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ON THE LOCALNESS OF THE EMBEDDING OF ALGEBRAS

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Dedicated to Manuela Sobral

Abstract: Let $\mathcal{B}$ be a category and let $\mathcal{A}$ be a subcategory of $\mathcal{B}$; given an object B of $\mathcal{B}$, we may ask whether there is an embedding $B \hookrightarrow A$ with $A \in \mathcal{A}$. In some cases the answer is well known. For instance, an abelian semigroup may be embedded in an abelian group if and only if it is cancellative. And every Lie algebra over a field K is embeddable in an associative K-algebra with identity. Many other examples are known. This text concentrates in the localness of the embeddability. That is, it studies conditions under which the following statement holds: $B \in \mathcal{B}$ is embeddable in an object of $\mathcal{A}$ whenever every finitely generated subobject of $\mathcal{B}$ is so.

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1. Introduction

The following problem has been investigated in Algebra: Let $\mathcal{B}$ be a category of algebras and let $\mathcal{A}$ be a subcategory of $\mathcal{B}$; given an object B of $\mathcal{B}$, determine if there is an embedding $B \hookrightarrow A$ with $A \in \mathcal{A}$. The following two results on this subject are well-known:

(a) An abelian semigroup may be embedded in an abelian group if and only if it is cancellative.
(b) Poincaré-Birkhoff-Witt Theorem: Every Lie algebra over a field K is embeddable in an associative K-algebra with identity.

There are many other examples on the embeddability of algebras in the literature. J. MacDonald studied the subject from a categorical point of view [7, 8, 9]; in particular, he obtained a categorical generalization of the Poincaré-Birkhoff-Witt Theorem. In [6] P. Johnstone gave a new approach to the characterization of the semigroups which can be embedded in a

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group, unifying previous existing results. More generally, he obtained a characterization of the categories which can be embedded in a groupoid.

However these studies have very different aspects, and a general categorical treatment of this problem that could encompass a larger number of known results of the above type seems to be very difficult. This text devotes just to its localness facet. That is, the aim of this note is to study conditions under which, for a subcategory $\mathcal{A}$ of a category $\mathcal{B}$, and an object B of $\mathcal{B}$, the following statement holds:

(E) *$B \in \mathcal{B}$ is embeddable in an object of $\mathcal{A}$ whenever every finitely generated subobject of B is so.*

This kind of result was already achieved by B. H. Neumann in [11], and the present paper was inspired by his work. The main result, stated in Theorem 3.2 of Section 3, is a categorical approach of the embedding theorem of Neumann.

Let Σ and Σ' be finitary signatures with $\Sigma' \subseteq \Sigma$, and let I be a set of finitary implications with respect to Σ. For $\mathcal{B} = \mathrm{Alg}(\Sigma')$ the category of Σ'-algebras and $\mathcal{A} = \mathrm{Alg}(\Sigma, I)$ its subcategory of Σ-algebras which satisfy the implications of I, it follows from Neumann's result that the statement (E) holds. This case is Leading Example of Section 2. The three definitions and the three lemmas of this section capture properties of the example which are going to have a rôle in the proof of the main result.

2. Leading Example

The above statement (E) is proved in Section 3, within an environment which includes the following

Leading Example. Let $\mathcal{B} = \mathrm{Alg}(\Sigma')$ be the category of algebras for a given finitary signature Σ', and let $\mathcal{A}$ be a quasivariety of the form $\mathrm{Alg}(\Sigma, I)$, where Σ is a finitary signature containing Σ' and I is a set of finitary implications with respect to Σ. Then, the inclusion functor of $\mathcal{A}$ into $\mathcal{B}$ factorizes through the inclusion functor of $\mathcal{A}$ into the category $\mathcal{C} = \mathrm{Alg}(\Sigma)$:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{B} \arrow[dr, "U'"] & \\
& \mathcal{C} \arrow[ur, hook] \arrow[rr, shift left, "U"] & & \mathbf{Set} \arrow[ll, shift left, "F"]
\end{tikzcd} \tag{1}$$

In the diagram, U' and U denote the usual forgetful functors, and F denotes the left adjoint of U. Thus, the two upper triangles of the above diagram commute.

In this section, we present some definitions and prove some lemmas that guarantee that the above leading example is encompassed by the hypotheses of the theorem of the next section, whose proof is of a categorical type.

General Assumptions. From now on, we assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are arbitrary categories, with $\mathcal{A}$ a full subcategory of $\mathcal{C}$ and $\mathcal{C}$ a subcategory of $\mathcal{B}$, and $U : \mathcal{C} \to \mathbf{Set}$ and $U' : \mathcal{B} \to \mathbf{Set}$ are functors such that the triangle on the right side of (1) is commutative. Moreover, we assume that:

- $\mathcal{C}$ is cocomplete, finitely complete, and has intersections;
- $\mathcal{B}$ has intersections;
- $U : \mathcal{C} \to \mathbf{Set}$ and $U' : \mathcal{B} \to \mathbf{Set}$ are faithful and preserve monomorphisms.

