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ON THE “SMITH IS HUQ” CONDITION IN S-PROTOMODULAR CATEGORIES

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Abstract: We study the so-called “Smith is Huq” condition in the context of S-protomodular categories: two S-equivalence relations centralise each other if and only if their normalisations commute. We prove that this condition is satisfied by every category of monoids with operations equipped with the class S of Schreier split epimorphisms. Some consequences in terms of characterisation of internal structures are explored.

Keywords: Smith-Pedicchio centralising relations, Huq commuting morphisms, S-protomodular categories, monoids with operations, Schreier split epimorphisms.
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1. Introduction

In the paper [16] the authors introduced a class of split epimorphisms, called *Schreier split epimorphisms*, in the context of monoids and, more generally, of *monoids with operations* (which generalise Porter’s groups with operations [22]). This class was used to describe crossed modules of monoids with operations in terms of internal categories, generalising some results obtained by Patchkoria [20] for the category of monoids.

The Schreier split epimorphisms were then widely studied in the paper [7] and in the monograph [6]. It turned out that this class of split epimorphisms possesses many properties which are typical of all split epimorphisms of groups, rings and, more generally, of any protomodular category [2, 1]. Among these properties we mention the Split Short Five Lemma for Schreier

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split epimorphisms, and the fact that every internal Schreier reflexive relation (i.e. a reflexive relation such that the first projection and the reflexivity morphism form a Schreier split epimorphism) is transitive. This lead to the notion of S-protomodular category [6, 8]. Roughly speaking, a S-protomodular category is a pointed category equipped with a class S of split epimorphisms which satisfies suitable properties (see the next section for the precise definition). When S is the class of all split epimorphisms, we recover the notion of a protomodular category.

S-protomodular categories have many classical properties of protomodular categories, restricted to the specified class S (see [8] for a detailed account of these properties). However, the category of monoids, and more generally every category of monoids with operations, have even stronger properties, that are known to be true for groups with operations, but not for a general protomodular category. In the papers [15, 18] it was shown that these properties depend on the so-called "Smith is Huq" condition: any two equivalence relations centralise each other in the Smith-Pedicchio sense [23, 21] if and only if their normalisations commute in the Huq sense [11].

The aim of the present paper is to study the "Smith is Huq" condition in the "relative" context of S-protomodular categories: any two S-equivalence relations (i.e. equivalence relations where the projections and the reflexivity morphism form split epimorphisms belonging to the class S) centralise each other if and only if their normalisations commute. We compare this condition with others, concerning properties of internal reflexive graphs and properties of the so-called *kernel functor*, which associates with any split epimorphism its kernel. In the "absolute" context of pointed protomodular categories (where S is the class of all split epimorphisms), the conditions mentioned above are all equivalent to the "Smith is Huq" condition, as proved in [18, 9, 19]. This is no more true for S-protomodular categories, where it is possible to prove only some implications.

We prove that every category of monoids with operations, equipped with the class S of Schreier split epimorphisms, satisfies the "Smith is Huq" condition. This fact explains why this context appeared to be "richer" than the general one of S-protomodular category, of which any Jonsson-Tarski variety [14] constitutes an example, as we prove in Section 2. Moreover, some consequences of the "Smith is Huq" condition in terms of characterisation of internal categories and groupoids are described.

2. S-protomodular categories

We recall now from [6] and [8] the notion of S-protomodular category. Let $\mathbb{C}$ be a pointed finitely complete category. By a *point* in $\mathbb{C}$ we mean a pair (f, s) of morphisms in $\mathbb{C}$ such that $fs = 1$; in other terms, f is a split epimorphism with a fixed section s. Let S be a class of points in $\mathbb{C}$ which is stable under pullbacks. Accordingly this class determines a subfibration $\P_{\mathbb{C}}^S$ of the fibration of points $\P_{\mathbb{C}} \colon Pt(\mathbb{C}) \to \mathbb{C}$. We recall that the fibration of points is the functor that associates with every split epimorphism its codomain. Let us denote by $SPt(\mathbb{C})$ the full subcategory of the category $Pt(\mathbb{C})$ of points of $\mathbb{C}$ whose objects are those which are in S:

$$\begin{tikzcd} SPt(\mathbb{C}) \arrow[rr, tail, "j"] \arrow[dr, "\P_{\mathbb{C}}^S"'] & & Pt(\mathbb{C}) \arrow[dl, "\P_{\mathbb{C}}"] \\ & \mathbb{C}. & \end{tikzcd}$$

Given a split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ in $\mathbb{C}$, we say that it is a *strongly split epimorphism* (see [4], and [17], where strongly split epimorphisms were introduced under the name of regular points) if the pair (k, s), where $k \colon X \to A$ is a kernel of f, is jointly strongly epimorphic. This means that A is the supremum of the two subobjects X and B, i.e. X and B are not both contained in a proper subobject of A.

Definition 2.1. *The category* $\mathbb{C}$ *is said to be* S-protomodular *when:*

1. *any object in* $SPt(\mathbb{C})$ *is a strongly split epimorphism;*
2. $SPt(\mathbb{C})$ *is closed under finite limits in* $Pt(\mathbb{C})$ *(in particular, it contains the terminal object* $0 \rightleftarrows 0$ *of* $Pt(\mathbb{C})$*).*

As shown in [8], Theorem 3.2, in a S-protomodular category every change-of-base functor, w.r.t. the fibration $\P_{\mathbb{C}}^S$ of points in the class S, is conservative. This implies, in particular, that the Split Short Five Lemma holds, when the split epimorphisms involved belong to S.

In [8] it is proved that every category of *monoids with operations* (introduced in [16]) is S-protomodular, when S is the class of *Schreier split* *epimorphisms* [16]. The definition of monoids with operations, recalled here below, is inspired by Porter's definition of *groups with operations* [22].

Definition 2.2. *Let Ω be a set of finitary operations such that the following conditions hold: if Ω_i is the set of i-ary operations in Ω, then:*

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(2) *There is a binary operation $+ \in \Omega_2$ (not necessarily commutative) and a constant $0 \in \Omega_0$ satisfying the usual axioms for monoids;*
(3) $\Omega_0 = \{0\}$;
(4) *Let $\Omega_2' = \Omega_2 \backslash \{+\}$; if $* \in \Omega_2'$, then $*^\circ$, defined by $x *^\circ y = y * x$, is also in Ω_2';*
(5) *Any $* \in \Omega_2'$ is left distributive w.r.t. $+$, i.e.:*

$$a * (b + c) = a * b + a * c;$$

(6) *For any $* \in \Omega_2'$ we have $b * 0 = 0$;*
(7) *Any $\omega \in \Omega_1$ satisfies the following conditions:*
   - $\omega(x + y) = \omega(x) + \omega(y)$;
   - *for any* $* \in \Omega_2'$, $\omega(a * b) = \omega(a) * b$.

Let moreover E be a set of axioms including the ones above. We will denote by $\mathbb{C}$ the category of (Ω, E)-algebras. We call the objects of $\mathbb{C}$ monoids with operations.

Examples of categories of monoids with operations are the categories of monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), join-semilattices with a bottom element, distributive lattices with a bottom element (or a top one). The algebraic structures covered by Porter's definition, such as groups, rings, associative algebras, Lie algebras and many others, can also be seen as examples of monoids with operations (although, in order to include these examples, condition (7) above should be slightly modified, see [16] for more details).

The definition of a Schreier split epimorphism was originally given only for monoids with operations. However, it is actually meaningful in every Jonsson-Tarski variety, so we now recall the definition in this more extended context. We start by recalling what a Jonsson-Tarski variety is.

