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SOME REMARKS ON PULLBACKS IN GUMM CATEGORIES

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Dedicated to Manuela Sobral on the occasion of her seventieth birthday

ABSTRACT: We extend some properties of pullbacks which are known to hold in a Mal'tsev context to the more general context of Gumm categories. The varieties of universal algebras which are Gumm categories are precisely the congruence modular ones. These properties lead to a simple alternative proof of the known property that central extensions and normal extensions coincide for any Galois structure associated with a Birkhoff subcategory of an exact Goursat category.

KEYWORDS: regular category, Mal'tsev category, Goursat category, Gumm category, congruence modularity, pullback properties.
AMS SUBJECT CLASSIFICATION (2010): 18C05, 08B10, 08C05, 18B10, 18E10.

Introduction

A categorical approach to the property of congruence modularity, well known in universal algebra, was proposed in [5, 6] via a categorical formulation of the so-called *Shifting Lemma* (recalled in Section 2). The categories satisfying this categorical property are called *Gumm categories*, since it was the mathematician H.P. Gumm who proved that, for a variety of universal algebras, the validity of the Shifting Lemma is equivalent to congruence modularity [13]. As examples of Gumm categories, we also have regular Mal'tsev categories [9, 8] and regular Goursat categories [8], which are defined by the property that any pair of equivalence relations R and S in $\mathbb{C}$ (on a same object) 3-permute: $RSR = SRS$.

In this context [3] D. Bourn established an interesting permutability result (see Theorem 2.2), that we use in the present paper to prove the following property of regular Gumm categories (Proposition 3.1). Given a commutative

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diagram

$$\begin{tikzcd}
Z \times_V U \arrow[r, two heads] \arrow[d, shift right] \arrow[dr, phantom, "1"] & X \arrow[r] \arrow[d, shift right] \arrow[dr, phantom, "2"] & U \arrow[d] \\
Z \arrow[r, two heads] \arrow[u, shift right] & Y \arrow[r] \arrow[u, shift right] & V
\end{tikzcd}$$

in a regular Gumm category such that the whole rectangle is a pullback and the left square $\boxed{1}$ is composed by vertical split epimorphisms and horizontal regular epimorphisms, then both squares $\boxed{1}$ and $\boxed{2}$ are pullbacks. This property is known to hold in any regular Mal'tsev category, and has been used, for example, in the categorical theory of central extensions [10, 7].

In the present article we also show that this property can be used to give a new proof of a remarkable property of exact Goursat categories, namely the fact that central extensions and normal extensions relative to any (admissible) Birkhoff subcategory $\mathbb{X}$ of $\mathbb{C}$ coincide [15]. Let us recall that a full reflective subcategory $\mathbb{X}$ of an exact category $\mathbb{C}$ is called a Birkhoff subcategory when $\mathbb{X}$ is closed in $\mathbb{C}$ under subobjects and regular quotients. In particular, a Birkhoff subcategory of a variety of universal algebras is just a subvariety. A Birkhoff subcategory $\mathbb{X}$ is admissible, from the point of view of Categorical Galois Theory, when the reflector $I\colon \mathbb{X} \to \mathbb{C}$ preserves pullbacks of regular epimorphisms in $\mathbb{X}$ along any morphism in $\mathbb{C}$. The notions of central extension and of normal extension are defined relatively to the choice of the admissible Birkhoff subcategory $\mathbb{X}$ of $\mathbb{C}$, as recalled in Section 4. It is precisely the useful property of pullbacks in regular Gumm categories stated above which allows one to find a simple proof of the coincidence of these two notions in the exact Goursat context (Theorem 4.2 and Corollary 4.3).

In [15] G. Janelidze and G.M. Kelly proved that every Birkhoff subcategory $\mathbb{X}$ of an exact category $\mathbb{C}$ with modular lattice of equivalence relations (on any object in $\mathbb{C}$) is always admissible. It was later shown by V. Rossi in [20] that the same admissibility property still holds in the more general context of Gumm categories which are *almost exact*, a notion introduced by G. Janelidze and M. Sobral in [16]. We conclude the article by relating our observations on Gumm categories with these results concerning the admissibility of Galois structures.

