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ZERO-INFLATED COMPOUND POISSON DISTRIBUTIONS IN INTEGER-VALUED GARCH MODELS

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Abstract: In this paper we introduce a wide class of integer-valued stochastic processes that allows to take into consideration, simultaneously, the main characteristics observed in count data namely zero inflation and overdispersion. This class includes, in particular, the CP-INGARCH model ([**7**]), the zero-inflated Poisson and the zero-inflated negative binomial INGARCH models, introduced by Zhu in [**18**]. The main probabilistic analysis of this process is here developed. Precisely, first and second order stationarity conditions are derived as well as the autocorrelation function. We also analyse the existence of higher-order moments and deduce the explicit form for the first four cumulants for some class of models, such as its skewness and kurtosis, reencountering published results of [**5**] and [**13**].

Keywords: integer-valued GARCH model, zero inflation, compound distributions.
AMS Subject Classification (2000): this environment is optional.

1. Introduction

Count time series are quite common in various scientific fields like medicine, economics, finance, tourism and queuing systems. The modelling of these time series has been receiving increasing attention and several integer-valued stochastic models have been recently proposed and developed in order to best describe and analyze this kind of data.

The change of the series variability is often observed in count time series which lead to the proposal of conditionally heteroskedastic models. The integer-valued process proposed by Ferland, Latour and Oraichi [**5**], denoted INGARCH(p, q) and inspired in the GARCH models of Bollerslev [**2**], takes into account this characteristic. In fact, Ferland, Latour and Oraichi [**5**] assume a Poissonian conditional distribution whose parameter evolves with the past of the process similarly to GARCH models. Following this idea, several models have been introduced in literature considering other deviates discrete distributions like the negative binomial ([**16**]), the generalized Poisson ([**17**]) or a general compound Poisson distribution ([**7**]).

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Excess of zeros is another fact often observed in count time series. The interest of this characteristic is clear because zero counts frequently have special status.

Neyman [**11**] and Feller [**3**] first introduced the concept of zero inflation to address the problem of excess of zeros. Since then, there have been extensive studies related to the development of zero-inflated probability processes, in particular Poisson models, essentially considered in econometric literature and in regression context. Application areas are diverse and have included situations that produce a low fraction of non-conforming units, road safety, species abundance and processes related to health where it is of interest the monitoring of a rare disease. [**12**] include several references and details.

A zero-inflated distribution can be viewed as a mixture of a degenerate distribution with mass at zero and a nondegenerate distribution. For example, the random variable X is zero-inflated Poisson (λ, ω) distributed ([**10**]) if its probability mass function can be written in the form

$$P(X=k) = \omega\delta_{k,0} + (1-\omega)\frac{\lambda^k e^{-\lambda}}{k!}, \ k = 0, 1, 2, ...,$$

where $0 < \omega < 1$, $\delta_{k,0}$ is the Kronecker delta, i.e., $\delta_{k,0}$ is 1 when $k = 0$ and is zero when $k \neq 0$. This distribution is denoted by ZIP.

In count time series context, [**1**], [**9**] and [**18**] are recent works dedicated to the proposal and study of zero-inflated models for integer-valued time series, namely the zero-truncated Poisson INAR(1) model, the zero-inflated INAR(1) model and the zero-inflated Poisson and Negative Binomial integer-valued GARCH models. For example, the ZIP-INGARCH(p, q) ([**18**]) is defined as

$$\begin{cases} X_t | \underline{X}_{t-1} : ZIP(\lambda_t, \omega), \ \forall t \in \mathbb{Z}, \\ \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases}$$

where $0 < \omega < 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$, $\beta_k \geq 0$, $j = 1, ..., p$, $k = 1, ..., q$, $p \geq 1$, $q \geq 0$ and $\underline{X}_{t-1}$ the σ-field generated by $\{X_{t-j}, j \geq 1\}$. The Zero-Inflated Negative Binomial INGARCH model is analogously defined by Zhu [**18**], giving weak stationarity conditions and the autocorrelation function and proposing an estimating procedure.

With the aim of enlarging and unifying the study of these several models we introduce in this paper an Eventually Zero-Inflated INteger-valued GARCH process with general Compound Poisson deviates. We include in this general class the CP-INGARCH models ([**7**]) corresponding to $\omega = 0$, and if $0 < \omega < 1$ we have, as particular cases, the ZIP-INGARCH and ZINB-INGARCH models.

Additionally to the zero-inflated characteristic, in most count data sets the conditional variance is greater in value than the conditional mean, often much greater. This characteristic is known as overdispersion. For example, Xu et al. [**15**] present a study of weekly dengue cases observed in Singapure where the conditional overdispersion is highly significant. Our proposal have also the aim of modelling zero inflation, overdispersion and conditional heteroskedasticity in the same framework.

The paper is organized as follows. In Section 2 we introduce the Eventually Zero-Inflated Compound Poisson INGARCH model. The wide range of this proposal is stressed referring the most important models recently studied that it includes ([**18**], [**7**]) and also presenting the general procedure to obtain new models. A necessary and sufficient condition of first-order stationarity is given. Conditions of second-order stationarity are discussed in Section 3 based on a vectorial state space representation of the general EZICP-INGARCH process. In Section 4 we focus on the EZICP-INGARCH$(1,1)$ model and establish a necessary and sufficient condition for the existence of higher-order moments and give explicitly the first four cumulants of X_t from which skewness and kurtosis of the process are deduced. Section 5 gives some discussions and in Appendices A and B we summarize some auxiliary forms and calculations.

2. The model

Let $X = (X_t, t \in \mathbb{Z})$ be a stochastic process with values in $\mathbb{N}_0$ and, for any $t \in \mathbb{Z}$, let $\underline{X}_{t-1}$ be the $\sigma-$field generated by $\{X_{t-j}, j \geq 1\}$.

Definition 2.1. *The process X is said to satisfy an Eventually Zero-Inflated Compound Poisson INteger-valued GARCH model with orders p and q, ($p, q \in \mathbb{N}$), briefly an EZICP-INGARCH(p,q), if, $\forall t \in \mathbb{Z}$, the characteristic function of $X_t|\underline{X}_{t-1}$, Φ, is given by*

$$\begin{cases} \Phi(u) = \omega + (1-\omega)\exp\left\{ i\frac{\lambda_t}{\varphi_t'(0)}\left[\varphi_t(u) - 1\right]\right\}, \ u \in \mathbb{R}, \\ \\ \lambda_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{k=1}^{q} \beta_k \lambda_{t-k}, \end{cases} \tag{1}$$

for some constants $0 \leq \omega < 1$, $\alpha_0 > 0$, $\alpha_j \geq 0$ ($j = 1, ..., p$), $\beta_k \geq 0$ ($k = 1, ..., q$), and where $(\varphi_t, t \in \mathbb{Z})$ is a family of characteristic functions on $\mathbb{R}$, $\underline{X}_{t-1}$-measurable associated to a family of discrete laws with support $\mathbb{N}_0$ and finite mean. i denotes the imaginary unit.

We observe that the conditional distribution of X_t is a mixture of the Dirac law at zero with a discrete compound Poisson law. The probability at zero is then inflated with value ω.

The EZICP-INGARCH(p,q) model with $q = 1$ and $\beta_1 = 0$ is denoted by EZICP-INARCH(p) and, as mentioned before, when $\omega = 0$ we recover the CP-INGARCH(p,q) model considered in **[7]**.

The conditional mean and conditional variance of X_t are given by

$$E(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t,\ V(X_t|\underline{X}_{t-1}) = (1-\omega)\lambda_t\left(-i\frac{\varphi_t''(0)}{\varphi_t'(0)} + \omega\lambda_t\right).$$

In this case, to assure the variance existence, we consider that the characteristic functions (φ_t) are derivable at zero up to order 2.

As can be seen, a wide class of processes is included in the EZICP-INGARCH model (1). In fact, let $X = (X_t, t \in \mathbb{Z})$ be a stochastic process defined by

$$X_t = \sum_{j=1}^{N_t} X_{t,j}$$

where N_t is a random variable that, conditionally to $\underline{X}_{t-1}$, follows a zero-inflated Poisson law and $X_{t,1}$,..., X_{t,N_t} are discrete random variables with support $\mathbb{N}_0$ that, conditionally to $\underline{X}_{t-1}$, are independent, independent of N_t and having characteristic function φ_t derivable at zero. If the parameters of the probability mass function of N_t are (λ_t^*, ω), that is,

$$P(N_t = n) = \omega\delta_{n,0} + (1-\omega)\frac{(\lambda_t^*)^n e^{-\lambda_t^*}}{n!},\ n = 0, 1, 2, ...,$$

where $\lambda_t^* = \frac{i\lambda_t}{\varphi_t'(0)}$ and $0 \leq \omega < 1$, then X satisfies the model (1) as we have

$$\begin{aligned}\Phi_{X_t|\underline{X}_{t-1}}(u) &= \sum_{n=0}^{\infty} E[\exp\{iu\,(X_{t,1} + ... + X_{t,N_t})\}|N_t = n] \cdot P(N_t = n)\\ &= \omega\sum_{n=0}^{\infty}\varphi_t^n(u)\delta_{n,0} + (1-\omega)e^{-\lambda_t^*}\sum_{n=0}^{\infty}\varphi_t^n(u)\frac{(\lambda_t^*)^n}{n!}\\ &= \omega + (1-\omega)\exp\left\{\lambda_t^*\left[\varphi_t(u) - 1\right]\right\}.\end{aligned}$$

Based on this construction, many particular models can be deduced.

Example 2.1. ***(a)*** *When $\omega = 0$, as we recover the CP-INGARCH model we obtain, in particular, the INGARCH (***[5]***), negative binomial INGARCH, generalized Poisson INGARCH (***[16]***,* ***[17]***) and negative binomial DINARCH (***[15]***)*

models. For $0 < \omega < 1$, we have the zero-inflated Poisson INGARCH *and the zero-inflated negative binomial* INGARCH *models (*[**18**]*).*

(b) *Let us consider independent random variables $(X_{t,j}, t \in \mathbb{Z})$ following a geometric law with parameter $p_t = \frac{r}{r+\lambda_t}$ and $r > 0$ arbitrarily fixed, that is, $\varphi_t(u) = \frac{p_t e^{iu}}{1-(1-p_t)e^{iu}}$, $u \in \mathbb{R}$, $t \in \mathbb{Z}$. If N_t is a random variable independent of $X_{t,j}$ and following a zero-inflated Poisson law with parameters (r, ω), $0 \leq \omega < 1$ then the process $X_t = \sum_{j=1}^{N_t} X_{t,j}$ satisfies, unless an additive parameter r, the model* (1)*. In this case, the model will be denoted by* EZIGEOMP-INGARCH(p, q)*. For $\omega = 0$, we obtain the* GEOMP-INGARCH *model studied in* [**7**]*.*

(c) *As in the previous example, let us consider a sequence of independent random variables $(X_{t,j}, t \in \mathbb{Z})$ following a geometric law with parameter $p \in]0, 1[$ and N_t following a zero-inflated Poisson law with parameters $\lambda_t^* = p\lambda_t$ and ω. Then, $X_t = \sum_{j=1}^{N_t} X_{t,j}$ also satisfies the model* (1)*. In this case, the model will be denoted by* EZIGEOMP2-INGARCH(p, q)*.*

(d) *If $(X_{t,j}, t \in \mathbb{Z})$ are independent random variables following a Poisson distribution with parameter $\theta > 0$ and N_t is independent of $X_{t,j}$ and follows a zero-inflated Poisson law with parameters $\lambda_t^* = \frac{\lambda_t}{\theta}$ and ω, the resulting process X satisfies the model* (1)*. When $\omega = 0$, the $X_t|\underline{X}_{t-1}$ law is the Neyman Type A distribution with parameters (λ_t^*, θ) (*[**10**]*) and so we will denote this model by* EZINTA-INGARCH(p, q)*.*

We note that since the characteristic functions φ_t are $\underline{X}_{t-1}$-measurable, they may be random functions, which means that the parameter involved in φ_t may depends on the previous observations of the process, via λ_t, as in the EZIGEOMP-INGARCH model.