We recall that, if $U : \mathcal{C} \to \mathbf{Set}$ is a faithful functor and $\mathcal{C}$ is a category with intersections, given an object C of $\mathcal{C}$ and a subset X of UC, with inclusion map $m : X \to UC$, we obtain the *subobject of C generated by X* by taking the intersection of all subobjects $n_A : A \to C$ of C such that $m : X \to UC$ factorizes through Un_A. When a subobject of C is generated by a finite set we say that it is *finitely generated*.

Description of special cointersections in **Set.** Given a non-empty set Z, consider the following situation in **Set**:

Let $\mathcal{F}$ be the set of all a nonempty finite subsets of Z, and, for every $X \in \mathcal{F}$, let

$$P_X \overset{\pi_1^X}{\underset{\pi_2^X}{\rightrightarrows}} X$$

be an equivalence relation, in such a way that, for every $X, Y \in \mathcal{F}$, with $X \subseteq Y$,

$$P_X = P_Y \cap (X \times X).$$

In particular, in the diagram

$$\begin{tikzcd}
P_X \arrow[r, shift left, "\pi_1^X"] \arrow[r, shift right, "\pi_2^X"'] \arrow[d, hook] & X \arrow[d, hook] \\
P_Y \arrow[r, shift left, "\pi_2^Y"] \arrow[r, shift right, "\pi_1^Y"'] & Y
\end{tikzcd}$$

the inside and outside squares commute.

Denoting by $m_X : X \hookrightarrow Z$ the inclusion maps, take the coequalizers $c_X = \mathrm{coeq}(m_X \pi_1^X, m_X \pi_2^X)$,

$$\begin{tikzcd}
P_X \arrow[r, shift left, "\pi_1^X"] \arrow[r, shift right, "\pi_2^X"'] & X \arrow[r, hook, "m_X"] & Z \arrow[r, "c_X"] & C_X ,
\end{tikzcd}$$

and the cointersection $c : Z \to C$ of these coequalizers:

$$\begin{tikzcd}
Z \arrow[r, "c"] \arrow[dr, "c_X"'] & C \\
 & C_X \arrow[u, "t_X"']
\end{tikzcd}$$

Then this cointersection has the following property:

$$\text{For every } X \in \mathcal{F} \text{ and every } u, v \in X,\ c(u) = c(v) \text{ iff } c_X(u) = c_X(v). \qquad (2)$$

Definition 2.1. An epimorphism of **Set** of the form $c : UT \to C$, with $T \in \mathcal{C}$, is said to be *U-separated*, if c is obtained as a special cointersection of the type described above, with $Z = UT$, such that there is a family of maps $f_X : X \to UA_X$, $X \in \mathcal{F}$, with $A_X \in \mathcal{C}$, fulfilling the following conditions, for every finite $X \in \mathcal{F}$:

(i) $(\pi_1^X, \pi_2^X) = \ker(X \xrightarrow{f_X} UA_X)$.

(ii) There exist a subobject of T in $\mathcal{C}$, $t_X : T_X \hookrightarrow T$, and a $\mathcal{C}$-morphism $h_X : T_X \to A_X$ that make the diagram

$$\begin{tikzcd}
X \arrow[rr, hook, "m_X"] \arrow[dr, hook] \arrow[dd, "f_X"'] & & UT \\
& UT_X \arrow[ur, hook, "Ut_X"'] \arrow[dl, "Uh_X"] & \\
UA_X & &
\end{tikzcd} \tag{3}$$

commutative, where m_X is the inclusion map.

Lemma 2.2. *In the context of Leading Example, U creates U-separated epimorphisms, i.e., if $c : UT \to C$ is a U-separated epimorphism, then there is a unique $\mathcal{C}$-morphism $\bar{c} : T \to \bar{C}$ in $\mathcal{C}$ such that $U\bar{c} = c$.*

Proof: Let $c : UT \to C$ be a U-separated epimorphism. Let $f_X : X \to UA_X$ be as in Definition 2.1, and put $c_X = \mathrm{coeq}(m_X\pi_1^X, m_X\pi_2^X)$. Then c is the cointersection in **Set** of all c_X and, as observed in (2) of Description 1, for every finite set $X \subseteq UT$, and for every $u, v \in X$,

$$c(u) = c(v) \quad \text{iff} \quad c_X(u) = c_X(v), \text{ i.e.,} \quad \text{iff} \quad f_X(u) = f_X(v).$$