Definition 2.3 ([14]). *A variety in the sense of universal algebra is a Jonsson-Tarski variety if the corresponding theory contains a unique constant* 0 *and a binary operation* $+$ *satisfying the equalities* $0+x=x+0=x$ *for any* x.

It was proved in [1] that a variety, seen as a category, is a unital category [3] (see the next section for a recall of the definition) if and only if it is a Jonsson-Tarski variety.

Definition 2.4. *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *in a Jonsson-Tarski variety is said to be a* Schreier split epimorphism *when, for any* $a \in A$, *there exists a unique* α *in the kernel* $Ker(f)$ *of* f *such that* $a = \alpha + sf(a)$.

The definition above can be expressed in the following equivalent form: a split epimorphism (f,s) as above is a Schreier split epimorphism if there exists a unique map $q_f \colon A \to Ker(f)$ (which is not a morphism, in general) such that $a = q_f(a) + sf(a)$ for any $a \in A$. This map q_f is called the *Schreier retraction* of the split epimorphism (f,s).

We omit the proof of the following proposition, since it is completely analogous to the proof, given in [8], in the more restricted context of monoids with operations.

Proposition 2.5. *If* $\mathbb{C}$ *is a Jonsson-Tarski variety and* S *is the class of Schreier split epimorphisms, then* $\mathbb{C}$ *is a* S*-protomodular category.*

The reason why, at the beginning, the definition of Schreier split epimorphisms was considered only in the context of monoids with operations is that some results, proved in [16] in this context, like the fact that crossed modules are equivalent to Schreier internal categories, are not valid in every Jonsson-Tarski variety. In the next sections we give an explanation for this fact, showing that monoids with operations are not only S-protomodular, but they satisfy a stronger assumption, which is a relative version of the so-called "Smith is Huq" condition.

2.1. S-graphs, S-relations and S-special morphisms. We recall from [8] some other notions that will be useful in the rest of the paper. Let $\mathbb{C}$ be a S-protomodular category.

Definition 2.6. *A reflexive graph*

$$C_1 \underset{c}{\overset{d}{\underset{\leftarrow e -}{\rightrightarrows}}} C_0$$

is a S-reflexive graph if the point (d, e) belongs to S. In particular, a reflexive relation

$$R \underset{r_2}{\overset{r_1}{\underset{\leftarrow e -}{\rightrightarrows}}} C_0$$

is a S-reflexive relation if the point (r_1, e) belongs to S.

Definition 2.7. *A morphism $f\colon X \to Y$ in $\mathbb{C}$ is called* S-special *when its kernel equivalence relation $R[f]$ is a S-equivalence relation. An object X is called* S-special *when the terminal morphism $\tau_X\colon X \to 1$ is S-special (or, in other terms, when the undiscrete relation on X is a S-equivalence relation).*

3. Commutativity in the sense of Huq and of Smith

In this section we recall the notions of commutativity of subobjects in the sense of Huq [11] and of S-reflexive relations in a S-protomodular category. We start by recalling from [3] the following.

Definition 3.1. *Let $\mathbb{C}$ be a pointed finitely complete category. It is a* unital *category if, for every pair X, Y of objects of $\mathbb{C}$, the morphisms $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ in the following diagram are jointly strongly epimorphic:*

$$X \underset{\langle 1,0\rangle}{\overset{\pi_X}{\leftrightarrows}} X \times Y \underset{\langle 0,1\rangle}{\overset{\pi_Y}{\rightleftarrows}} Y.$$

Again, this means that the product $X \times Y$ is the supremum of the two subobjects X and Y, i.e. X and Y are not both contained in a proper subobject of $X \times Y$.

In this context it is possible to define the commutativity of two morphisms (and, in particular, of two subobjects) in the sense of Huq:

Definition 3.2. *Let $\mathbb{C}$ be a unital category. Two morphisms $f\colon X \to Z$ and $g\colon Y \to Z$, with the same codomain, are said to cooperate, or to commute, if there exists a (necessarily unique) morphism $\varphi\colon X \times Y \to Z$ making the two* *triangles in the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, "{\langle 1,0\rangle}"] \arrow[dr, "f"'] & X \times Y \arrow[d, dotted, "\varphi"] & Y \arrow[l, "{\langle 0,1\rangle}"'] \arrow[dl, "g"] \\ & Z. & \end{tikzcd}$$

The morphism φ is called the cooperator *of f and g.*

When X and Y are subobjects of an object Z, we denote by $[X,Y]=0$ the fact that they commute.

The notion of centralisation of equivalence relations was introduced by Smith [23] for Mal'tsev varieties and then extended to the categorical context of Mal'tsev categories by Pedicchio [21]. We recall that a finitely complete category is a Mal'tsev category [10] if every internal reflexive relation is an equivalence relation. As shown in [3], a finitely complete category is Mal'tsev if and only if every fibre $Pt_X(\mathbb{C})$ of the fibration of points $¶_{\mathbb{C}}\colon Pt(\mathbb{C}) \to \mathbb{C}$ is unital. In a Mal'tsev category, the equivalence relations R on an object X, coinciding with the reflexive relations on X, are just the subobjects of the object $(\pi_1, \Delta_X)\colon X\times X \rightleftarrows X$ in the fibre $Pt_X(\mathbb{C})$:

$$\begin{tikzcd} R \arrow[r, tail, "{(r_1,r_2)}"] \arrow[dr, shift right, "r_1"'] & X \times X \arrow[d, shift right, "\pi_1"'] \\ & X. \arrow[u, shift right, "\Delta_X"'] \arrow[ul, shift right, "e"'] \end{tikzcd}$$

Two equivalence relations R and W on X centralise each other in a Mal'tsev category $\mathbb{C}$ when the subobjects $(r_2,r_1)\colon R \rightarrowtail X\times X$ and $(w_1,w_2)\colon W \rightarrowtail X\times X$ commute in the unital fiber $Pt_X(\mathbb{C})$. In set-theoretical terms, the cooperator $R\times_X W \to X\times X$ in the fiber is necessarily of the form $\phi(xRyWz) = (x, p(xRyWz))$, with the two equations $p(xRxWy)=y$ and $p(xRyWy)=x$. The morphism $p\colon R\times_X W \to X$ satisfying these two equations, which is characteristic of the fact that R and W centralise each other (see [5]), is called the *connector* between the relations R and W.

It was proved in [8] that, in a S-protomodular category $\mathbb{C}$, any fiber $Pt_X(\mathbb{C})$ is $SPt_X(\mathbb{C})$-unital. This means that, given a product diagram in $Pt_X(\mathbb{C})$:

$$\begin{tikzcd}
A \arrow[r, shift right, "\langle 1,tf\rangle"'] \arrow[dr, shift right, "f"'] & A \times_X B \arrow[l, shift right, "\pi_A"'] \arrow[r, shift left, "\pi_B"] \arrow[d, shift right] & B \arrow[l, shift left, "\langle sg,1\rangle"] \arrow[dl, shift right, "g"'] \\
& X, \arrow[ul, shift right, "s"'] \arrow[u, shift right] \arrow[ur, shift right, "t"'] &
\end{tikzcd}$$

where one of the two points belongs to S, the morphisms $\langle 1, tf\rangle$ and $\langle sg, 1\rangle$ are jointly strongly epimorphic.

