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ON CONVERGENCE RATES FOR WEIGHTED SUMS OF ASSOCIATED RANDOM VARIABLES

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ABSTRACT: We study the convergence of weighted sums of associated random variables assuming only the existence of moments of order $p < 2$. We use a truncation technique together with coupling with independent variables, which allows a relaxation of the assumptions on the weights. Moreover, this coupling allows not only for the proof of almost sure results and but enables to identify convergence rates. The assumptions on p, that now include the case $p < 1$, excluded from earlier results for positively associated variables, depend on the asymptotic behaviour of the weights, as usual. We give a direct comparison with the characterizations previously available, showing that the scope of applicability of our results does not overlap with known conditions for the same asymptotic results.

KEYWORDS: weighted sums, associated random variables, almost sure convergence, convergence rates.
AMS SUBJECT CLASSIFICATION (2000): 60F15.

1. Introduction

Many linear statistics are written as weighted sums of random variables, raising thus the interest in the characterization of the asymptotics of such sums, conveniently normalized. Since Baum and Katz [2] proved an almost sure result for constant weights with a normalization sequence $n^{-1/p}$, where p describes the moment condition on the variables, many authors studied this problem. Chow [4] and Cuzick [5] obtained conditions for the convergence for weighted sums with independent variables, later extended by Cheng [3], Bai and Cheng [1], or Sung [12] relaxing the moment assumption. This convergence has also been considered for dependent variables. Louhichi [9] obtained sufficient conditions for the convergence with constant weights but requiring only the existence of low, less than 2, order moments. These results were, more recently, extended for weighted sums in Oliveira [10] and Çağın and Oliveira [6], using an approach similar to Louhichi's [9]. Here we follow

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the method used in Oliveira [11] for the proof of exponential inequalities, extending results that Ioannides and Roussas [8] proved in a more reduced framework, to prove conditions for the almost sure convergence and, in some cases, of its rate. These conditions depend on the covariances and link p with the behaviour of the weighting coefficients. In Sect. 2 we describe the framework and useful results, Sect. 4 presents the main results, and compares with results in Çağın and Oliveira [6], while Sect. 3 states versions of these results in a reduced setting, but proves the main steps for the final theorems.

2. Definitions and preliminary results

Let us assume that the X_n, $n \geq 1$, are centered and associated random variables and denote $S_n = X_1 + \cdots + X_n$. Let $a_{n,i}$, $i = 1, \ldots, n$, $n \geq 1$, be non negative real numbers and define, for some $\alpha > 1$, $A_{n,\alpha}^\alpha = n^{-1} \sum_{i=1}^n |a_{n,i}|^\alpha$. We will be interested in the convergence of $T_n = \sum_{i=}^n a_{n,i} X_i$ assuming that

$$A_\alpha = \sup_n A_{n,\alpha} < \infty. \tag{1}$$

This is the only condition on the weights throughout this paper, thus relaxing the assumption on the weights when compared to Oliveira [10] or Çağın and Oliveira [6]. Remark that, due to the nonnegativity of the weights, the variables T_n, $n \geq 1$, are associated. Define the usual Cox-Grimmett coefficients

$$u(n) = \sup_{k \geq 1} \sum_{j:|k-j| \geq n} \mathrm{Cov}(X_j, X_k). \tag{2}$$

If the random variables are stationary, then $u(n) = 2 \sum_{j=n+1}^\infty \mathrm{Cov}(X_1, X_j)$.

Consider p_n a sequence of natural numbers such that $p_n < \frac{n}{2}$, r_n the largest integer less or equal to $\frac{n}{2p_n}$, and define the variables

$$Y_{n,j} = \sum_{i=(j-1)p_n+1}^{jp_n} a_{n,i} X_i, \qquad j = 1, \ldots, 2r_n.$$

These random variables are associated, due to the nonnegativity of the weights. Moreover, if the variables X_n are uniformly bounded by $c > 0$, then it is obvious that $|Y_{n,j}| \leq c A_\alpha n^{1/\alpha} p_n$. Finally, put

$$T_{n,od} = \sum_{j=1}^{r_n} Y_{n,2j-1} \quad \text{and} \quad T_{n,ev} = \sum_{j=1}^{r_n} Y_{n,2j}.$$

We prove first an easy but useful upper bound.