We show that the equivalence relation in UT given by

$$u \sim v \qquad \text{iff} \qquad c(u) = c(v)$$

is indeed a congruence. Let then be given elements $x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n'$ of UT with $x_i \sim x_i'$, and let $\theta \in \Sigma_n$. We want to show that $\theta_T(x_1, x_2, \dots, x_n) \sim \theta_T(x_1', x_2', \dots, x_n')$. Let X be a finite subset of UT containing $x_1, x_2, \dots, x_n$, $x_1', x_2', \dots, x_n'$, $\theta_T(x_1, x_2, \dots, x_n)$ and $\theta_T(x_1', x_2', \dots, x_n')$. Then, $f_X(x_i) = f_X(x_i')$ for all i's. Let the morphisms

$$A_X \xleftarrow{h_x} T_X \xhookrightarrow{t_X} T$$

be as described in (ii) of Definition 2.1. Without loss of generality we may consider t_X as an inclusion in T of its subalgebra T_X. Then we have:

$$\begin{aligned}
f_X(\theta_T(x_1, \dots, x_n)) &= h_X(\theta_T(x_1, \dots, x_n)), \text{ since } h_X \text{ restricted to } X \text{ gives } f_X \\
&= \theta_A(h_X(x_1), \dots, h_X(x_n)), \text{ because } h_X \text{ is a homomorphism} \\
&= \theta_A(f_X(x_1), \dots, f_X(x_n)), \text{ since } h_X \text{ restricted to } X \text{ gives } f_X \\
&= \theta_A(f_X(x_1'), \dots, f_X(x_n')) \\
&= f_X(\theta_T(x_1', \dots x_n')), \text{ by using the same arguments as before.}
\end{aligned}$$

Consequently, $\theta_T(x_1, x_2, \ldots, x_n) \sim \theta_T(x'_1, x'_2, \ldots, x'_n)$. ■

Remark 2.3. Observe that, in the leading example, $\mathcal{C}$ has coequalizers, and the right adjoint U is faithful, preserves epimorphisms and reflects isomorphisms. As a consequence, Lemma 2.2 gives, as a corollary, the well-known fact that U creates coequalizers of kernel-pairs: given a $\mathcal{C}$-morphism $h : T \to T'$, let $c : UT \to C$ be the coequalizer of the kernel pair of Uh. For every nonempty finite set $X \subseteq UT$, put $f_X = Uh \cdot m_X$, where m_X is the inclusion of X into UT, $h_X = h$, and $t_X = \mathrm{id}_T$; then $c : UT \to C$ is clearly under the conditions of Definition 2.1. Consequently, there is a unique epimorphism $\bar{c} : T \to \bar{C}$ such that $U\bar{c} = c$. Now it is easily seen that $\bar{c}$ is the coequalizer of the kernel pair of h in $\mathcal{C}$.

Definition 2.4. We say that U *locally detects $\mathcal{C}$-morphisms* if we have

$$(\ UB \xrightarrow{g} UC\ ) = U(\ B \xrightarrow{h} C\ ),\ \text{for some } \mathcal{C}\text{-morphism } h,$$

whenever $g : UB \to UC$ is a map fulfilling the following "local" condition:

For every finite set $X \subseteq UB$, there exists a $\mathcal{C}$-object D and a monomorphism $d : X \to UD$ such that:

(i) There is a $\mathcal{C}$-morphism $\bar{g} : D \to C$ making the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "d"'] & UB \arrow[r, "g"] & UC \\ & UD \arrow[ur, "U\bar{g}"'] & \end{tikzcd} \tag{4}$$

commutative.

(ii) For B_X denoting the subobject of B generated by X, the family of morphisms

$$\left\{Uf \mid \begin{tikzcd} X \arrow[r, hook] \arrow[dr, "d"'] & UB_X \arrow[r, "Um"] & UA \\ & UD \arrow[ur, "Uf"'] & \end{tikzcd} \text{ commutes, with } m \text{ a } \mathcal{C}\text{-monomorphism}\right\} \tag{5}$$

separates every pair of points of UD separated by $U\bar{g}$ (i.e., $U\bar{g}(u) \neq U\bar{g}(v) \Rightarrow Uf(u) \neq Uf(v)$, for some f).

Lemma 2.5. *(a) In the context of Leading Example, U locally detects $\mathcal{C}$-morphisms (and, analogously, U' locally detects $\mathcal{B}$-morphisms).*

(b) For every faithful functor $U : \mathcal{C} \to \mathbf{Set}$, where $\mathcal{C}$ has intersections, if U locally detects $\mathcal{C}$-morphisms, then it reflects isomorphisms.

Proof: (a) Let $g : UB \to UC$ be a map with $B, C \in \mathcal{C}$ and under the conditions of Definition 2.4. In order to simplify the writing, we assume, without loss of generality, that d is an inclusion map. Let $\theta \in \Sigma_n$, and let $b_1, \ldots, b_n \in B$. We want to show that $g(\theta_B(b_1, \ldots, b_n)) = \theta_C(g(b_1), \ldots, g(b_n))$.