Thanks to this fact, in a S-protomodular category we can keep the same definition of reflexive relations centralising each other as for Mal'tsev categories, provided that one of the relations, let us choose W, is a S-reflexive relation:

Definition 3.3. *Given a reflexive relation R and a S-reflexive relation W on the same object X in a S-protomodular category $\mathbb{C}$, we say that R and W* centralise each other *when there is a (necessarily unique) morphism $p\colon R \times_X W \to X$, where $R \times_X W$ is defined by the following pullback:*

$$\begin{tikzcd}
R \times_X W \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & W \arrow[l, shift left, "\sigma_R"] \arrow[d, shift right, "w_1"'] \\
R \arrow[r, shift left, "r_2"] \arrow[u, shift right, "\sigma_W"'] & X \arrow[l, shift left, "e_R"] \arrow[u, shift right, "e_W"']
\end{tikzcd}$$

such that $p\sigma_R = w_2$ and $p\sigma_W = r_1$. In set-theoretical terms, this means that we have both $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphisms σ_R and σ_W, defined by the universal property of the pullback, are explicitly given by $\sigma_R(yWz) = yRyWz$ and $\sigma_W(xRy) = xRyWy$. We denote this situation by $[R, W] = 0$.

Since W is a S-reflexive relation, the split epimorphism (w_1, e_W) is in S, and consequently the pair (σ_R, σ_W) is jointly strongly epimorphic. This implies that the connector p is unique.

In the context of pointed Mal'tsev categories, the so-called "Smith is Huq" condition is the following:

(SH) Two effective equivalence relations centralise each other if and only if their normalisations commute,

where the normalisation of an equivalence relation is the “equivalence class of 0”, i.e. it is the composite $r_2 k$, where the relation is given by

$$R \underset{r_2}{\overset{r_1}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} X$$

and k is a kernel of r_1. It is always a monomorphism. It is always true that, when two effective equivalence relations centralise each other, then their normalisations commute. The converse, however, is not always true, not even in semi-abelian categories [13], as the well-known counterexample of digroups shows. We are now interested in the study of the (SH) condition in the context of pointed S-protomodular categories.

4. The “Smith is Huq” condition for S-protomodular categories

Throughout this section $\mathbb{C}$ will be a pointed S-protomodular category with the two following additional conditions:

- every product projection, i.e. every point of the form $X \times Y \underset{\pi_Y}{\overset{\langle 0,1 \rangle}{\rightleftarrows}} Y$ belongs to the class S (we observe that this implies, in particular, that the category is unital);
- S is closed under composition.

It is immediate to see that the first condition is satisfied by any Jonsson-Tarski variety with the class S of Schreier split epimorphisms. This is not the case for the second one. However, Proposition 2.3.2 in [6] shows that it is satisfied in the case of monoids. It is then easy to see that Schreier split epimorphisms are closed under composition in any category of monoids with operations: in order to do that, it suffices to observe that a split epimorphism (f, s) in a category $\mathbb{C}$ of monoids with operations is a Schreier split epimorphism if and only if $(U(f), U(s))$ is a Schreier split epimorphism in the category Mon of monoids, where U is the forgetful functor $U : \mathbb{C} \to Mon$, forgetting everything but the monoid structure of the objects of $\mathbb{C}$.

Before stating the first important result of this section, we need to recall the following.

Definition 4.1. *An internal reflexive graph*

$$C_1 \overset{d}{\underset{c}{\rightleftarrows}} C_0 \tag{1}$$

is multiplicative *if it is equipped with a morphism $m\colon C_2 \to C_1$, where C_2 is the pullback of d along c, such that the following equalities are satisfied:*

$$m\langle ed, 1\rangle = m\langle 1, ec\rangle = 1,$$

where the morphisms $C_1 \overset{\langle 1,ec\rangle}{\underset{\langle ed,1\rangle}{\rightrightarrows}} C_2$ *are induced by the universal property of the pullback.*

Theorem 4.2. *Under the hypotheses above, consider the following conditions:*

(a) *Every reflexive graph*

$$\begin{tikzcd} X \arrow[dr, "k=ker\ d"] & & \\ & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e"'] \\ Y \arrow[ur, "l=ker\ c"'] & & \end{tikzcd}$$

such that both points (d,e) and (c,e) belong to S and $[X,Y]=0$ is multiplicative.

(b) *Two effective S-equivalence relations centralise each other as soon as their normalisations commute.*

(c) *Every reflexive graph*

$$\begin{tikzcd} X \arrow[dr, "k=ker\ d"] & & \\ & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e"'] \\ Y \arrow[ur, "l=ker\ c"'] & & \end{tikzcd}$$

such that both morphisms d and c are S-special and $[X,Y]=0$ is multiplicative.

Then we have the following chain of implications: $(a) \Longrightarrow (b) \Longrightarrow (c)$.

Proof: We start by proving that (a) implies (b). Consider the following diagram:

$$\begin{tikzcd}
 & & Y \arrow[r, equal] \arrow[d, "\langle 0,l\rangle"'] & Y \arrow[d, "l"] \\
 & R[d,c] \arrow[r, shift right, "\pi_2^{d,c}"'] \arrow[d, shift right, "\pi_1^{d,c}"'] & R[c] \arrow[l, shift right, dotted, "\Delta_2^{d,c}"'] \arrow[r, shift right, "\pi_2^c"'] \arrow[d, shift right, "\pi_1^c"'] & C_1 \arrow[l, shift right, "\Delta^c"'] \arrow[d, "c"] \\
X \arrow[r, "\langle k,0\rangle"] \arrow[d, equal] & R[d] \arrow[u, shift right, dotted, "\Delta_1^{d,c}"'] \arrow[r, shift right, "\pi_2^d"'] \arrow[d, shift right, "\pi_1^d"'] & C_1 \arrow[u, shift right, "\Delta^c"'] \arrow[l, shift right, "\Delta^d"'] \arrow[r, "c"'] \arrow[d, "d"] & C_0' \\
X \arrow[r, "k"'] & C_1 \arrow[u, shift right, "\Delta^d"'] \arrow[r, "d"'] & C_0, &
\end{tikzcd}$$

where $R[d]$ and $R[c]$ are two effective S-equivalence relations (kernel pairs of d and c, respectively), k is a kernel of d, l is a kernel of c, $R[d,c]$ is the pullback of π_2^d along π_1^c, Δ^d and Δ^c are the diagonal morphisms, while $\Delta_1^{d,c}$ and $\Delta_2^{d,c}$ are induced by the universal property of the pullback $R[d,c]$. Since $R[d]$ and $R[c]$ are S-equivalence relations, we have that the points (π_1^d, Δ^d) and (π_1^c, Δ^c) belong to the class S. By symmetry of the relations, also the points (π_2^d, Δ^d) and (π_2^c, Δ^c) belong to S. Suppose, moreover, that there exists a cooperator $\varphi\colon X \times Y \to C_1$.

We build the following reflexive graph:

$$\begin{tikzcd}
R[d,c] \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & C_1, \arrow[l, "s"']
\end{tikzcd} \tag{2}$$

where

$$\mathrm{dom} = \pi_1^d \pi_1^{d,c}, \qquad \mathrm{cod} = \pi_2^c \pi_2^{d,c}, \qquad s = \Delta_1^{d,c} \Delta^d = \Delta_2^{d,c} \Delta^c.$$

Since the class S is closed under composition, we have that both (dom, s) and (cod, s) belong to S. In order to apply the hypothesis (a), we need to prove that $[\mathrm{Ker}(\mathrm{dom}), \mathrm{Ker}(\mathrm{cod})] = 0$. We first observe that, in set-theoretical terms, the elements of $R[d,c]$ can be represented as triples:

$$\cdot \xleftarrow{x} \cdot \xrightarrow{f} \cdot \xleftarrow{y} \cdot$$

Then $\operatorname{dom}(x, f, y) = x$, $\operatorname{cod}(x, f, y) = y$ and hence $\operatorname{Ker}(\operatorname{dom}) = P(c, ck)$ and $\operatorname{Ker}(\operatorname{cod}) = P(d, dl)$ are given by the following pullbacks:

$$\begin{tikzcd} P(c,ck) \arrow[r, "p_2^c"] \arrow[d, "p_1^c"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0', \end{tikzcd} \qquad \begin{tikzcd} P(d,dl) \arrow[r, "p_2^d"] \arrow[d, "p_1^d"'] & Y \arrow[d, "dl"] \\ C_1 \arrow[r, "d"'] & C_0. \end{tikzcd}$$