Figure 1 and Figure 2 present trajectories and the basic descriptives of EZIP and EZIGEOMP-INGARCH$(1, 1)$ models, with $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$, considering different values for ω, namely $\omega = 0$, 0.2, 0.4, illustrating clearly the zero-inflated characteristic of these models.

We observe that, whenever $\omega \geq 0$, the model (1) is overdispersed because

$$\frac{V(X_t|\underline{X}_{t-1})}{E(X_t|\underline{X}_{t-1})} = -i\frac{\varphi_t''(0)}{\varphi_t'(0)} + \omega\lambda_t \geq 1.$$

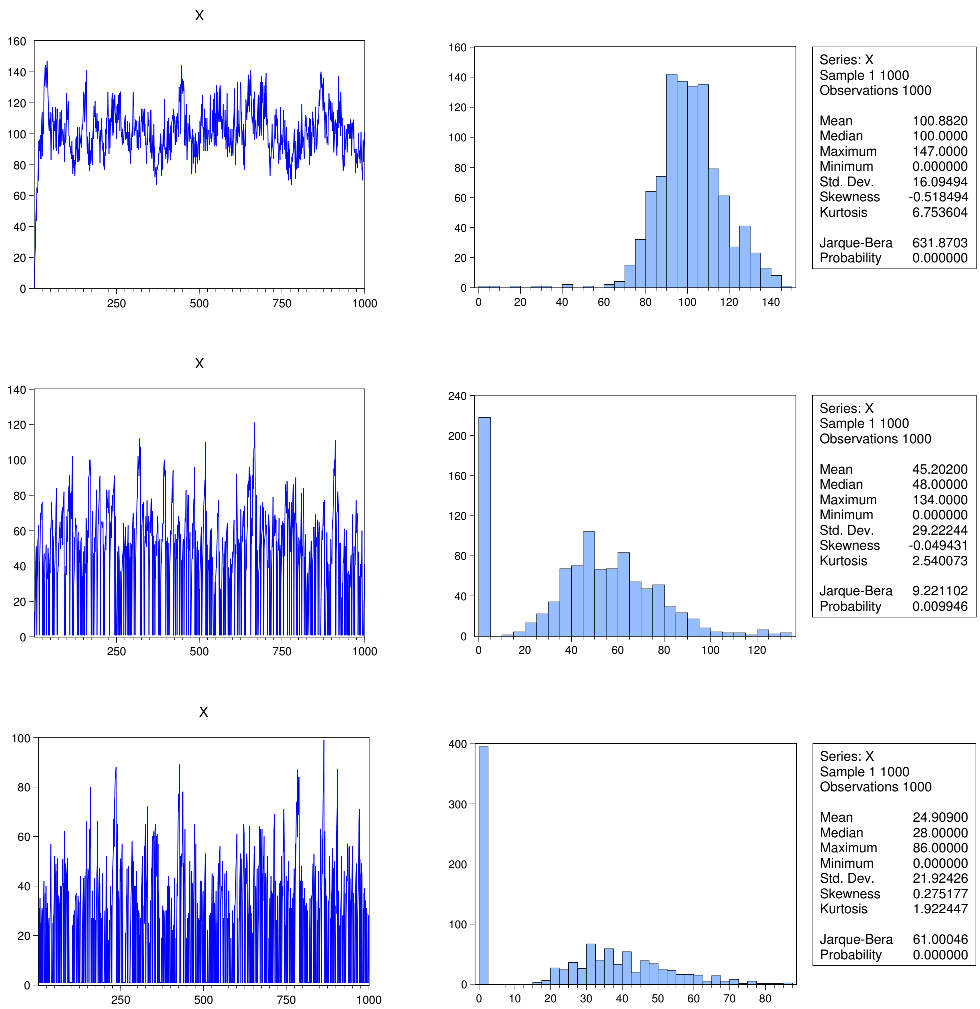

FIGURE 1. Trajectories and descriptives of EZIP-INGARCH(1, 1) model with $\omega = 0$ (on top), $\omega = 0.2$ (middle) and $\omega = 0.4$ (below): $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$.

We highlight that the definition of the model (1) is still valuable when the parameter ω takes negative values, provided that

$$0 \leq \omega + (1-\omega)P(X_t = 0|\underline{X}_{t-1}) \leq 1,$$

which is equivalent to

$$\max\left\{-1, -\frac{P(X_t = 0|\underline{X}_{t-1})}{1 - P(X_t = 0|\underline{X}_{t-1})}\right\} \leq \omega \leq 0.$$

For instance, when the conditional distribution is a mixture of a degenerate distribution with mass at zero and a Poisson or a negative binomial law as

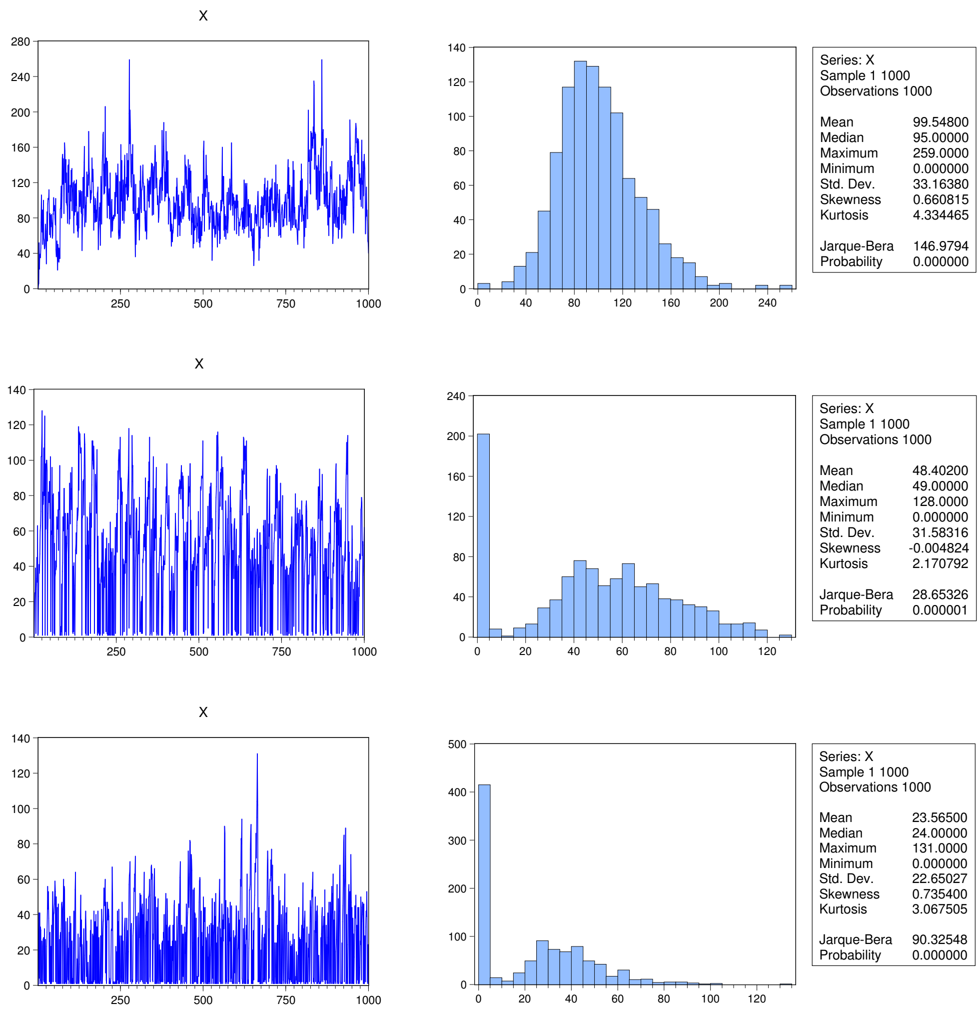

FIGURE 2. Trajectories and descriptives of EZIGEOMP-INGARCH$(1,1)$ model with $\omega = 0$ (on top), $\omega = 0.2$ (middle) and $\omega = 0.4$ (below): $\alpha_0 = 10$, $\alpha_1 = 0.4$, $\beta_1 = 0.5$.

defined in [**18**], we obtain, respectively,

$$\max\left\{-1, \frac{-e^{-\lambda_t}}{1-e^{-\lambda_t}}\right\} \leq \omega \leq 0$$

and

$$\max\left\{-1, \frac{-\left(\frac{1}{1+a\lambda_t^c}\right)^{\lambda_t^{1-c}/a}}{1-\left(\frac{1}{1+a\lambda_t^c}\right)^{\lambda_t^{1-c}/a}}\right\} \leq \omega \leq 0, \;\; c = 0, 1.$$

To consider negative values for ω corresponds to a deflation at point zero, and for the referred mixed Poisson distribution leads to models with under-dispersion.

3. First and second-order stationarity

The study of first and second-order stationarity of these processes follows the approach developed for the compound Poisson INGARCH processes in [**7**]. In the following we summarize the main conclusions of this study. We note that the results obtained are not affected by the form of the conditional law but mainly by the evolution of λ_t.

In what concerns first-order stationarity we consider $\mu_t = E(X_t)$ and we deduce from the difference equation

$$\mu_t = (1-\omega)\alpha_0 + \sum_{j=1}^{p}(1-\omega)\mu_{t-j} + \sum_{k=1}^{q}\beta_k\mu_{t-k},$$

that X is first-order stationary if, and only if $(1-\omega)\sum_{j=1}^{p}\alpha_j + \sum_{k=1}^{q}\beta_k < 1$. Moreover, under this condition, the processes (X_t) and (λ_t) are both first-order stationary and we have

$$\mu = E(X_t) = (1-\omega)E(\lambda_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)\sum_{j=1}^{p}\alpha_j - \sum_{k=1}^{q}\beta_k}.$$

In order to obtain second-order stationarity conditions for the Eventually Zero-Inflated CP-INGARCH model (1) we assume that the family of characteristic functions $(\varphi_t, t \in \mathbb{Z})$ is derivable at zero up to order 2, to assure the existence of the corresponding distribution variance. Furthermore, as for the CP-INGARCH(p,q) processes, we restrict our study to the subclass of EZICP-INGARCH models with φ_t satisfying the following condition:

Hypothesis H1: $-i\frac{\varphi_t''(0)}{\varphi_t'(0)} = v_0 + v_1\lambda_t$,

with $v_0 \geq 0$, $v_1 \geq 0$, not simultaneously zero. We stress that this particular case includes all the models presented in example 2.1 **(a)** as well as a wide class of models not studied in literature like the models introduced in example 2.1 **(b)** (for which $v_0 = 1$ and $v_1 = 2/r$), **(c)** (with $v_0 = (2-p)/p$ and $v_1 = 0$) and **(d)** (with $v_0 = 1+\theta$ and $v_1 = 0$).