Let $X = \{b_1, \ldots, b_n, \theta_B(b_1, \ldots, b_n)\} \subseteq B$, and consider a diagram as in (4). Then $u = \theta_B(b_1, \ldots, b_n)$ and $v = \theta_D(b_1, \ldots, b_n)$ belong to D. We show that $\bar{g}(u) = \bar{g}(v)$. For that, taking into account the hypothesis on the map $\bar{g}$, it suffices to show that for every homomorphism $f : D \to A$ with Uf belonging to the family described in (5), $f(u) = f(v)$. Indeed we have that $f(u) = f(\theta_B(b_1, b_2, \ldots, b_n)) = m(\theta_{B_X}(b_1, b_2, \ldots, b_n))$, because of the commutativity of (5), and, since $\theta_{B_X}(b_1, b_2, \ldots, b_n) = \theta_B(b_1, b_2, \ldots, b_n)$ and $m : B_X \hookrightarrow A$ is a homomorphism, we get $f(u) = \theta_A(m(b_1), m(b_2), \ldots, m(b_n))$. Using (5) again and the fact of $f : D \to A$ being a homomorphism, we conclude then that

$$f(u) = \theta_A(f(b_1), f(b_2), \ldots, f(b_n)) = f(\theta_D(b_1, b_2, \cdots, b_n) = f(v).$$

Consequently, $\bar{g}(u) = \bar{g}(v)$, and it follows that

$$\begin{aligned} g(\theta_B(b_1, \ldots, b_n)) &= \bar{g}(\theta_B(b_1, \ldots, b_n)) \\ &= \bar{g}(\theta_D(b_1, \ldots, b_n)) \\ &= \theta_C(\bar{g}(b_1), \ldots, \bar{g}(b_n)) = \theta_C(g(b_1), \ldots, g(b_n)). \end{aligned}$$

(b) Let $f : C \to B$ be a $\mathcal{C}$-morphism such that Uf is an isomorphism, and let $g : UB \to UC$ be the inverse of Uf. For every finite set $X \subseteq UB$, let $j_X : X \to UB$ be the inclusion map. Then, we obtain a triangle as in (4), by putting $D = C$, $d = gj_X$ and $\bar{g} = \mathrm{id}_C$. Moreover, for every pair of elements u and v of UC, with $U\mathrm{id}_C(u) \neq U\mathrm{id}_C(v)$, we have $Uf(u) \neq Uf(v)$; thus, the following commutative triangle, where $i_X : B_X \to B$ is the subobject of B generated by X,

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "gj_X"'] & UB_X \arrow[r, "Ui_X"] & UB \\ & UC \arrow[ur, "Uf"'] & \end{tikzcd}$$

assures that (ii) of Definition 2.4 is also satisfied.

Consequently, as U locally detects $\mathcal{C}$-morphisms, we have that $g = Uh$ for some $h : B \to C$. Since U is faithful, we conclude that f is an isomorphism with $f^{-1} = h$. ■

Definition 2.6. Let $U : \mathcal{C} \to \mathbf{Set}$ be a faithful functor, and let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$. An object C of $\mathcal{C}$ is said to have a *local $\mathcal{A}$-behaviour* if, for every nonempty finite set $X \subseteq UC$, there is a $\mathcal{C}$-morphism $h : D \to C$ and a $\mathcal{C}$-monomorphism $s : X \to UD$ fulfilling the following conditions:

(i) The diagram

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, hook, "s"'] & & UC \\ & UD \arrow[ur, "Uh"'] & \end{tikzcd} \tag{6}$$

is commutative.

(ii) For every finite set Z such that s factors through Z into two monomorphisms,

$$X \hookrightarrow Z \overset{n_Z}{\hookrightarrow} UD$$

there is some morphism $f : D \to A$ in $\mathcal{C}$, with $A \in \mathcal{A}$, such that $\ker(Uf \cdot n_Z) = \ker(Uh \cdot n_Z)$.

Lemma 2.7. *For $\mathcal{A}$ and $\mathcal{C}$ as in Leading Example, every $C \in \mathcal{C}$ with a local $\mathcal{A}$-behaviour belongs to $\mathcal{A}$.*

Proof: Let C be an object of $\mathcal{C}$ with local $\mathcal{A}$-behaviour. Consider an implication of I,

$$(u_i(x) = v_i(x),\ i = 1,\dots,k) \Rightarrow (u(x) = v(x)) \tag{7}$$

where $x = (x_1,\dots,x_n)$, and $u_i(x)$, $v_i(x)$, $u(x)$ and $v(x)$ are terms on the variables $x_1,\dots,x_n$. Given $C \in \mathcal{C}$, let $c_1,\dots,c_n \in C$, and put $c = (c_1,\dots,c_n)$. We write

$$u^C(c)$$

to denote the element of C obtained from $u(x)$ by replacing every x_i by c_i, and every operation symbol $\theta \in \Sigma$ by the operation θ_C.