The cooperator

$$\psi\colon \operatorname{Ker}(\operatorname{dom}) \times \operatorname{Ker}(\operatorname{cod}) \to R[d, c]$$

is defined, using the cooperator $\varphi\colon X \times Y \to C_1$ in the following way:

$$\psi(\ \cdot \xleftarrow{0} \cdot \xrightarrow{f} \cdot \xleftarrow{g} \cdot\ ,\ \cdot \xleftarrow{h} \cdot \xrightarrow{a} \cdot \xleftarrow{0} \cdot\ ) = \ \cdot \xleftarrow{h} \cdot \xrightarrow{\varphi(f,a)} \cdot \xleftarrow{g} \cdot$$

In other terms, the morphism ψ is obtained by using repeatedly the universal property of the pullbacks as in the following diagrams: first we get the morphisms α and β as in the diagrams below:

$$\begin{tikzcd} P(c,ck) \times P(d,dl) \arrow[rr, "p_2^c \times p_2^d"] \arrow[d, "p_2"'] \arrow[dr, dotted, "\alpha"] & & X \times Y \arrow[d, "\varphi"] \\ & R[d] \arrow[r] \arrow[d] & C_1 \arrow[d, "d"] \\ P(d,dl) \arrow[r, "p_1^d"'] & C_1 \arrow[r, "d"'] & C_0, \end{tikzcd}$$

$$\begin{tikzcd} P(c,ck) \times P(d,dl) \arrow[rr, "p_2^c \times p_2^d"] \arrow[d, "p_1"'] \arrow[dr, dotted, "\beta"] & & X \times Y \arrow[d, "\varphi"] \\ & R[c] \arrow[r] \arrow[d] & C_1 \arrow[d, "c"] \\ P(c,ck) \arrow[r, "p_1^c"'] & C_1 \arrow[r, "c"'] & C_0', \end{tikzcd}$$

and then ψ is induced by α and β:

$$\begin{tikzcd} P(c,ck)\times P(d,dl) \arrow[dr, dotted, "\psi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & R[d,c] \arrow[r] \arrow[d] & R[c] \arrow[d] \\ & R[d] \arrow[r] & C_1. \end{tikzcd}$$

Now, thanks to the hypothesis (a), we get that the graph (2) is multiplicative. The needed connector $p\colon R[d,c] \to C_1$, in order to conclude the proof, is determined by:

$$\cdot \xleftarrow{h} \cdot \xrightarrow{p(f,g,h)} \cdot \xleftarrow{f} \cdot \; = m\big(\, \cdot \xleftarrow{g} \cdot \xrightarrow{f} \cdot \xleftarrow{f} \cdot \,,\; \cdot \xleftarrow{h} \cdot \xrightarrow{h} \cdot \xleftarrow{g} \cdot \,\big),$$

where m is the multiplication in the graph (2).

The proof that (b) implies (c) is much easier: given a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

such that d and c are S-special and $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0$, it suffices to consider the kernel equivalence relations $R[d]$ and $R[c]$: they are S-equivalence relations, because d and c are S-special. Hypothesis (b) gives a connector $p\colon R[d,c] \to C_1$; the multiplication m we are looking for is then given by:

$$m(f,g) = p\,(\, \cdot \xleftarrow{g} \cdot \xrightarrow{1} \cdot \xleftarrow{f} \cdot \,).$$

■

We observe that the proof of the previous theorem is analogous to the one of Theorem 2.3 in [18] for the case of pointed protomodular categories, i.e. when the class S is the class of all points. In the "absolute" context considered in [18], conditions (a) and (c) above are equivalent (because every morphism is S-special), hence the three conditions of the previous theorem become equivalent. The "relative" context of pointed S-protomodular categories appears then to be more diversified. The following example, in the category of monoids with the class S of Schreier split epimorphisms, exhibits a multiplicative reflexive graph (actually, an internal category) such that both the domain and codomain morphisms form, with the unit morphism, points belonging to S, but none of them is S-special.

Example 4.3. Any commutative monoid M can be seen as an internal category (with only one object) in the category of monoids:

$$M \rightleftarrows 1.$$

It is immediate to see that both the domain and codomain morphisms (which coincide in this example) form, with the unit morphism (which is, in this case, the unique morphism $1 \to M$), a Schreier split epimorphism. However, they are special morphisms if and only if M is a group (see [6] for more details).

We will prove in the next section that all the conditions of Theorem 4.2 are satisfied in every category of monoids with operations, equipped with the class S of Schreier split epimorphisms.

The equivalence recalled above of Theorem 2.3 in [18] was used in the same paper to give a characterisation of semi-abelian categories [13] in which the "Smith is Huq" condition (SH) is satisfied: it happens if and only if every star-multiplicative graph (see the definition below) is multiplicative ([18], Theorem 3.8). The proof of this result uses the fact that, in a semi-abelian category, a reflexive graph is star-multiplicative if and only if the kernels of the domain and codomain morphisms commute ([18], Proposition 3.7). The situation in a S-protomodular category is much more complicated, as the following results will show.

We first recall from [12] and [15] the following definitions.

Definition 4.4. *A reflexive graph (1), with $k\colon X \to C_1$ a kernel of d and $l\colon Y \to C_1$ a kernel of c is* star-multiplicative *if there exists a morphism $\xi\colon P(d, ck) \to X$, where $P(d, ck)$ is the pullback:*

$$\begin{array}{ccc} P(d,ck) & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle ck} \\ C_1 & \xrightarrow[d]{} & C_0, \end{array}$$

such that

$$\xi\langle k, 0\rangle = 1_X = \xi\langle eck, 1_X\rangle.$$

It is star-divisible *if there exists a morphism* $\omega\colon X \times X \to C_1$ *such that*

$$\omega\langle 1_X, 0\rangle = k \quad \textit{and} \quad \omega\langle 1_X, 1_X\rangle = eck.$$

Proposition 4.5. *If $\mathbb{C}$ is a pointed S-protomodular category and (1) is a S-reflexive graph such that X (the kernel of d) is a S-special object, then (1) is star-multiplicative if and only if it is star-divisible.*

Proof: Suppose first that (1) is star-multiplicative, with star-multiplication ξ. In the following pullback, the lower split epimorphism belongs to S, hence the upper one belongs to S, too:

$$\begin{tikzcd}
P(d, ck) \arrow[r, shift right, "p_2"'] \arrow[d, "p_1"'] & X \arrow[l, shift right, "\langle eck, 1\rangle"'] \arrow[d, "ck"] \\
C_1 \arrow[r, shift right, "d"'] & C_0. \arrow[l, shift right, "e"']
\end{tikzcd}$$

Consider then the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\langle k, 0\rangle"] \arrow[d, equal] & P(d, ck) \arrow[r, shift right, "p_2"'] \arrow[d, "\langle \xi, p_2\rangle"'] & X \arrow[l, shift right, "\langle eck, 1\rangle"'] \arrow[d, equal] \\
X \arrow[r, "\langle 1, 0\rangle"'] & X \times X \arrow[r, shift right, "\pi_2"'] & X. \arrow[l, shift right, "\langle 1, 1\rangle"']
\end{tikzcd}$$