Lemma 2.1. *Assume the variables X_n, $n \geq 1$, are associated, stationary, bounded (by $c > 0$) and $u(0) < \infty$. Then* $\mathrm{E}(S_n^2) \leq 2c^* n$, *where* $c^* = c^2 + u(0)$.

Proof: Using the stationarity, it follows easily that $\mathrm{E}(S_n^2) = n\mathrm{Var}(X_1) + 2\sum_{j=1}^{n-1}(n-j)\mathrm{Cov}(X_1, X_{j+1}) \leq 2nc^2 + 2nu(0)$. ■

The next result is an extension of Lemma 3.1 in Oliveira [11].

Lemma 2.2. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary, bounded (by $c > 0$), $u(0) < \infty$ and the nonnegative weights satisfy (1). If $d_n \geq 1$ and $0 < \lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$, then*

$$\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{n,2j-1}}\right) \leq \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n\right)$$

and

$$\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{n,2j}}\right) \leq \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n\right).$$

Proof: As remarked above, as the variables X_n are bounded, we have that $|Y_{n,j}| \leq cA_\alpha n^{1/\alpha} p_n$. So, using a Taylor expansion it follows that

$$\mathrm{E}\left(e^{\lambda Y_{n,2j-1}}\right) \leq 1 + \lambda^2 \mathrm{E}\left(Y_{n,2j-1}^2\right) \sum_{k=2}^{\infty} (cA_\alpha \lambda n^{1/\alpha} p_n)^{k-2}.$$

Now, $\mathrm{E}\left(Y_{n,2j-1}^2\right) = \sum_{\ell,\ell'} a_{n,k} a_{n,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}) \leq n^{2/\alpha} A_\alpha^2 \mathrm{E}(S_{p_n}^2)$, due to the stationarity and the nonnegativity of the weights and covariances. So, applying Lemma 2.1, it follows that

$$\mathrm{E}\left(e^{\lambda Y_{n,2j-1}}\right) \leq 1 + \frac{2\lambda^2 c^* A_\alpha^2 n^{2/\alpha} p_n}{1 - cA_\alpha \lambda n^{1/\alpha} p_n} \leq \exp\left(2\lambda^2 c^* A_\alpha^2 n^{2/\alpha} p_n d_n\right).$$

To conclude the proof multiply the upper bounds above and remember that $2r_n p_n \leq n$. ■

A basic tool for the analysis of convergence and rates is the following inequality due to Dewan and Prakasa Rao [7].

Theorem 2.3. *Assume $X_1, \ldots, X_n$ are centered, associated and uniformly bounded (by $c > 0$). Then, for every $\lambda > 0$,*

$$\left| \mathrm{E}e^{\lambda \sum_j X_j} - \prod_j \mathrm{E}e^{\lambda X_j} \right| \leq \frac{1}{2} \lambda^2 e^{c\lambda n} \sum_{j \neq k} \mathrm{Cov}(X_j, X_k). \tag{3}$$

3. The case of uniformly bounded variables

We assume first that there exists some $c > 0$ such that, with probability 1, $|X_n| \leq c$, for every $n \geq 1$. This allows for a direct use of the results proved above. We start by deriving an upper bound for the tail probabilities for the summations defined above.

