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EFFECTIVE ÉTALE-DESCENT MORPHISMS IN M-ORDERED SETS

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Abstract: Following the results obtained in **Preord** and in **Cat**, we characterize the effective étale-descent morphisms in M-**Ord**, the category of M-ordered sets for a given monoid M. Furthermore we show that in M-**Ord** every effective descent morphism is effective for étale-descent (while the converse is false), and we generalize it to a more general context of relational algebras.

1. Introduction

In [10] G. Janelidze and M. Sobral gave a complete characterization of the morphisms in the category **Preord** of preordered sets which are effective for descent with respect to the class of étale morphisms, i.e., discrete fibrations. In [14] M. Sobral characterized the effective descent morphisms in the category **Cat** of small categories with respect to the class of discrete fibrations. These two works suggested the study of descent theory for the class of étale morphisms in M-**Ord**, the category of M-ordered sets for a given monoid M.

In this paper we give a complete characterization of the effective étale-descent morphisms in M-**Ord**. This characterization allows us to state that an effective descent morphism in M-**Ord** is effective for étale-descent. The converse is false and we give an appropriate counter-example in the surjective case. Being M-**Ord** presented as a category of relational algebras, we conclude investigating the relation between effective descent and effective étale-descent morphisms in $\mathbf{RelAlg}(\mathbb{T})$, the category of relational algebras for a suitable monad $\mathbb{T}$.

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2. Basic notions and results

Given a monoid $(M, \circ, e_M)$, consider the monad

$$\mathbb{M} = (M \times (-), \mu, \eta)$$

on **Set**, with $\mu_X : M \times M \times X \to M \times X$ defined by $(m, n, x) \mapsto (m \circ n, x)$ and $\eta_X : X \to M \times X$ by $x \mapsto (e_M, x)$, for each set X. The Barr extension [1] $\overline{\mathbb{M}} : \mathbf{Rel} \to \mathbf{Rel}$ is an extension of the monad $\mathbb{M}$, and it is given by

$$(m, x)(\overline{\mathbb{M}}r)(n, y) \iff m = n \quad \text{and} \quad x(r)y,$$

where $r : X \to Y$ is a relation, $x \in X$, $y \in Y$, and $m, n \in M$. The category $\mathbf{Alg}(\overline{\mathbb{M}}, 2)$ of reflexive and transitive $(\overline{\mathbb{M}}, 2)$-algebras is the category M-**Ord** of *M-ordered sets* and *equivariant maps.* For a relation $a : M \times X \to X$ one writes $x \xrightarrow{m} y$ instead of $(m, x)(a)y$, that is x is related to y with weight m. As remarked in [2, Section V.1.4], this arrow notation for the structure of an $(\overline{\mathbb{M}}, 2)$-category (X, a) emphasizes that X is actually the object set of a small category, denoted again by X, with hom-sets

$$X(x, y) = \{(x, m, y) \mid m \in M \quad \text{and} \quad x \xrightarrow{m} y\}$$

for $x, y \in X$; moreover this small category comes equipped with a faithful functor

$$p : X \to M, \quad (x, m, y) \mapsto m,$$

with M considered as a one-object category. In this direction identity morphisms and composition in an M-ordered set X are given by

$$x \xrightarrow{e_M} x \quad \text{and} \quad (x \xrightarrow{m} y \quad \& \quad y \xrightarrow{n} z \Longrightarrow x \xrightarrow{n \circ m} z),$$

while a morphism $f : X \to Y$ must satisfy

$$x \xrightarrow{m} y \Longrightarrow f(x) \xrightarrow{m} f(y)$$

for all $x, y \in X$ and $m \in M$. Defining a norm to be a functor from the small category X to the monoid M, [2, Proposition 1.4.2] actually reveals that M-ordered sets can be identified as those small categories over M whose norm is faithful.

Now let $\mathbb{E}$ be the class of étale morphisms in M-**Ord**. As introduced in [5], the étale morphisms in the context of relational algebras are defined as the pullback stable discrete fibrations. For a cartesian monad, as $\mathbb{M}$ is, discrete fibrations are pullback stable, hence étale morphisms and discrete fibrations coincide. Using the arrow notation, an equivariant map $f : X \to Y$ in M-**Ord** is then an étale morphism if and only if:

$$\forall x_0 \in X, \quad \forall y_1 \in Y, \quad \forall m \in M : y_1 \xrightarrow{m} f(x_0) \Longrightarrow \exists! x_1 \in f^{-1}(y_1) : x_1 \xrightarrow{m} x_0.$$

The problem concerning the characterization of effective étale-descent morphisms in M-**Ord** can be stated as follows: given an equivariant map $p : E \to B$ of M-ordered sets, denote by $\mathbb{E}(B)$ the slice category of étale morphisms over B and by $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ the pullback functor along p. We have then a commutative (up to isomorphism) diagram

$$\begin{tikzcd}
\mathbb{E}(B) \arrow[rr, "K^p"] \arrow[dr, "p^*"'] & & \mathrm{Des}_{\mathbb{E}}(p) \arrow[dl, "U^p"] \\
& \mathbb{E}(E) &
\end{tikzcd}$$

where $\mathrm{Des}_{\mathbb{E}}(p)$ is the category of $\mathbb{E}$-descent data for p, and U^p and K^p are the forgetful and the comparison functor, respectively. By definition, the equivariant map $p : E \to B$ is an *(effective) étale-descent morphism* if the comparison functor K^p is full and faithful (an equivalence of categories). For a more accurate understanding of descent theory we refer the reader to the papers [11] and [12]. Following the results given in [9] and [10] for **Preord** and in [14] for **Cat**, we are going to give a complete characterization of effective étale-descent morphisms in M-**Ord**.

