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STILL MORE ABOUT SUBFITNESS

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Dedicated to Eva Colebunders on the occasion of her 65th birthday

ABSTRACT: Several features of subfitness are analyzed. Isbell's Spatiality Theorem leads to the concept of T_1-spatiality which is compared with the T_D-spatiality. Subfitness is put into relation with other weak separation axioms (and with the existence of nearness), both in spaces and locales, and seeming discrepancies are explained. A new characteristic of subfitness by means of codensity is presented.

KEYWORDS: Frame (locale), sublocale, sublocale lattice, subfitness, T_1-spatiality, T_D-spatiality, codensity, nearness.

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Introduction

In a previous paper ([15]), continuing in a way a paper by Simmons ([19]), we analyzed several features of the subfitness axiom. Defined by Isbell in [9], it was right away dismissed because of its unsatisfactory categorial behaviour. But a categorial behaviour of an isolated concept is not everything and one can show that subfitness is in fact a very useful one. In [15] we presented

- several results on subfitness as a supportive property (e.g. in the context of weak Hausdorff properties it not only makes one of them conservative, as shown in [5] — see also [4]; moreover, augmented by subfitness many weak Hausdorff properties coincide),
- some features of the relation between subfitness and fitness,
- a related concept of prefitness and a new characteristic of fitness.

When preparing a talk for the ACT V Conference (Brussels, September 2014), aiming to advertise the usefulness of subfitness we obtained some further new facts and insights we would like to present here.

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A brief analysis of the Isbell's spatialization theorem (the only feature of subfitness the author seemed to really like) leads us to a concept of T_1-spatiality. This, a.o. compared with the concept of T_D-spatiality ([3]), is a subject of Section 2.

In Section 3 we discuss the relation of subfitness with some other weak separation axioms, in particular with the symmetry, also in the context of spaces where the subfitness, although not often employed, is also relevant ([6]).

In Section 4 we discuss the *codensity* (in analogy with density, a subspace resp. sublocale is codense if the intersection with each non-void *closed* subobject is non-void). In particular we show that a locale is subfit iff it has no non-trivial codense subobject.

Comparing the results for spaces and for locales we encounter several seeming discrepancies. They are explained in Section 5 scrutinizing the relations of induced sublocales and subspaces in spatial frames.

Originally, the authors' intention was just to supplement the previous paper [15] by the facts appearing in the ACT V talk as mentioned above. It should be stated, however, that while doing this we have encountered new problems (and re-encountered an old, unsolved, one) deserving further investigation. The subject is not exhausted.

1. Preliminaries

1.1. In a poset $(X, \leq)$ we use the standard notation ${\downarrow}A = \{x \mid \exists a \in A,\ x \leq a\}$ and similarly ${\uparrow}A$. The suprema (joins) are denoted, as usual, by $\vee$, $\bigvee$, and infima (meets) by $\wedge$, $\bigwedge$. The elements 0 resp. 1 is the smallest resp. the largest element, and a^* is the pseudocomplement of a (if it exists).

1.2. A *frame* resp. *co-frame* is a complete lattice L satisfying the distributive law

$$a \wedge (\bigvee B) = \bigvee\{a \wedge b \mid b \in B\} \quad \text{resp.} \quad a \vee (\bigwedge B) = \bigwedge\{a \vee b \mid b \in B\}$$

for all $a \in L$ and $B \subseteq L$. Thus, the mappings $(x \mapsto x \wedge b) : L \to L$ preserve suprema and hence we have the right Galois adjoints $(x \mapsto (b \to x)) : L \to L$,

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c,$$

making the frame a Heyting algebra.

A typical frame is the lattice

$$\Omega(X)$$

of all open sets of a topological space X. A *frame homomorphism* $h\colon L \to M$ preserves all joins and finite meets; if $f\colon X \to Y$ is a continuous map we have a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}[U]$.

1.2.1. The Ω above is a contravariant functor from the category **Top** of topological spaces into the category of frames, **Frm**. It becomes covariant if we consider the opposite category

$$\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}.$$

It is of advantage to treat the category **Loc** as a concrete one with the opposite arrows to frame homomorphisms $h : L \to M$ represented as their right Galois adjoints $f : M \to L$; these will be referred to as *localic maps*. Emphasizing this point of view we often speak of frames as of *locales*.

1.2.2. The functor $\Omega : \mathbf{Top} \to \mathbf{Loc}$ has a right adjoint $\Sigma : \mathbf{Loc} \to \mathbf{Top}$ called the *spectrum*. For us it will be of advantage to use the description of ΣL as the set $\{p \in L \mid p \text{ prime}\}$ endowed with the topology $\{\Sigma_a \mid a \in L\}$ where $\Sigma_a = \{p \mid a \nleq p\}$.

1.3. One thinks of a frame L as of a generalized space. One of several representations of a (generalized) subspace of L is that of a *sublocale*. It is a subset $S \subseteq L$ such that

(S1) $M \subseteq S \Rightarrow \bigwedge M \in S$, and
(S2) $x \in L, s \in S \Rightarrow x \to s \in S$.

S is a frame in the order of L and inherits its Heyting structure, and the embedding $j_S\colon S \subseteq L$ is a localic map. The corresponding frame homomorphism (the left adjoint)

$$\nu_S\colon L \to S$$

is given by $\nu_S(x) = \bigwedge\{s \in S \mid s \geq x\}$. The system of all sublocales constitutes a co-frame

$$\mathcal{S}(L)$$

with the order given by inclusion, meet coinciding with the intersection and the join given by

$$\bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\};$$

the top is L and the bottom is the set $\mathsf{O} = \{1\}$; the latter, representing the void subspace, will be referred to as the void, or empty, sublocale.

Another representation of sublocales we will sometimes use is that by frame congruences $E = \{(a, b) \in L \times L \mid \nu_S(a) = \nu_S(b)\}$.