Lemma 3.1. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary and bounded (by $c > 0$) and $u(0) < \infty$. If the nonnegative weights satisfy (1), $d_n \geq 1$ and $0 < \lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$, then, for every $\varepsilon > 0$ and n large enough,*

$$\begin{aligned} \mathrm{P}(T_{n,od} > n^{1/p}\varepsilon) \;\leq\; & \frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon\right) u(p_n) \\ & + \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p} \varepsilon\right). \end{aligned} \tag{4}$$

An analogous inequality for $\mathrm{P}(T_{n,ev} > n^{1/p}\varepsilon)$ also holds.

Proof: If we apply (3) to $T_{n,od}$ we find

$$\left| \mathrm{E} e^{\lambda T_{n,od}} - \prod_j \mathrm{E} e^{\lambda Y_{n,2j-1}} \right| \leq \frac{1}{2}\lambda^2 \exp\left(cA_\alpha r_n p_n n^{1/\alpha}\lambda\right) \sum_{j \neq j'} \mathrm{Cov}(Y_{n,j}, Y_{n,j'}). \tag{5}$$

Now, it is obvious that each $0 \leq a_{n,i} \leq n^{1/\alpha} A_{n,\alpha}$, thus

$$\mathrm{Cov}(Y_{n,j}, Y_{n,j'}) \leq \sum_{\ell,\ell'} a_{n,\ell} a_{n,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}) \leq n^{2/\alpha} A_\alpha^2 \sum_{\ell,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}).$$

Put $Y^*_{n,j} = \sum_{\ell=(j-1)p_n+1}^{jp_n} X_\ell$, $j = 1, \ldots, r_n$. Then we have just verified that

$$\mathrm{Cov}(Y_{n,j}, Y_{n,j'}) \leq n^{2/\alpha} A_\alpha^2 \mathrm{Cov}(Y^*_{n,j}, Y^*_{n,j'}).$$

Using twice the stationarity of the random variables we obtain

$$\sum_{\ell \neq \ell'} \mathrm{Cov}(Y^*_{n,j}, Y^*_{n,j'}) = 2 \sum_{j=1}^{r_n - 1} (r_n - j) \mathrm{Cov}(Y^*_{n,1}, Y^*_{n,2j-1})$$

and

$$
\begin{aligned}
&\mathrm{Cov}(Y^*_{n,1}, Y^*_{n,2j-1}) \\
&\quad \leq \sum_{\ell=0}^{p_n-1} (p_n-\ell)\mathrm{Cov}(X_1, X_{2jp_n+\ell+1}) + \sum_{\ell=1}^{p_n-1} (p_n-\ell)\mathrm{Cov}(X_\ell, X_{2jp_n+1}) \\
&\quad \leq p_n \sum_{\ell=(2j-1)p_n+2}^{(2j+1)p_n} \mathrm{Cov}(X_1, X_\ell).
\end{aligned}
$$

Inserting this inequality in (5) we find

$$
\begin{aligned}
&\left| \mathrm{E}e^{\lambda T_{n,od}} - \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}} \right| \\
&\qquad \leq \frac{1}{2}\lambda^2 n^{2/\alpha} A_\alpha^2 r_n p_n \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda\right) \sum_{\ell=p_n+2}^{2r_n-1} \mathrm{Cov}(X_1, X_\ell) \\
&\qquad \leq \frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda\right) u(p_n+2).
\end{aligned}
$$

We can now use this together with Markov's inequality to find that, for every $\varepsilon > 0$,

$$
\begin{aligned}
\mathrm{P}(T_{n,od} > n^{1/p}\varepsilon) &\leq e^{-\lambda n^{1/p}\varepsilon} \left| \mathrm{E}e^{\lambda T_{n,od}} - \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}} \right| + e^{-\lambda n^{1/p}\varepsilon} \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}} \\
&\leq \frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon\right) u(p_n+2) \\
&\qquad + \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right),
\end{aligned}
$$

and remember that $u(p_n+2) \leq u(p_n)$, due to the nonnegativity of the covariances. ■

3.1. Almost sure convergence. We prove two different versions of the almost sure of $n^{-1/p}T_n$, depending on the Cox-Grimmett coefficients being decreasing at polynomial or geometric rate.