Since ξ is a star-multiplication, the diagram gives rise to a morphism of split extensions: the upper one is in S, as already observed; the lower one also is, because X is a S-special object. Then the Split Short Five Lemma for split extensions in S (which is a consequence of Theorem 3.2 in [8]) implies that $\langle \xi, p_2\rangle$ is an isomorphism. We can then define the needed star-division by putting

$$\omega = p_1\langle \xi, p_2\rangle^{-1}\colon X \times X \to C_1.$$

Indeed:

$$\omega\langle 1_X, 0\rangle = p_1\langle \xi, p_2\rangle^{-1}\langle 1_X, 0\rangle = p_1\langle k, 0\rangle = k$$

and

$$\omega\langle 1_X, 1_X\rangle = p_1\langle \xi, p_2\rangle^{-1}\langle 1_X, 1_X\rangle = p_1\langle eck, 1\rangle = eck.$$

Conversely, suppose that (1) is star-divisible, with star-division ω. The morphism $\langle \omega, \pi_2 \rangle$ in the diagram below is determined by the universal property of the pullback:

$$\begin{tikzcd}
X \times X \arrow[drr, bend left, "\pi_2"] \arrow[dr, dotted, "{\langle \omega,\pi_2 \rangle}"] \arrow[ddr, bend right, "\omega"'] & & \\
& P(d,ck) \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d, "ck"] \\
& C_1 \arrow[r, "d"'] & C_0,
\end{tikzcd}$$

indeed:

$$d\omega\langle 1_X, 0\rangle = dk = 0 = ck\pi_2\langle 1_X, 0\rangle,$$
$$d\omega\langle 1_X, 1_X\rangle = deck = ck = ck\pi_2\langle 1_X, 1_X\rangle,$$

and the morphisms $\langle 1_X, 0\rangle$ and $\langle 1_X, 1_X\rangle$ are jointly strongly epimorphic, because X is a S-special object. The fact that ω is a star-division turns $\langle \omega, \pi_2\rangle$ a morphism of split extensions:

$$\begin{tikzcd}
X \arrow[r, "{\langle 1,0\rangle}"'] \arrow[d, equal] & X \times X \arrow[r, shift right, "\pi_2"'] \arrow[d, "{\langle \omega,\pi_2\rangle}"'] & X \arrow[l, shift right, "{\langle 1,1\rangle}"'] \arrow[d, equal] \\
X \arrow[r, "{\langle k,0\rangle}"] & P(d,ck) \arrow[r, shift right, "p_2"'] & X. \arrow[l, shift right, "{\langle eck,1\rangle}"']
\end{tikzcd}$$

As observed in the first part of the proof, both split extensions belong to S. Once again, the Split Short Five Lemma implies that $\langle \omega, \pi_2\rangle$ is an isomorphism. We get then a star-multiplication by putting

$$\xi = \pi_1\langle \omega, \pi_2\rangle^{-1} \colon P(d,ck) \to X.$$

Indeed:

$$\xi\langle k, 0\rangle = \pi_1\langle \omega, \pi_2\rangle^{-1}\langle k, 0\rangle = \pi_1\langle 1_X, 0\rangle = 1_X$$

and

$$\xi\langle eck, 1\rangle = \pi_1\langle \omega, \pi_2\rangle^{-1}\langle eck, 1\rangle = \pi_1\langle 1_X, 1_X\rangle = 1_X.$$

■

Remark In a S-protomodular category, on a S-reflexive graph (1) there is at most one star-multiplication, because, as observed in the previous proof, the split epimorphism $P(d,ck) \underset{p_2}{\overset{\langle eck,1\rangle}{\leftrightarrows}} X$ belongs to S, and hence $\langle k, 0\rangle$ and

$\langle eck, 1\rangle$ are jointly strongly epimorphic. If, moreover, X is S-special, then there is at most one star-division, because $\langle 1_X, 0\rangle$ and $\langle 1_X, 1_X\rangle$ are jointly strongly epimorphic.

In order to compare the fact that a S-reflexive graph is star-multiplicative (or star-divisible) with the fact that the kernels of the domain and codomain morphisms commute, we need the following.

Lemma 4.6. *Let (1) be a star-multiplicative (or, equivalently, star-divisible) S-reflexive graph such that X, the kernel of d, is a S-special object. The following conditions are equivalent:*

(i) *the square*
$$\begin{tikzcd} P(d,ck) \arrow[r, "\xi"] \arrow[d, "p_1"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$
is a pullback;

(ii) *the square*
$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$
is a pullback.

Proof: Suppose that Condition (i) is satisfied. Consider the following diagram:

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\langle \xi, p_2\rangle^{-1}"'] & X \arrow[d, equal] \\ P(d,ck) \arrow[r, "\xi"] \arrow[d, "p_1"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0. \end{tikzcd}$$

The lower square is a pullback by hypothesis. The upper square commutes by definition of the morphism $\langle \xi, p_2\rangle \colon P(d, ck) \to X \times X$, because $\pi_1\langle \xi, p_2\rangle = \xi$. It is a pullback because $\langle \xi, p_2\rangle^{-1}$ is an isomorphism. Then the whole rectangle is a pullback.

The proof that (ii) implies (i) is similar: it suffices to consider the diagram:

$$\begin{tikzcd} P(d,ck) \arrow[r, "\langle \omega,\pi_2\rangle^{-1}"] \arrow[d, "p_1"'] & X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, equal] & C_1 \arrow[r, "c"'] & C_0. \end{tikzcd}$$

■

Proposition 4.7. *If (1) is a star-divisible S-reflexive graph, $X = Ker(d)$ is a S-special object and the square* $\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$ *is a pullback, then $[X,Y] = 0$, where $Y = Ker(c)$.*

Proof: Complete the pullback in the statement with the horizontal kernels:

$$\begin{tikzcd} X \arrow[r, "\langle 0,1\rangle"] \arrow[d, "i"'] & X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ Y \arrow[r, "l"'] & C_1 \arrow[r, "c"'] & C_0. \end{tikzcd}$$

Since the right hand side square is a pullback, i is an isomorphism. Its inverse j is such that

$$\omega\langle 0,1\rangle j = \omega\langle 0,j\rangle = l.$$

Then we can define the cooperator $\varphi\colon X \times Y \to C_1$ by putting $\varphi = \omega(1 \times j)$. It is in fact a cooperator, since

$$\varphi\langle 1,0\rangle = \omega(1 \times j)\langle 1,0\rangle = \omega\langle 1,0\rangle = k$$

and

$$\varphi\langle 0,1\rangle = \omega(1 \times j)\langle 0,1\rangle = \omega\langle 0,j\rangle = l.$$

■

Corollary 4.8. *If (1) is a star-multiplicative S-reflexive graph, $X = Ker(d)$ is a S-special object and the square* $\begin{tikzcd} P(d,ck) \arrow[r, "\xi"] \arrow[d, "p_1"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$ *is a pullback, then $[X,Y] = 0$, where $Y = Ker(c)$.*

The converses of Proposition 4.7 and Corollary 4.8 seem to be false in general. In order to have that a S-reflexive graph with commuting kernels is star-divisible (or star-multiplicative), we need stronger hypotheses. First we need to introduce a generalisation of the notion of S-special morphism.

Definition 4.9. *A morphism $f\colon A \to B$ in a S-protomodular category $\mathbb{C}$ is said to be* t-special*, where $t\colon Z \to A$ is another morphism in $\mathbb{C}$, if the upper split epimorphism in the following pullback belongs to S:*

$$\begin{tikzcd} P \arrow[r, shift left, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[l, shift left, "\langle t,1\rangle"] \arrow[d, "ft"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Observe that, even when f has a section s such that $(f, s) \in S$, it is not necessarily t-special, because the section $\langle t, 1\rangle$ is not the one induced by s through the universal property of the pullback. A morphism $f\colon A \to B$ is S-special if and only it is 1_A-special.