3. The equivalence of (indexed) categories

The first step to reach such a characterization is to generalize to M-**Ord** the standard equivalence of categories $\mathbb{E}(B) \simeq \mathbf{Set}^{B^{\mathrm{op}}}$, given in [8, Chapter 13, Proposition 30], for $\mathbb{E}$ the class of discrete fibrations in **Graph**, **Cat** or **Grpd**.

Proposition 3.1. *Let $\mathbb{E}$ be the class of étale morphisms in M-**Ord**, and let B be an M-ordered set. Then the categories $\mathbb{E}(B)$ and $\mathbf{Set}^{B^{\mathrm{op}}}$ are equivalent.*

Proof. We start by defining a functor $F : \mathbb{E}(B) \to \mathbf{Set}^{B^{\mathrm{op}}}$. Given an étale morphism $f : A \to B$, the functor $F(f) : B^{\mathrm{op}} \to \mathbf{Set}$ is described as follows: each element $b \in B$ is mapped to

$$F(f)(b) = f^{-1}(b) = \{a \in A \mid f(a) = b\}$$

and, given an element $a \in F(f)(b)$, if $b' \xrightarrow{m} b$ in B then, by definition of étale morphism, there exists a unique $a' \in f^{-1}(b')$ such that $a' \xrightarrow{m} a$. Hence the assignment $a \mapsto a'$ defines the function from the set $F(f)(b)$ to the set $F(f)(b')$. Remark that we may have $F(f)(b) = \emptyset$ for some $b \in B$, in which case every $b' \xrightarrow{m} b$ is represented by the empty function $\emptyset \to F(f)(b')$. If $g : C \to B$ is another étale morphism over A with corresponding functor $F(g)$, and if $h : A \to C$ is an equivariant map such that $g \circ h = f$, then h induces functions $h_b : F(f)(b) \to F(g)(b)$; the family $\{h_b\}_{b \in B}$ defines actually a natural transformation $F(f) \Rightarrow F(g)$.
Conversely, we define now a functor $G : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbb{E}(B)$. Given $\alpha : B^{\mathrm{op}} \to \mathbf{Set}$, define an étale morphism over B in the following way: the objects of the domain $\widetilde{B}_\alpha$ are the pairs (b, v), where $b \in B$ and v in an element of the set $\alpha(b)$, while there is a morphism from (b, v) to (b', v'), say $(m, \alpha(m))$, if $b \xrightarrow{m} b'$ in B for $m \in M$ and $\alpha(m) : \alpha(b') \to \alpha(b)$ maps $v' \mapsto v$. The domain $\widetilde{B}_\alpha$ comes equipped with an M-valued norm, i.e., a functor $\widetilde{B}_\alpha \to M$, defined via

$$(b, v) \xrightarrow{(m,\alpha(m))} (b', v') \mapsto (\bullet \xrightarrow{m} \bullet).$$

This functor is faithful making $\widetilde{B}_\alpha$ an M-ordered set. Then the morphism $G(\alpha) : \widetilde{B}_\alpha \to B$ defined via

$$(b, v) \xrightarrow{(m,\alpha(m))} (b', v') \mapsto (b \xrightarrow{m} b')$$

gives rise to an étale morphism. If $\alpha' : B^{\mathrm{op}} \to \mathbf{Set}$ is another functor and $\beta : \alpha \Rightarrow \alpha'$ is a natural transformation between them, given by functions $\{\beta_b : \alpha(b) \to \alpha'(b)\}_{b \in B}$, we get an equivariant map $G(\beta) : \widetilde{B}_\alpha \to \widetilde{B}_{\alpha'}$ by mapping

$$(b, v) \xrightarrow{(m,\alpha(m))} (b', v') \mapsto (b, \beta_b(v)) \xrightarrow{(m,\alpha'(m))} (b', \beta_{b'}(v')).$$

This $G(\beta)$ defines a morphism in $\mathbb{E}(B)$, giving then the claimed functor $G : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbb{E}(B)$.
It is straightforward to verify that $F \circ G$ is naturally isomorphic to $1_{\mathbf{Set}^{B^{\mathrm{op}}}}$ and that $G \circ F$ is naturally isomorphic to $1_{\mathbb{E}(B)}$. $\square$

In [12] the general framework of descent theory given in Section 2 is generalized to a context of $\mathbf{C}$-indexed categories, i.e., pseudo-functors $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to$