Theorem 3.2. *Assume the random variables X_n, $n \geq 1$, are centered, associated, stationary and bounded (by $c > 0$). Assume that $p < 1$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} \geq 1$ and $u(n) \sim n^{-a}$, for some $a > 0$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$.*

Proof: Consider the decomposition of T_n into the blocks $Y_{n,j}$ defined previously, taking $p_n \sim n^\theta$, for some $\theta \in (0,1)$. It is obviously enough to prove that both $n^{-1/p}T_{n,od}$ and $n^{-1/p}T_{n,ev}$ converge almost surely to 0. As these terms are analogous we will concentrate on the former, starting from (4). A minimization of the exponent on the second term of the upper bound in (4) leads to the choice

$$\lambda = \frac{\varepsilon}{2c^* A_\alpha^2} \frac{n^{1/p-1-2/\alpha}}{d_n}, \tag{6}$$

meaning that

$$\exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right) = \exp\left(-\frac{\varepsilon^2 n^{2/p-1-2/\alpha}}{4c^* A_\alpha^2 d_n}\right).$$

Assume that, for some $\beta > 1$,

$$\frac{\varepsilon^2 n^{2/p-1-2/\alpha}}{4c^* A_\alpha^2 d_n} = \beta \log n \quad \Leftrightarrow \quad d_n = \frac{\varepsilon^2}{4c^* A_\alpha^2 \beta} \frac{n^{2/p-1-2/\alpha}}{\log n}. \tag{7}$$

As $\frac{1}{p} - \frac{1}{\alpha} > 1$, it follows that, for n large enough, we have $d_n > 1$ as required in Lemma 2.2. In order to use Lemma 2.2 we also need to verify that the condition on λ is satisfied: $\lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$. Replacing the above choices for λ and d_n, this condition on λ is satisfied if

$$\varepsilon^{-1} \leq \frac{1}{2cA_\alpha \beta} \frac{n^{1/p-1/\alpha}}{n^\theta \log n}. \tag{8}$$

As $\theta < 1 \leq \frac{1}{p} - \frac{1}{\alpha}$ this upper bound grows to infinity, so this inequality is satisfied for n large enough.

We consider now the first term in (4), the term involving the Cox-Grimmett coefficients. The exponent in this term is

$$cn^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon = \frac{c\varepsilon}{2c^* A_\alpha} \frac{n^{1/p-1/\alpha}}{d_n} - \frac{\varepsilon^2}{2c^* A_\alpha^2} \frac{n^{2/p-1-2/\alpha}}{d_n}.$$

The second term above is, up to multiplication by 2, the exponent that was found after the optimization with respect to λ of the exponent on the second term of (4). So, to control the upper bound (4) we can factor this part of the exponential, leaving to control, after substituting the expression for d_n,

$$\frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\frac{2cA_\alpha\beta}{\varepsilon} n^{1/\alpha-1/p+1} \log n\right) u(p_n). \tag{9}$$

As the term that we factored defines a convergent series, it is enough to verify that (9) is bounded. Further, the polynomial term in (9) is clearly dominated by the exponential, thus we may drop it, verifying only that

$$\exp\left(\frac{2cA_\alpha\beta}{\varepsilon}n^{1/\alpha-1/p+1}\log n\right)u(p_n)\leq c_0, \tag{10}$$

for some $c_0>0$. Taking logarithms and taking into account the choice for $p_n\sim n^\theta$, the above inequality is equivalent to $\frac{2cA_\alpha\beta}{\varepsilon}n^{1/\alpha-1/p+1}\log n-a\theta\log n$ having a finite upper bound. But then, this a consequence of the assumption on p and α, as $\frac{1}{p}-\frac{1}{\alpha}\geq 1$ implies that the exponent on the first term is not positive, so this term converges to 0. ∎

We may relax somewhat the assumptions on p and α if the covariances decrease faster.