1. Preliminaries

In the present paper the term *regular category* [1] will be used for a finitely complete category such that any kernel pair has a coequaliser and, moreover, regular epimorphisms are stable under pullbacks. Any morphism $f\colon A \to B$ in a regular category $\mathbb{C}$ has a factorisation $f = m \cdot p$, with p a regular epimorphism and m a monomorphism. It is well known that such factorisations are necessarily stable under pullbacks in a regular category. The subobject determined by the monomorphism m in the factorisation of $f = m \cdot p$ is usually called the *image* of f.

A relation R from A to B is a subobject $\langle r_1, r_2\rangle\colon R \rightarrowtail A \times B$. The opposite relation, denoted by R°, is the relation from B to A given by the subobject $\langle r_2, r_1\rangle\colon R \rightarrowtail B \times A$. A morphism $f : A \to B$ can be identified with the relation $\langle 1_A, f\rangle\colon A \rightarrowtail A \times B$; its opposite relation $\langle f, 1_A\rangle\colon A \rightarrowtail B \times A$ is usually referred to as f°. Given another relation $\langle s_1, s_2\rangle\colon S \rightarrowtail B \times C$ from B to C, the composite relation of R and S is a relation SR from A to C, obtained as the image of the induced arrow $\langle r_1\pi_1, s_2\pi_2\rangle\colon R \times_B S \rightarrowtail A \times C$, where $(R \times_B S, \pi_1, \pi_2)$ is the pullback of r_2 and s_1. With the above notation, we can write any relation $\langle r_1, r_2\rangle\colon R \rightarrowtail A \times B$ as $R = r_2 r_1^\circ$. The following properties are well known, and also easy to check (see [8], for instance):

Lemma 1.1. *Let $f : A \to B$ be any morphism in a regular category $\mathbb{C}$. Then:*
(1) *$ff^\circ f = f$ and $f^\circ f f^\circ = f^\circ$;*
(2) *$ff^\circ = 1_B$ if and only if f is a regular epimorphism.*

A relation R from an object A to A is called a *relation on* A. Such a relation is *reflexive* if $1_A \leqslant R$, *symmetric* if $R^\circ \leqslant R$, and *transitive* when $RR \leqslant R$. A relation R on A is called an *equivalence relation* when it is reflexive, symmetric and transitive. Any kernel pair $\langle f_1, f_2\rangle\colon R_f \rightarrowtail A \times A$ of a morphism $f : A \to B$ is an equivalence relation, called an *effective* equivalence relation. By using the composition of relations, it can be written either as $R_f = f^\circ f$, or as $R_f = f_2 f_1^\circ$. Of course, if $f = m \cdot p$ is the (regular epimorphism, monomorphism) factorisation of an arbitrary morphism f, then $R_f = R_p$, so that an effective equivalence relation is always the kernel pair of a regular epimorphism.

2. Gumm categories

A lattice $(L, \vee, \wedge)$ is called *modular* when, for $x, y, z \in L$, one has

$$x \leqslant z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z.$$

A variety $\mathcal{V}$ of universal algebras is called *congruence modular* when every lattice of congruences (= effective equivalence relations) on any algebra in $\mathcal{V}$

is modular. It is well known from [13] that a variety $\mathcal{V}$ is congruence modular if and only if the following property, called the Shifting Lemma, holds in $\mathcal{V}$:

Shifting Lemma
Given congruences R, S and T on the same algebra X such that $R \wedge S \leqslant T$, whenever x, y, t, z are elements in X with $(x, y) \in R \wedge T$, $(x, t) \in S$, $(y, z) \in S$ and $(t, z) \in R$, it then follows that $(t, z) \in T$:

$$\begin{tikzcd}
x \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=80, "T"'] & t \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=80, "T"] \\
y \arrow[r, dash, "S"'] & z
\end{tikzcd}$$

This notion has been extended to a categorical context in [6]. Indeed, the property expressed by the Shifting Lemma can be equivalently reformulated in any finitely complete category $\mathbb{C}$ by asking that a specific class of internal functors are discrete fibrations, as we are now going to recall. For any object X in $\mathbb{C}$ and any equivalence relations R, S, T on X with

$$R \wedge S \leq T \leq R$$

there is a canonical inclusion $(i, j)\colon T\square S \to R\square S$ of equivalence relations, depicted as

$$\begin{tikzcd}
T\square S \arrow[r, tail, "j"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & R\square S \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\
T \arrow[r, tail, "i"'] & R,
\end{tikzcd} \tag{1}$$

where $T\square S$ (respectively, $R\square S$) is the largest double equivalence relation on T and S (respectively, on R and S) and π_1 and π_2 are the projections on T (respectively, on R).