A sufficient condition of second-order stationarity of X is easily deduced from the vectorial state space representation presented below.

Proposition 3.1. *Suppose that the process X is first-order stationary, satisfies the hypothesis* **H1** *and follows an* EZICP-INGARCH(p,q) *model. The vector of dimension $p+q-1$ given by*

$$W_t = \begin{bmatrix} E(X_t^2) \\ E(X_t X_{t-1}) \\ \cdots \\ E(X_t X_{t-(p-1)}) \\ E(\lambda_t \lambda_{t-1}) \\ \cdots \\ E(\lambda_t \lambda_{t-(q-1)}) \end{bmatrix},$$

$t \in \mathbb{Z}$, satisfies an autoregressive equation of order $\max(p,q)$*:*

$$W_t = B_0 + \sum_{k=1}^{\max(p,q)} B_k W_{t-k}, \tag{2}$$

where B_0 is a real vector of dimension $p+q-1$ and B_k $(k=1,\ldots,\max(p,q))$ are real squared matrices of order $p+q-1$.

Proof. This result is a consequence of the recursive equations satisfied by $E(X_t^2)$, $E(X_t X_{t-k})$ and $E(\lambda_t \lambda_{t-k})$ that may be deduced as the involved variables are nonnegative. We focus on the case $p=q$, whereas the other cases can be obtained from this one setting additional parameters to 0. So, following the same steps of Proposition 3.1 of **[7]**, we obtain

$$\begin{aligned} E(X_t^2) = C + (1+v_1) \Bigg[&\sum_{i=1}^{p} \left((1-\omega)\alpha_i^2 + \frac{2(1-\omega)\alpha_i\beta_i + \beta_i^2}{1+v_1} \right) E(X_{t-i}^2) \\ &+2 \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \alpha_j ((1-\omega)\alpha_i + \beta_i) E(X_{t-i} X_{t-j}) \\ &+2(1-\omega) \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \beta_j ((1-\omega)\alpha_i + \beta_i) E(\lambda_{t-i}\lambda_{t-j}) \Bigg] \end{aligned} \tag{3}$$

$$\begin{aligned} E(X_t X_{t-k}) = &\left[\alpha_0 - \frac{v_0 \beta_k}{1+v_1} \right] (1-\omega)\mu \\ &+(1-\omega) \left[\alpha_k + \frac{\beta_k}{1+v_1} \right] E(X_{t-k}^2) + \sum_{j=k+1}^{p} (1-\omega)^2 \beta_j E(\lambda_{t-j}\lambda_{t-k}) \\ &+ \sum_{j=1}^{k-1} ((1-\omega)\alpha_j + \beta_j) E(X_{t-j} X_{t-k}) + \sum_{j=k+1}^{p} (1-\omega)\alpha_j E(X_{t-j} X_{t-k}) \end{aligned} \tag{4}$$

$$E(\lambda_t \lambda_{t-k}) = \left[\alpha_0 - \frac{v_0(\alpha_k + \beta_k)}{1+v_1} \right] \frac{\mu}{1-\omega}$$

$$+\frac{\alpha_k+\beta_k}{(1-\omega)(1+v_1)}E(X^2_{t-k})+\sum_{j=k+1}^{p}\beta_j E(\lambda_{t-j}\lambda_{t-k})$$
$$+\sum_{j=k+1}^{p}\frac{\alpha_j}{1-\omega}E(X_{t-j}X_{t-k})+\sum_{j=1}^{k-1}((1-\omega)\alpha_j+\beta_j)E(\lambda_{t-j}\lambda_{t-k}) \quad (5)$$

with $C = v_0\mu + (1+v_1)\left[2\alpha_0\mu - (1-\omega)\alpha_0^2\right] - v_0\mu\sum_{i=1}^{p}(2(1-\omega)\alpha_i\beta_i+\beta_i^2)$ positive and independent of t and $k \geq 1$. Using the above expressions it is clear that $W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}$, with B_0 the vector and B_k $(k = 1, ..., p)$ the matrices presented in Appendix A. $\square$

In the following theorem we present the referred sufficient condition for weak stationarity of the process under study.

Theorem 3.1. *Let X be a first-order stationary process following a EZICP-INGARCH(p, q) model such that* **H1** *is satisfied. This process is weakly stationary if*

$$P(L) = I_{p+q-1} - \sum_{k=1}^{\max(p,q)} B_k L^k$$

is a polynomial matrix such that $\det P(z)$ *has all its roots outside the unit circle, where I_{p+q-1} is the identity matrix of order $p+q-1$ and B_k $(k = 1, ..., \max(p, q))$ are the squared matrices of the autoregressive equation* (2). *Moreover, if e_j is the order j row of the identity matrix,*

$$Cov(X_t, X_{t-j}) = e_{j+1}[P(1)]^{-1}B_0 - \mu^2, \ j = 0, ..., p-1,$$
$$Cov(\lambda_t, \lambda_{t-j}) = e_{p+j}[P(1)]^{-1}B_0 - \frac{\mu^2}{(1-\omega)^2}, \ j = 1, ..., q-1.$$

Let us consider a first-order stationary EZICP-INGARCH process with $p = q = 1$. The previous study leads us to the following weak stationarity characterization.

Theorem 3.2. *Consider a first-order stationary EZICP-INGARCH$(1, 1)$ model satisfying* **H1**. *A necessary and sufficient condition for weak stationarity is* $(1-\omega)(1+v_1)\alpha_1^2 + 2(1-\omega)\alpha_1\beta_1 + \beta_1^2 < 1$.

Proof. From expression (3), we obtain, in this case, the non-homogeneous difference equation of first order $E(X_t^2) - [(1-\omega)(1+v_1)\alpha_1^2 + 2(1-\omega)\alpha_1\beta_1 + \beta_1^2]E(X^2_{t-1}) = C$, where $C = v_0\mu + (1+v_1)[2\alpha_0\mu - (1-\omega)\alpha_0^2] - v_0\mu(\beta_1^2 + 2(1-\omega)\alpha_1\beta_1) > 0$.

If $(1-\omega)(1+v_1)\alpha_1^2+2(1-\omega)\alpha_1\beta_1+\beta_1^2<1$, then the above equation has an independent of t solution, *i.e.*, the process is second-order stationary. On the other hand, if the process is second-order stationary, $[1-(1-\omega)(1+v_1)\alpha_1^2+2(1-\omega)\alpha_1\beta_1+\beta_1^2]E(X_t^2)=C$. □

A necessary condition of weak stationarity for an EZICP-INGARCH(p,q) model can be obtained by generalizing the results of **[7]**, as we consider here a larger class of conditional distributions. In fact, for a second-order stationary process following an EZICP-INGARCH(p,q) model satisfying **H1** and such that $\alpha_0(1-\omega)(1+v_1)>v_0$, it is necessary that the roots of the equation $1-C_1z-...-C_rz^r=0$ lie outside the unit circle, where for $v=1,...,r-1$,

$$C_v=(1-\omega)(1+v_1)\alpha_v^2+2(1-\omega)\alpha_v\beta_v+\beta_v^2-2(1-\omega)(1+v_1)\times$$

$$\left[\sum_{\substack{(j,k)\in\{1,\dots,p\}\times\{1,\dots,q\}:\\ k-j=v}}\left(\alpha_j+\frac{\beta_j}{1-\omega}\right)\sum_{u=1}^{p+q-2}\left(\alpha_k d_{vu}+(1-\omega)\beta_k d_{v+r-1,u}\right)b_{u0}\right],$$

$$C_r=(1-\omega)(1+v_1)\alpha_r^2+2(1-\omega)\alpha_r\beta_r+\beta_r^2,$$

with $r=\max(p,q)$, $B=(b_{ij})$ and $B^{-1}=(d_{ij})$ are squared matrices and $b=(b_{i0})$ is a vector, both of order $p+q-2$ and easily deduced from the expressions (3), (4) and (5).

Let us point out that when X follows a EZICP-INARCH(p) model we do not need to ensure $\alpha_0(1-\omega)(1+v_1)>v_0$ and the coefficients, the matrix and the vector entries reduce to the form

$$C_u=(1-\omega)\left(1+v_1\right)\left[\alpha_u^2-\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j d_{vu}b_{u0}\right],\ C_p=(1-\omega)\left(1+v_1\right)\alpha_p^2,$$

$b_{l0}=(1-\omega)\alpha_l$, $b_{ll}=(1-\omega)\sum_{|i-l|=l}\alpha_i-1$ and $b_{lu}=(1-\omega)\sum_{|i-l|=u}\alpha_i$, $u\neq l$, for $u,l=1,...,p-1$.

We conclude this section presenting a result from which the autocorrelation function of the EZICP-INGARCH model can be obtained. The result extends those presented in Theorem 4 of **[18]** and is obtained using the same arguments.

Theorem 3.3. *Suppose that X follows a second-order stationary EZICP-INGARCH(p,q) process. The autocovariances $\Gamma(k)=Cov(X_t,X_{t-k})$ and $\widetilde{\Gamma}(k)=Cov(\lambda_t,\lambda_{t-k})$ satisfy the linear equations*

$$\Gamma(k)=(1-\omega)\sum_{i=1}^{p}\alpha_i\cdot\Gamma(k-i)+\sum_{j=1}^{\min(k-1,q)}\beta_j\cdot\Gamma(k-j)+(1-\omega)^2\sum_{j=k}^{q}\beta_j\cdot\widetilde{\Gamma}(j-k),\ k\geq1,$$

$$\widetilde{\Gamma}(k) = (1-\omega) \sum_{i=1}^{\min(k,p)} \alpha_i \cdot \widetilde{\Gamma}(k-i) + \frac{1}{1-\omega} \sum_{i=k+1}^{p} \alpha_i \cdot \Gamma(i-k) + \sum_{j=1}^{q} \beta_j \cdot \widetilde{\Gamma}(k-j), \; k \geq 0,$$

assuming that $\sum_{j=k}^{q} \beta_j \widetilde{\Gamma}(j-k) = 0$ *if* $k > q$ *and* $\sum_{i=k+1}^{p} \alpha_i \Gamma(i-k) = 0$ *if* $k+1 > p$.

We point out the ARMA(p,q)-like serial dependence structure for X stated by this result.

4. The moments of Eventually Zero-Inflated Compound Poisson INGARCH(1,1) model

In this section, we focus on model (1) with $p = q = 1$, *i.e.*,

$$\Phi(u) = \omega + (1-\omega) \exp\left\{ i \frac{\lambda_t}{\varphi'(0)} \left[\varphi(u) - 1\right] \right\}, u \in \mathbb{R}, \; \lambda_t = \alpha_0 + \alpha_1 X_{t-1} + \beta_1 \lambda_{t-1}.$$

In the following example we obtain its autocorrelation function deduced from Theorem 3.3.