1.3.1. Open resp. closed subspaces are represented by open resp. closed sublocales

$$\mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \text{ resp. } \mathfrak{c}(a) = \uparrow a = \{x \mid x \geq a\}.$$

$\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other. Here are a few rules (see e.g. [13, 12]):

- $\mathfrak{o}(0) = \mathsf{O}, \mathfrak{o}(1) = L, \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b), \mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$,
- $\mathfrak{c}(0) = L, \mathfrak{c}(1) = \mathsf{O}, \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b), \mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$,
- $\mathfrak{o}(a) \cap \mathfrak{c}(b) \neq \mathsf{O}$ iff $a \nleq b$, and $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ iff $a \vee b = 1$.

1.3.2. Similarly like in spaces and subspaces,

open sublocales in a sublocale S are the $\mathfrak{o}_S(\nu_S(a)) = S \cap \mathfrak{o}(a)$, and similarly we have the closed sublocales of S, $\mathfrak{c}_S(\nu_S(a)) = S \cap \mathfrak{c}(a)$.

Due to (S1) one has an extremely simple formula for the closure $\overline{S}$ (the smallest closed sublocale containing S):

$$\overline{S} = \uparrow \bigwedge S.$$

1.3.3. Observation. $\mathfrak{o}(a) \cap S \neq \mathsf{O}$ *iff* $\mathfrak{o}(a) \cap \overline{S} \neq \mathsf{O}$ (since $\mathfrak{o}(a) \cap S = \mathsf{O}$ iff $S \subseteq \uparrow a$ iff $\overline{S} \subseteq \uparrow a$).

1.3.4. An important property of a complemented S is that for any system T_i, $i \in I$, of sublocales one has

$$S \cap \bigvee T_i = \bigvee (S \cap T_i) \tag{1.3.4}$$

(note that this is exceptional: $\mathcal{S}(L)$ is a co-frame, not a frame; in fact this law characterizes complementarity — see [13, VI.4.4.3]).

1.4. A frame L is *subfit* if

$$\forall a, b \in L, \quad a \nleq b \ \Rightarrow \ \exists c, \ a \vee c = 1 \neq b \vee c.$$

Equivalently, L is subfit iff

each open sublocale in L is a join of closed ones.

In fact, this is the original definition, the first order formula above came later (see e.g. [18]).

Subfitness is not a hereditary property, but we have

1.4.1. Proposition. *A complemented sublocale of a subfit frame is subfit.*

Proof: Let $\mathfrak{o}_S(a)$ be open in S, thus, by 1.3.2, $\mathfrak{o}_S(a) = \mathfrak{o}(a) \cap S$ (for $a \in S$, $\nu_S(a) = a$). We have $\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(b) \mid \mathfrak{c}(b) \subseteq \mathfrak{o}(a)\}$ and hence, by (1.3.4),

$$\mathfrak{o}_S(a) = \mathfrak{o}(a) \cap S = \bigvee\{\mathfrak{c}(b) \cap S \mid \mathfrak{c}(b) \subseteq \mathfrak{o}(a)\} = \\ = \bigvee\{\mathfrak{c}_S(\nu_S(b)) \mid \mathfrak{c}(b) \subseteq \mathfrak{o}(a)\}. \quad \blacksquare$$

For more about frames see e.g. [10, 13, 12, 16].

1.5. A *cover* of a frame L is a subset $C \subseteq L$ such that $\bigvee C = 1$. For a cover C and an element $x \in L$ set

$$Cx = \bigvee\{c \in C \mid c \wedge x \neq 0\}.$$

A system $\mathcal{C}$ of covers is *admissible* if

$$\forall a \in L,\ a = \bigvee\{x \mid \exists C \in \mathcal{C}, Cx \leq a\}.$$

We say that a cover A *refines* a cover B and write $A \leq B$ if for every $a \in A$ there exists a $b \in B$ such that $a \leq b$.

1.5.1 A *nearness* (see e.g. [2]) on L is an admissible system of covers $\mathcal{A}$ such that

(N1) if $A \in \mathcal{A}$ and $A \leq B$ then $B \in \mathcal{A}$, and

(N2) if $A, B \in \mathcal{A}$ then $A \wedge B = \{a \wedge b \mid a \in A, b \in B\} \in \mathcal{A}$.

This extends the concept of a *regular* nearness, as defined for spaces by Herrlich [7], to the point-free context. In [8] the admissibility was modified to fit the general Herrlich's concept of nearness. This will be discussed in 5.2 below. For a further generalization see [14].

2. Isbell's Spatiality Theorem; T_1-spatiality

2.1. A frame L is *max-bounded* if for each $x \in L$ there is a maximal $p \in L$ such that $x \leq p$.

2.2. T_1-spatial frames. A frame L is *T_1-spatial* if for each $x \in L$,

$$x = \bigwedge\{p \mid p \text{ maximal},\ x \leq p\}.$$

Thus, if L is T_1-spatial then it can be represented as $\Omega(\Sigma_{\max}(L))$ where $\Sigma_{\max}(L)$ is the subspace of $\Sigma(L)$ carried by all the maximal $p \in L$ (note that all the maximal elements are prime, but of course not all the prime elements are maximal).

2.2.1. Note. Obviously the space $\Sigma_{\max}(L)$ is T_1 so that a T_1-spatial frame is representable by a T_1-space. On the other hand, the representability of L as $\Omega(X)$ with a T_1-space X seems to be so far a slightly weaker condition: immediately it implies only representing L by a subspace of $\Sigma(L)$ consisting of mutually incomparable primes. But see 2.6.1 below.

2.3. Proposition. *A max-bounded frame is T_1-spatial iff it is subfit.*

Proof: $\Leftarrow$: Let $a \not\leq b$. Take a c such that $a \vee c = 1 \neq b \vee c$, and a maximal $p \geq b \vee c$. Then $p \not\geq a$ (else $p \geq a \vee c = 1$) so that $a \not\leq p \geq b$.