Definition 2.1. [6] A finitely complete category $\mathbb{C}$ is called a *Gumm category* when any inclusion $(i, j)\colon T\square S \to R\square S$ as in (1) is a discrete fibration. This means that any of the commutative squares in (1) is a pullback.

Let us recall that a *Mal'tsev* category $\mathbb{C}$ is a finitely complete category such that every reflexive relation in $\mathbb{C}$ is an equivalence relation. A regular category $\mathbb{C}$ is a Mal'tsev category when the composition of (effective) equivalence relations on any object in $\mathbb{C}$ is 2-permutable: $RS = SR$, where R and S are (effective) equivalence relations on a same object (see [9, 8]). The strictly weaker 3-permutability property for (effective) equivalence relations, $RSR = SRS$, defines the notion of regular *Goursat categories* [8]. Goursat categories, thus in particular Mal'tsev categories, have the property that every lattice of equivalence relations (on the same object) is modular (Proposition 3.2 in [8]). This fact implies that any regular Mal'tsev category and, more generally, any regular Goursat category is a Gumm category. Thanks to the characterization theorem in [13] one knows that a variety of universal algebras is congruence modular if and only if it is a Gumm category.

The following property of regular Gumm categories, due to D. Bourn, will play a fundamental role in the next section.

Theorem 2.2. (Theorem 7.12 of [3]) *Let $\mathbb{C}$ be a regular Gumm category. Consider equivalence relations R, S and T on a same object such that $RS = SR$ and $R \wedge S \leqslant T \leqslant R$. Then $TS = ST$.*

3. Pullback properties in regular Gumm categories

In this section we extend some useful properties of pullbacks from the context of Mal'tsev categories to that of Gumm categories.

The first observation concerns a generalisation of Proposition 3.4 in [7]:

Proposition 3.1. *Let $\mathbb{C}$ be a regular Gumm category. Consider a commutative diagram*

$$\begin{tikzcd}
Z \times_V U \arrow[r, two heads, "x"] \arrow[d, shift right, "\varphi"'] \arrow[dr, phantom, "1"] & X \arrow[r, "u"] \arrow[d, shift right, "f"'] \arrow[dr, phantom, "2"] & U \arrow[d, "w"] \\
Z \arrow[r, two heads, "y"'] \arrow[u, shift right, "\sigma"'] & Y \arrow[r, "v"'] \arrow[u, shift right, "s"'] & V,
\end{tikzcd} \tag{2}$$

such that the whole rectangle is a pullback and the left square $\boxed{1}$ is composed by vertical split epimorphisms and horizontal regular epimorphisms. Then both squares $\boxed{1}$ and $\boxed{2}$ are pullbacks.

Proof: The comparison morphism $\langle \varphi, x \rangle \colon Z \times_V U \to Z \times_Y X$ is clearly a monomorphism. We are now going to show that it is also a regular epimorphism, i.e. that $\varphi x^\circ = y^\circ f$. This will imply that the square $\boxed{1}$ is a pullback.

Consider the effective equivalence relations $R = R_{ux}, S = R_\varphi$ and $T = R_x$ (to use the same notations as in Theorem 2.2). The fact that the whole rectangle $\boxed{1}\boxed{2}$ is a pullback, implies that $R_{ux} \wedge R_\varphi = 1$ and $R_{ux}R_\varphi = R_\varphi R_{ux}$. Since $R_{ux} \wedge R_\varphi = 1 \leqslant R_x \leqslant R_{ux}$, we conclude that $R_x R_\varphi = R_\varphi R_x$ by Theorem 2.2. Equivalently, one has the equality $x^\circ x \varphi^\circ \varphi = \varphi^\circ \varphi x^\circ x$.

Since the vertical morphisms in $\boxed{1}$ are split epimorphisms, then the comparison morphism $R_x \to R_y$ is also a split epimorphism, thus the image of R_x along φ is R_y: $\varphi(R_x) = R_y$ or, equivalently, $\varphi x^\circ x \varphi^\circ = y^\circ y$. One then has the following equalities:

$$\begin{aligned} \varphi x^\circ &= \varphi\varphi^\circ\varphi x^\circ x x^\circ \\ &= \varphi x^\circ x \varphi^\circ \varphi x^\circ \\ &= y^\circ y \varphi x^\circ \\ &= y^\circ f x x^\circ \\ &= y^\circ f. \end{aligned}$$