Example 4.1. *Supposing that* X *follows an* EZICP-INGARCH$(1,1)$ *model, from Theorem* 3.3, *we have*

$$\Gamma(k) = [(1-\omega)\alpha_1 + \beta_1]^{k-1} \cdot \frac{(1-\omega)\alpha_1[1 - (1-\omega)\alpha_1\beta_1 - \beta_1^2]}{1 - 2(1-\omega)\alpha_1\beta_1 - \beta_1^2} \cdot \Gamma(0), \; k \geq 1,$$

$$\widetilde{\Gamma}(k) = [(1-\omega)\alpha_1 + \beta_1]^k \cdot \widetilde{\Gamma}(0), \; k \geq 1,$$

from which the autocorrelation functions of X *and* λ *can be obtained.*

Under the hypothesis **H1**, *the value of* $\Gamma(0)$ *can be deduced using the expression derived in Theorem* 3.1. *Indeed,* $\Gamma(0) = V(X_t) = [P(1)]^{-1} B_0 - \mu^2$, *where*

$$P(1) = 1 - B_1 = 1 - (1-\omega)(1+v_1)\alpha_1^2 - 2(1-\omega)\alpha_1\beta_1 - \beta_1^2,$$

$$B_0 = v_0\mu + (1+v_1)[2\alpha_0\mu - (1-\omega)\alpha_0^2] - v_0\mu[2(1-\omega)\alpha_1\beta_1 + \beta_1^2],$$

$$\mu = \frac{(1-\omega)\alpha_0}{1 - (1-\omega)\alpha_1 - \beta_1}.$$

Thus,

$$\begin{aligned} \Gamma(0) &= \frac{v_0\mu + (1+v_1)[2\alpha_0\mu - (1-\omega)\alpha_0^2] - v_0\mu[2(1-\omega)\alpha_1\beta_1 + \beta_1^2]}{1 - (1-\omega)(1+v_1)\alpha_1^2 - 2(1-\omega)\alpha_1\beta_1 - \beta_1^2} - \mu^2 \\ &= \frac{[1 - 2(1-\omega)\alpha_1\beta_1 - \beta_1^2]}{1 - (1-\omega)(1+v_1)\alpha_1^2 - 2(1-\omega)\alpha_1\beta_1 - \beta_1^2} \left(v_0\mu + \frac{v_1 + \omega}{1-\omega}\mu^2 \right). \end{aligned}$$

Using this result when $\omega = 0$ *we recover, in particular, the expressions stated in Example 1 of* **[13]** *for the Poisson law considering* $v_0 = 1$, $v_1 = 0$, *and the expressions stated in Example 1 of* **[16]** *and* **[17]** *for the negative binomial and generalized Poisson laws considering, respectively,* $v_0 = 1$, $v_1 =$

$1/r$ and $v_0 = 1/(1-\kappa)^2$, $v_1 = 0$. If we consider $v_0 = (1-p)/p$ and $v_1 = 0$ we obtain expressions for the case where the conditional distribution of the process is the geometric Poisson law. For $\omega \neq 0$, we recover the results stated in the examples 1 and 2 of [**18**], *respectively, for the zero-inflated Poisson law, where $v_0 = 1$, $v_1 = 0$, and for the zero-inflated negative binomial distribution with $v_0 = 1 + a$, $v_1 = 0$ (when $c = 0$) and $v_0 = 1$, $v_1 = a$ (when $c = 1$).*

In the following, let us assume that φ_t is derivable as many times as necessary. We start by stating a necessary and sufficient condition for the existence of all the moments of the process when φ_t is a deterministic function. This result includes Proposition 6 of [**5**] in which $\varphi_t(u)$ is independent of t and equal to e^{iu}.

Theorem 4.1. *The moments of an EZICP-INGARCH$(1,1)$ model with φ_t deterministic are all finite if, and only if, $(1-\omega)(\alpha_1+\beta_1) < 1$.*

Proof. According to [**8**], since $X_t|\underline{X}_{t-1}$ is a compound random variable where the counting distribution is the zero-inflated Poisson law, its mth moment is given by

$$E[X_t^m|\underline{X}_{t-1}] = (1-\omega)\sum_{r=0}^{m}\frac{1}{r!}\frac{\lambda_t^r}{(\varphi_t'(0))^r}\sum_{j=0}^{r}\binom{r}{j}\frac{(-1)^{r-j}}{i^{m-r}}\left(\varphi_t^j\right)^{(m)}(0),$$

with $\varphi_t^j = \prod_{k=1}^{j}\varphi_t$, $\varphi_t^j(u) = 0$ if $j = 0$ and $\left(\varphi_t^j\right)^{(m)}$ the mth derivative of φ_t^j, namely,

$$\left(\varphi_t^j\right)^{(m)}(u) = \sum_{n=m-j}^{m-1}\left\{\frac{j!}{(j-m+n)!}\,\varphi_t^{j-m+n}(u)\;\times\right.$$

$$\left.\sum_{\substack{k_1+\ldots+k_m=m-n\\ k_1+2k_2+\ldots+mk_m=m\\ k_r\in\mathbb{N}_0}}(m;k_1,\ldots,k_m)\,[\varphi_t'(u)]^{k_1}\ldots[\varphi_t^{(m)}(u)]^{k_m}\right\},\; m\geq j, \tag{6}$$

where $(m;k_1,\ldots,k_m) = \frac{m!}{k_1!k_2!\ldots k_m!(1!)^{k_1}(2!)^{k_2}\ldots(m!)^{k_m}}$. For a proof of expression (6), see Appendix B.1. So,

$$E[X_t^m] = (1-\omega)\sum_{r=0}^{m}\sum_{j=0}^{r}\frac{1}{r!}\binom{r}{j}\frac{(-1)^{r-j}\left(\varphi_t^j\right)^{(m)}(0)}{i^{m-r}(\varphi_t'(0))^r}\,E[\lambda_t^r]. \tag{7}$$

As λ_{t-1}^{n-l} is $\underline{X}_{t-2}$-measurable we have

$$E[X_{t-1}^l\lambda_{t-1}^{n-l}|\underline{X}_{t-2}] = (1-\omega)\sum_{v=0}^{l}\frac{1}{v!}\frac{\lambda_{t-1}^{v+n-l}}{(\varphi_{t-1}'(0))^v}\sum_{x=0}^{v}\binom{v}{x}\frac{(-1)^{v-x}}{i^{l-v}}\left(\varphi_{t-1}^x\right)^{(l)}(0),$$

and, using the fact that

$$\lambda_t^r = (\alpha_0 + \alpha_1 X_{t-1} + \beta_1 \lambda_{t-1})^r = \sum_{n=0}^{r} \binom{r}{n} \alpha_0^{r-n} \sum_{l=0}^{n} \binom{n}{l} \alpha_1^l \beta_1^{n-l} X_{t-1}^l \lambda_{t-1}^{n-l},$$

$$E[\lambda_t^r | \underline{X}_{t-2}] = (1-\omega) \sum_{n=0}^{r} \binom{r}{n} \alpha_0^{r-n} \sum_{l=0}^{n} \binom{n}{l} \sum_{v=0}^{l} \frac{\alpha_1^l \beta_1^{n-l}}{v!(\varphi'_{t-1}(0))^v}$$
$$\times \sum_{x=0}^{v} \binom{v}{x} \frac{(-1)^{v-x}}{i^{l-v}} \left(\varphi_{t-1}^x\right)^{(l)}(0) \lambda_{t-1}^{v+n-l}. \tag{8}$$

Let $\Lambda_t = (\lambda_t^m, ..., \lambda_t)'$. In the algebraic expression of $E[\lambda_t^r | \underline{X}_{t-2}]$, for $r = 1, ..., m$, all the powers of λ_{t-1} are $\leq r$. Therefore, a constant vector $\mathbf{d}$ and an upper triangular matrix $\mathbf{D} = (d_{ij})$, $i, j = 1, ..., m$, exist such that the following equation is satisfied:

$$\begin{bmatrix} E[\lambda_t^m | \underline{X}_{t-2}] \\ \vdots \\ E[\lambda_t^2 | \underline{X}_{t-2}] \\ E[\lambda_t | \underline{X}_{t-2}] \end{bmatrix} = (1-\omega) \begin{bmatrix} \alpha_0^m \\ \vdots \\ \alpha_0^2 \\ \alpha_0 \end{bmatrix} + \begin{bmatrix} (1-\omega)(\alpha_1+\beta_1)^m & \cdots & d_{1,m-1} & d_{1m} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & (1-\omega)(\alpha_1+\beta_1)^2 & d_{m-1,m} \\ 0 & \cdots & 0 & (1-\omega)(\alpha_1+\beta_1) \end{bmatrix} \begin{bmatrix} \lambda_{t-1}^m \\ \vdots \\ \lambda_{t-1}^2 \\ \lambda_{t-1} \end{bmatrix}$$

$$\Leftrightarrow E[\Lambda_t | \underline{X}_{t-2}] = \mathbf{d} + \mathbf{D} \Lambda_{t-1}.$$

Indeed, let us prove that the diagonal entries of the matrix $\mathbf{D}$ are those given above. The kth diagonal entry of the matrix $\mathbf{D}$ corresponds to the case where in equation (8), we consider $r = m - k + 1$. Thus, to obtain the coefficient of λ_{t-1}^{m-k+1}, we look at the terms corresponding to $n = m - k + 1$ and $l = v$. Then,

$$d_{kk} = (1-\omega) \sum_{l=0}^{m-k+1} \binom{m-k+1}{l} \frac{\alpha_1^l \beta_1^{m-k+1-l}}{l!(\varphi'_{t-1}(0))^l} \sum_{x=0}^{l} \binom{l}{x} (-1)^{l-x} \left(\varphi_{t-1}^x\right)^{(l)}(0).$$

Using (6) we obtain

$$\sum_{x=0}^{l} \binom{l}{x} (-1)^{l-x} \left(\varphi_{t-1}^x\right)^{(l)}(0) = \sum_{x=0}^{l} \binom{l}{x} (-1)^{l-x} \sum_{j=l-x}^{l-1} \frac{x!}{(x-l+j)!} \varphi_{t-1}^{x-l+j}(0)$$
$$\times \sum_{\substack{k_1+...+k_l=l-j \\ k_1+2k_2+...+lk_l=l}} (l; k_1, ..., k_l) \left[\varphi'_{t-1}(0)\right]^{k_1} ... [\varphi_{t-1}^{(l)}(0)]^{k_l},$$

and therefore, for any arbitrarily fixed $k_1, ..., k_l \in \mathbb{N}_0$ such that $k_1 + ... + k_l = l - j$ and $k_1 + 2k_2 + ... + lk_l = l$, the coefficient of $\left[\varphi'_{t-1}(0)\right]^{k_1} \ldots [\varphi_{t-1}^{(l)}(0)]^{k_l}$ is given by

$$\left[\sum_{x=0}^{l} \binom{l}{x} (-1)^{l-x} \frac{x!}{(x-(k_1+...+k_l))!}\right] (l; k_1, ..., k_l)$$

$$= \left[\sum_{x=k_1+...+k_l}^{l} \frac{(-1)^{l-x}}{(l-x)!(x-(k_1+...+k_l))!} \right] l!(l;k_1,...,k_l)$$

$$= \frac{(-1)^{l-(k_1+...+k_l)}}{(l-(k_1+...+k_l))!} \left[\sum_{m=0}^{l-(k_1+...+k_l)} \binom{l-(k_1+...+k_l)}{m} (-1)^{-m} \right] l!(l;k_1,...,k_l).$$

When $k_1 = l$, $k_2 = \ldots = k_l = 0$, we obtain the coefficient $l!(l;l,0,...,0) = l!$. Otherwise, the coefficient is zero. Therefore, we finally conclude that

$$\sum_{x=0}^{l} \binom{l}{x} (-1)^{l-x} \left(\varphi_{t-1}^{x}\right)^{(l)}(0) = l! \, [\varphi'_{t-1}(0)]^l, \tag{9}$$

and then the kth diagonal entry of the matrix $\mathbf{D}$ is

$$d_{kk} = (1-\omega) \sum_{l=0}^{m-k+1} \binom{m-k+1}{l} \alpha_1^l \beta_1^{m-k+1-l} = (1-\omega)(\alpha_1+\beta_1)^{m-k+1}.$$

So, we conclude that the eigenvalues of $\mathbf{D}$ (which coincide with its diagonal entries because it is a triangular matrix) are inside the unit circle if, and only if, $(1-\omega)(\alpha_1+\beta_1) < 1$. Using this fact and proceeding as in Proposition 6 of [**5**], we can write

$$E[\Lambda_t | \underline{X}_{t-k}] = (I_m - \mathbf{D})^{-1}(I_m - \mathbf{D}^{k-1})\mathbf{d} + \mathbf{D}^{k-1}\Lambda_{t-(k-1)},$$

where I_m is the identity matrix of order m.