$\Rightarrow$: Let $a \not\leq b$. Choose a maximal c such that $a \not\leq c \geq b$. Then $a \vee c > c$ and hence by maximality $a \vee c = 1$, and $b \vee c = c \neq 1$. ■

2.3.1. Corollary. (Isbell's Spatiality Theorem) *A compact subfit frame is T_1-spatial.*

(By Zorn's Lemma, a compact frame is obviously max-bounded.)

2.3.2. Note. The representation by maximal elements does not necessarily mean a representation by the whole of the spectrum. Of course, in such a representation of a T_1-space *all* of the maximal elements have to be present. But the spectrum can have more points (there may be non-maximal primes constituting together with the maximal ones the sobrification of the space).

This can easily happen even to a compact space. Consider an infinite set X with the topology of complements of finite sets plus $\emptyset$. It is a compact T_1 (and hence subfit) space, but not a sober one. In the spectrum there is an extra point ω (corresponding to the void set which is in this $\Omega(X)$ prime) such that $\overline{\{\omega\}} = X$ (so that the sobrification is not T_1 !).

On the other hand, the representation of a regular compact frame by the maximal elements coincides with the spectrum, since the space in question is here necessarily sober.

2.4. T_D-spatiality (see [3]). A frame L is T_D-spatial if $L \cong \Omega(X)$ with a T_D-space X.

Write

$$a \lessdot b$$

for immediate precedence, that is, for the situation where $a < b$ and if $a \leq x \leq b$ then either $a = x$ or $x = b$. The following characteristic is in [3].

2.4.1. Proposition. *A frame is T_D-spatial iff for any $a < b$ there are u, v with $a \leq u \lessdot v \leq b$.*

2.4.2. Note. This is how the condition in 2.4.1 implies plain spatiality: Define $h : L \to \{u, v\}$ by setting $h(x) = (x \vee u) \wedge v$. Then because of the immediate precedence we have really $h(x) \in \{u, v\}$, h is obviously a frame homomorphism, and we have $h(u) = u$ and $h(v) = v$.

2.5. A frame is *step-bounded* if for each $a < 1$ there are u, v such that

$$a \leq u \lessdot v.$$

2.5.1. Lemma. *A T_1-spatial frame satisfies the formula from 2.4.1.*

Proof: Let $a < b$. Pick a maximal p such that $a \leq p \not\geq b$. Thus, $a \leq b \wedge p < b$. Let $b \wedge p < x \leq b$. Then $x \not\leq p$ since otherwise $x \leq b \wedge p$; consequently $x \vee p \neq p$, and by maximality $x \vee p = 1$. Thus,

$$b = b \wedge (x \vee p) = (b \wedge x) \vee (b \wedge p) = x \vee (b \wedge p) = x$$

so that $b \wedge p \lessdot b$. ∎

2.5.2. Proposition. *A subfit step-bounded frame is T_1-spatial.*

Proof: For $a < 1$ choose u, v with $a \leq u \lessdot v$. By subfitness there is a c such that $v \vee c = 1 \neq u \vee c$.

We will prove that $u \vee c$ is maximal. Consider an x with $u \vee c < x$. We have

$$u \leq x \wedge v \leq v$$

so that either $u = x \wedge v$ or $v = x \wedge v$. If $u = x \wedge v$ then we have a contradiction

$$x = x \wedge (v \vee c) = (x \wedge v) \vee (x \wedge c) = u \vee (x \wedge c) = (u \vee x) \wedge (u \vee c) < x.$$

Hence $x \wedge v = v$. That is, $x \geq v$, and we see that $x \geq u \vee c \vee v = 1$, and $u \vee c \geq a$ is maximal.

Thus, L is max-bounded and using subfitness again we conclude by 2.3 that it is T_1-spatial. ∎

2.6. Theorem. *The following statements about a frame L are equivalent:*

(1) *L is T_1-spatial.*
(2) *L is T_D-spatial and subfit.*
(3) *L is step-bounded and subfit.*

Proof: (1)⇒(2) follows from 2.3 and 2.5.1. Note that 2.3 is applicable because L, being T_1-spatial, is indeed max-bounded.
(2)⇒(3) is trivial.
(3)⇒(1) is in 2.5.2. ■

Now we can justify the definition in 2.2.

2.6.1. Corollary. *A frame is T_1-spatial iff it is isomorphic to an $\Omega(X)$ with a T_1-space X.*

(⇒ is trivial. On the other hand, an $\Omega(X)$ with a T_1-space X is obviously subfit and a T_D-frame.)

2.6.2. Notes. 1. The equivalence (1)≡(2) corresponds to the (hopefully) standard classical fact that for a space, T_1 is equivalent to T_D & (subfit). ($T_1 \Rightarrow T_D$&(subfit) obviously. Let X be T_D&(subfit) and let $x \neq y$; consider by T_D an open $U \ni x$ such that $U \smallsetminus \{x\}$ is open, and by subfitness a V such that $U \cup V = X$ and $(U \smallsetminus \{x\}) \cup V = X \smallsetminus \{x\}$. Then $x \notin V \ni y$.)
2. On the other hand, the equivalence (1)≡(3) is a new fact characterizing (a special kind of) spatiality by first order formulas in the language of the order.

2.7. Remark. Recall Theorem 3.4 from [8] stating that for a T_1-space X, X is sober iff each T_0-spatial sublocale is T_1 iff it has no Sierpiński sublocale. In view of T_1-spatiality this fact is, perhaps, more transparent. The sobriety of a representation of a T_1-spatial frame amounts to the system of all maximal elements coinciding with the spectrum, that is, with each prime element being maximal.