Consequently, the arrow $\langle \varphi, x\rangle \colon Z \times_V U \to Z \times_Y X$ is an isomorphism, and the square $\boxed{1}$ is a pullback. The square $\boxed{2}$ is then also a pullback, since the change-of-base functor along a regular epimorphism in a regular category reflects isomorphisms (see Proposition 2.7 of [15], for instance). ■

From the proposition above we shall deduce a general result in the context of almost exact Gumm categories, which extends Lemma 1.1 in [10]. Recall that a regular category $\mathbb{C}$ is called an *almost exact category* [16] when any regular epimorphism in $\mathbb{C}$ is an effective descent morphism. This property can be expressed as follows [17]: for any regular epimorphism $f \colon X \twoheadrightarrow Y$ and any vertical discrete fibration $(g, h) \colon R \to R_f$ in $\mathbb{C}$

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[l] \arrow[d, "g"] & \\ R_f \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & X \arrow[l] \arrow[r, two heads, "f"] & Y \end{tikzcd}$$

with R an equivalence relation on X', then R is an effective equivalence relation.

Proposition 3.2. *Let $\mathbb{C}$ be an almost exact Gumm category. Consider a commutative diagram*

$$\begin{tikzcd} Z \times_V U \arrow[r, two heads, "x"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "1"] & X \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "2"] & U \arrow[d, "w"] \\ Z \arrow[r, two heads, "y"'] & Y \arrow[r, "v"'] & V, \end{tikzcd} \tag{3}$$

such that the whole rectangle is a pullback and the left square $\boxed{1}$ *is a pushout of regular epimorphisms. Then both squares* $\boxed{1}$ *and* $\boxed{2}$ *are pullbacks.*

Proof: Consider the following commutative diagram where $\boxed{3}$ gives the image factorisation of $\langle x\varphi_1, x\varphi_2\rangle \colon R_\varphi \to X \times X$, and the morphism ρ is induced by the equality $wur_1 = wur_2$:

$$\begin{tikzcd} R_\varphi \arrow[r, two heads, "p"] \arrow[d, shift right=2, "\varphi_1"'] \arrow[d, shift left=2, "\varphi_2"] \arrow[dr, phantom, "3"] & R \arrow[r, "\rho"] \arrow[d, shift right=2, "r_1"'] \arrow[d, shift left=2, "r_2"] \arrow[dr, phantom, "4"] & R_w \arrow[d, shift right=2, "w_1"'] \arrow[d, shift left=2, "w_2"] \\ Z \times_V U \arrow[r, two heads, "x"'] \arrow[d, "\varphi"'] \arrow[u] \arrow[dr, phantom, "1"] & X \arrow[r, "u"'] \arrow[d, "f"] \arrow[u] \arrow[dr, phantom, "2"] & U \arrow[d, "w"] \arrow[u] \\ Z \arrow[r, two heads, "y"'] & Y \arrow[r, "v"'] & V. \end{tikzcd}$$

Since $\boxed{1}\boxed{2}$ is a pullback, then (any of the commutative squares) $\boxed{3}\boxed{4}$ is also a pullback. We can apply Proposition 3.1 to $\boxed{3}\boxed{4}$ to conclude that (any of the commutative squares) $\boxed{3}$ and $\boxed{4}$ are pullbacks.

Note that $R = x(R_\varphi)$ is an equivalence relation: it is necessarily reflexive and symmetric being the image of the equivalence relation R_φ along a regular epimorphism x. It is also transitive: indeed, as in the proof of Proposition 3.1, the assumptions still guarantee that $R_\varphi R_x = R_x R_\varphi$, and this implies that

$$\begin{aligned} RR &= x\varphi^\circ\varphi x^\circ x\varphi^\circ\varphi x^\circ \\ &= xx^\circ x\varphi^\circ\varphi\varphi^\circ\varphi x^\circ \\ &= x\varphi^\circ\varphi x^\circ \\ &= R. \end{aligned}$$

Since regular epimorphisms are effective for descent in $\mathbb{C}$, the equivalence relation R is effective, and it is then the kernel pair of its coequaliser. Moreover, the fact that the square $\boxed{1}$ is a pushout easily implies that this coequaliser is $f\colon\ X \twoheadrightarrow Y$, and $R = R_f = f^\circ f$. To complete the proof, one applies the Barr-Kock Theorem [1] twice to conclude that $\boxed{1}$ and $\boxed{2}$ are pullbacks. ■

Remark 3.3. Any efficiently regular category in the sense of [4] is almost-exact, so that the result above is true in particular in efficiently regular Gumm categories. For instance, this is the case for any category of topological Mal'tsev algebras [19], then in particular for the category of topological groups. Also any almost abelian category in the sense of [21] is almost exact (see [12]). Another example of almost exact category is provided by the category of regular epimorphisms in an exact Goursat category [16].