Consequently, since $\mathbf{D}^{k-1} \to 0$ when $k \to \infty$, we have

$$E[\Lambda_t] = \lim_{k\to\infty} E[\Lambda_t | \underline{X}_{t-k}] = (I_m - \mathbf{D})^{-1}\mathbf{d},$$

and then from (7) all the moments of X_t of order $\leq m$ are finite. □

Now let us consider that the characteristic function φ_t satisfies the condition:

Hypothesis H2: φ_t is deterministic and independent of t,

which is equivalent to say that $v_1 = 0$ in hypothesis **H1**. Henceforward we simply denote φ_t as φ. We stress that this particular case still includes a wide class of new models not studied in literature as those introduced in example 2.1 **(c)** and **(d)**.

Consider a first-order stationary EZICP-INARCH(1) model. Since it is too tedious to derive higher-order cummulants, as an illustration we obtain only its first two. To do this, let us start by recalling that if Φ_{X_t} denotes the characteristic function of X_t, its cumulant genera- ting function is given by $\kappa_{X_t}(z) = \ln(\Phi_{X_t}(z))$, and the coefficient $\kappa_j(X_t)$ of the series expansion $\kappa_{X_t}(z) =$

$\sum_{j=1}^{\infty} \kappa_j(X_t) \cdot (iz)^j/j!$ is referred to as the cumulant with $\kappa_j(X_t) = (-i)^j \kappa_{X_t}^{(j)}(0)$. Using the expression of the characteristic function of the conditional distribution and the fact that $\lambda_t = \alpha_0 + \alpha_1 X_{t-1}$ we obtain, for $z \in \mathbb{R}$,

$$\Phi_{X_t}(z) = E\left(e^{izX_t}\right) = E\left[E\left(e^{izX_t}|\underline{X}_{t-1}\right)\right] = E\left[\omega + (1-\omega)\exp\left(i\frac{\lambda_t}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\right]$$

$$= E\left[\omega + (1-\omega)\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\exp\left(\frac{i\alpha_1 X_{t-1}}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\right]$$

$$= \omega + (1-\omega)\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\Phi_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right) = \omega + (1-\omega)A(z), \quad (10)$$

hence, the cumulant generating function of X_t is given by

$$\kappa_{X_t}(z) = \ln\left(\omega + (1-\omega)A(z)\right).$$

Taking derivatives on both sides, it follows that

$$\kappa'_{X_t}(z) = \frac{\Phi'_{X_t(z)}}{\Phi_{X_t(z)}} = \frac{(1-\omega)A'(z)}{\omega + (1-\omega)A(z)} = \frac{(1-\omega)\frac{A'(z)}{A(z)}}{(1-\omega) + \frac{\omega}{A(z)}},$$

$$\kappa''_{X_t}(z) = \frac{\Phi''_{X_t(z)}}{\Phi_{X_t(z)}} - \left[\frac{\Phi'_{X_t(z)}}{\Phi_{X_t(z)}}\right]^2 = \frac{(1-\omega)\frac{A''(z)}{A(z)}}{(1-\omega) + \frac{\omega}{A(z)}} - \left(\kappa'_{X_t}(z)\right)^2,$$

where

$$A'(z) = \frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\cdot\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\Phi_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\cdot\Phi'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$\Longrightarrow \frac{A'(z)}{A(z)} = \frac{i\alpha_0\varphi'(z)}{\varphi'(0)} + \frac{\alpha_1\varphi'(z)}{\varphi'(0)}\cdot\kappa'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right),$$

$$A''(z) = \left[\frac{i\alpha_0\varphi''(z)}{\varphi'(0)} + \left(\frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\Phi_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\left[\frac{\alpha_1\varphi''(z)}{\varphi'(0)} + 2i\alpha_0\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\Phi'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^2\cdot\exp\left(\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\cdot\Phi''_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$\Longrightarrow \frac{A''(z)}{A(z)} = \frac{i\alpha_0\varphi''(z)}{\varphi'(0)} + \left(\frac{i\alpha_0\varphi'(z)}{\varphi'(0)}\right)^2 + \left[\frac{\alpha_1\varphi''(z)}{\varphi'(0)} + 2i\alpha_0\alpha_1\left(\frac{\varphi'(z)}{\varphi'(0)}\right)^2\right]\cdot\kappa'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\left(\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right)^2\cdot\left[\kappa''_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right) + \left(\kappa'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\right)^2\right]$$

Inserting $z=0$ into the previous equations and noting that $A(0)=1$, we obtain

$$\kappa'_{X_t}(0)=(1-\omega)\left[i\alpha_0+\alpha_1\cdot\kappa'_{X_{t-1}}(0)\right]\Rightarrow \kappa_1(X_t)=\frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1}=\mu,$$

$$\kappa''_{X_t}(0)=(1-\omega)\frac{\varphi''(0)}{\varphi'(0)}\left[i\alpha_0+\alpha_1\cdot\kappa'_{X_{t-1}}(0)\right]+(1-\omega)(i\alpha_0)^2$$
$$+2i\alpha_0\left[\kappa'_{X_t}(0)-i(1-\omega)\alpha_0\right]+(1-\omega)\alpha_1^2\cdot\kappa''_{X_{t-1}}(0)-(1-\omega)\alpha_1^2\mu^2+\mu^2$$
$$\Leftrightarrow\left[1-(1-\omega)\alpha_1^2\right]\cdot\kappa''_{X_t}(0)=i\frac{\varphi''(0)}{\varphi'(0)}\mu+2\alpha_0\mu+(1-\omega)\alpha_0^2+(1-\omega)\left(\frac{i\mu}{1-\omega}-i\alpha_0\right)^2+\mu^2$$
$$\Rightarrow\kappa_2(X_t)=\frac{v_0\mu+\frac{\omega\mu^2}{1-\omega}}{1-(1-\omega)\alpha_1^2}=V(X_t),$$

as stated in Example 4.1.

When $\omega=0$, we are able to study completely the first four cumulants using the technique given in **[13]**. So we derive these expressions for a CP-INARCH(1) process, under the hypothesis **H2**. The skewness and the flatness of the distribution of the process are now available. To do that, we consider the notations

$$v_0=-i\,\frac{\varphi''(0)}{\varphi'(0)},\ d_0=-\frac{\varphi'''(0)}{\varphi'(0)},\ c_0=i\frac{\varphi^{(iv)}(0)}{\varphi'(0)},\ f_k=\alpha_0/\prod_{j=1}^{k}(1-\alpha_1^j),\ k\in\mathbb{N}.$$

Theorem 4.2. *Let X be a first order stationary CP-INARCH(1) process admitting fourth order moment and such that the hypothesis* **H2** *is satisfied with φ derivable up to order 4. Then, the first four cumulants of X_t are given by*

$$\kappa_1(X_t)=f_1,\ \kappa_2(X_t)=v_0f_2,\ \kappa_3(X_t)=f_3[d_0(1-\alpha_1)^2+3v_0^2\alpha_1^2],$$
$$\kappa_4(X_t)=f_4[c_0(1-\alpha_1^2)(1-\alpha_1^3)+v_0^3(3\alpha_1^2+15\alpha_1^5)+v_0d_0(4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)].$$

Proof. Using the expression (10) with $\omega=0$, we deduce the cumulant generating function of X_t

$$\kappa_{X_t}(z)=\frac{i\alpha_0}{\varphi'(0)}\left[\varphi(z)-1\right]+\kappa_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right).$$

Taking derivatives on both sides, it follows that

$$\kappa'_{X_t}(z)=\frac{i\alpha_0}{\varphi'(0)}\varphi'(z)+\frac{\alpha_1}{\varphi'(0)}\varphi'(z)\cdot\kappa'_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right),$$

$$\kappa^{(n)}_{X_t}(z)=\frac{i\alpha_0}{\varphi'(0)}\varphi^{(n)}(z)+\sum_{j=1}^{n-1}a_{n-1,j}\cdot\kappa^{(j)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$
$$+\left[\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right]^n\cdot\kappa^{(n)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right),\ n=2,3,4,\tag{11}$$

where the second formula is proved by induction in Appendix B.2.

The coefficients $a_{n-1,j}$ are given by

$$a_{n-1,1} = \frac{\alpha_1}{\varphi'(0)}\varphi^{(n)}(z),$$

$$a_{n-1,j} = \left[\frac{\alpha_1}{\varphi'(0)}\right]^j \sum_{\substack{k_1+\ldots+k_n=j \\ k_1+2k_2+\ldots+nk_n=n \\ k_r\in\mathbb{N}_0}} (n;k_1,\ldots,k_n)\,[\varphi'(z)]^{k_1}\ldots[\varphi^{(n)}(z)]^{k_n},\ j\geq 2.$$