3. Subfitness and some related separation axioms, in particular symmetry

3.1. Weak subfitness and prefitness. Here are two separation axioms related to subfitness, the *weak subfitness*

$$\forall a > 0\ \exists c,\ c \neq 1 \text{ and } a \vee c = 1, \tag{wsfit}$$

and *prefitness*,

$$\forall a > 0\ \exists c,\ c^* \neq 0 \text{ and } a \vee c = 1. \qquad \text{(prefit)}$$

3.1.1. Notes. 1. Weak subfitness is obtained from (sfit) considering just $b = 0$.

2. Prefitness is more usually formulated by stating that for each $a > 0$ there is an $x \neq 0$ such that $x \prec a$ (that is, $x^* \vee a = 1$). We have chosen an equivalent formulation making it particularly obvious that

(prefit) implies (wsfit).

Although these two concepts look formally very close in actual fact they are worlds apart. The weak subfitness is weaker than subfitness and this is still weaker than T_1. On the other hand, prefitness is already close to regularity (where for $a \not\leq b$ one has an $x \prec a$ such that $x \not\leq b$) and it is not implied by subfitness. What is perhaps more interesting is that prefitness does not imply subfitness either (see [15]).

3.2. Recall 2.3. Since a finite space is T_1 only if it is discrete we immediately obtain that

a finite distributive lattice is a Boolean algebra iff it is subfit.

This is a part of a much more general statement. The point is that the dual of a (finite) distributive lattice is again a (finite) distributive lattice and that, while (wsfit) is in our context a very weak axiom indeed, its dual

$$b < 1 \ \Rightarrow\ \exists c,\ c \neq 0, \text{and } b \wedge c = 0 \qquad \text{(dual.wsfit)}$$

is (again in our context) a very strong one.

3.2.1. Proposition. *A pseudocomplemented distributive lattice* (*in particular, a frame*) *is a Boolean algebra iff it is dually weakly subfit.*

Proof: Suppose the pseudocomplement x^* of some $x \in L$ is not a complement, that is, $x \vee x^* \neq 1$. If we have (dual.wsfit) there is a $c \neq 0$ such that $c \wedge (x \vee x^*) = (c \wedge x) \vee (c \wedge x^*) = 0$, hence $c \wedge x = 0$ so that $c \leq x^*$ and since also $c \wedge x^* = 0$ we have a contradiction $c = 0$. ■

3.2.2. Since a Boolean algebra is fit (indeed regular) we have

Corollary. *For finite frames the subfitness is hereditary.*

(That is, unlike in 1.4.1 we do not need the complementarity.)

3.2.3. Note. Thus, a finite frame that is not subfit (that is, a Boolean algebra) cannot be a sublocale of a *finite* subfit frame. But with infinite extensions the situation is different. Consider the following example.

In the set $\omega + 1 = \{0, 1, \ldots, \omega\}$ take the topology consisting of the empty set and the complement of finite sets that contain ω. The obtained space is easily seen to be subfit, but it contains (a.o.) the Sierpiński space

$$\big(\{0, \omega\}, \{\emptyset, \{\omega\}, \{0, \omega\}\}\big).$$

3.3. Back to subfitness, in particular in spaces. We have the obvious

3.3.1. Fact. *A frame L is subfit if and only if each of its closed sublocales is weakly subfit.*

(If $a \not\leq b$ we have $a \vee b \neq b = 0_{\uparrow b}$ and hence there is a c, $b \leq c \neq 1$ such that $a \vee b \vee c = a \vee c = 1$; on the other hand $b \vee c = c \neq 1$.)

Now let us have a space X. In the definition of weak subfitness consider $a = X \smallsetminus A_1$ and set $A_2 = X \smallsetminus c$. From 3.3.1 we easily infer

3.3.2. Proposition. *A space X is weakly subfit iff for each non-empty closed $A_1 \subset X$ there is a non-empty closed A_2 such that $A_1 \cap A_2 = \emptyset$.*

Consequently, X is subfit if and only if

$$\begin{array}{c}\textit{for every closed } A_1, B \textit{ with } \emptyset \neq A_1 \subset B \textit{ there is a}\\ \textit{closed } A_2 \textit{ such that } \emptyset \neq A_2 \subseteq B \textit{ and } A_1 \cap A_2 = \emptyset.\end{array} \tag{$*$}$$

From the formula $(*)$ we obtain an easy proof of the Isbell-Simmons characterization of subfit spaces.

3.3.3. Theorem. (Isbell-Simmons) *A space is subfit if and only if*

$$\textit{for each } x \in X \textit{ and open } U \ni x \textit{ there is a } y \in \overline{\{x\}} \textit{ with } \overline{\{y\}} \subseteq U. \tag{I-S}$$

Proof: $(*) \Rightarrow$ (I-S): Take an $x \in U$. If $\overline{\{x\}} \not\subseteq U$ set $B = \overline{\{x\}}$, $A = X \smallsetminus U$ and $A_1 = \overline{\{x\}} \cap A$ to obtain $\emptyset \neq A_1 \subset B$ and a closed A_2 such that $\emptyset \neq A_2 \subseteq \overline{\{x\}}$ and $A_1 \cap A_2 = \emptyset$. Then $A_2 \cap A = A_2 \cap \overline{\{x\}} \cap A = \emptyset$, and for $y \in A_2$, $\overline{\{y\}} \subseteq A_2 \subseteq X \smallsetminus A = U$.

(I-S) $\Rightarrow (*)$: Let $\emptyset \neq A_1 \subset B$. Choose an $x \in B \smallsetminus A_1$ and set $U = X \smallsetminus A_1$. Now if $y \in \overline{\{x\}}$ and $\overline{\{y\}} \subseteq U$ then $\overline{\{y\}} \cap A_1 = \emptyset$ and $\overline{\{y\}} \subseteq \overline{\{x\}} \subseteq B$. Set $A_2 = \overline{\{y\}}$. ∎

3.4. The symmetry axiom. Later on we will discuss a general concept of a nearness in frames, closely connected with the subfitness. Recall that in the classical context a space admits a nearness iff it is symmetric in the sense that

$$x \in \overline{\{y\}} \;\Leftrightarrow\; y \in \overline{\{x\}}. \tag{symm}$$

Note. In classical topology a property equivalent to symmetry appeared for spaces already in 1951 ([11]), in the form of the statement (2) in 3.4.1 below, under the name of *weak regularity*. This term is somewhat surprising for a property weaker than T_1 (although stronger than subfitness, see 3.4.2 below).