Theorem 3.3. *Assume the random variables X_n, $n\geq 1$, are centered, associated, strictly stationary and bounded (by $c>0$). Assume that $p<2$ and $\alpha>1$ are such that $\frac{1}{p}-\frac{1}{\alpha}>\frac{1}{2}$ and $u(n)\sim\rho^{-n}$, for some $\rho\in(0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n\longrightarrow 0$.*

Proof: Follow the proof of Theorem 3.2, choosing $\max(0,\frac{1}{p}-\frac{1}{\alpha}+1)<\theta<\frac{1}{p}-\frac{1}{\alpha}$, until (10). Remark that the assumption on p and α ensures that such a choice for θ is possible. Now the boundedness required in (10) is equivalent to $\frac{2cA_\alpha\beta}{\varepsilon}n^{1/\alpha-1/p+1}\log n-n^\theta\log\rho$ being bounded above. But this follows from $\theta>\frac{1}{p}-\frac{1}{\alpha}+1$ and $\rho\in(0,1)$. ∎

3.2. Convergence rates. A small modification of the previous arguments allows, for the case of geometric decreasing Cox-Grimmett coefficients, the identification of a convergence rate for the almost sure convergence just proved.

Theorem 3.4. *Assume the random variables X_n, $n\geq 1$, are centered, associated, strictly stationary and bounded (by $c>0$). Assume that $p<2$ and $\alpha>1$ are such that $\frac{1}{p}-\frac{1}{\alpha}>\frac{1}{2}$ and $u(n)\sim\rho^{-n}$, for some $\rho\in(0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n\longrightarrow 0$ with convergence rate $\frac{\log n}{n^{1/p-1/\alpha-1/2-\delta}}$, for arbitrarily small $\delta>0$.*

Proof: We again start as in the proof of Theorem 3.2 choosing $\theta = \frac{1}{2} + \delta$, with $0 < \delta < \frac{1}{p} - \frac{1}{\alpha} - \frac{1}{2}$ and $p_n \sim n^\theta$. Now, on (7), allow ε to depend on n:

$$\varepsilon_n^2 = \frac{4\beta c^* A_\alpha^2 d_n \log n}{n^{2/p-1-2/\alpha}}.$$

The verification of the assumptions of Lemma 2.2, given above by (8), becomes now:

$$\frac{n^{1/p-1/2-1/\alpha}}{2(\beta c^*)^{1/2} A_\alpha d_n^{1/2} (\log n)^{1/2}} \leq \frac{1}{2c\beta A_\alpha} \frac{n^{1/p-1/\alpha}}{n^\theta \log n},$$

which is equivalent to $d_n \geq \frac{c^2\beta}{c^*} n^{2\theta-1} \log n \sim n^{2\delta} \log n$. Thus, as we are interested in a slow growing sequence, we choose $d_n \sim n^{2\delta} \log n$. So, $\varepsilon_n^2 \sim n^{2\delta-2/p+2/\alpha+1} (\log n)^2 \longrightarrow 0$, given the choice for δ. To complete the proof, it is enough to bound $\exp(cn^{1+1/\alpha}\lambda) u(p_n)$. It is easily verified that $n^{1+1/\alpha}\lambda \sim n^{1/2-\delta}$, so the term we need to bound is of order $n^{1/2-\delta} + n^\theta \log \rho = n^{\frac{1}{2}-\delta} + n^{1/2+\delta} \log \rho$. But, this is an immediate consequence of $\rho \in (0,1)$ and $\delta > 0$, so the proof is concluded. ∎

Remark 3.5. *The above argument does not hold if the decrease rate of the Cox-Grimmett coefficients is only polynomial. Indeed, in this case we would be driven to bound $n^{1/2-\delta} + a(\frac{1}{2} + \delta) \log n$, which is always unbounded as $\frac{1}{2} + \delta > 0$.*