Suppose that

$$u_i^C(c) = v_i^C(c), i = 1,\dots,k.$$

We want to prove that then $u^C(c) = v^C(c)$. Put

$$X = \{c_1,\dots,c_n\} \cup \{u_i^C(c), i = 1,\dots,k\} \cup \{u^C(c), v^C(c)\}.$$

By hypothesis we have a commutative diagram as in (6). Without loss of generality, we may assume that all arrows denoted by $\hookrightarrow$ are inclusion maps. Then,

$$h(u_i^D(c)) = u_i^C(h(c_1), \dots, h(c_n)) = u_i^C(c_1, \dots, c_n) = u_i^C(c),$$

and, analogously, $h(v_i^D(c)) = v_i^C(c)$. Consequently, $h(u_i^D(c)) = h(v_i^D(c))$. Consider the subset of UD given by

$$Z = X \cup \{u_i^D(c), i = 1, \dots, k\} \cup \{v_i^D(c), i = 1, \dots, k\} \cup \{u^D(c), v^D(c)\},$$

and let $f : D \to A$ be as in (ii) of Definition 2.6. By hypothesis, $\ker(Uf \cdot n_Z) = \ker(Uh \cdot n_Z)$, then the equality $h(u_i^D(c)) = h(v_i^D(c))$ implies

$$f(u_i^D(c)) = f(v_i^D(c)).$$

And then, since f is a homomorphism,

$$u_i^A(f(c_1), \dots, f(c_n)) = v_i^A(f(c_1), \dots, f(c_n)),\ i = 1, \dots, k.$$

Hence, since A fulfils the given implication (7),

$$u^A(f(c_1), \dots, f(c_n)) = v^A(f(c_1), \dots, f(c_n)).$$

This is the same as $f(u^D(c)) = f(v^D(c))$. But, again by the hypothesis that $\ker(Uf \cdot n_Z) = \ker(Uh \cdot n_Z)$, this implies that

$$h(u^D(c)) = h(v^D(c)).$$

That is, taking into account (6),

$$u^C(c) = v^C(c).$$

■

3.Main result

Before stating and proving the main result, we need the following properties on right adjoints over **Set**. Part (a) of the lemma is showed in Manes [10]:

Lemma 3.1. *Let $F \dashv U : \mathcal{C} \to$ **Set** be a non trivial adjunction with U faithful. (By non trivial, we mean that there is some $C \in \mathcal{C}$ such that UC has at least two elements.) Then:*

(a) the unit η is pointwise injective and F preserves monomorphisms.

(b) If, moreover, U preserves directed colimits, then, given sets X and Z, with X finite, and a monomorphism $m : X \to UFZ$, there exists a finite subset E of

Z and a monomorphism $n : X \to UFE$ such that $UFd \cdot n = m$, where d is the inclusion map of E into Z.

(c) If, in addition to the assumption of (b), F preserves intersections, then, given sets X and Z as in (b), there is a smallest set under the conditions of the set E.

Proof: (a) Given $X \in \mathbf{Set}$, and two different elements $x, y \in X$, let C be an object of $\mathcal{C}$ such that UC has at least two elements, a and b. Define $h : X \to UC$ by $h(x) = a$ and $h(z) = b$ for all $z \neq x$. Now let $h^\#$ be the morphism in $\mathcal{C}$ such that $h^\# \cdot \eta_X = h$. Since $h(x) \neq h(y)$, then $\eta_X(x) \neq \eta_X(y)$.

Let now $m : X \to Y$ be an injective map. If $X \neq \emptyset$, then m is a split mono, thus the same happens to Fm; if $X = FX = \emptyset$, Fm is clearly a monomorphism; if $X = \emptyset$, $Y \neq \emptyset$ and $FX \neq \emptyset$, consider the diagram

$$\begin{tikzcd} \emptyset \arrow[r, "\eta_\emptyset"] \arrow[d, "m"'] & F\emptyset \arrow[d, "Fm"', shift right] \\ Y \arrow[r, "\eta_Y"'] \arrow[ur, dotted, "t"] & FY \arrow[u, dotted, "t^\#"', shift right] \end{tikzcd}$$

where t is any map from Y to $F\emptyset$. Then we have $t^\# Fm\eta_\emptyset = t^\# \eta_Y m = tm = \eta_\emptyset$. Thus $t^\# Fm = \mathrm{id}_{F\emptyset}$, so Fm is a mono.

(b) Let Z be a set, and let Z_i, $i \in I$, be the family of all finite subsets of Z. Then we know that, by the hypothesis, for $d_i : Z_i \to Z$ the corresponding inclusions, the maps $UFd_i : UFZ_i \to UFZ$ form a directed colimit (in fact, a directed union). Hence, for $m : X \to UFZ$ with X finite, there is some $i \in I$ and a map $n : X \to UFZ_i$ such that $UFd_i \cdot n = m$. The morphism n is clearly monomorphic.

(c) It is obvious. ■

Let $F \dashv U : \mathcal{C} \to \mathcal{D}$ be an adjunction between categories, with unit η. A morphism $f : X \to Y$ of $\mathcal{D}$ is said to be a *trivial covering* with respect to the adjunction, if the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\ Y \arrow[r, "\eta_Y"] & UFY \end{tikzcd}$$

is a pullback.