4. An application to Galois Theory

In this section we give an application of the results from Section 3 in Categorical Galois Theory [14, 15].

A commutative square of regular epimorphism

$$\begin{tikzcd}
P \arrow[r, two heads, "x"] \arrow[d, two heads, "a"'] & X \arrow[d, two heads, "u"] \\
A \arrow[r, two heads] & U
\end{tikzcd}$$

is called *right saturated* [11] when the comparison morphism $\bar{x}\colon R_a \to R_u$ is also a regular epimorphism.

Proposition 4.1. *Let $\mathbb{C}$ be a regular Gumm category. Consider a commutative cube*

$$\begin{tikzcd}
P \arrow[rr, two heads, "x"] \arrow[dr, two heads, "a"] \arrow[dd, shift right, "\varphi"'] & & X \arrow[dr, two heads, "u"] \arrow[dd, dashed, shift right, "f"'] & \\
& A \arrow[rr, two heads] \arrow[dd, shift right, "g"'] & & U \arrow[dd, shift right, "w"'] \\
Z \arrow[rr, dashed, two heads, "y"' near end] \arrow[dr, two heads, "b"'] \arrow[uu, shift right, "\sigma"'] & & Y \arrow[dr, dashed, two heads, "v"] \arrow[uu, dashed, shift right, "s"'] & \\
& B \arrow[rr, two heads] \arrow[uu, shift right, "t"'] & & V \arrow[uu, shift right, "i"']
\end{tikzcd} \tag{4}$$

of vertical split epimorphisms and regular epimorphisms. If the left and back faces are pullbacks and the top and bottom faces are right saturated, then the front and right faces are also pullbacks.

Proof: We take the kernel pairs of a, b, u and v and the induced morphisms between them:

$$
\begin{tikzcd}
R_a \arrow[rr, two heads, "\bar{x}"] \arrow[dr, shift left, "a_2"] \arrow[dr, shift right, "a_1"'] \arrow[dd, shift right, "\bar{\varphi}"'] & & R_u \arrow[dr, shift left, "u_2"] \arrow[dr, shift right, "u_1"'] \arrow[dd, dashed, shift right, "\bar{f}"'] & \\
& P \arrow[rr, two heads, "x"] \arrow[dd, shift right, "\varphi"'] & & X \arrow[dd, shift right, "f"'] \\
R_b \arrow[uu, shift right, "\bar{\sigma}"'] \arrow[rr, dashed, two heads, "\bar{y}"'] \arrow[dr, shift left, "b_2"] \arrow[dr, shift right, "b_1"'] & & R_v \arrow[uu, dashed, shift right, "\bar{s}"'] \arrow[dr, dashed, shift left, "v_2"] \arrow[dr, dashed, shift right, "v_1"'] & \\
& Z \arrow[uu, shift right, "\sigma"'] \arrow[rr, two heads, "y"'] & & Y. \arrow[uu, shift right, "s"']
\end{tikzcd}
$$

Note that $\bar{x}$ and $\bar{y}$ are regular epimorphisms since the top and bottom faces of the cube (4) are right saturated. The left and front faces above are pullbacks, so that the rectangle formed by the back and right faces is a pullback. We can apply Proposition 3.1 to conclude that both the back and right faces above are pullbacks. By the Barr-Kock Theorem the right face of diagram (4) is then a pullback, hence so is the front face of (4). ■

As a consequence of Proposition 4.1 we give a new proof of Theorem 4.8 of [15] stating that every central and split extension is a trivial extension for the Galois structure associated with any Birkhoff subcategory of an exact Goursat category. Let us briefly recall the main definitions, and we refer to [15] for more details.