4. The general case

For general sequences of associated random variables we need an extension of Lemma 3.1. For this purpose we will introduce a truncation on the random variables, which can be analysed using the results in the previous section, and control the remaining tails. Let c_n, $n \geq 1$, be a sequence of nonnegative real numbers such that $c_n \longrightarrow +\infty$ and define, for each $i, n \geq 1$,

$$\begin{aligned} X_{1,i,n} &= -c_n \mathbb{I}_{(-\infty,-c_n)}(X_i) + X_i \mathbb{I}_{[-c_n,c_n]}(X_i) + c_n \mathbb{I}_{(c_n,+\infty)}(X_i), \\ X_{2,i,n} &= (X_i - c_n)\mathbb{I}_{(c_n,+\infty)}(X_i), \qquad X_{3,i,n} = (X_i + c_n)\mathbb{I}_{(-\infty,-c_n)}(X_i), \end{aligned} \tag{11}$$

where $\mathbb{I}_A$ represents the characteristic function of the set A. Notice that the above transformations are monotonous, so these new families of variables are still associated. Morevoer, it is obvious that, for each $n \geq 1$ fixed, the variables $X_{1,1,n}, \ldots, X_{1,n,n}$ are uniformly bounded. Consider, as before, a sequence of natural numbers p_n such that, for each $n \geq 1$, $p_n < \frac{n}{2}$ and define

r_n as the largest integer less or equal to $\frac{n}{2p_n}$. For $q=1,2,3$, and $j=1,\ldots,2r_n$, define

$$Y_{q,j,n}=\sum_{\ell=(j-1)p_n+1}^{jp_n} a_{n,i}\Big(X_{q,\ell,n}-\mathrm{E}(X_{q,\ell,n})\Big), \tag{12}$$

and

$$T_{q,n,od}=\sum_{j=1}^{r_n} Y_{q,2j-1,n}, \qquad T_{q,n,ev}=\sum_{j=1}^{r_n} Y_{q,2j,n}, \tag{13}$$

For $q=2,3$, assuming the variables are identically distributed, we have the following upper bound,

$$\begin{aligned}\mathrm{P}&\left(\left|\sum_{i=1}^n a_{n,i}\Big(X_{q,i,n}-\mathrm{E}(X_{q,i,n})\Big)\right|>n^{1/p}\varepsilon\right)\\ &\leq n\mathrm{P}\left(|X_{q,1,n}-\mathrm{E}(X_{q,1,n})|>\frac{n^{1/p-1}\varepsilon}{A_\alpha}\right)\\ &\leq \frac{n^{3-2/p}A_\alpha^2}{\varepsilon^2}\mathrm{Var}(X_{q,1,n})\leq \frac{n^{3-2/p}A_\alpha^2}{\varepsilon^2}\mathrm{E}(X_{q,1,n}^2).\end{aligned}$$

The following result is an easy extension of Lemma 4.1 in [11].

Lemma 4.1. *Let $X_1,X_2,\ldots$ be strictly stationary random variables such that there exists $\delta>0$ satisfying $\sup_{|t|\leq\delta}\mathrm{E}(e^{tX_1})\leq M_\delta<+\infty$. Then, for $t\in(0,\delta]$,*

$$\mathrm{P}\left(\left|\sum_{i=1}^n a_{n,i}\Big(X_{q,i,n}-\mathrm{E}(X_{q,i,n})\Big)\right|>n^{1/p}\varepsilon\right)\leq\frac{2M_\delta A_\alpha^2 n^{3-2/p}e^{-tc_n}}{t^2\varepsilon^2}, \qquad q=2,3. \tag{14}$$