Proposition 4.10. *Let (1) be a S-reflexive graph such that also the point (c, e) belongs to S, X is S-special, d is l-special and c is k-special, where $k\colon X \to C_1$ and $l\colon Y \to C_1$ are kernels of d and c, respectively. If $[X, Y] = 0$, then (1) is star-divisible (and hence star-multiplicative, too) and the square*

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

is a pullback.

Proof: We denote by $P(c, ck)$ and $P(d, dl)$ the following pullbacks:

$$\begin{tikzcd} P(c,ck) \arrow[r, "p_2^c"] \arrow[d, "p_1^c"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} P(d,dl) \arrow[r, "p_2^d"] \arrow[d, "p_1^d"'] & Y \arrow[d, "dl"] \\ C_1 \arrow[r, "d"'] & C_0. \end{tikzcd}$$

First we observe that $c\varphi = ck\pi_X$, where $\varphi\colon X \times Y \to C_1$ is the cooperator. This is easily seen by precomposing both sides with $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$

(that are jointly strongly epimorphic). Thus we obtain an induced morphism $\langle\varphi,\pi_X\rangle\colon X\times Y\to P(c,ck)$. It is actually a morphism of split extensions:

$$\begin{tikzcd}
Y \arrow[r, "{\langle 0,1\rangle}"] \arrow[d, equal] & X\times Y \arrow[r, shift left, "\pi_X"] \arrow[d, "{\langle\varphi,\pi_X\rangle}"] & X \arrow[l, shift left, "{\langle 1,0\rangle}"] \arrow[d, equal] \\
Y \arrow[r, "{\langle l,0\rangle}"'] & P(c,ck) \arrow[r, shift left, "p_2^c"] & X. \arrow[l, shift left, "{\langle k,1\rangle}"]
\end{tikzcd}$$

Since, by our hypotheses, $(\pi_X,\langle 1,0\rangle)$ and $(p_2^c,\langle k,1\rangle)$ belong to S (the first because product projections are in S, the latter because c is k-special), the Split Short Five Lemma implies that $\langle\varphi,\pi_X\rangle$ is an isomorphism. Similarly we obtain an isomorphism $\langle\varphi,\pi_Y\rangle$:

$$\begin{tikzcd}
X \arrow[r, "{\langle 1,0\rangle}"] \arrow[d, equal] & X\times Y \arrow[r, shift left, "\pi_Y"] \arrow[d, "{\langle\varphi,\pi_Y\rangle}"] & Y \arrow[l, shift left, "{\langle 0,1\rangle}"] \arrow[d, equal] \\
X \arrow[r, "{\langle k,0\rangle}"'] & P(d,dl) \arrow[r, shift left, "p_2^d"] & Y. \arrow[l, shift left, "{\langle l,1\rangle}"]
\end{tikzcd}$$

For simplicity, let us denote $\alpha=\langle\varphi,\pi_X\rangle^{-1}$ and $\beta=\langle\varphi,\pi_Y\rangle^{-1}$. Observe that the following conditions hold:

$$\begin{aligned}
\varphi\alpha &= p_1^c\\
\pi_X\alpha &= p_2^c\\
\varphi\beta &= p_1^d\\
\pi_Y\beta &= p_2^d
\end{aligned}$$

Now we show that in this case, that is when the two kernels commute, the two objects X and Y are isomorphic in a strong sense: there are morphisms $i\colon X\to Y$ and $j\colon Y\to X$ such that $ij=1_Y$, $ji=1_X$, $ckj=dl$ and $ck=dli$. Indeed, defining

$$\begin{aligned}
i &= \pi_Y\alpha\langle eck,1_X\rangle\\
j &= \pi_X\beta\langle edl,1_Y\rangle,
\end{aligned}$$

where $\langle eck,1_X\rangle\colon X\to P(c,ck)$ and $\langle edl,1_Y\rangle\colon Y\to P(d,dl)$ are induced by the universal property of the pullback, we show that $ckj=dl$ by using the fact that $c\varphi=ck\pi_X$ and observing that

$$ckj=ck\pi_X\beta\langle edl,1\rangle=c\varphi\beta\langle edl,1\rangle=cp_1^d\langle edl,1\rangle=cedl=dl;$$

in a similar manner we prove $ck = dli$. To prove $ij = 1_Y$ we first observe that

$$\beta\langle edl, 1_Y\rangle = \alpha\langle edl, j\rangle \tag{3}$$

or, equivalently, that $\langle\varphi, \pi_X\rangle\beta\langle edl, 1_Y\rangle = \langle edl, j\rangle$. Indeed we have

$$\begin{aligned}\langle\varphi, \pi_X\rangle\beta\langle edl, 1_Y\rangle &= \langle\varphi\beta, \pi_X\beta\rangle\langle edl, 1_Y\rangle = \langle p_1^d\langle edl, 1_Y\rangle, \pi_X\beta\langle edl, 1_Y\rangle\rangle \\ &= \langle edl, j\rangle.\end{aligned}$$

From here we observe that composing both sides of the equation (3) with π_Y gives on the one hand 1_Y, while on the other hand we get ij:

$$\pi_Y\beta\langle edl, 1_Y\rangle = p_2^d\langle edl, 1_Y\rangle = 1_Y,$$

$$\pi_Y\alpha\langle edl, j\rangle = \pi_Y\alpha\langle eckj, j\rangle = \pi_Y\alpha\langle eck, 1_X\rangle j = ij.$$

This shows that $ij = 1_Y$, a similar argument can be used to show that $ji = 1_X$. It is now an easy task to check that $\langle 1, i\rangle\colon X \to X \times Y$ is the same as $\alpha\langle eck, 1_X\rangle$, since

$$\pi_X\alpha\langle eck, 1_X\rangle = 1_X$$

and

$$\pi_Y\alpha\langle eck, 1_X\rangle = i$$

and similarly we have $\beta\langle edl, 1_Y\rangle = \langle j, 1_Y\rangle$. Now we can define the star-division as $\omega = \varphi(1 \times i)\colon X \times X \to C_1$. It is in fact a star-division, since

$$\omega\langle 1, 0\rangle = \varphi(1 \times i)\langle 1, 0\rangle = \varphi\langle 1, 0\rangle = k$$

and

$$\omega\langle 1, 1\rangle = \varphi(1 \times i)\langle 1, 1\rangle = \varphi\langle 1, i\rangle = \varphi\alpha\langle eck, 1_X\rangle = p_1^c\langle eck, 1_X\rangle = eck.$$

Finally, to prove that the square

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

is a pullback we use the facts that i is an isomorphism, the split epimorphisms $(\pi_1, \langle 1, 1\rangle)$ and $(p_2^c, \langle eck, 1\rangle)$ are both in S (the first because X is S-special, the second because (c, e) is in S) and the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\langle 0,1\rangle"] \arrow[d, "i"'] & X \times X \arrow[r, shift left, "\pi_1"] \arrow[d, "\langle \omega,\pi_1\rangle"] & X \arrow[l, shift left, "\langle 1,1\rangle"] \arrow[d, equal] \\ Y \arrow[r, "\langle l,0\rangle"'] & P(c, ck) \arrow[r, shift left, "p_2^c"] & X \arrow[l, shift left, "\langle eck,1\rangle"] \end{tikzcd}$$

to show that $\langle \omega, \pi_1 \rangle$ is an isomorphism. ■

5. Variations on the “Smith is Huq” condition

Before proving that, in every category of monoids with operations, the (SH) condition, relatively to S-equivalence relations (condition (b) of Theorem 4.2), is satisfied, we study some variations of this condition. The first one (condition (i) in Theorem 5.1 below) was already considered, in the absolute case (i.e. when S is the class of all points) in [9].