CAT, for a given category **C** with pullbacks. Given then a **C**-indexed category $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ described by

$$\begin{tikzcd} X \arrow[r, mapsto] & \mathbb{A}^X \arrow[d, "f^*"] \\ Y \arrow[u, "f"] \arrow[r, mapsto] & \mathbb{A}^Y, \end{tikzcd}$$

let $p : E \to B$ be a morphism in **C** and let $Eq(p)$ be the internal category in **C** induced by the kernel pair of p

$$\begin{tikzcd} E \times_B E \times_B E \arrow[r, shift left=2, "\pi_{23}"] \arrow[r, "\pi_{13}" description] \arrow[r, shift right=2, "\pi_{12}"'] & E \times_B E \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & E, \arrow[l, "e" description] \end{tikzcd}$$

with $e =< 1_E, 1_E >$. The factorization of p through $Eq(p)$

$$\begin{tikzcd} & Eq(p) \arrow[dr, "\overline{p}"] & \\ E \arrow[ur, "\delta"] \arrow[rr, "p"'] & & B, \end{tikzcd}$$

where δ and $\overline{p}$ are internal functors given respectively by $\delta_0 = 1_E$ and $\delta_1 = e$, $\overline{p}_0 = p$ and $\overline{p}_1 = p\pi_1 = p\pi_2$, gives rise to a commutative (up to natural isomorphism) diagram in **CAT**

$$\begin{tikzcd} & \mathrm{Des}_{\mathbb{A}}(p) \arrow[dl, "\delta^*"'] & \\ \mathbb{A}^E & & \mathbb{A}^B, \arrow[ll, "p^*"] \arrow[ul, "K^p=\overline{p}^*"'] \end{tikzcd}$$

where

$$\mathrm{Des}_{\mathbb{A}}(p) := \mathbb{A}^{Eq(p)}$$

is the category of $\mathbb{A}$-descent data relative to p. By definition p is called an *(effective)* $\mathbb{A}$*-descent morphism* if the comparison functor $K^p = \overline{p}^*$ is full and faithful (an equivalence of categories).

The equivalence stated in Proposition 3.1 turns out to be an equivalence of M-**Ord**-indexed categories, which is crucial for our purpose. In fact we prove now a general result which can be applied in our case.

Theorem 3.2. *Let* **C** *be a category with pullbacks and let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two* **C***-indexed categories. If* $\mathbb{A}$ *and* $\mathbb{B}$ *are equivalent as* **C***-indexed categories,* *then a morphism $p : E \to B$ in $\mathbf{C}$ is (effective) for $\mathbb{A}$-descent if and only if it is (effective) for $\mathbb{B}$-descent.*

Before proving the theorem we just recall that by an equivalence of $\mathbf{C}$-indexed categories we mean an indexed functor $F : \mathbb{A} \to \mathbb{B}$, in the sense of [13, Section 3], in which each functor $F_X : \mathbb{A}^X \to \mathbb{B}^X$, for $X \in Ob(\mathbf{C})$, is an equivalence of categories.

Proof. Let $\mathbb{A}, \mathbb{B} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ be two $\mathbf{C}$-indexed categories described respectively by

$$\begin{tikzcd} X \arrow[r, mapsto] & \mathbb{A}^X \arrow[d, "f^*_{\mathbb{A}}"] \\ Y \arrow[u, "f"] \arrow[r, mapsto] & \mathbb{A}^Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, mapsto] & \mathbb{B}^X \arrow[d, "f^*_{\mathbb{B}}"] \\ Y \arrow[u, "f"] \arrow[r, mapsto] & \mathbb{B}^Y \end{tikzcd}$$

with corresponding natural isomorphisms

$$i^X_{\mathbb{A}} : Id_{\mathbb{A}^X} \to (1_X)^*_{\mathbb{A}} \qquad i^X_{\mathbb{B}} : Id_{\mathbb{B}^X} \to (1_X)^*_{\mathbb{B}}$$

$$j^{f,g}_{\mathbb{A}} : f^*_{\mathbb{A}} g^*_{\mathbb{A}} \to (gf)^*_{\mathbb{A}} \qquad j^{f,g}_{\mathbb{B}} : f^*_{\mathbb{B}} g^*_{\mathbb{B}} \to (gf)^*_{\mathbb{B}},$$

and let F be an equivalence of $\mathbf{C}$-indexed categories from $\mathbb{A}$ to $\mathbb{B}$ given by the following data:

(a) for each object X in $\mathbf{C}$, an equivalence of categories $F_X : \mathbb{A}^X \to \mathbb{B}^X$,
(b) for each morphism $f : Y \to X$ in $\mathbf{C}$, a natural isomorphism

$$\tau_f : f^*_{\mathbb{B}} \circ F_X \Rightarrow F_Y \circ f^*_{\mathbb{A}},$$

such that, for each pair of composable morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$, the diagram

$$\begin{tikzcd} g^*_{\mathbb{B}} f^*_{\mathbb{B}} F_X \arrow[r, "g^*_{\mathbb{B}}\tau_f"] \arrow[d, "j^{g,f}_{\mathbb{B}} F_X"'] & g^*_{\mathbb{B}} F_Y f^*_{\mathbb{A}} \arrow[r, "\tau_g f^*_{\mathbb{A}}"] & F_Z g^*_{\mathbb{A}} f^*_{\mathbb{A}} \arrow[d, "F_Z j^{g,f}_{\mathbb{A}}"] \\ (fg)^*_{\mathbb{B}} F_X \arrow[rr, "\tau_{fg}"'] & & F_Z (fg)^*_{\mathbb{A}} \end{tikzcd}$$

commutes. We show how to construct an equivalence $\Phi : \mathrm{Des}_{\mathbb{A}}(p) \to \mathrm{Des}_{\mathbb{B}}(p)$ between the categories of descent data of $\mathbb{A}$ and $\mathbb{B}$. Given an object (A, ξ) in $\mathrm{Des}_{\mathbb{A}}(p)$, where A is an object in $\mathbb{A}^E$ and $\xi : \pi^*_{2_{\mathbb{A}}} \to \pi^*_{1_{\mathbb{A}}}$ is a morphism in