4.1. Almost sure convergence and rates. We may now prove the extensions of the results proved for uniformly bounded sequences of random variables. The main argument in the proofs in Sect. 3 was the control of the exponent in the exponential upper bounds found. The bound obtained in (14) is, essentially, of the same form, depending on the choice of the truncation sequence. So, we will obtain the same characterizations for the almost sure convergence and for its rate, as in the case of uniformly bounded sequences of random variables. Remark that, due to the association of the variables,

$$\begin{aligned}
&\operatorname{Cov}(X_{1,1,n}, X_{1,j,n}) \\
&\quad = \int\!\!\int_{[-c_n,c_n]^2} \mathrm{P}(X_1 > u, X_j > v) - \mathrm{P}(X_1 > u)\mathrm{P}(X_j > v)\, dudv \\
&\quad \le \int\!\!\int_{\mathbb{R}^2} \mathrm{P}(X_1 > u, X_j > v) - \mathrm{P}(X_1 > u)\mathrm{P}(X_j > v)\, dudv = \operatorname{Cov}(X_1, X_j).
\end{aligned}$$

Obviously, this inequality holds even if $\operatorname{Cov}(X_1, X_j)$ is not finite.

Theorem 4.2. *Assume the random variables X_n, $n \ge 1$, are centered, associated and strictly stationary. Assume that $p < 1$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > 1$ and $u(n) \sim n^{-a}$, for some $a > 0$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p} T_n \longrightarrow 0$.*

Proof: To control the tail terms, that is, $T_{q,n,od}$ and $T_{q,n,ev}$, for $q = 2, 3$, choose the truncation sequence $c_n = \log n$ and $t = \beta > 4 - \frac{2}{p}$. Thus according to Lemma 4.1, the probabilities depending on these variables are bounded above by a convergent series. Concerning the remaining term, follow the proof of Theorem 3.2 but keep in mind that the constants c and c^* now depend on n. According to the comment immediately after Lemma 2.1, we have $c_n^* = c_n^2 + u(0) \sim (\log n)^2$. Thus, instead of (6), we find the choice

$$\lambda = \frac{n^{1/p-1-2/\alpha}\varepsilon}{2c_n^* A_\alpha^2 d_n} \sim \frac{n^{1/p-1-2/\alpha}\varepsilon}{(\log n)^2 d_n},$$

and

$$\frac{n^{2/p-1-2/\alpha}\varepsilon^2}{4c_n^* A_\alpha^2 d_n} = \beta \log n \quad \Leftrightarrow \quad d_n = \frac{\varepsilon^2}{4c_n^* \beta A_\alpha^2} \frac{n^{2/p-1-2/\alpha}}{\log n} \sim \frac{n^{2/p-1-2/\alpha}}{(\log n)^3}.$$

The condition on λ required by Lemma 2.2 translates now into

$$\varepsilon^{-1} \le \frac{n^{1/p-1/\alpha}}{2c_n \beta A_\alpha n^\theta \log n} \sim \frac{n^{1/p-1/\alpha-\theta}}{(\log n)^2}.$$

Thus, up to a logarithmic factor, we find an upper bound with the same behaviour as the one found in (8), so the argument used in course of proof of Theorem 3.2 still applies. Remark also that the present choice for d_n also only changes with respect to the one made in the proof of Theorem 3.2 by the introduction of a logarithmic factor in the denominator. Thus the fact that d_n becomes larger that 1, for n large enough, is not affected. The same holds for the term corresponding to (10). Indeed, the exponent we need to control takes now the form $c_n n^{1+1/\alpha}\lambda \sim n^{1-1/p+1/\alpha}(\log n)^2$, that is, the same we found before multiplied by a logarithmic factor that, as is easily verified, does not affect the remaining argument of the proof. ■

For sake of completeness we state the results corresponding to Theorems 3.3 and 3.4. We do not include proofs as these are modifications of the corresponding ones exactly as done for Theorem 4.2.