Since the axiom of symmetry is not often discussed it may be in order to recall a few equivalent characteristics.

An *inflation* of a space X is obtained as follows. Take a disjoint system $\Phi = \{\phi(x) \mid x \in X\}$ of sets and endow the union

$$X\Phi = \bigcup\{\phi(x) \mid x \in X\}$$

with the topology consisting of the open sets

$$U\Phi = \bigcup\{\phi(x) \mid x \in U\}$$

with U open in X.

3.4.1. Proposition. *The following statements on a space X are equivalent.*

(1) *X is symmetric.*
(2) *For every open $U \subseteq X$ and every $x \in U$, $\overline{\{x\}} \subseteq U$.*
(3) *X is an inflation of a T_1-space.*
(4) *Each open subset of X is a union of closed ones.*

Proof: (1)⇒(2): Let $x \in U$ open and $y \in \overline{\{x\}}$. Then $x \in \overline{\{y\}}$ and hence $y \in U$.

(2)⇒(3): Define

$$x \sim y \quad \equiv_{\mathrm{df}} \quad \overline{\{x\}} = \overline{\{y\}}.$$

Then $\sim$ is an equivalence with the equivalence class of x equal to $\overline{\{x\}}$: indeed, if x is not in $\overline{\{y\}}$ then x is in the open $X \smallsetminus \overline{\{y\}}$, hence $\overline{\{x\}} \subseteq X \smallsetminus \overline{\{y\}}$ and $\overline{\{x\}}$ and $\overline{\{y\}}$ are disjoint.

Now we see that

$$X/\!\sim$$

is a T_1-space (the $\sim$-preimage of $(X/\sim) \smallsetminus \{\overline{\{y\}}\}$ is the open $X \smallsetminus \overline{\{x\}}$), and that X is an inflation of $X/\sim$ (with $\phi(\overline{\{x\}}) = \overline{\{x\}}$).

(3)⇒(4): If $X\Phi$ is an inflation of a T_1 space X then each $\phi(x)$ is closed by T_1, and an open U is a union of such sets by the definition of inflation.

(4)⇒(1): Let $x \notin \overline{\{y\}}$. Then $x \in U = X \smallsetminus \overline{\{y\}} = \bigcup_i A_i$ with A_i closed, and for some i, $x \in A_i$, and hence $\overline{\{x\}} \subseteq A_i$ and finally $y \notin \overline{\{x\}}$. ■

3.4.2. Comparing 3.4.1(2) with 3.3.3 we immediately see that

$$\text{(symm) is stronger than (sfit).}$$

It is strictly stronger: take $\mathbb{N} = \{0, 1, 2, \dots\}$ and declare a non-void $U \subseteq \mathbb{N}$ for open if $0 \in U$ and $\mathbb{N} \smallsetminus U$ is finite. The resulting space is subfit but not symmetric.

4. Codensity

4.1. Recall that a subspace Y of a space X (more generally, a sublocale S of a locale L) is dense if for each open $U \neq \emptyset$ (each $a \neq 0$), $U \cap Y \neq \emptyset$ ($\mathfrak{o}(a) \cap S \neq \mathsf{O}$).

Dually, we will say that a subspace Y of a space X (more generally, a sublocale S of a locale L) is *codense* if for each *closed* $A \neq \emptyset$ (each $a \neq 1$), $A \cap Y \neq \emptyset$ ($\mathfrak{c}(a) \cap S \neq \mathsf{O}$).

4.2. A frame homomorphism $h : L \to M$ is said to be codense if $h(a) = 1$ implies $a = 1$.

4.2.1. Proposition. *Let $f : M \to L$ be the localic map associated with a frame homomorphism $h : L \to M$ (that is, the right Galois adjoint of h). Then $f[M]$ is codense in L iff h is codense.*

Proof: We have

$$\mathfrak{c}(a) \cap f[M] = \{f(x) \mid a \le f(x)\} = \{f(x) \mid h(a) \le x\}.$$

Thus, $\mathfrak{c}(a) \cap f[M] \neq \mathsf{O} = \{1\}$ iff $f(h(a)) < 1$ and hence the codensity of $f[M]$ amounts to the implication $a < 1 \ \Rightarrow\ f(h(a)) < 1$, that is, $f(h(a)) = 1 \ \Rightarrow\ a = 1$. Since $1 \le f(h(a))$ iff $1 = h(1) \le h(a)$ this is equivalent to $h(a) = 1 \ \Rightarrow\ a = 1$. ■

4.3. In analogy with the semiclosed sublocales from [15],

$$\mathfrak{sc}(a) = \bigwedge\{\mathfrak{o}(u) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(u)\},$$

we define the *semiopen sublocales*

$$\mathfrak{so}(a) = \bigvee\{\mathfrak{c}(u) \mid \mathfrak{c}(u) \subseteq \mathfrak{o}(a)\}.$$

Thus we have

$$\begin{aligned}\mathfrak{so}(a) &= \bigvee\{\mathfrak{c}(u) \mid \mathfrak{c}(u) \cap \mathfrak{c}(a) = \mathsf{O}\}\\ &= \bigvee\{\mathfrak{c}(u) \mid \mathfrak{c}(u \vee a) = \mathsf{O}\} = \bigvee\{\mathfrak{c}(u) \mid u \vee a = 1\}.\end{aligned}$$

These equalities are obvious and we will use any of the formulas without further mentioning.