$\mathbb{A}^{E\times_B E}$ such that the diagrams

$$\begin{tikzcd}
e^*_{\mathbb{A}}\pi^*_{2_{\mathbb{A}}}A \arrow[rr, "e^*_{\mathbb{A}}\xi"] \arrow[dr, "\cong"'] & & e^*_{\mathbb{A}}\pi^*_{1_{\mathbb{A}}}A \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
& \pi^*_{23_{\mathbb{A}}}\pi^*_{1_{\mathbb{A}}}A \arrow[r, "\cong"] & \pi^*_{12_{\mathbb{A}}}\pi^*_{2_{\mathbb{A}}}A \arrow[dr, "\pi^*_{12_{\mathbb{A}}}\xi"] & \\
\pi^*_{23_{\mathbb{A}}}\pi^*_{2_{\mathbb{A}}}A \arrow[ur, "\pi^*_{23_{\mathbb{A}}}\xi"] \arrow[dr, "\cong"'] & & & \pi^*_{12_{\mathbb{A}}}\pi^*_{1_{\mathbb{A}}}A \\
& \pi^*_{13_{\mathbb{A}}}\pi^*_{2_{\mathbb{A}}}A \arrow[r, "\pi^*_{13_{\mathbb{A}}}\xi"'] & \pi^*_{13_{\mathbb{A}}}\pi^*_{1_{\mathbb{A}}}A \arrow[ur, "\cong"'] &
\end{tikzcd}$$

commute respectively in $\mathbb{A}^E$ and in $\mathbb{A}^{E\times_B E\times_B E}$, we define $\Phi(A,\xi) := (F_E(A), \overline{\xi})$, where $\overline{\xi}$ is defined by the following diagram

$$\begin{tikzcd}
F_{E\times_B E}(\pi^*_{2_{\mathbb{A}}}(A)) \arrow[d, "F_{E\times_B E}(\xi)"'] & \pi^*_{2_{\mathbb{B}}}(F_E(A)) \arrow[l, "\tau^A_{\pi_2}"'] \arrow[d, "\overline{\xi}"] \\
F_{E\times_B E}(\pi^*_{1_{\mathbb{A}}}(A)) & \pi^*_{1_{\mathbb{B}}}(F_E(A)). \arrow[l, "\tau^A_{\pi 1}"]
\end{tikzcd}$$

One can verify that $\overline{\xi}$ is actually a descent data for $F_E(A)$.
We define now Φ on morphisms. Given $h : (A,\xi) \to (A',\xi')$ in $\mathrm{Des}_{\mathbb{A}}(p)$, that is a morphism $h : A \to A'$ in $\mathbb{A}^E$ such that the diagram

$$\begin{tikzcd}
\pi^*_{2_{\mathbb{A}}}(A) \arrow[r, "\pi^*_2 h"] \arrow[d, "\xi"'] & \pi^*_{2_{\mathbb{A}}}(A') \arrow[d, "\overline{\xi}"] \\
\pi^*_{1_{\mathbb{A}}}(A) \arrow[r, "\pi^*_1 h"'] & \pi^*_{1_{\mathbb{A}}}(A')
\end{tikzcd}$$

commutes in $\mathbb{A}^{E\times_B E}$, define

$$\Phi(h) := F_E(h) : F_E(A) \to F_E(A').$$

One can easily verify that the diagram

$$\begin{tikzcd} \pi^*_{2_{\mathbb{B}}} F_E(A) \arrow[r, "\pi^*_{2_{\mathbb{B}}} F_E(h)"] \arrow[d, "\overline{\xi}"'] & \pi^*_{2_{\mathbb{B}}} F_E(A') \arrow[d, "\overline{\xi'}"] \\ \pi^*_{1_{\mathbb{B}}} F_E(A) \arrow[r, "\pi^*_{1_{\mathbb{B}}} F_E(h)"'] & \pi^*_{1_{\mathbb{B}}} F_E(A') \end{tikzcd}$$

commutes in $\mathbb{B}^{E\times_B E}$.
The equivalence of categories Φ makes the following diagram commutative up to isomorphism

$$\begin{tikzcd} \mathbb{A}^B \arrow[r, "K^p_{\mathbb{A}}"] \arrow[d, "F_B"'] & \mathrm{Des}_{\mathbb{A}}(p) \arrow[d, "\Phi"] \\ \mathbb{B}^B \arrow[r, "K^p_{\mathbb{B}}"'] & \mathrm{Des}_{\mathbb{B}}(p), \end{tikzcd}$$

allowing us to conclude that the morphism $p : E \to B$ in the category $\mathbf{C}$ is (effective) for $\mathbb{A}$-descent if and only if it is (effective) for $\mathbb{B}$-descent. $\square$