Let $\mathbb{C}$ be a pointed finitely complete category. For any object $X \in \mathbb{C}$ the kernel functor

$$\mathrm{Ker}_X \colon Pt_X(\mathbb{C}) \to \mathbb{C}$$

is the functor associating, with every point, its kernel.

Theorem 5.1. *Let $\mathbb{C}$ be a S-protomodular category. Consider the following conditions:*

(i) *For every $X \in \mathbb{C}$ the kernel functor $Ker_X \colon Pt_X(\mathbb{C}) \to \mathbb{C}$ reflects the commutativity of normal subobjects, whenever the domains of the two subobjects are points belonging to S;*

(ii) *two S-equivalence relations centralise each other as soon as their normalisations commute.*

We have that condition (i) implies condition (ii).

Proof: As we observed in Section 3, two S-equivalence relations R and W on the same object X can be seen as two subobjects of $(\pi_1, \Delta_X) \colon X \times X \rightleftarrows X$ in $Pt_X(\mathbb{C})$:

$$\begin{tikzcd} R \arrow[r, tail, "(r_2,r_1)"] \arrow[dr, shift left, "r_2"] & X \times X \arrow[d, shift right, "\pi_1"'] & W \arrow[l, tail, "(w_1,w_2)"'] \arrow[dl, shift right, "w_1"'] \\ & X. \arrow[ul, shift left, "e_R"] \arrow[u, shift right, "\Delta_X"'] \arrow[ur, shift right, "e_W"'] & \end{tikzcd}$$

Observe that, being R an equivalence relation (and hence symmetric), the points (r_1, e_R) and (r_2, e_R) are isomorphic. Since the first is in S, the second is in S, too. Moreover, r_1 and r_2 have isomorphic kernels. The relations R and W as above are actually normal subobjects of (π_1, Δ_X), indeed they are the normalisations of the following equivalence relations in $Pt_X(\mathbb{C})$:

$$\begin{tikzcd}
X \times R \arrow[rr, shift left=2, "1_X \times r_2"] \arrow[rr, shift right=2, "1_X \times r_1"'] \arrow[dr, shift left, "\pi_X"'] & & X \times X \arrow[ll] \arrow[rr] \arrow[d, shift right, "\pi_1"'] & & X \times W \arrow[ll, shift right=2, "1_X \times w_2"'] \arrow[ll, shift left=2, "1_X \times w_1"] \arrow[dl, shift left, "\pi_X"] \\
& & X. \arrow[ul, shift left, "\langle 1_X, e_R \rangle"'] \arrow[u, shift right, "\Delta_X"'] \arrow[ur, shift left, "\langle 1_X, e_W \rangle"] & &
\end{tikzcd}$$

Recalling that two S-equivalence relations on X centralise each other if and only if, when they are seen as subobjects in $Pt_X(\mathbb{C})$, they commute, the fact that condition (i) implies (ii) is immediate. ■

Since the kernel functors always preserve the commutativity of any pair of subobjects, it is easy to see that condition (i) above is equivalent to the condition that, for every morphism $p\colon E \to B$ in $\mathbb{C}$, the pullback functor

$$p^*\colon Pt_B(\mathbb{C}) \to Pt_E(\mathbb{C}),$$

which sends every split epimorphism over B into its pullback along p, reflects the commutativity of normal subobjects, whenever the domains of the two subobjects are points belonging to S.

It was proved in [6], Theorem 2.4.6, that, in the category Mon of monoids, the kernel functors reflect the commutativity not only of pairs of normal subobjects, but of any pair of morphisms, provided that their domains are points belonging to the class S of Schreier split epimorphisms. This fact, which obviously implies both conditions in the previous theorem, is true in any category of monoids with operations. Before proving this, we reformulate the above mentioned condition about the reflection of commutativity by the kernel functors in other terms, following the same idea as in Proposition 4.1 in [19].

Proposition 5.2. *Let $\mathbb{C}$ be a S-protomodular category. The following conditions are equivalent:*

(i) *For every $X \in \mathbb{C}$ the kernel functor $Ker_X\colon Pt_X(\mathbb{C}) \to \mathbb{C}$ reflects the commutativity of any pair of morphisms, whenever their domains are points belonging to S;*

(ii) *given any diagram of the form*

$$\begin{tikzcd}
A \arrow[r, shift right, "f"'] \arrow[dr, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[r, shift left, "s"] \arrow[d, "\beta"] & C \arrow[l, shift left, "g"] \arrow[dl, "\gamma"] \\
& D, &
\end{tikzcd} \tag{4}$$

where (f, r) and (g, s) are points in S, and $\alpha r = \beta = \gamma s$, if the morphisms αk and γl commute (where $k\colon X \to A$ is a kernel of f, and $l\colon Y \to C$ is a kernel of g), there exists a morphism $\varphi\colon A \times_B C \to D$, where $A \times_B C$ is the following pullback

$$\begin{tikzcd}
A \times_B C \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \\
A \arrow[r, shift right, "f"'] \arrow[u, shift right, "e_1"'] & B, \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$.

Proof: Condition (i) is the particular case of (ii), where the morphism β in diagram (4) has a retraction p making α and γ morphisms of points. The only thing that needs to be proved is that the morphism φ is a morphism of points, whose domain is $fp_1 = gp_2\colon A \times_B C \to B$, with section $e_1 r = e_2 s$. We first observe that

$$p\varphi e_1 = p\alpha = f = fp_1 e_1,$$

and, similarly, $p\varphi e_2 = fp_1 e_2$. Being e_1 and e_2 jointly strongly epimorphic, we get that $p\varphi = fp_1 = gp_2$. Moreover

$$\varphi e_1 r = \alpha r = \beta.$$

This proves that (ii) implies (i). To prove the converse, we rewrite diagram (4) as:

$$\begin{tikzcd}
A \arrow[r, "{\langle \alpha, f \rangle}"] \arrow[dr, shift left, "f"] & D \times B \arrow[d, shift right, "\pi_B"'] & C \arrow[l, "{\langle \gamma, g \rangle}"'] \arrow[dl, shift left, "g"] \\
& B, \arrow[ul, shift left, "r"] \arrow[u, shift right, "{\langle \beta, 1 \rangle}"'] \arrow[ur, shift left, "s"] &
\end{tikzcd}$$

and we look at it as a cospan $(\langle \alpha, f\rangle, \langle \gamma, g\rangle)$ in $Pt_B(\mathbb{C})$. Since (f,r) and (g,s) belong to S, it is immediate to see that (i) implies (ii). ∎

Corollary 5.3. *If the equivalent conditions of Proposition 5.2 are satisfied, then all the conditions of Theorem 4.2 are satisfied.*

Proof: We only need to prove Condition (a), since the others are implied by it. In order to do that, consider a S-reflexive graph (1) such that both points (d,e) and (c,e) belong to S and the kernels of d and c commute. Then Condition (ii) of Proposition 5.2, applied to the diagram

$$\begin{tikzcd} C_1 \arrow[r, shift right, "d"'] \arrow[dr, "1_{C_1}"'] & B \arrow[l, shift right, "e"'] \arrow[r, shift left, "e"] \arrow[d, "e"] & C \arrow[l, shift left, "c"] \arrow[dl, "1_{C_1}"] \\ & C_1, & \end{tikzcd}$$

gives the desired multiplication. ∎

In order to prove that in every category of monoids with operations the equivalent conditions of Proposition 5.2 are satisfied, we need the following.