4.4. Proposition. *In every frame L, $\mathfrak{so}(a) \wedge \mathfrak{c}(a) = \mathsf{O}$ and $\mathfrak{so}(a) \vee \mathfrak{c}(a)$ is codense.*

Proof: The first equality follows from the fact that $\mathfrak{so}(a) \subseteq \mathfrak{o}(a)$. Now let $\mathfrak{c}(b) \neq \mathsf{O}$ (that is, $b \neq 1$). If $\mathfrak{c}(b) \wedge \mathfrak{c}(a) = \mathsf{O}$ then $b \vee a = 1$ and $\mathfrak{c}(b) \subseteq \mathfrak{so}(a)$. ∎

4.5. Proposition. *The following statements about a frame L are equivalent:*

(1) *L is subfit.*
(2) *Every codense sublocale $S \subseteq L$ is equal to L.*
(3) *For every $a \in L$, $\mathfrak{so}(a) = \mathfrak{o}(a)$.*
(4) *Each open sublocale in L is a join of closed ones.*

Proof: (1)⇒(2): Let S be codense and let $a \in L$. Set $s = \bigwedge\{t \mid t \in S,\ a \leq t\}$. Suppose $s \vee c = 1$ and $a \vee c \leq t \in S$. Then $a \leq t$ and hence $s \leq t$, and since $c \leq t$ we have $1 = s \vee c \leq t$. By codensity, $a \vee c = 1$, so that $s \vee c = 1$ implies $a \vee c = 1$ and by subfitness $s \leq a$ and $a = s \in S$.

(2)⇒(3): By 4.4, $\mathfrak{so}(a) \wedge \mathfrak{c}(a) = \mathsf{O}$ and $\mathfrak{so}(a) \vee \mathfrak{c}(a) = L$ so that $\mathfrak{so}(a) = \mathfrak{o}(a)$, the unique complement.

(3)⇒(1): Let (3) hold and let $a \not\leq b$. Then $\mathfrak{c}(b) \not\subseteq \mathfrak{c}(a)$ and since $\mathfrak{c}(b) = \mathfrak{c}(b) \cap (\mathfrak{so}(a) \vee \mathfrak{c}(a))$ we have $\mathfrak{c}(b) \cap \mathfrak{so}(a) \neq \mathsf{O}$ and hence for some c with $c \vee a = 1$, $\mathfrak{c}(b) \cap \mathfrak{c}(c) \neq \mathsf{O}$, that is, $b \vee c \neq 1$.

(4): this standard characteristic of subfitness is a trivial reformulation of (3). ∎

4.6. Here is a related characteristic of weak subfitness (where $\mathrm{int}(S)$ denotes the interior of a sublocale S, that is, the largest open sublocale contained in S); slightly surprisingly, it comes in terms of semiclosed sublocales rather than in terms of semiopen ones.

Proposition. *A frame L is weakly subfit iff* $\mathrm{int}(\mathfrak{sc}(a) \wedge \mathfrak{o}(a)) = \mathsf{O}$ *for every* $a \in L$.

Proof: Assume L is weakly subfit and let $\mathfrak{o}(b) \subseteq \mathfrak{sc}(a) \wedge \mathfrak{o}(a)$. Then $b \leq a$ and

$$\mathfrak{o}(u) \geq \mathfrak{c}(a) \Rightarrow \mathfrak{o}(u) \geq \mathfrak{o}(b),$$

that is,

$$u \vee a = 1 \Rightarrow u \geq b. \quad (*)$$

Therefore, by weak subfitness, $b > 0$ would imply the existence of an $u \neq 1$ such that $b \vee u = 1$, which is absurd: by $(*)$, from $1 = b \vee u \leq a \vee u$ would follow $u \geq b$ and then $1 = u \vee b = u$.

Hence $b = 0$.

Conversely, let $a > 0$. Since $\mathrm{int}(\mathfrak{sc}(a) \wedge \mathfrak{o}(a)) = \mathsf{O}$, we have in particular that $\mathfrak{o}(a) \nsubseteq \mathfrak{sc}(a)$. This means that there is some u such that $\mathfrak{c}(a) \subseteq \mathfrak{o}(u)$ (i.e., $a \vee u = 1$) and $\mathfrak{o}(a) \nsubseteq \mathfrak{o}(u)$ (i.e., $a \nleq u$ and so $u \neq 1$). ∎

4.7. Proposition. *Any codense sublocale of a normal frame is normal.*

Proof: Let $j_S \colon S \subseteq L$ be a codense sublocale of a normal frame L. We use the symbol $\sqcup$ for denoting the joins in the frame S. If $a \sqcup b = 1$ in S then $a \vee b = 1$ in L (otherwise, $a \vee b \leq s \neq 1$ for some $s \in S$). By normality, there are $u, v \in L$ satisfying $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$. Then, immediately, $\nu_S(u) \wedge \nu_S(v) = 0$, $a \sqcup \nu_S(u) \geq \nu_S(a) \sqcup \nu_S(u) = 1$ and $b \sqcup \nu_S(v) \geq \nu_S(b) \sqcup \nu_S(v) = 1$. ∎

5. Some seeming paradoxes

5.1. Recall 3.4.1. A space X is *symmetric* iff each open subset of X is a union of closed ones. By 4.5, a frame is *subfit* iff each open sublocale is a join of closed ones.

Strange. But by 3.4.2, subfitness is strictly weaker than symmetry, even for spaces.