This general result can be applied in our context of M-ordered sets as claimed. In fact, given $\mathbb{A}, \mathbb{B} : M\text{-}\mathbf{Ord}^{\mathrm{op}} \to \mathbf{CAT}$ defined respectively by

$$\begin{tikzcd} X \arrow[r, mapsto] & \mathbb{E}(X) \arrow[d, "f^*"] \\ Y \arrow[u, "f"] \arrow[r, mapsto] & \mathbb{E}(Y) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, mapsto] & \mathbf{Set}^{X^{\mathrm{op}}} \arrow[d, "\mathbf{Set}^{f^{\mathrm{op}}}"] \\ Y \arrow[u, "f"] \arrow[r, mapsto] & \mathbf{Set}^{Y^{\mathrm{op}}}, \end{tikzcd}$$

where $\mathbb{E}$ is the class of étale morphisms in M-$\mathbf{Ord}$, f^* is the pullback functor and $\mathbf{Set}^{f^{\mathrm{op}}}$ is the composition functor, using Proposition 3.1 one can verify that there exists an equivalence of M-$\mathbf{Ord}$-indexed categories from $\mathbb{A}$ to $\mathbb{B}$. Thanks to Theorem 3.2 we can then state that an equivariant map $p : E \to B$ in M-$\mathbf{Ord}$ is an effective $\mathbb{A}$-descent morphism, i.e., it is effective for étale-descent, if and only if it is an effective $\mathbb{B}$-descent morphism.

4. The characterization of effective étale-descent morphisms

The category $\mathrm{Des}_{\mathbb{B}}(p)$ can be described as the category of pairs (X, ξ) where $X : E^{\mathrm{op}} \to \mathbf{Set}$ is a functor and $\xi = (\xi_{e,e'})_{e,e' \in E\times_B E}$ is a family of functions

$\xi_{e,e'} : X(e') \to X(e)$ defined for $e, e' \in E$ with $p(e) = p(e')$ such that

$$\xi_{e,e} = 1_{X(e)} \qquad \xi_{e',e''} \circ \xi_{e,e'} = \xi_{e,e''},$$

for each $e, e', e'' \in E$ with $p(e) = p(e') = p(e'')$, and each diagram

$$\begin{tikzcd} X(e_0') \arrow[r, "\xi_{e_0,e_0'}"] \arrow[d, "X(e_1' \xrightarrow{m} e_0')"'] & X(e_0) \arrow[d, "X(e_1 \xrightarrow{m} e_0)"] \\ X(e_1') \arrow[r, "\xi_{e_1,e_1'}"'] & X(e_1) \end{tikzcd}$$

commutes for all $e_1' \xrightarrow{m} e_0'$ and $e_1 \xrightarrow{m} e_0$ in E with $m \in M$, $p(e_0) = p(e_0')$ and $p(e_1) = p(e_1')$. The comparison functor

$$K^p_{\mathbb{B}} : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathrm{Des}_{\mathbb{B}}(p)$$

is given by

$$G \mapsto (G \circ p^{\mathrm{op}}, 1),$$

where 1 is the family of identity morphisms $1_{e,e'} : G(p(e)) \to G(p(e'))$.
The internal category $Eq(p)$ in M-**Ord** can be presented as a double category in the following way:

(1) objects are elements of E;
(2) vertical arrows are morphisms of E, that is $e_0 \xrightarrow{m} e_1$;
(3) horizontal arrows are elements of $E \times_B E$;
(4) squares of 2-cells are morphisms of $E \times_B E$, that is squares of the form

$$\begin{tikzcd} e_0 \arrow[r] \arrow[d, "m"'] & e_0' \arrow[d, "m"] \\ e_1 \arrow[r] & e_1' \end{tikzcd}$$

for e_0, e_0', e_1, e_1' in E with $p(e_0) = p(e_0')$, $p(e_1) = p(e_1')$ and $e_0 \xrightarrow{m} e_1$, $e_0' \xrightarrow{m} e_1'$ in E for $m \in M$.

This allows us to present an object (X, ξ) of the category $\mathrm{Des}_{\mathbb{B}}(p)$ as a double functor from the double category $Eq(p)$ to the double category $S(\mathbf{Set})$ of commutative squares in **Set**. The functor

$$S : \mathbf{CAT} \to \mathbf{DoubleCAT},$$

which sends each category $\mathbf{C}$ to the double category of commutative squares of $\mathbf{C}$, has a left adjoint Z described in [10, section 1] as a quotient of a pushout. The category $Z(Eq(p))$ can be then contructed as follows. Consider

- $Eq(p)_0$ to be the discrete category with objects as in $Eq(p)$;
- $Eq(p)_h$ and $Eq(p)_v$ to be categories with the same objects and the morphisms to be, respectively, the horizontal and the vertical arrows of $Eq(p)$;
- $Eq(p)_+$ the pushout in **Cat** of the embeddings $Eq(p)_0 \to Eq(p)_h$ and $Eq(p)_0 \to Eq(p)_v$;

then, for every square of 2-cell in $Eq(p)$

$$\begin{tikzcd} e_0 \arrow[r] \arrow[d, "m"'] & e_0' \arrow[d, "m"] \\ e_1 \arrow[r] & e_1' \end{tikzcd}$$

the pairs

$$\begin{tikzcd} e_0 \arrow[r] & e_0' \arrow[d, "m"] & & e_0' \arrow[d, "m"] \\ & e_1' & e_1 \arrow[r] & e_1' \end{tikzcd}$$

become morphisms in $Eq(p)_+$ from e_0 to e_1', and we construct $Z(Eq(p))$ as the quotient category $Eq(p)_+/\sim$ under the smallest equivalence relation $\sim$ for which