A morphism $f : X \to Y$ being a trivial covering just means that f is split over the identity morphism $\mathrm{id}_Y : Y \to Y$ in the sense of [5].

Theorem 3.2. *Let the following diagram of functors be commutative and under General Assumptions (see Section 2):*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{B} \arrow[dr, "U'"] & \\ & \mathcal{C} \arrow[ur, hook] \arrow[rr, "U"] & & \mathbf{Set} \end{tikzcd}$$

Moreover, assume that:

(H0) *U is non trivial, has a left adjoint F, U preserves directed colimits, F preserves intersections, and monomorphisms are trivial covers with respect to the adjunction $F \dashv U$;*
(H1) *U creates U-separated epimorphisms;*
(H2) *U' locally detects $\mathcal{B}$-morphisms;*
(H3) *$\mathcal{A}$ is closed in $\mathcal{C}$ under objects with local $\mathcal{A}$-behaviour.*

Then, $B \in \mathcal{B}$ is a subobject of some object of $\mathcal{A}$ whenever every finitely generated subobject of $\mathcal{B}$ is so.

Proof: Let $B \in \mathcal{B}$ be such that every finitely generated subobject of B is a subobject of some object of $\mathcal{A}$. If $U'B = \emptyset$, the fact that U' reflects isomorphisms, by (b) of Lemma 2.5, implies that B is its subobject generated by the emptyset. Then B is trivially a subobject of an object of $\mathcal{A}$. Let us assume now that $U'B \neq \emptyset$

1. *An inverse limit of non-empty finite sets.* Let

$$m_X : X \hookrightarrow UFU'B, \;\; X \in \mathcal{F},$$

denote all inclusions of a nonempty finite subset X into $UFU'B$. This is obviously a directed union in **Set**. For every $X \in \mathcal{F}$, let E_X be the smallest (finite) subset of $U'B$ such that the following diagram of monomorphisms, with d_X the corresponding inclusion map, is commutative:

$$\begin{tikzcd} X \arrow[r, hook, "n_X"] \arrow[dr, hook, "m_X"'] & UFE_X \arrow[d, hook, "UFd_X"] \\ & UFU'B \end{tikzcd} \tag{1}$$

(The existence of this set E_X is assured by the above lemma.)

For each E_X, there is a subobject B_X of B generated by E_X (recall that, by Assumptions 0, $\mathcal{B}$ has intersections); let $e_X : E_X \to U'B_X$ be the inclusion map. By hypothesis, there is some $A \in \mathcal{A}$, with B_X a subobject of A in $\mathcal{B}$. Let $a : B_X \hookrightarrow A$ be the corresponding monomorphism, and let $\bar{a}$ be the unique morphism of $\mathcal{C}$ such that $U\bar{a} \cdot \eta_{E_X} = U'a \cdot e_X$:

$$\begin{tikzcd}
 & & X \arrow[d, hook, "n_X"] \arrow[ddd, bend left, "\phi"] \\
E_X \arrow[rr, hook, "\eta_{E_X}"] \arrow[dr, hook, "e_X"'] & & UFE_X \arrow[dd, "U\bar{a}"] \\
 & U'B_X \arrow[dr, "U'a"'] & \\
 & & U'A = UA
\end{tikzcd} \tag{2}$$

Put $\phi = U\bar{a} \cdot n_X$, form the kernel pair of ϕ in **Set**, and take the coequalizer c_X of the composition of that kernel pair with m_X:

$$\begin{tikzcd}
\mathrm{Ker}(\phi) \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "m_X"] \arrow[d, "\phi"'] & UFU'B \arrow[r, "c_X"] & C_X \\
 & A & &
\end{tikzcd}$$

Let now define a functor

$$\mathcal{F}^{\mathrm{op}} \xrightarrow{K} \mathbf{Set}$$

from the dual of the directed category $\mathcal{F}$, formed by all non-empty finite subsets of $U'B$ and inclusions between them, to **Set**. For every X, KX is the set of all kernel-pairs of maps of the form $\phi = U\bar{a} \cdot n_X$, where $A \in \mathcal{A}$, and the morphisms $n_X : X \hookrightarrow UFE_X$ and $\bar{a} : FE_X \to A$ are obtained as in (2). Moreover, given an inclusion $i : X \hookrightarrow Y$, Ki sends each kernel pair of a morphism $\phi' = U\tilde{a} \cdot n_Y : Y \to UA$, obtained by an anlogous way to ϕ in (2), to the kernel pair of the morphism $\phi = \phi' i : X \to UA$. (This last morphism is of the form $U\bar{a} \cdot n_X$, with $\bar{a} = \tilde{a} \cdot Fu$ for $u : E_X \to E_Y$ the inclusion map.) Since all $X \in \mathcal{F}$ are finite, the same holds to all KX. Consequently, by the well-known result that states that the projective limit of non-empty compact spaces is non-empty, we conclude that the limit of K is nonempty (being a finite set, KX is a compact discrete space). Let

$$(P_X)_{X\in\mathcal{F}}$$

be an element of this limit. For every $X \in \mathcal{F}$, let $P_X \underset{\pi_2^X}{\overset{\pi_1^X}{\rightrightarrows}} X$ denote the corresponding projections. In particular, every P_X is the kernel pair of a morphism

$$X \xrightarrow{f_X} UA_X\,, \quad \text{with } A_X \in \mathcal{A},$$

which is obtained as ϕ in (2). Put $c_X = \mathrm{coeq}(m_X\pi_1^X, m_X\pi_2^X)$, and let $c : UFU'B \to C$ be the cointersection of all these c_X.