Theorem 4.3. *Assume the random variables X_n, $n \geq 1$, are centered, associated and strictly stationary. Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > \frac{1}{2}$ and $u(n) \sim \rho^{-n}$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$.*

Theorem 4.4. *Assume the random variables X_n, $n \geq 1$, are centered, associated and strictly stationary. Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > \frac{1}{2}$ and $u(n) \sim \rho^{-n}$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$ with convergence rate $\frac{\log n}{n^{1/p-1/\alpha-1/2-\delta}}$, for arbitrarily small $\delta > 0$.*

The above statements include an assumption on the Cox-Grimmett coefficients of the original untruncated variables. In fact, this assumption, which implies the existence of second order moments, may be relaxed, as we only need the coefficients corresponding to the truncated variables defined as, assuming already the stationarity of the variables,

$$u^*(n) = 2\sum_{j=n+1}^{\infty} \mathrm{Cov}(X_{1,1,n}, X_{1,j,n}).$$

Taking into account the inequality between the covariances, it is obvious that $u^*(n) \leq u(n)$. Of course, this choice for the statements would imply a definition for the truncating sequence on the statement.

4.2. Comparing with previous results. Theorem 4.2 above extends Corollary 3.5 in Çağın and Oliveira [6]. Indeed, in [6] it is assumed that $p > 1$ due to the technicalities of the proof, while here we are assuming $p < 1$. This later case would imply, with respect to the framework of [6], the need to assume the existence of moments of order 2, which was what was trying to be avoided in [6]. In the present case, as we are dealing with bounded variables or using truncation, this is not a problem. In order to be somewhat more precise on the relations of the present results and those in [6] we need some more notation, extending the truncation in (11). Given $v>0$ and $i\geq 1$, define $X_{1,i,v}=-v\mathbb{I}_{(-\infty,-v)}(X_i)+X_i\mathbb{I}_{[-v,v]}(X_i)+v\mathbb{I}_{(v,+\infty)}(X_i)$ and $G_i(v)=\mathrm{Cov}(X_{1,1,v},X_{1,i,v})$. Now the assumption on the Cox-Grimmett coefficients in Theorem 4.2 rewrites as

$$u^*(n)=2\sum_{j=n+1}G_j(\log n)\sim n^{-a},\quad a>0.$$

A translation of this decay rate directly into the covariances is achieved if we assume that $G_j(v)\sim e^{-(a+1)v}$, thus a geometric decay rate for the covariances. Moreover, we may still verify that Corollary 3.5 in Çağın and Oliveira [6] does not overlap with Theorem 4.2, considering the version with an assumption on the truncated Cox-Grimmett coefficients $u^*(n)$ instead. In fact, the result in [6] assumes that

$$\sum_{n=1}^{\infty}\int_{(n+1)^{(\alpha-2p)/(\alpha p)}}v^{\frac{\alpha(p-1)}{\alpha-2p}-2}G_n(v)\,dv<\infty.$$

If $G_j(v)\sim e^{-(a+1)v}$, this condition is equivalent to

$$\sum_{n=1}^{\infty}\int_{(a+1)(n+1)^{(\alpha-2p)/(\alpha p)}}t^{\frac{\alpha(p-1)}{\alpha-2p}-2}e^{-t}\,dt<\infty.$$

The convergence of the series above is equivalent to the finiteness of the integral

$$\int_1^{\infty}\int_{(a+1)(n+1)^{(\alpha-2p)/(\alpha p)}}t^{\frac{\alpha(p-1)}{\alpha-2p}-2}e^{-t}\,dt\,dx,$$

which, after inverting the integration order is bounded above by

$$\int_{2^{(\alpha-2p)/(\alpha p)}(a+1)}^{\infty}t^{(2\alpha p-\alpha)/(\alpha-2p)-2}e^{-t}\,dt\leq\Gamma\left(\frac{2\alpha p-\alpha}{\alpha-2p}-1\right),$$

where Γ represents the Euler Gamma function, and this is finite if the argument is positive, that is, if $\alpha p>\alpha-p$ or, equivalently, if $\frac{1}{p}-\frac{1}{\alpha}<1$, the reverse inequality of what is assumed in Theorem 4.2.

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