$$\begin{tikzcd} e_0 \arrow[r] & e_0' \arrow[d, "m"] & \sim & e_0 \arrow[d, "m"] & \\ & e_1' & & e_1 \arrow[r] & e_1', \end{tikzcd}$$

for all such pairs. Observe that a morphism in $Z(Eq(p))$ from a point e_0 to a point e_n' can be then given by an equivalent class of a morphism in $Eq(p)_+$, say an n-zigzag z

$$\begin{tikzcd}
e_0 \arrow[r] & e_0' \arrow[d, "m_1"] \\
& e_1 \arrow[r] & e_1' \arrow[d, "m_2"] \\
& & e_2 \arrow[r] & e_2' \arrow[d, "m_2"] \\
& & & {} \arrow[r, dashed] & {} \arrow[d, dashed] \\
& & & & {} \arrow[r, dashed] & {} \arrow[d, "m_n"] \\
& & & & & e_n \arrow[r] & e_n'
\end{tikzcd}$$

where $p(e_i) = p(e_i')$ for $i = 0, \cdots, n$, and $e_i' \xrightarrow{m_{i+1}} e_{i+1}$ in E for $i = 0, \cdots, n-1$. The notation for such an n-zigzag as above will be $z = [e_n, e_n']m_n \cdots m_1[e_0, e_0']$. The equivariant map $p : E \to B$ can be then factorized in **Cat** through the category $Z(Eq(p))$

$$\begin{tikzcd}
E \arrow[rr, "p"] \arrow[dr, "\psi"'] & & B \\
& Z(Eq(p)) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

where ψ is defined as the identity on objects and $\psi(e_0 \xrightarrow{m} e_1) = [e_0 \xrightarrow{m} e_1]$ on morphisms, while φ on objects acts as p and the image of an equivalent class of an n-zigzag $z = [e_n, e_n']m_n \cdots m_1[e_0, e_0']$ via φ is

$$\varphi([z]) = p(e_0) \xrightarrow{m_1} p(e_1') \xrightarrow{m_2} \cdots \xrightarrow{m_n} p(e_n') = p(e_0) \xrightarrow{m_n \circ \cdots \circ m_1} p(e_n').$$

Thanks to the adjoint situation

$$\begin{tikzcd}
\mathbf{CAT} \arrow[r, shift right, "S"'] \arrow[r, phantom, "\perp"] & \mathbf{DoubleCAT} \arrow[l, shift right, "Z"']
\end{tikzcd}$$

an object (X, ξ) of $\mathrm{Des}_{\mathbb{B}}(p)$, i.e., a double functor from $Eq(p)$ to $S(\mathbf{Set})$, is the same as a functor from $Z(Eq(p))$ to $\mathbf{Set}$, and the diagram

$$\begin{tikzcd} \mathbf{Set}^{B^{\mathrm{op}}} \arrow[rr, "K_{\mathbb{B}}^{p}"] \arrow[dr, "\mathbf{Set}^{p^{\mathrm{op}}}"'] & & \mathrm{Des}_{\mathbb{B}}(p) \arrow[dl, "U_{\mathbb{B}}^{p}"] \\ & \mathbf{Set}^{E^{\mathrm{op}}} & \end{tikzcd}$$

can be identified, up to equivalence, to the diagram

$$\begin{tikzcd} \mathbf{Set}^{B^{\mathrm{op}}} \arrow[rr, "\mathbf{Set}^{\varphi^{\mathrm{op}}}"] \arrow[dr, "\mathbf{Set}^{p^{\mathrm{op}}}"'] & & \mathbf{Set}^{Z(Eq(p))^{\mathrm{op}}} \arrow[dl, "\mathbf{Set}^{\psi^{\mathrm{op}}}"] \\ & \mathbf{Set}^{E^{\mathrm{op}}}. & \end{tikzcd}$$

This allows us to state that a morphism $p : E \to B$ in M-$\mathbf{Ord}$ is effective for étale-descent if and only the functor $\mathbf{Set}^{\varphi^{\mathrm{op}}} : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbf{Set}^{Z(Eq(p))^{\mathrm{op}}}$ is an equivalence of categories. The general argument given in [14, Theorem 2] can be applied in our situation.

Theorem 4.1. *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-$\mathbf{Ord}$ if and only if $\varphi : Z(Eq(p)) \to B$ is a full and faithful lax epimorphism in $\mathbf{Cat}$.*

Corollary 4.2. *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-$\mathbf{Ord}$ if and only if*

(a) *for each $p(e) \xrightarrow{k} p(e')$ in B with $k \in M$ there exists a zigzag $z = [e_n, e'_n]m_n \cdots m_1[e_0, e'_0]$ in $Z(Eq(p))$ with $k = m_n \circ \cdots \circ m_1$, and such a zigzag is unique up to equivalence;*

(b) *every point $b \in B$ is in relation to a point of the image via a right-invertible element of the monoid, i.e., for each $b \in B$ there exist $e \in E$, $n, m \in M$ such that $p(e) \xrightarrow{n} b$ and $b \xrightarrow{m} p(e)$ with $n \circ m = e_M$.*

Remarks 4.3.