We are going to show that the morphism

$$U'B \xrightarrow{\eta_{U'B}} UFU'B \xrightarrow{c} C$$

is the underlying map of a monomorphism $B \to \bar{C}$ in $\mathcal{B}$ with $\bar{C} \in \mathcal{A}$, what proves the theorem.

2. *Applying (H1).* The fact that $(P_X)_{X\in\mathcal{F}} \in \mathrm{Lim}K$ assures that for every $X, Y \in \mathcal{F}$, with $X \subseteq Y$, $P_X = P_Y \cap (X \times X)$. Then the cointersection $c : UFU'B \to C$ is of the special type described in Description 1, with $Z = UFU'B$. By definition, the family of maps

$$f_X : X \to UA_X \quad (X \in \mathcal{F})$$

fulfils condition (i) of Definition 2.1. Moreover, it fulfils also condition (ii): in diagram (3), put $T = FU'B$, $T_X = FE_X$, $t_X = Fd_X$ and $h_X = \bar{a}$.

Therefore, by the hypothesis (H1), the regular epimorphism

$$c : UFU'B \to C$$

which is the cointersection in **Set** of all coequalizers $c_X = \mathrm{coeq}(m_X\pi_1^X, m_X\pi_2^X)$ is created by U. That is, there is a unique epimorphism $\bar{c} : FU'B \to \bar{C}$ in $\mathcal{C}$ such that $U\bar{c} = c$.

3. *Applying (H2).* We show now that the morphism

$$U'B \xrightarrow{\eta_{U'B}} UFU'B \xrightarrow{U\bar{c}} UC = U'C$$

may be lifted to a morphism of $\mathcal{B}$. Since U' locally detects $\mathcal{B}$-morphisms, it suffices to show that the morphism $U\bar{c}\cdot\eta_{U'B} : U'B \to U'C$ is under the conditions of Definition 2.4. Let X be a finite subset of $U'B$, with $n : X \hookrightarrow U'B$

the inclusion map. Then, we have the following commutative triangle:

$$\begin{tikzcd}
X \arrow[r, hook, "n"] \arrow[dr, hook, "\eta_X"'] & U'B \arrow[r, hook, "\eta_{U'B}"] & UFU'B \arrow[r, "U\bar{c}"] & U\bar{C} = U'\bar{C} \\
& UFX = U'FX \arrow[urr, "U'(\bar{c}\cdot Fn)"'] & &
\end{tikzcd}$$

This diagram plays, for U', the role of (4) of Definition 2.4. We show that it has the property illustrated with (5). Indeed, let u and v be two elements of UFX, and assume that, for every morphism $f : FX \to A$ and every monomorphism $m : B_X \to A$ in $\mathcal{B}$, making the diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, hook, "\eta_X"'] & U'B_X \arrow[r, "U'm"] & U'A \\
& U'FX \arrow[ur, "U'f"'] &
\end{tikzcd} \tag{3}$$

commutative, $U'f(u) = U'f(v)$. We want to prove that then $U(\bar{c}Fn)(u) = U(\bar{c}Fn)(v)$.

Let Z be the set obtained from X by adding the elements u and v; more precisely, let $Z = \eta_X[X] \cup \{u, v\}$. In order to simplify the notation we look at Z as a subsete of $UFU'B$. We observe that, under the present assumptions, $E_Z = X$. Indeed, since E_Z is the smallest subset of $U'B$ such that there is n_Z with $m_Z \cdot UFd_Z = n_Z$ (see (1)), E_X is contained in X; let $i : E_X \to X$ be the inclusion map. By hypothesis (H0), the square of the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "t"] \arrow[drr, hook] \arrow[ddr, "\mathrm{id}_X"'] & & \\
& E_Z \arrow[r, hook, "\eta_{E_Z}"'] \arrow[d, hook, "i"] & UFE_Z \arrow[d, hook, "UFi"] \\
& X \arrow[r, hook, "\eta_X"'] & UFX
\end{tikzcd}$$

is a pullback. Hence, there is a map $t : X \to E_Z$ with $it = \mathrm{id}_X$. Since i is an inclusion, we conclude that $X = E_X$. Consequently, the diagram (2), for f_Z

in the place of ϕ, takes the following form, where $\bar{a} = f$:

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, hook] & & U'B_X \arrow[rr, hook, "U'a"] & & UA_Z = U'A_Z \\
& Z \arrow[urrr, "f_Z"'] \arrow[dr, hook] & & & \\
& & UFX = U'FX \arrow[uurr, "Uf=U'f"'] & &
\end{tikzcd} \tag{4}$$

Since the outside triangle of this diagram is of the type of (3), we are assuming that $U'f(u) = U'f(v)$. In particular, $f_Z(u) = f_Z(v)$. This implies that $c_Z(UFn(u)) = c_Z(UFn(v))$, and, thus, $U'(\bar{c} \cdot Fn)(u) = U'(\bar{c} \cdot Fn)(v)$.