Inserting $z=0$ into the previous equations, one obtains

$$\kappa'_{X_t}(0) = i\alpha_0 + \alpha_1\cdot\kappa'_{X_{t-1}}(0) \Rightarrow \kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1},$$

$$\kappa_n(X_t) = \sum_{j=1}^{n-1} b_{n-1,j}\cdot\kappa_j(X_{t-1}) + \alpha_1^n\cdot\kappa_n(X_{t-1}),\ n=2,3,4,$$

where the coefficients $b_{n-1,j}$ are given by

$$b_{n-1,1} = (-i)^{n-1}\frac{\varphi^{(n)}(0)}{\varphi'(0)},\ b_{n-1,n-1} = -i\ \frac{n(n-1)}{2}\frac{\varphi''(0)}{\varphi'(0)}\alpha_1^{n-1},$$

$$b_{n-1,j} = (-i)^{n-j}\left[\frac{\alpha_1}{\varphi'(0)}\right]^j \sum_{\substack{k_1+\ldots+k_n=j \\ k_1+2k_2+\ldots+nk_n=n \\ k_r\in\mathbb{N}_0}} (n;k_1,\ldots,k_n)\,[\varphi'(0)]^{k_1}\ldots[\varphi^{(n)}(0)]^{k_n},$$

for $1<j<n-1$. From here, it follows that

$$(1-\alpha_1^2)\cdot\kappa_2(X_t) = b_{1,1}\cdot\kappa_1(X_t) \Rightarrow \kappa_2(X_t) = -i\frac{\varphi''(0)}{\varphi'(0)}\frac{\alpha_0}{(1-\alpha_1)(1-\alpha_1^2)},$$

$$\left(1-\alpha_1^3\right)\cdot\kappa_3(X_t) = -\frac{\varphi'''(0)}{\varphi'(0)}\cdot\kappa_1(X_t) - 3i\alpha_1^2\ \frac{\varphi''(0)}{\varphi'(0)}\cdot\kappa_2(X_t)$$

$$\Rightarrow \kappa_3(X_t) = -\alpha_0\ \frac{(1-\alpha_1^2)\frac{\varphi'''(0)}{\varphi'(0)} + 3\alpha_1^2\left[\frac{\varphi''(0)}{\varphi'(0)}\right]^2}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

$$\left(1-\alpha_1^4\right)\kappa_4(X_t) = i\frac{\varphi^{(iv)}(0)}{\varphi'(0)}\cdot\kappa_1(X_t) - \alpha_1^2\left[4\frac{\varphi'''(0)}{\varphi'(0)} + 3\left(\frac{\varphi''(0)}{\varphi'(0)}\right)^2\right]\cdot\kappa_2(X_t) - 6i\alpha_1^3\ \frac{\varphi''(0)}{\varphi'(0)}\cdot\kappa_3(X_t)$$

$$\Rightarrow \kappa_4(X_t) = i\alpha_0\ \frac{(1-\alpha_1^2)(1-\alpha_1^3)\frac{\varphi^{(iv)}(0)}{\varphi'(0)} + (3\alpha_1^2+15\alpha_1^5)\left[\frac{\varphi''(0)}{\varphi'(0)}\right]^3 + (4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)\frac{\varphi''(0)\varphi'''(0)}{(\varphi'(0))^2}}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

that ends the proof, using the notation indicated above. □

Observation 4.1. *As a consequence of Theorem 4.2, X is an asymmetric process around the mean and is leptokurtic since its skewness and kurtosis are, respectively, given by*

$$S_{X_t} = \frac{(1-\alpha_1^2)d_0 + 3\alpha_1^2 v_0^2}{v_0(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{v_0\alpha_0}},$$

$$K_{X_t} = 3 + \frac{(1-\alpha_1^2)(1-\alpha_1^3)c_0 + (3\alpha_1^2+15\alpha_1^5)v_0^3 + (4\alpha_1^2+6\alpha_1^3-10\alpha_1^5)v_0 d_0}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)v_0^2}.$$

In the following we illustrate the expressions displayed for the skewness and kurtosis of a CP-INARCH(1) process considering some particular compound Poisson distributions.

Example 4.2. ***(1) Poisson law:***
We have $\varphi(u) = e^{iu}$ and so $\varphi^{(n)}(0) = i^n$, $v_0 = d_0 = c_0 = 1$ and then

$$\kappa_2(X_t) = \frac{\alpha_0}{(1-\alpha_1)(1-\alpha_1^2)},\ \kappa_3(X_t) = \frac{\alpha_0 + 2\alpha_0\alpha_1^2}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

$$\kappa_4(X_t) = \frac{\alpha_0(1+6\alpha_1^2+5\alpha_1^3+6\alpha_1^5)}{(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

with skewness and kurtosis of X_t, respectively, given by

$$S_{X_t} = \frac{1+2\alpha_1^2}{1+\alpha_1+\alpha_1^2}\sqrt{\frac{1+\alpha_1}{\alpha_0}},\ K_{X_t} = 3 + \frac{1+6\alpha_1^2+5\alpha_1^3+6\alpha_1^5}{\alpha_0(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)},$$

In this case, if $\left\{ \begin{matrix} n \\ j \end{matrix} \right\}$ represents a Stirling number of the second kind, as an alternative to the recursive form established by **[13]**, *the cumulants of the INARCH(1) model can be determined recursively from*

$$\kappa_1(X_t) = \frac{\alpha_0}{1-\alpha_1},\ \kappa_n(X_t) = (1-\alpha_1^n)^{-1}\cdot\sum_{j=1}^{n-1} b_{n-1,j}\cdot\kappa_j(X_t),\ n \geq 2,$$

where the coefficients $b_{n-1,j}$ are given by

$$b_{n-1,1} = 1,\ b_{n-1,n-1} = \frac{n(n-1)}{2}\ \alpha_1^{n-1},\ b_{n-1,j} = \left\{ \begin{matrix} n \\ j \end{matrix} \right\}\ \alpha_1^j,\ 1 < j < n-1.$$

(2) generalized Poisson law:
In this case φ is the characteristic function of the variables $X_{t,j}$, $j = 1, ..., N_t$ having the Borel law with parameter κ. For $0 < \kappa < 1$, all the moments of the Borel distribution exist and $\varphi^{(k)}(0) = i^k E(X_{t,1}^k)$. As

$$E(X_{t,1}) = (1-\kappa)^{-1},\ E(X_{t,1}^2) = (1-\kappa)^{-3},$$

$$E(X_{t,1}^3) = (2\kappa+1)(1-\kappa)^{-5},\ E(X_{t,1}^4) = (6\kappa^2+8\kappa+1)(1-\kappa)^{-7},$$

we have

$$\varphi'(0) = i(1-\kappa)^{-1},\ \varphi''(0) = -(1-\kappa)^{-3},$$
$$\varphi'''(0) = -i(2\kappa+1)(1-\kappa)^{-5},\ \varphi^{(iv)}(0) = (6\kappa^2+8\kappa+1)(1-\kappa)^{-7},$$
$$v_0 = \frac{1}{(1-\kappa)^2},\ d_0 = \frac{2\kappa+1}{(1-\kappa)^4},\ c_0 = \frac{6\kappa^2+8\kappa+1}{(1-\kappa)^6}.$$

Thus we obtain the cumulants

$$\kappa_2(X_t) = \frac{\alpha_0}{(1-\kappa)^2(1-\alpha_1)(1-\alpha_1^2)},\ \kappa_3(X_t) = \frac{\alpha_0(1-\alpha_1^2)(2\kappa+1)+3\alpha_0\alpha_1^2}{(1-\kappa)^4(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$
$$\kappa_4(X_t) = \alpha_0\,\frac{6\kappa^2+8\kappa+1-6\alpha_1^2(\kappa^2+1)-\alpha_1^3(6\kappa^2-4\kappa-5)+6\alpha_1^5(\kappa^2-2\kappa+1)}{(1-\kappa)^6(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)(1-\alpha_1^4)},$$

and the skewness and the kurtosis of X_t

$$S_{X_t} = \frac{(1-\alpha_1^2)(2\kappa+1)+3\alpha_1^2}{(1-\kappa)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{1+\alpha_1}{\alpha_0}},$$
$$K_{X_t} = 3 + \frac{6\kappa^2+8\kappa+1-6\alpha_1^2(\kappa^2+1)-\alpha_1^3(6\kappa^2-4\kappa-5)+6\alpha_1^5(\kappa^2-2\kappa+1)}{\alpha_0(1-\kappa)^2(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

Note that using the fact that the generalized Poisson distribution is a compound Poisson law instead of the procedure adopted by Zhu **[17]** *made much easier the deduction of their cumulants. Furthermore, in* **[17]** *only the first two cumulants are presented given the complexity of the calculations involved. In Figure 3 the trajectory and the basic descriptives of* 1000 *observations of a* $GP\text{-}INARCH(1)$ *process are presented for which is evident the closeness of the theoretical values* $S_{X_t} \simeq 1.0362$ *and* $K_{X_t} = 4.2527$*, according to the above formulas, and the empirical ones.*

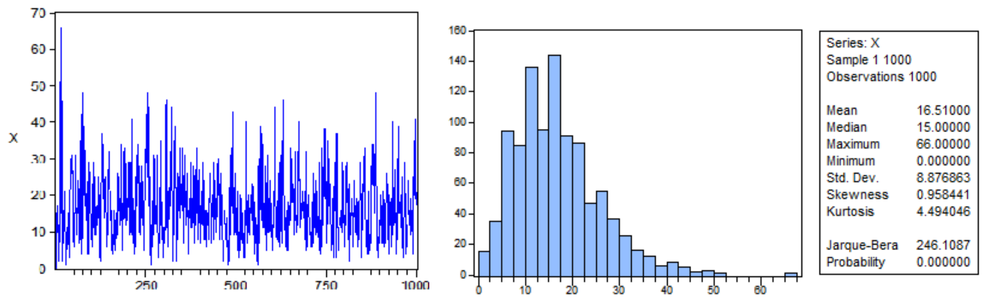

FIGURE 3. Trajectory and descriptives of GP-INARCH(1) model: $\alpha_0 = 10,\ \alpha_1 = 0.4,\ \kappa = 0.5$.

(3) geometric Poisson law *(like in Example* 2.1 *(c)):*
For this distribution we have

$$\varphi'(0) = \frac{i}{p},\ \varphi''(0) = \frac{p-2}{p^2},\ \varphi'''(0) = -i\,\frac{6-6p+p^2}{p^3},\ \varphi^{(iv)}(0) = \frac{16-16p+2p^2-p^3}{p^4},$$

$$v_0 = \frac{2-p}{p},\ d_0 = \frac{6-6p+p^2}{p^2},\ c_0 = \frac{16-16p+2p^2-p^3}{p^3},$$

from where we deduce, for instance, the cumulants

$$\kappa_2(X_t) = \frac{\alpha_0(2-p)}{p(1-\alpha_1)(1-\alpha_1^2)},\ \kappa_3(X_t) = \alpha_0\ \frac{6(1+\alpha_1^2)-6(1-\alpha_1^2)p+(1+2\alpha_1^2)p^2}{p^2(1-\alpha_1)(1-\alpha_1^2)(1-\alpha_1^3)},$$

and the skewness and the kurtosis of X_t, respectively,

$$S_{X_t} = \frac{6-6p+6p^2+2p^2\alpha_1^2}{(2p-p^2)(1+\alpha_1+\alpha_1^2)}\sqrt{\frac{p(1+\alpha_1)}{\alpha_0(2-p)}},$$

$$K_{X_t} = 3+\frac{(1-\alpha_1^2)(1-\alpha_1)(16-16p+12p^2-p^3)}{\alpha_0 p(2-p)^2(1+\alpha_1^2)}+\frac{\alpha_1^2(3+15\alpha_1^3)(2-p)}{\alpha_0 p(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}$$
$$-\frac{2\alpha_1^2(1-\alpha_1)(5\alpha_1^2+5\alpha_1+2)(p^2-6p+6)}{\alpha_0 p(2-p)(1+\alpha_1+\alpha_1^2)(1+\alpha_1^2)}.$$

5. Conclusion

A new class of models which includes the main INGARCH processes present in literature is proposed and developed in this paper enlarging and unifying the analysis of those processes, and accomplishing the practical goal of modeling simultaneously different stylized facts that have been recorded in real count data. In fact, considering a mixture of a Dirac at zero with a general discrete compound Poisson as conditional distribution of INGARCH processes, we define the Eventually Zero-inflated Compound Poisson INGARCH model, denoted EZICP-INGARCH , that may capture in the same framework characteristics of zero inflation and overdispersion. A general procedure to obtain new models is developed showing the main nature of the processes that are solution of the model equations, namely the fact that they may be expressed as a random sum of random variables. Conditions for stationarity of these models are established and also illustrated for particular important cases. Furthermore, for EZICP-INGARCH$(1,1)$ processes, conditions assuring higher order moments existence and closed-form expressions for the cumulants up to order 4 are deduced, from which the skewness and kurtosis of the processes are derived.

