.

[6] B. Banaschewski, J. Gutiérrez García and J. Picado, Extended real functions in pointfree topology, *J. Pure Appl. Algebra* **216** (2012) 905–922.

[7] B. Banaschewski, S. S. Hong and A. Pultr, On the completion of nearness frames, *Quaest. Math.* **21** (1998) 19–37.

[8] B. Banaschewski and A. Pultr, A constructive view of complete regularity, *Kyungpook Math. J.* **43** (2003) 257–262.

[9] R. L. Blair and E. van Douwen, Nearly realcompact spaces, *Topology Appl.* **47** (1992) 209–221.

[10] T. Dube, On the ideal of functions with compact support in pointfree function rings, *Acta Math. Hungar.* **129** (2010) 205–226.

[11] T. Dube and J. Walters-Wayland, Coz-onto frame maps and some applications, *Appl. Categ. Struct.* **15** (2007) 119–133.

[12] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960.

[13] J. Gutiérrez García and T. Kubiak, General insertion and extension theorems for localic real functions, *J. Pure Appl. Algebra* **215** (2011) 1198–1204.

[14] J. Gutiérrez García, T. Kubiak and J. Picado, Pointfree forms of Dowker's and Michael's insertion theorems, *J. Pure Appl. Algebra* **213** (2009) 98–108.

[15] J. Gutiérrez García, T. Kubiak and J. Picado, Localic real functions: a general setting, *J. Pure Appl. Algebra* **213** (2009) 1064–1074.

[16] J. Gutiérrez García, T. Kubiak and J. Picado, *Perfectness in locales*, in preparation.

[17] J. Gutiérrez García and J. Picado, Rings of real functions in pointfree topology, *Topology Appl.* **158** (2011) 2264–2278.

[18] J. Gutiérrez García and J. Picado, Insertion and extension results for point-free complete regularity, *Bull. Belg. Math. Soc. Simon Stevin* **20** (2013) 675–687.
[19] R. W. Heath and E. A. Michael, A property of the Sorgenfrey line, *Compositio Math.* **23** (1971) 185–188.
[20] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* **31** (1972) 5–32.
[21] J. R. Isbell, I. Kříž, A. Pultr and J. Rosický, Remarks on localic groups, in: *Categorical Algebra and its Applications* (Proc. Int. Conf. Louvain-La-Neuve 1987, ed. by F. Borceux), Lecture Notes in Math. **1348**, pp. 154-172.
[22] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, Cambridge, 1982.
[23] A. Joyal, *Nouveaux fondaments de l'analyse*, Lectures Montréal 1973 and 1974 (unpublished).
[24] J. Picado and A. Pultr, Sublocale sets and sublocale lattices, *Arch. Math.* (*Brno*) **42** (2006) 409–418.
[25] J. Picado and A. Pultr, *Locales treated mostly in a covariant way*, Textos de Matemática, Vol. **41** University of Coimbra, 2008.
[26] J. Picado and A. Pultr, *Frames and Locales: Topology without points*, Frontiers in Mathematics, vol. **28**, Springer Basel, 2012.
[27] M. Weir, *Hewitt-Nachbin Spaces*, Mathematics Studies, vol. **17**, North-Holland, 1975.

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