5.2. Existence of a nearness. Recall the nearness defined by a system of covers as in 1.5. The admissibility is based on the "cover stars"

$$Au = \bigvee\{a \mid a \in A,\ a \wedge u \neq 0\}$$

of elements of L. This in fact corresponds to the stars of open sets

$$\mathrm{st}(\mathcal{C}, U) = \bigcup\{C \mid C \in \mathcal{C},\ C \cap U \neq \emptyset\},$$

and hence the nearness in 1.5.1 is in fact just an extension of Herrlich's *regular nearness* ([7]), not of the general one, the admissibility of which is based on the stars of points

$$\mathrm{st}(\mathcal{C}, x) = \bigcup\{C \mid C \in \mathcal{C},\ x \in C\}.$$

Now in a frame we do not have (to have) points, but this is not really an obstacle: equivalently, we can base the admissibility of the (general) nearness on stars of subsets, $\mathrm{st}(\mathcal{C}, M) = \bigcup\{C \mid C \in \mathcal{C},\ C \cap M \neq \emptyset\}$, with the same result, and this can be imitated by using the stars of sublocales S,

$$AS = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \cap S \neq \mathsf{O}\}$$

and declaring a system of covers $\mathcal{A}$ (satisfying (N1) and (N2) from 1.5.1) for a *general nearness* on L if

$$\forall a \in L, \quad \mathfrak{o}(a) = \bigvee\{S \in \mathcal{S}(L) \mid \exists A \in \mathcal{A},\ AS \subseteq \mathfrak{o}(a)\}.$$

5.2.1. The extension is correct. General sublocales are, of course, many more than subspaces, but by 1.3.3,

$$\textit{for any cover } A \textit{ and any sublocale } S,\ AS = A\overline{S}$$

so that the admissibility reduces to stars of *closed* sublocales only, and they correspond precisely to closed subsets.

5.2.2. Theorem. *A frame admits a generalized nearness iff it is subfit.*

Proof: By 5.2.1, L admits a generalized nearness iff

$$\forall a \in L, \quad \mathfrak{o}(a) = \bigvee\{\overline{S} \mid S \in \mathcal{S}(L), \exists A \in \mathcal{A},\ A\overline{S} \subseteq \mathfrak{o}(a)\}.$$

Hence each open sublocale is a join of closed ones and L is subfit by 3.4.1.

Conversely, if L is subfit, then $\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(b) \mid \mathfrak{c}(b) \subseteq \mathfrak{o}(a)\}$ (by 3.4.1). Now $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ iff $\mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \vee b) = L$ iff $a \vee b = 1$ which makes $\{a, b\}$ a cover. Hence $\{a, b\}\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ (as $\mathfrak{o}(b) \cap \mathfrak{c}(b) = \mathsf{O}$) and see that each system of covers containing all the finite covers is admissible. ∎

Strange. It is well known that a space admits a nearness iff it is symmetric. But here we have admissible nearnesses on subfit spaces, more general than symmetric ones.

5.3. Proposition. *The following statements about a space X are equivalent:*
(1) *X is T_1.*
(2) *Each subset of X is a union of closed ones.*
(3) *There is no non-trivial codense subspace Y of X.*

Proof: (1)⇒(2) is trivial.
(2)⇒(3): If $Y \neq X$ choose a non-void closed subset $A \subseteq X \smallsetminus Y$ to obtain $A \cap Y = \emptyset$.
(3)⇒(1): If X is not T_1 there is an x with $\overline{\{x\}} \neq \{x\}$. Then for every closed set A containing x we have $A \cap (X \smallsetminus \{x\}) \neq \emptyset$. Hence $X \smallsetminus \{x\}$ is codense. ■

Strange. This may seem even more peculiar than the previous two observations: we are speaking only of subspaces and not of general sublocales, hence we should have the corresponding separation condition weaker, if anything, while we have a stronger one.

5.4. The explanation of the discrepancies above is not hard. The point is in an imperfect pointfree representation of subspaces as sublocales in an "insufficiently separated" space.

For the purposes of the following paragraph it will be of advantage to represent sublocales as (frame) congruences. For a subset $Y \subseteq X$ define a congruence E_Y by setting

$$E_Y = \{(U, V) \mid U, V \in \Omega(X),\ U \cap Y = V \cap Y\}.$$

The correspondence $Y \mapsto E_Y$ is not always one-to-one. To be precise:

the correspondence $Y \mapsto E_Y$ is one-to-one iff X is a T_D-space

(see [1]). However, it is easy to see that

$$T_D \ \&\ (\text{sfit}) = T_1 \quad \text{and} \quad T_D \ \&\ (\text{symm}) = T_1.$$

Hence, our space is either T_D and then (sfit), (symm) and T_1 coincide, or it is not T_D, and then claiming that two sublocales representing subspaces coincide does not say enough on the relation on the subspaces themselves. The latter case is more interesting and we will discuss it in the following subsections.

5.5. Let us say that subsets Y, Z are *congruence-equivalent* (briefly, *cong-equivalent*) and write

$$Y \approx Z$$

if $E_Y = E_Z$. More explicitly

$$Y \approx Z \quad \equiv_{\mathrm{def}} \quad (\forall \text{ open } U, V, \ \ U \cap Y = V \cap Y \text{ iff } U \cap Z = V \cap Z).$$

5.5.1. Observation. *If $Y \approx Z$ and W is open then $W \cap Y \approx W \cap Z$.*

$(U \cap (W \cap Y) = (U \cap W) \cap Y$ iff $(U \cap W) \cap Z = U \cap (W \cap Z)$.)

If X is not T_D we cannot infer the non-existence of a codense subset from subfitness. But we have at least the following

5.5.2. Proposition. *Let Y be codense in a subfit X. Then $Y \approx X$.*

Proof: Suppose $U \cap Y = V \cap Y$ and $U \neq V$, say let there be an $x \in U \smallsetminus V$. Take an $y \in \overline{\{x\}}$ such that $A = \overline{\{y\}} \subseteq U$. Since V is open, $\overline{\{x\}} \cap V = \emptyset$ and hence also $A \cap V = \emptyset$ and $A \cap (V \cap Y) = \emptyset$. On the other hand, $A \cap U \cap Y = A \cap Y \neq \emptyset$, by codensity. ■

The converse holds in every space:

5.5.3. Proposition. *Let Y be a subspace of a space X such that $Y \approx X$. Then Y is codense.*

Proof: $Y \approx X$ means that

$$A \cap Y = B \cap Y \text{ iff } A = B \text{ for every open } A, B.$$

In particular, for $B = X$, we have

$$Y \subseteq A \text{ iff } A = X \text{ for every open } A.$$

Now suppose Y is not codense. Then there is some closed $F \neq \emptyset$ such that $Y \cap F = \emptyset$. Therefore $X \smallsetminus F$ is an open set containing Y. Hence $F = \emptyset$, a contradiction. ■