These results are useful in other probabilistic developments of these models as, in particular, the study of the Taylor property (**[6]**) or other type of applications **[14]**.

As illustrated in the EZIP-INGARCH process, we point out that this proposal may also include underdispersed models analyzing in each case the possibility of negative values for the additional weight, ω, on zero, that is, models with deflation in zero. Finally, we stress that, using the same methodology and slightly heavier calculations, this study is valid when the inflation takes place in a nonzero point.

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Appendix A. Autoregressive equation of W_t

From (3), (4) and (5) it follows that the vector W_t satisfies the autoregressive equation of order p, $W_t = B_0 + \sum_{k=1}^{p} B_k W_{t-k}$ where $B_0 = (b_j)$ is such that

$$b_j = \begin{cases} C, & j = 1 \\ (1-\omega)\mu \left[\alpha_0 - \frac{v_0\beta_{j-1}}{1+v_1}\right], & j = 2, ..., p \\ \frac{\mu}{1-\omega}\left[\alpha_0 - \frac{v_0(\alpha_{j-p}+\beta_{j-p})}{1+v_1}\right], & j = p+1, ..., 2p-1 \end{cases}$$

and B_k $(k = 1, ..., p)$ are the squared matrices having generic element $b_{ij}^{(k)}$ given by:

- row $i = 1$:

$$b_{1j}^{(k)} = \begin{cases} (1-\omega)(1+v_1)\alpha_k^2 + 2(1-\omega)\alpha_k\beta_k + \beta_k^2, & j = 1 \\ 2(1+v_1)[(1-\omega)\alpha_k + \beta_k]\alpha_{j+k-1}, & j = 2, ..., p \\ 2(1-\omega)(1+v_1)[(1-\omega)\alpha_k + \beta_k]\beta_{j+k-p}, & j = p+1, ..., 2p-1 \end{cases}$$

- row $i = k+1$, $(k \neq p)$:

$$b_{k+1,j}^{(k)} = \begin{cases} (1-\omega)\left[\alpha_k + \frac{\beta_k}{1+v_1}\right], & j = 1 \\ (1-\omega)\alpha_{j+k-1}, & j = 2, ..., p \\ (1-\omega)^2\beta_{j+k-p}, & j = p+1, ..., 2p-1 \end{cases}$$

- row $i = k+p$:

$$b_{k+p,j}^{(k)} = \begin{cases} \frac{\alpha_k+\beta_k}{(1-\omega)(1+v_1)}, & j = 1 \\ \frac{\alpha_{j+k-1}}{1-\omega}, & j = 2, ..., p \\ \beta_{j+k-p}, & j = p+1, ..., 2p-1 \end{cases}$$

- row $i = k+j$:

$$b_{k+j,j}^{(k)} = \begin{cases} (1-\omega)\alpha_k + \beta_k, & j = 2, ..., p-k, p+1, ..., 2p-1-k \\ 0 & j = p-k+1, ..., p \end{cases}$$

and for any other case $b_{ij}^{(k)} = 0$, where we consider $\alpha_j = \beta_j = 0$, for $j > p$.

For the simpler EZICP-INARCH(p) model, the previous matrices assume the following form:

- row $i = 1$:

$$b_{1j}^{(k)} = \begin{cases} (1-\omega)(1+v_1)\alpha_k^2, & \text{if } j=1 \\ 2(1-\omega)(1+v_1)\alpha_k\alpha_{j+k-1}, & \text{if } j=2,...,p \end{cases}$$

- row $i \neq 1$:

$$b_{ij}^{(k)} = \begin{cases} (1-\omega)\alpha_{j+k-1}, & \text{if } i=k+1,\ j=1,...,p \\ (1-\omega)\alpha_k, & \text{if } i=k+j,\ j=2,...,p \\ 0, & \text{otherwise} \end{cases}$$

with $B_0 = (v_0\mu + \alpha_0(1+v_1)(2\mu-(1-\omega)\alpha_0), (1-\omega)\alpha_0\mu, ..., (1-\omega)\alpha_0\mu)$ a vector of dimension p and $W_t = \big(E(X_t^2), E(X_tX_{t-1}), ..., E(X_tX_{t-(p-1)})\big)$.

Appendix B. Proofs

B.1. Proof of expression (6) in Theorem 4.1. For $j, m \in \mathbb{N}_0$,

$$\left(\varphi_t^j\right)^{(m)}(u) = \sum_{n=\max\{0,m-j\}}^{m-1} \frac{j!}{(j-m+n)!}\ \varphi_t^{j-m+n}(u) \times$$

$$\sum_{\substack{k_1+...+k_m=m-n \\ k_1+2k_2+...+mk_m=m \\ k_r\in\mathbb{N}_0}} (m;k_1,...,k_m)\,[\varphi_t'(u)]^{k_1}...[\varphi_t^{(m)}(u)]^{k_m},\ u\in\mathbb{R},$$

considering $\left(\varphi_t^j\right)^{(0)}(u) = 1, \forall j \geq 0$ and where $(m;k_1,...,k_m) = \frac{m!}{k_1!k_2!...k_m!(1!)^{k_1}(2!)^{k_2}...(m!)^{k_m}}$.

Proof. Without loss of generality, let us consider $m \geq j$. For $m=1$ the result is valid since $\left(\varphi_t^j\right)'(u) = j\varphi_t^{j-1}(u)\varphi_t'(u)$. Now, let us assume that the formula has been shown for an arbitrarily fixed value of m and let us prove that for $m+1$ follows

$$\left(\varphi_t^j\right)^{(m+1)}(u) = \sum_{n=m+1-j}^{m} \frac{j!}{(j-m-1+n)!}\ \varphi_t^{j-m-1+n}(u) \times$$

$$\sum_{\substack{k_1+...+k_{m+1}=m+1-n \\ k_1+2k_2+...+(m+1)k_{m+1}=m+1 \\ k_r\in\mathbb{N}_0}} (m+1;k_1,...,k_{m+1})\,[\varphi_t'(u)]^{k_1}...[\varphi_t^{(m+1)}(u)]^{k_{m+1}}.$$

We have

$$\left(\varphi_t^j\right)^{(m+1)}(u) = \frac{d}{du}\left(\sum_{n=m-j}^{m-1} \frac{j!\ \varphi_t^{j-m+n}(u)}{(j-m+n)!} \sum_{\substack{k_1+...+k_m=m-n \\ k_1+...+mk_m=m}} (m;k_1,...,k_m)\,[\varphi_t'(u)]^{k_1}...[\varphi_t^{(m)}(u)]^{k_m}\right)$$

$$= \sum_{n=m-j+1}^{m-1} \frac{j!(j-m+n)}{(j-m+n)!}\ \varphi_t^{j-m+n-1}(u) \sum_{\substack{k_1+...+k_m=m-n \\ k_1+...+mk_m=m}} (m;k_1,...,k_m)\,[\varphi_t'(u)]^{k_1+1}\,[\varphi_t''(u)]^{k_2}...[\varphi_t^{(m)}(u)]^{k_m}$$

$$+\sum_{n=m-j}^{m-1}\frac{j!\ \varphi_t^{j-m+n}(u)}{(j-m+n)!}\left(\sum_{\substack{k_1+...+k_m=m-n\\k_1+...+mk_m=m}}k_1(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1-1}\left[\varphi_t''(u)\right]^{k_2+1}\left[\varphi_t'''(u)\right]^{k_3}\dots[\varphi_t^{(m)}(u)]^{k_m}\right.$$
$$+\sum_{\substack{k_1+...+k_m=m-n\\k_1+...+mk_m=m}}k_2(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}\left[\varphi_t''(u)\right]^{k_2-1}\left[\varphi_t'''(u)\right]^{k_3+1}[\varphi_t^{(iv)}(u)]^{k_4}\dots[\varphi_t^{(m)}(u)]^{k_m}$$
$$\left.+\dots+\sum_{\substack{k_1+...+k_m=m-n\\k_1+...+mk_m=m}}k_m(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}\dots[\varphi_t^{(m)}(u)]^{k_m-1}\varphi_t^{(m+1)}(u)\right)$$

$$=\sum_{n=m-j+1}^{m-1}\frac{j!\ \varphi_t^{j-m+n-1}(u)}{(j-m+n-1)!}\left[\sum_{\substack{k_1+...+k_m=m-n\\k_1+...+mk_m=m}}(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1+1}\left[\varphi_t''(u)\right]^{k_2}\dots[\varphi_t^{(m)}(u)]^{k_m}\right.$$
$$+\sum_{\substack{k_1+...+k_m=m-n+1\\k_1+...+mk_m=m}}k_1(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1-1}\left[\varphi_t''(u)\right]^{k_2+1}\left[\varphi_t'''(u)\right]^{k_3}\dots[\varphi_t^{(m)}(u)]^{k_m}$$
$$+\sum_{\substack{k_1+...+k_m=m-n+1\\k_1+...+lk_m=m}}k_2(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}\left[\varphi_t''(u)\right]^{k_2-1}\left[\varphi_t'''(u)\right]^{k_3+1}[\varphi_t^{(iv)}(u)]^{k_4}\dots[\varphi_t^{(m)}(u)]^{k_m}$$
$$\left.+\dots+\sum_{\substack{k_1+...+k_m=m-n+1\\k_1+...+mk_m=m}}k_m(m;k_1,...,k_m)\left[\varphi_t'(u)\right]^{k_1}\dots[\varphi_t^{(m)}(u)]^{k_m-1}\varphi_t^{(m+1)}(u)\right]+j\ \varphi_t^{j-1}(u)\varphi_t^{(m+1)}(u)$$

where the last term results from the second sum when $n=m-1$, since in this case one obtains $(m;0,\dots,0,1)\times 0+0+\dots+0+(m;0,\dots,0,1)\varphi_t^{(m+1)}(u)$. Thus,