(1) The uniqueness (up to equivalence) for zigzags in the condition (a) of Corollary 4.2 comes from the faithfulness of the morphism $\varphi : Z(Eq(p)) \to B$ in Theorem 4.1. This condition can be expressed by the fact that $Z(Eq(p))$ is an M-ordered set: consider the following

(commutative) diagram

$$\begin{tikzcd} Z(Eq(p)) \arrow[rr, "\varphi"] \arrow[dr, "q\circ\varphi"'] & & B \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

where $q : B \to M$ is the faithful M-valued norm of the M-ordered set B. The morphism φ is faithful if and only if the M-valued norm $q \circ \varphi$ for $Z(Eq(p))$ is faithful.

(2) The characterization of the effective étale-descent morphisms leads to the characterization of the morphisms which are effective for descent with respect to the class of discrete op-fibrations $\mathbb{F}$. In fact the dual of Proposition 3.1 states that the slice category $\mathbb{F}(B)$ is equivalent to the category $\mathbf{Set}^B$, for a given M-ordered set B. Being also an equivalence of M-**Ord**-indexed categories, we conclude that the effective descent morphisms in M-**Ord** with respect to the classe of discrete fibrations, i.e., étale maps, and discrete op-fibrations are the same by the fact that φ is a full and faithful lax epimorphism if and only if the same holds for φ^{op}.

(3) Of course we get also a characterization of the étale-descent morphisms, that is the morphisms for which the functor $\mathbf{Set}^{\varphi^{\mathrm{op}}}$ is full and faithful. They are precisely the morphisms for which condition (b) of Corollary 4.2 is satisfied [14, Theorem 1, Corollary 3].

(4) As remarked in [2, Remarks 1.4.3, (2)], in the trivial case $M = 1$ everything collapses to **Preord**, also identified as the full subcategory of **Cat** given by small categories X for which $X \to 1$ is faithful. Because of that, the characterization of (effective) étale-descent morphisms in M-**Ord** generalizes the characterization of (effective) étale-descent morphisms in **Preord** given in [10].

5. Effective descent and effective étale-descent morphisms in M-Ord

A morphism $p : E \to B$ in a category $\mathbf{C}$ with pullbacks is called an *effective descent morphism* if the pullback functor $p^* : \mathbf{C} \downarrow B \to \mathbf{C} \downarrow E$ is monadic. A complete characterization of the effective descent morphisms in M-**Ord** is given in [6, Theorem 1.8]. They are the equivariant maps

$p : E \to B$ satisfying the following condition: for each $b_0, b_1, b_2 \in B$ such that $b_0 \xrightarrow{m} b_1 \xrightarrow{n} b_2$, for $m, n \in M$, there exist $e_0, e_1, e_2 \in E$ such that $p(e_0) = b_0, p(e_1) = b_1, p(e_2) = b_2$ and

$$e_0 \xrightarrow{m} e_1 \xrightarrow{n} e_2.$$

Lemma 5.1. *If $p : E \to B$ is an effective descent morphism, then every 2-zigzag in $Z(Eq(p))$ is equivalent to a 1-zigzag.*

Proof. Let $[t, e']n[z, y]m[e, x]$ be a 2-zigzag in $Z(Eq(p))$. Take $b_0 = p(e) = p(x), b_1 = p(y) = p(z)$ and $b_2 = p(t) = p(e')$; then

$$b_0 \xrightarrow{m} b_1 \xrightarrow{n} b_2.$$

Since p is effective descent, by definition there exist $e_0, e_1, e_2 \in E$ such that $p(e_0) = b_0, p(e_1) = b_1, p(e_2) = b_2$ and

$$e_0 \xrightarrow{m} e_1, e_1 \xrightarrow{n} e_2.$$

After that we have

$$\begin{aligned} [t, e']n[z, y]m[e, x] &\sim [t, e']n[e_1, z][y, e_1]m[e, x] \\ &\sim [t, e'][e_2, t](n \circ m)[x, e_0][e, x] \\ &\sim [e_2, e'](n \circ m)[e, e_0] \end{aligned}$$

as claimed. □

Lemma 5.2. *The morphism $\varphi : Z(Eq(p)) \to B$ is faithful on 1-zigzags.*

Proof. Let $[y, e']m[e, x]$ and $[v, e']n[e, u]$ be two 1-zigzags in $Z(Eq(p))$ such that $m = n$. Then by construction of the category $Z(Eq(p))$ they are equivalent; in fact

$$[y, e']m[e, x] \sim [v, e'][y, v]m[e, x] \sim [v, e']m[x, u][e, x] \sim [v, e']m[e, u]$$

as desired. □

The following theorem is an immediate consequence of Lemma 5.1, Lemma 5.2 and of fact that an n-zigzag in $Z(Eq(p))$ is a composition of n 1-zigzags.

Theorem 5.3. *Every effective descent morphism in M-**Ord** is effective for étale-descent.*

Effective descent morphisms are necessarly surjective, while there are non-surjective effective étale-descent morphisms in M-**Ord**. Hence the converse of Theorem 5.3 is false. For surjective maps the problem is more interesting, although thr answer is the same. An appropriate counter-example in the surjective case of Theorem 5.3 can be given as follows. Let $p : E \to B$ be the following equivariant map:

$$\begin{array}{ccccc} e_0 & & & & b_0 \\ \downarrow{\scriptstyle m} & & & & \downarrow{\scriptstyle m} \\ e_{11} & e_{12} & & \xrightarrow{p} & b_1 \\ & \downarrow{\scriptstyle n} & & & \downarrow{\scriptstyle n} \\ & e_2 & & & b_2 \end{array}$$

where $p(e_0) = b_0, p(e_{11}) = p(e_{12}) = b_1$ and $p(e_2) = b_2$. By Corollary 4.2, an easy inspection reveals that p is effective for étale-descent but it is not an effective descent morphism.