When $\mathbb{C}$ is an exact category and $\mathbb{X}$ a full replete subcategory of $\mathbb{C}$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathbb{X}, \arrow[l, shift left, hook]
\end{tikzcd}
\tag{5}
$$

one calls $\mathbb{X}$ a *Birkhoff* subcategory of $\mathbb{C}$ when $\mathbb{X}$ is stable in $\mathbb{C}$ under subobjects and regular quotients. Equivalently, all X-components η_X of the unit $\eta \colon 1_{\mathbb{C}} \Rightarrow I$ of the adjunction are regular epimorphisms (the right adjoint is assumed to be a full inclusion and will not be mentioned explicitely), and the naturality square

$$
\begin{tikzcd}
X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & IX \arrow[d, "If"] \\
Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}
\tag{6}
$$

is a pushout for any regular epimorphism $f \colon X \twoheadrightarrow Y$.

A regular epimorphism $f\colon X \twoheadrightarrow Y$ is called a *trivial extension* when the naturality square (6) is a pullback. It is called a *central extension* when it is "locally" trivial: there exists a regular epimorphism $y\colon Z \twoheadrightarrow Y$ such that the pullback of f along y is a trivial extension.

Theorem 4.2. (Theorem 4.8 of [15]) *Let $\mathbb{C}$ be an exact Goursat category, and $\mathbb{X}$ a Birkhoff subcategory of $\mathbb{C}$. Then every central and split extension is necessarily a trivial extension.*

Proof: Let $f\colon X \twoheadrightarrow Y$ be both a central extension and a split epimorphism. By definition, there exists a regular epimorphism $y\colon Z \twoheadrightarrow Y$ such that the pullback of f along y is a trivial extension. So, in the following commutative cube

$$\begin{tikzcd}
P \arrow[rr, two heads, "x"] \arrow[dr, "\eta_P"] \arrow[dd, shift right, "\varphi"'] & & X \arrow[dr, "\eta_X"] \arrow[dd, dashed, shift right, "f"'] & \\
& IP \arrow[rr, two heads] \arrow[dd, shift right, "I\varphi"'] & & IX \arrow[dd, shift right, "If"'] \\
Z \arrow[uu, shift right, "\sigma"'] \arrow[rr, dashed, two heads, "y"] \arrow[dr, "\eta_Z"] & & Y \arrow[uu, dashed, shift right, "s"'] \arrow[dr, dashed, "\eta_Y"] & \\
& IZ \arrow[uu, shift right, "I\sigma"'] \arrow[rr, two heads] & & IY, \arrow[uu, shift right, "Is"']
\end{tikzcd}$$

the back face is a pullback by construction, and the left face is a pullback by the assumption that φ is a trivial extension. Note that the top and bottom faces are pushouts of regular epimorphisms and are then right saturated (by Proposition 7.1 of [8]). By Proposition 4.1 we conclude that the front and right faces are pullbacks. It follows that f is a trivial extension. ∎

A regular epimorphism $f\colon X \twoheadrightarrow Y$ is called a *normal extension* if the pullback of f along itself is a trivial extension. By definition any normal extension is central, but the converse is false in general, as various counter-examples given in [15] show. In the exact Goursat context the notions of central and normal extensions coincide:

Corollary 4.3. [15] *Consider a Birkhoff adjunction* (5) *where $\mathbb{C}$ is an exact Goursat category. Then every central extension is normal.*

Proof: This follows immediately from Theorem 4.2 and the fact that central extensions are pullback stable (this follows from the fact that adjunction is "admissible" in the sense of Categorical Galois Theory, see the Remark 4.4 here below). ∎

Remark 4.4. A Birkhoff subcategory $\mathbb{X}$ of an exact category $\mathbb{C}$ is called *admissible* when I preserves pullbacks of the form

$$\begin{tikzcd} A \arrow[r, "n"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "w"] \\ B \arrow[r, "m"'] & V, \end{tikzcd} \tag{7}$$

where $w\colon U \twoheadrightarrow V$ is a regular epimorphism of $\mathbb{X}$ (see Proposition 3.3 of [15]). In [20] V. Rossi proved that any Birkhoff subcategory of an almost exact Gumm category is admissible, extending a result due to G. Janelidze and G.M. Kelly [15]. The proof of the more general Proposition 3.2 above is actually similar to the one given in [20]. In order to deduce the admissibility result from Proposition 3.2 it suffices to decompose the square (7) above as

$$\begin{tikzcd} A \arrow[r, two heads, "\eta_A"] \arrow[d, "\varphi"'] & IA \arrow[r, "In"] \arrow[d, "I\varphi"] & U \arrow[d, "w"] \\ B \arrow[r, two heads, "\eta_B"'] & IB \arrow[r, "Im"'] & V \end{tikzcd}$$

to conclude that both squares are pullbacks.

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