$$\left(\varphi_t^j\right)^{(m+1)}(u)=\sum_{n=m-j+1}^{m-1}\frac{j!\ \varphi_t^{j-m+n-1}(u)}{(j-m+n-1)!}\left[\sum_{\substack{c_1+...+c_m=m+1-n\\c_1+...+mc_m=m+1}}(m;c_1-1,c_2,...,c_m)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}\right.$$
$$+\sum_{\substack{c_1+...+c_m=m+1-n\\c_1+...+mc_m=m+1}}(c_1+1)\ (m;c_1+1,c_2-1,c_3,...,c_m)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}$$
$$+\sum_{\substack{c_1+...+c_m=m+1-n\\c_1+...+mc_m=m+1}}(c_2+1)\ (m;c_1,c_2+1,c_3-1,c_4,...,c_m)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}$$
$$\left.+\dots+\sum_{\substack{c_1+...+c_{m+1}=m+1-n\\c_1+...+(m+1)c_{m+1}=m+1}}(c_m+1)\ (m;c_1,...,c_{m-1},c_m+1)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}\varphi_t^{(m+1)}(u)\right]$$
$$+\ j\ \varphi_t^{j-1}(u)\varphi_t^{(m+1)}(u).$$
$$=\sum_{n=m-j+1}^{m-1}\frac{j!\ \varphi_t^{j-m+n-1}(u)}{(j-m+n-1)!}\left[\sum_{\substack{c_1+...+c_m=m+1-n\\c_1+...+mc_m=m+1}}(m+1;c_1,...,c_m,0)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}\right.$$
$$\left.+\sum_{\substack{c_1+...+c_m=m+1-n\\c_1+...+mc_m=m+1}}(m+1;c_1,...,c_m,1)\left[\varphi_t'(u)\right]^{c_1}\dots[\varphi_t^{(m)}(u)]^{c_m}\varphi_t^{(m+1)}(u)\right]$$
$$+\ j\ \varphi_t^{j-1}(u)\ (m+1;0,...,0,1)\varphi_t^{(m+1)}(u),$$

using the fact that

$$(c_i+1)\ (m;c_1,...,c_{i-1},c_i+1,c_{i+1}-1,...,c_m)=\frac{(i+1)c_{i+1}}{m+1}(m+1;c_1,...,c_m,0),\ i=1,...,m-1,$$

$$(m;c_1-1,c_2,...,c_m)=\frac{c_1}{m+1}(m+1;c_1,...,c_m,0),$$

$$(c_m+1)\ (m;c_1,...,c_{m-1},c_m+1)=(m+1;c_1,...,c_m,1), \tag{12}$$

and hence

$$(m;c_1-1,c_2,...,c_m)+\sum_{i=1}^{m-1}(c_i+1)\ (m;c_1,...,c_{i-1},c_i+1,c_{i+1}-1,...,c_m)$$
$$=(m+1;c_1,...,c_m,0)\left[\frac{c_1+2c_2+...+mc_m}{m+1}\right]=(m+1;c_1,...,c_m,0).$$

Finally, we obtain

$$\left(\varphi_t^j\right)^{(m+1)}(u)=\sum_{n=m-j+1}^{m}\frac{j!}{(j-m+n-1)!}\ \varphi_t^{j-m+n-1}(u)\ \times$$
$$\sum_{\substack{c_1+...+c_{m+1}=m+1-n\\ c_1+...+(m+1)c_{m+1}=m+1}}(m+1;c_1,...,c_{m+1})\left[\varphi_t'(u)\right]^{c_1}\ldots[\varphi_t^{(m+1)}(u)]^{c_{m+1}}.$$

□

B.2. Proof of expression (11) in Theorem 4.2. Suppose φ derivable as many times as necessary and X admitting moments of all orders.

Then, for $n\geq 2$,

$$\kappa_{X_t}^{(n)}(z)=\frac{i\alpha_0}{\varphi'(0)}\varphi^{(n)}(z)+\sum_{j=1}^{n-1}a_{n-1,j}\cdot\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$
$$+\left[\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right]^n\cdot\kappa_{X_{t-1}}^{(n)}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right), \tag{13}$$

where the coefficients $a_{n-1,j}$ are given by

$$a_{n-1,1}=\frac{\alpha_1}{\varphi'(0)}\varphi^{(n)}(z),$$

$$a_{n-1,j}=\left[\frac{\alpha_1}{\varphi'(0)}\right]^j\sum_{\substack{k_1+...+k_n=j\\ k_1+2k_2+...+nk_n=n\\ k_r\in\mathbb{N}_0}}(n;k_1,...,k_n)\left[\varphi'(z)\right]^{k_1}...[\varphi^{(n)}(z)]^{k_n},\ j\geq 2.$$

Proof. Let us prove this formula by induction.

Taking derivative in the expression of κ_{X_t}', one obtains

$$\kappa_{X_t}''(z)=\frac{i\alpha_0}{\varphi'(0)}\varphi''(z)+\frac{\alpha_1}{\varphi'(0)}\varphi''(z)\cdot\kappa_{X_{t-1}}'\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\left[\frac{\alpha_1}{\varphi'(0)}\varphi'(z)\right]^2 \cdot \kappa''_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right),$$

so, (13) is valid for $n=2$ since $a_{11}=\frac{\alpha_1}{\varphi'(0)}\varphi''(z)$. Now, assume that formula (13) has been shown for an $n\geq 2$ and consider $a_{n-1,0}=0$. It then follows that

$$\kappa^{(n+1)}_{X_t}(z)=\frac{d}{dz}\left(\frac{i\alpha_0}{\varphi'(0)}\varphi^{(n)}(z)+\sum_{j=1}^{n-1}a_{n-1,j}\cdot\kappa^{(j)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\right.$$

$$\left.+\left[\frac{\alpha_1\varphi'(z)}{\varphi'(0)}\right]^n\cdot\kappa^{(n)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)\right)$$

$$=\frac{i\alpha_0}{\varphi'(0)}\varphi^{(n+1)}(z)+\sum_{j=1}^{n}a_{n-1,j-1}\frac{\alpha_1}{\varphi'(0)}\varphi'(z)\cdot\kappa^{(j)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\sum_{j=1}^{n-1}\left(\frac{\alpha_1}{\varphi'(0)}\right)^j\left[\sum_{\substack{k_1+\ldots+k_n=j\\k_1+\ldots+nk_n=n}}k_1(n;k_1,...,k_n)\left[\varphi'(z)\right]^{k_1-1}\left[\varphi''(z)\right]^{k_2+1}\ldots[\varphi^{(n)}(z)]^{k_n}\right.$$

$$+\sum_{\substack{k_1+\ldots+k_n=j\\k_1+\ldots+nk_n=n}}k_2(n;k_1,...,k_n)\left[\varphi'(z)\right]^{k_1}\left[\varphi''(z)\right]^{k_2-1}\left[\varphi'''(z)\right]^{k_3+1}\ldots[\varphi^{(n)}(z)]^{k_n}$$

$$\left.+\ldots+\sum_{\substack{k_1+\ldots+k_n=j\\k_1+\ldots+nk_n=n}}k_n(n;k_1,...,k_n)\left[\varphi'(z)\right]^{k_1}\ldots[\varphi^{(n)}(z)]^{k_n-1}\varphi^{(n+1)}(z)\right]\cdot\kappa^{(j)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+n\left(\frac{\alpha_1}{\varphi'(0)}\right)^n\left[\varphi'(z)\right]^{n-1}\varphi''(z)\cdot\kappa^{(n)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)+\left[\frac{\alpha_1}{\varphi'(0)}\varphi'(z)\right]^{n+1}\cdot\kappa^{(n+1)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$=\frac{i\alpha_0}{\varphi'(0)}\varphi^{(n+1)}(z)+\sum_{j=1}^{n-1}\left(\frac{\alpha_1}{\varphi'(0)}\right)^j\left[\sum_{\substack{c_1+\ldots+c_n=j\\c_1+\ldots+nc_n=n+1}}\frac{c_1}{n+1}(n+1;c_1,...,c_n,0)\left[\varphi'(z)\right]^{c_1}\ldots[\varphi^{(n)}(z)]^{c_n}\right.$$

$$+\sum_{\substack{c_1+\ldots+c_n=j\\c_1+\ldots+nc_n=n+1}}\frac{2c_2}{n+1}(n+1;c_1,...,c_n,0)\left[\varphi'(z)\right]^{c_1}\ldots[\varphi^{(n)}(z)]^{c_n}+\ldots$$

$$+\sum_{\substack{c_1+\ldots+c_n=j\\c_1+\ldots+nc_n=n+1}}\frac{nc_n}{n+1}(n+1;c_1,...,c_n,0)\left[\varphi'(z)\right]^{c_1}\ldots[\varphi^{(n)}(z)]^{c_n}$$

$$\left.+\sum_{\substack{c_1+\ldots+c_{n+1}=j\\c_1+\ldots+nc_n=n}}(n+1;c_1,...,c_n,1)\left[\varphi'(z)\right]^{c_1}\ldots[\varphi^{(n)}(z)]^{c_n}\varphi^{(n+1)}(z)\right]\cdot\kappa^{(j)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\left(\frac{\alpha_1}{\varphi'(0)}\right)^n(n+1;n-1,1,0,...,0)\left[\varphi'(z)\right]^{n-1}\varphi''(z)\cdot\kappa^{(n)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right)$$

$$+\left[\frac{\alpha_1}{\varphi'(0)}\varphi'(z)\right]^{n+1}\cdot\kappa^{(n+1)}_{X_{t-1}}\left(\frac{\alpha_1}{\varphi'(0)}\left[\varphi(z)-1\right]\right),$$

using (12) and the fact that

$$a_{n-1,j-1}\frac{\alpha_1}{\varphi'(0)}\varphi'(z)=\left(\frac{\alpha_1}{\varphi'(0)}\right)^j\sum_{\substack{k_1+...+k_n=j-1\\k_1+...+nk_n=n}}(n;k_1,...,k_n)\left[\varphi'(z)\right]^{k_1+1}...[\varphi^{(n)}(z)]^{k_n}$$

$$=\left(\frac{\alpha_1}{\varphi'(0)}\right)^j\sum_{\substack{c_1+...+c_n=j\\c_1+...+nc_n=n+1}}(n;c_1-1,c_2,...,c_n)\left[\varphi'(z)\right]^{c_1}...[\varphi^{(n)}(z)]^{c_n}$$

$$=\left(\frac{\alpha_1}{\varphi'(0)}\right)^j\sum_{\substack{c_1+...+c_n=j\\c_1+...+nc_n=n+1}}\frac{c_1}{n+1}(n+1;c_1,...,c_n,0)\left[\varphi'(z)\right]^{c_1}...[\varphi^{(n)}(z)]^{c_n},$$

$$\Rightarrow a_{n-1,n-1}\frac{\alpha_1}{\varphi'(0)}\varphi'(z)=\left(\frac{\alpha_1}{\varphi'(0)}\right)^n\frac{n-1}{n+1}\,(n+1;n-1,1,0,...,0)\left[\varphi'(z)\right]^{n-1}\varphi''(z)$$

$$=\frac{(n+1)n}{2}\left(\frac{\alpha_1}{\varphi'(0)}\right)^n\left[\varphi'(z)\right]^{n-1}\varphi''(z).$$

This implies that

$$\kappa_{X_t}^{(n+1)}(z)=\frac{i\alpha_0}{\varphi_t'(0)}\varphi_t^{(n+1)}(z)+\left[\frac{\alpha_1}{\varphi_t'(0)}\varphi_t'(z)\right]^{n+1}\cdot\kappa_{X_{t-1}}^{(n+1)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right)$$

$$+\sum_{j=1}^{n}\left(\frac{\alpha_1}{\varphi_t'(0)}\right)^j\sum_{\substack{c_1+...+c_{n+1}=j\\c_1+...+(n+1)c_{n+1}=n+1}}(n+1;c_1,...,c_{n+1})\left[\varphi_t'(z)\right]^{c_1}\dots[\varphi_t^{(n+1)}(z)]^{c_{n+1}}$$

$$\times\,\kappa_{X_{t-1}}^{(j)}\left(\frac{\alpha_1}{\varphi_t'(0)}\left[\varphi_t(z)-1\right]\right),$$

which ends the proof. □

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