5.5.4. In a space we have a statement similar to 4.4 for any subspace, not only for an open one. Set

$$\mathfrak{s}(Y) = \bigcup\{A \mid A \text{ closed}, \ A \subseteq Y\}.$$

We have

Lemma. $\mathfrak{s}(Y) \cup (X \smallsetminus Y)$ *is codense.*

Proof: Let C be non-void closed. Consider $\mathfrak{s}(Y) \cup (X \smallsetminus Y) \cap C$. If $C \cap (X \smallsetminus Y) = \emptyset$ then $C \subseteq Y$ and hence $C \subseteq \mathfrak{s}(Y)$. ■

Now the cong-equivalence relation yields a new characteristic of subfit spaces:

5.5.5. Proposition. *A space is subfit iff $U \approx \mathfrak{s}(U)$ for every open U.*

Proof: The implication "⇒" follows from 5.5.4 and 5.5.1. For the converse implication we use the Isbell-Simmons Theorem. So let U be an open set, $x \in U$. The condition $U \approx \mathfrak{s}(U)$ means that

$$A \cap U = B \cap U \text{ iff } A \cap \mathfrak{s}(U) = B \cap \mathfrak{s}(U) \text{ for every open } A, B.$$

In particular, for $B = U$ we have

$$U \subseteq A \text{ iff } \mathfrak{s}(U) \subseteq A \text{ for every open } A. \tag{5.5.5}$$

Applying (5.5.5) to $U \nsubseteq X \smallsetminus \overline{\{x\}}$ we conclude that $\mathfrak{s}(U) \nsubseteq X \smallsetminus \overline{\{x\}}$, that is, $\mathfrak{s}(U) \cap \overline{\{x\}} \neq \emptyset$. Hence there is some $y \in \overline{\{x\}}$ and some closed $F \subseteq U$ containing y. ■

Note. This of course also concerns the admissibility of a nearness in a subfit space.

5.6. Remark. The reader probably knows that the stronger property of fitness is characterized, equivalently, by *closed* sublocales being meets of open ones, and by *all* sublocales being meets of open ones. Comparing 3.4.1(4) and 5.3(2) we see that we cannot expect an analogous strengthening of the characteristic for subfitness: the two conditions are not equivalent even just for spaces. The question naturally arises for which frames all sublocales are joins of closed ones. It turns out that this condition is very prohibitive; for instance it is not satisfied in the compact interval $\langle 0, 1\rangle$ so that it cannot be guaranteed even by a strong separation (moreover supported by compactness) — not to speak of subfitness. It is an interesting topic and it is being investigated. It does not have, however, much to do with subfitness and therefore it does not fit into this article.

References

[1] B. Banaschewski and A. Pultr, *Variants of openness*, Appl. Categ. Structures 2 (1994) 331–350.

[2] B. Banaschewski and A. Pultr, *Cauchy points of uniform and nearness frames*, Quaest. Math. 19 (1996) 101–127.

[3] B. Banaschewski and A. Pultr, *Pointfree aspects of the T_D axiom of classical topology*, Quaest. Math. 33 (2010) 369–385.

[4] C. H. Dowker and D. P. Strauss, *Separation axioms for frames*, Colloq. Math. Soc. Janos Bolyai 8 (1974) 223–240.

[5] C. H. Dowker and D. Strauss, *T_1- and T_2-axioms for frames*, in: Aspects of topology, London Math. Soc. Lecture Note Ser. 93, pp. 325-335, Cambridge Univ. Press, Cambridge, 1985.

[6] J. Gutiérrez García, J. Picado and M. A. de Prada Vicente, *Monotone normality and stratifiability from a pointfree point of view*, Topology and its Applications 168 (2014) 46–65.

[7] H. Herrlich, *A concept of nearness*, Gen. Topology Appl. 5 (1974) 191–212.

[8] H. Herrlich and A. Pultr, *Nearness, subfitness and sequential regularity*, Appl. Categ. Structures 8 (2000) 67–80.

[9] J. R. Isbell, *Atomless parts of spaces*, Math. Scand. 31 (1972) 5–32.

[10] P. T. Johnstone, *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.

[11] K. Morita, *On the simple extension of a space with respect to a uniformity I*, Proc. Japan Acad. 27 (1951) 65–72.

[12] J. Picado and A. Pultr, *Locales treated mostly in a covariant way*, Textos de Matemática, Vol. 41, University of Coimbra, 2008.

[13] J. Picado and A. Pultr, *Frames and Locales: topology without points*, Frontiers in Mathematics, Vol. 28, Springer, Basel, 2012.

[14] J. Picado and A. Pultr, (*Sub*)*Fit biframes and non-symmetric nearness*, Topology Appl. 168 (2014) 66–81.

[15] J. Picado and A. Pultr, *More on subfitness and fitness*, Appl. Categ. Structures, to appear (published online: 13 February 2014).

[16] J. Picado, A. Pultr and A. Tozzi, *Locales*, in: Categorical Foundations - Special Topics in Order, Topology, Algebra and Sheaf Theory (ed. by M.C. Pedicchio and W. Tholen), Encyclopedia of Mathematics and its Applications, Vol. 97, pp. 49–101, Cambridge Univ. Press, Cambridge, 2003.

[17] A. Pultr, *Frames*, in: Handbook of Algebra (ed. by M. Hazewinkel), vol. 3, pp. 791–858, Elsevier, 2003.

[18] H. Simmons, *The lattice theoretic part of topological separation properties*, Proc. Edinburgh Math. Soc. (2) 21 (1978) 41–48.

[19] H. Simmons, *Regularity, fitness, and the block structure of frames*, Appl. Categ. Structures 14 (2006) 1–34.

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