Therefore, since U' locally detects $\mathcal{B}$-morphisms, the morphism $U\bar{c} \cdot \eta_{U'B} : U'B \to U'\bar{C}$ is of the form

$$U\bar{c} \cdot \eta_{U'B} = U'\hat{c} \qquad \text{for some} \qquad \hat{c} : B \to \bar{C}.$$

4. We prove that $U'\hat{c}$ is a monomorphism. Since U' is faithful, it follows that $\hat{c}$ is also a monomorphism. Let $u, v \in U'B$, and put $X = \{u,v\}$. The finite set X is, up to isomorphism, a subset of $UFU'B$, via the universal map $\eta_{U'B}$. And the corresponding E_X coincides with X because $X \subseteq U'B$ (the argument being the same as the one in part 3 to show that $E_Z = X$). We know that $c \cdot \eta_{U'B}(u) = c \cdot \eta_{U'B}(v)$ iff $f_X \cdot \eta_{U'B}(u) = f_X \cdot \eta_{U'B}(v)$. But, since $X = E_X$, we have that the morphism f_X, obtained as ϕ in (2), is of the form $f_X = U\bar{a}n_X = U\bar{a}\eta_X = U'ae_X$ Thus, f_X is indeed a monomorphism. Consequently, $c \cdot \eta_{U'B}(u) = c \cdot \eta_{U'B}(v)$ iff $u = v$.

5. *Applying (H3).* Finally, we prove that $\bar{C}$ belongs to $\mathcal{A}$, what finishes the proof of the theorem. For that, taking into account hypothesis (H3), it suffices to show that $\bar{C}$ has a local $\mathcal{A}$-behaviour. Given a nonempty finite subset X of $C = U\bar{C}$, form the pullback of the inclusion map $k : X \hookrightarrow C$ and $c : UFU'B \to C = U\bar{C}$:

$$\begin{tikzcd}
\bar{X} \arrow[r, hook, "\bar{k}"] \arrow[d, "r"'] & UFU'B \arrow[d, "c"] \\
X \arrow[r, hook, "k"] & C
\end{tikzcd}$$

Since we are in **Set**, c is a split epimorphism, and so is r. Hence, there is some $s : X \to \bar{X}$ such that $rs = \mathrm{id}_X$. Then the map

$$(m_X : X \to UFU'B) = (\ X \xhookrightarrow{s} \bar{X} \xhookrightarrow{\bar{k}} UFU'B\ )$$

may be seen as the inclusion m_X of X into $UFU'B$. Moreover, $k = krs = c\bar{k}s = cm_X$. Hence, for E_X defined as in (1), we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, hook, "k"] \arrow[dr, hook, "m_X"] \arrow[ddr, hook, "n_X"'] & & C = U\bar{C} \\
& UFU'B \arrow[ur, "U\bar{c}"] & \\
& UFE_X \arrow[u, hook, "UFd_X"'] \arrow[uur, "Uh"'] &
\end{tikzcd}$$

with $h = \bar{c}Fd_X$. The outside triangle plays the role of triangle (6). Let now have a finite set Z such that n_X factors as a composition of monomorphisms through Z, say $X \hookrightarrow Z \xhookrightarrow{n_Z} UFE_X$. Without loss of generality, we may consider the first monomorphism as an inclusion of X into Z ans Z a subset of $UFU'B$. Then, clearly, we have $E_Z = E_X$. Consequently, there is a homomorphism $\bar{a} : FE_X = FE_Z \to A_Z$ such that $f_Z = U\bar{a}\cdot n_Z$. It follows that $\mathrm{Ker}(U\bar{a}\cdot n_Z) = \mathrm{Ker}(Uh\cdot n_Z)$. Indeed, Given $u, v \in Z$, $U\bar{a}\cdot n_Z(u) = U\bar{a}\cdot n_Z(v)$ iff $f_Z(u) = f_Z(v)$, which is equivalent to have $c_Z(u) = c_Z(v)$, and, then, equivalent to $U\bar{c}(u) = U\bar{c}(v)$. And, finally, since $UFd_X \cdot n_Z$ is the inclusion of Z into B, the last equality is equivalent to $(Uh \cdot n_Z)(u) = (Uh \cdot n_Z)(v)$.

Therefore, by hypothesis (H3), $A \in \mathcal{C}$, and this finishes the proof. ∎

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