6. Effective descent and effective étale-descent morphisms in categories of relational algebras

Theorem 5.3 can be generalized to a larger context of relational algebras, including also the known results given in [11] for **Top** $\cong$ **RelAlg**$(\mathbb{U})$, where $\mathbb{U}$ is the ultrafilter monad, and in [10] for **Preord** $\cong$ **RelAlg**$(\mathbb{I})$, where $\mathbb{I}$ is the identity monad.

Recall that a functor T : **Set** $\to$ **Set** has the Beck-Chevalley (BC) property (in the sense of [4]), if T preserves (BC)-squares, where a (BC)-square is a diagram

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

such that $f^\circ \circ g = k \circ h^\circ$, with f° and h° the opposite relations of f and h, respectively.

Theorem 6.1. [3, Theorem 2.4] *Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on* **Set** *and $\overline{\mathbb{T}}$ its Barr extension. Assume that every naturality square of η with respect* *to relations with finite fibres is a (BC)-square. Then the following conditions are equivalent, for a morphism $f : (X, a) \to (Y, b)$ in* $\mathbf{RelAlg}(\mathbb{T})$*:*

(i) *f is final;*
(ii) *f is a pullback stable regular epimorphism in* $\mathbf{RelAlg}(\mathbb{T})$*;*
(iii) *f is a descent morphism in* $\mathbf{RelAlg}(\mathbb{T})$.

Theorem 6.2. [6, Proposition 5.2] *Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on* **Set** *and $\overline{\mathbb{T}}$ its Barr extension. Given a pullback diagram in* $\mathbf{RelAlg}(\mathbb{T})$

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

with f a final morphism:

(a) *If π_1 is a discrete (co)fibration, then g is a discrete (co)fibration.*
(b) *If π_1 has closed image, then g has closed image.*
(c) *If π_1 has separated fibres, then g has separated fibres.*

Theorem 6.3. *Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on* **Set** *and $\overline{\mathbb{T}}$ its Barr extension. Let* $\mathbf{RelAlg}(\mathbb{T})$ *the category of relational $\mathbb{T}$-algebras (or lax algebras, or $(\mathbb{T}, \mathbf{2})$-categories, see* [4] *and* [7]*). If $\mathbb{T}$ satisfies the following conditions:*

(1) *$\mathbb{T}$ has the (BC) property,*
(2) *η has (BC) for relations with finite fibres,*

an effective descent morphism $p : E \to B$ in $\mathbf{RelAlg}(\mathbb{T})$ *is effective for étale-descent.*

Proof. Let $p : E \to B$ be an effective descent morphism in $\mathbf{RelAlg}(\mathbb{T})$. We use [11, Proposition 2.6] to show that p is also effective with respect to the class of étale morphisms, i.e., pullback stable discrete fibrations. Consider then a pullback diagram in $\mathbf{RelAlg}(\mathbb{T})$

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "g"] \\ E \arrow[r, "p"'] & B, \end{tikzcd}$$

where π_1 is an étale morphism. The relational structure on

$$E \times_B A = \{(e, a) \in E \times A \mid p(e) = \alpha(a)\}$$

is defined by

$$\omega \to (e,a) \iff T\pi_1(\omega) \to e \quad \text{and} \quad T\pi_2(\omega) \to a,$$

for any $\omega \in T(E \times_B A)$ and $(e,a) \in E \times_B A$. We want to prove that g is an étale morphism as well.

By Theorem 6.1 and Theorem 6.2(a), g is a discrete fibration. To prove that every pullback of g is a discrete fibration we consider the following diagram

$$\begin{tikzcd}
& E \times_B A \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"'] & & A \arrow[dd, "g"] \\
& & X \times_B A \arrow[ur, "pr_2"'] \arrow[dd, "pr_1"', pos=0.2] & \\
& E \arrow[rr, "p"'] & & B \\
X \times_B E \arrow[ur, "\pi'_2"] \arrow[rr, "\pi'_1"'] & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

where the three faces are pullbacks. We want to prove that pr_1 is a discrete fibration. First of all observe that since effective descent morphisms are pullback stable π'_1 is an effective descent morphism. Building the pullback on the left-side, i.e., the pullback of π_1 along π'_2, by universality we get a cube such that all faces are pullbacks.

$$\begin{tikzcd}
& E \times_B A \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"] & & A \arrow[dd, "g"] \\
(X \times_B E) \times_E (E \times_B A) \arrow[ur, "pr'_2"] \arrow[rr] \arrow[dd, "pr'_1"'] & & X \times_B A \arrow[ur, "pr_2"'] \arrow[dd, "pr_1"', pos=0.2] & \\
& E \arrow[rr, "p"'] & & B \\
X \times_B E \arrow[ur, "\pi'_2"] \arrow[rr, "\pi'_1"'] & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

Now, since π_1 is an étale morphism, pr'_1 is a discrete fibration and, using the same argument that we used to prove that g is a discrete fibration, we conclude that pr_1 is a discrete fibration as well. $\square$

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