Lemma 5.4. *If* $\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, shift right, dotted, "q_f"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "s"'] \end{tikzcd}$ *is a Schreier split epimorphism in a category of monoids with operations, the Schreier retraction* q_f *satisfies the following equalities:*

(a) $kq_f(s(b)+k(x))+s(b) = s(b)+k(x)$ *for any* $b \in B$ *and any* $x \in X$;
(b) $q_f(a+a') = q_f(a) + q_f(sf(a)+q_f(a'))$;
(c) $q_f(a * a') = q_f(a) * q_f(a') + sf(a) * q_f(a') + q_f(a) * sf(a')$
for any binary operation $* \in \Omega_2$, *different from the monoid operation* $+$.

Proof: The equalities (a) and (b) were proved in [6], Proposition 2.1.5. The equality (c) was proved, in the particular case of semirings, in [6], Proposition 6.0.11; the proof of the general case is completely analogous. Observe that $sf(a)*q_f(a')$ and $q_f(a)*sf(a')$ belong to the kernel of f, so they are elements of X. ∎

Proposition 5.5. *Let $\mathbb{C}$ be a category of monoids with operations. Consider the following diagram in $\mathbb{C}$:*

$$
\begin{tikzcd}
& & Y \arrow[d, shift right, "l"'] & \\
& A\times_B C \arrow[d, shift right, "p_1"'] \arrow[r, shift right, "p_2"'] & C \arrow[l, shift right, "e_2"'] \arrow[u, dotted, shift right, "q_g"'] \arrow[d, shift right, "g"'] \arrow[ddr, bend left, "\gamma"] & \\
X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q_f"'] \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] \arrow[drr, bend right, "\alpha"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"'] \arrow[dr, "\beta"] & \\
& & & D,
\end{tikzcd}
$$

where (f,r) and (g,s) are Schreier split epimorphisms, with kernels k and l, respectively, and $\alpha r = \beta = \gamma s$. If αk and γl commute, then there exists a morphism $\varphi\colon A\times_B C \to D$ such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$.

Proof: We first observe that saying that αk and γl commute means that, for every $x \in X$ and every $y \in Y$, $\alpha k(x) + \gamma l(y) = \gamma l(y) + \alpha k(x)$, and $\alpha k(x) * \gamma l(y) = \gamma l(y) * \alpha k(x) = 0$ for any other binary operation $* \in \Omega_2$. The morphism φ we are looking for is defined by:

$$\varphi(a,c) = \alpha k q_f(a) + \gamma(c).$$

It is easy to see that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$, indeed:

$$\varphi e_1(a) = \varphi(a, sf(a)) = \alpha k q_f(a) + \gamma s f(a) = \alpha(k q_f(a) + r f(a)) = \alpha(a),$$

and

$$\varphi e_2(c) = \varphi(rg(c), c) = \alpha k q_f r g(c) + \gamma(c) = \gamma(c),$$

since $q_f r = 0$ ([6], Proposition 2.1.5). It remains to prove that φ is a morphism. We have:

$$\varphi((a,c)+(a',c')) = \varphi(a+a', c+c') = \alpha k q_f(a+a') + \gamma(c+c') =$$
$$= \alpha k(q_f(a) + q_f(rf(a) + kq_f(a'))) + \gamma(c) + \gamma(c') =$$
$$= \alpha k q_f(a) + \alpha k(q_f(rf(a) + kq_f(a'))) + \gamma l q_g(c) + \gamma s g(c) + \gamma(c')$$

Since αk and γl commute, the last expression is equal to:

$$\alpha k q_f(a) + \gamma l q_g(c) + \alpha k(q_f(rf(a) + kq_f(a'))) + \alpha r f(a) + \gamma(c') =$$
$$= \alpha k q_f(a) + \gamma l q_g(c) + \alpha(k(q_f(rf(a) + kq_f(a'))) + rf(a)) + \gamma(c') =$$
$$= \alpha k q_f(a) + \gamma l q_g(c) + \alpha(rf(a) + kq_f(a')) + \gamma(c') =$$

$$= \alpha k q_f(a) + \gamma l q_g(c) + \gamma s g(c) + \alpha k q_f(a') + \gamma(c') =$$
$$= \alpha k q_f(a) + \gamma(c) + \alpha k q_f(a') + \gamma(c') = \varphi(a,c) + \varphi(a',c')$$

If $*$ is any other binary operation, we have:

$$\varphi(a,c) * \varphi(a',c') = (\alpha k q_f(a) + \gamma(c)) * (\alpha k q_f(a') + \gamma(c')) =$$
$$= \alpha k q_f(a) * \alpha k q_f(a') + \gamma(c) * \alpha k q_f(a') + \alpha k q_f(a) * \gamma(c') + \gamma(c) * \gamma(c') =$$
$$= \alpha k q_f(a) * \alpha k q_f(a') + \gamma(l q_g(c) + s g(c)) * \alpha k q_f(a') +$$
$$+ \alpha k q_f(a) * \gamma(l q_g(c') + s g(c')) + \gamma(c) * \gamma(c')$$

Using the distributivity of $*$ w.r.t. $+$ and the fact that αk and γl commute, which gives that $\gamma l q_g(c) * \alpha k q_f(a') = 0 = \alpha k q_f(a) * \gamma l q_g(c')$, we obtain that the last expression is equal to:

$$\alpha k q_f(a) * \alpha k q_f(a') + \gamma s g(c) * \alpha k q_f(a') + \alpha k q_f(a) * \gamma s g(c') + \gamma(c) * \gamma(c') =$$
$$= \alpha k q_f(a) * \alpha k q_f(a') + \alpha r f(a) * \alpha k q_f(a') + \alpha k q_f(a) * \alpha r f(a') + \gamma(c) * \gamma(c') \overset{(\bullet)}{=}$$
$$= \alpha k (q_f(a) * q_f(a') + r f(a) * q_f(a') + q_f(a) * r f(a')) + \gamma(c) * \gamma(c') =$$
$$= \alpha k q_f(a * a') + \gamma(c) * \gamma(c') = \varphi(a * a', c * c') = \varphi((a,c) * (a',c'))$$

where the equality $(\bullet)$ holds because $rf(a) * q_f(a')$ and $q_f(a) * rf(a')$, being in the kernel of f, can be seen as elements of X. This completes the proof. ∎

Combining the previous proposition with Corollary 5.3, we get the following

Corollary 5.6. *In every category of monoids with operations, two Schreier equivalence relations centralise each other as soon as their normalisations commute.*

6. Consequences of the "Smith is Huq" condition

As a consequence of the "Smith is Huq" condition for S-equivalence relations, we mention the following fact, concerning the characterisation of internal categories and internal groupoids. The following proposition was proved in [8] (Proposition 7.5 there):

Proposition 6.1. *Let $\mathbb{C}$ be a S-protomodular category. Consider a S-reflexive graph in $\mathbb{C}$ such that d is a S-special morphism:*

$$C_1 \underset{c}{\overset{d}{\underset{\xleftarrow{e}}{\rightrightarrows}}} C_0.$$

The following conditions are equivalent:

(1) *the graph is underlying an internal S-category;*
(2) *the graph is underlying an internal S-groupoid;*
(3) *the kernel equivalence relations of d and c centralise each other.*

If $\mathbb{C}$ is a S-protomodular category in which the "Smith is Huq" condition for S-equivalence relations is satisfied, the previous propositions can be reformulated in the following way:

Proposition 6.2. *Let $\mathbb{C}$ be a S-protomodular category in which the "Smith is Huq" condition for S-equivalence relations is satisfied. Consider a S-reflexive graph in $\mathbb{C}$ such that d is a S-special morphism:*

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} C_0.$$

The following conditions are equivalent:
(1) *the graph is underlying an internal S-category;*
(2) *the graph is underlying an internal S-groupoid;*
(3) *the kernels of d and c commute.*

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