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A NOTE ON THE ROLLING ELLIPSOID AND THE C. NEUMANN PROBLEM

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Abstract: We study rolling maps of the ellipsoid rolling upon its affine tangent space along a geodesic. It is known that the Jacobi geodesic problems is related to the mechanical problem of Carl Neumann. We derive new integrals of motion for this system. We derive a simple formula for the angular velocity of the rolling ellipsoid in terms of the Gauss map. We provide an elementary proof of the Gauss curvature of an ellipsoid, for any dimension.

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1. Introduction

Let $\boldsymbol{\mathcal{E}}^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \langle x, \mathbf{D}^{-2}x\rangle = 1 \right\}$, $\mathbf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_{n+1}) \succ 0$, be an n-ellipsoid isometrically embedded in Euclidean space $\mathbb{R}^{n+1}$, $n > 1$, and let $\mathbb{SE}(n+1)$ denote the special Euclidean group, the group of orientation preserving isometries of the embedding space $\mathbb{R}^{n+1}$. Suppose $\boldsymbol{\mathcal{E}}^n$ rolls, without slip or twist, upon its affine tangent space. Given a piecewise smooth curve γ in $\boldsymbol{\mathcal{E}}^n$, i.e., a map from an interval $I \subset \mathbb{R}$ to $\boldsymbol{\mathcal{E}}^n$, there is a unique rolling map $\boldsymbol{\chi}\colon I \to \mathbb{SE}(n+1)$. The rotational part $\boldsymbol{\chi}_*$ of the rolling map is a group action $\mathbb{SO}(n+1)$, whilst the angular velocity in the *body coordinate system* $(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_*\colon I \to \mathfrak{h} \subset \mathfrak{so}(n+1)$ is a curve in an n-vector sub-space $\mathfrak{h}$ of the Lie algebra $\mathfrak{so}(n+1)$. This paper considers the angular velocity curves in $\mathfrak{h}$ when $\boldsymbol{\mathcal{E}}^n$ rolls along a geodesic.

The problem of geodesics on an ellipsoid has been studied by many mathematicians, notably by Legendre, Bessel and Gauss, to mention just a few names. It wasn't until the XIX century, when C. G. J. Jacobi had shown that Hamilton-Jacobi equations for a general case of a triaxial ellipsoid are separable in the elliptic coordinates. Two constants of motion of a free point in $\boldsymbol{\mathcal{E}}^n$ are well known

$$E = \langle \dot\gamma, \dot\gamma\rangle \quad \text{and} \quad J = \langle \mathbf{D}^{-2}\gamma, \mathbf{D}^{-2}\gamma\rangle \, \langle \mathbf{D}^{-1}\dot\gamma, \mathbf{D}^{-1}\dot\gamma\rangle.$$

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The first one comes from the energy conservative laws and the second one is the celebrated *Joachimsthal integral*. The Jacobi problem of geodesics has been one of those problems that stimulated many areas of mathematics. In his paper [10] Horst Knörrer studied a link between geodesics on quadrics and the Carl Neumann problem, a classical problem of a point moving on the unit sphere in the presence of a quadratic potential. In this paper we investigate a relationship between, what is now called, the Knörrer system of geodesics on the ellipsoid and the rolling maps. As a result we find a new form of integrals of motions of $\boldsymbol{\mathcal{E}}^n$, where $\boldsymbol{\mathcal{E}}^n$ rolls without a slip or twist along geodesics, that involve curves in the vector subspace $\mathfrak{h} \subset \mathfrak{so}(n+1)$.

Various generalisations of the original geodesic problem in $\boldsymbol{\mathcal{E}}^2$ are found in the literature. For example, the *generalised Jacobi* problem studied in [1] considers a point moving on a surface of an ellipsoid $\boldsymbol{\mathcal{E}}^2$ under the influence of a quadratic potential. In their study of the flow generated by the generalised Jacobi problem the authors derive necessary and sufficient conditions for this generalisation to be integrable with a meromorphic first integral. When the ellipsoid is the unit sphere, this problem reduces to C. Neumann. Then, as in the case of geodesic motion, this problem is separable in the elliptic coordinates. On the other hand, if the potential vanishes then the problem is reduced to free motion of a point leading to geodesics and Joachimsthal integral.

New types of geodesics appear in the pseudo-Riemannian geometry [5]. The induced metric from the Minkowski embedding space on the ellipsoid degenerates along two curves called the *tropics*. The induced metric is Riemannian in the *polar caps* and Lorentzian in the *equatorial belt* bounded by the tropics. There are the three types of geodesics γ: space-like (positive energy $\langle \dot\gamma, \dot\gamma \rangle > 0$), time-like (negative energy $\langle \dot\gamma, \dot\gamma \rangle < 0$) or light-like (zero energy or null $\langle \dot\gamma, \dot\gamma \rangle = 0$). In the Minkowski space, the null geodesics separate the space-like geodesics from the time-like geodesics. There exists Joachimsthal integral for this problem.

A class of integrable Hamiltonian systems are studied in [8] through the Maximum Principle. The author considers a class of left-invariant variational problems on a Lie group, whose Lie algebra admits a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$. This framework provides insight to the isospectral methods applicable to the Neumann mechanical and Jacobi geodesic problems.

Fedorov and Jovanović in [4] study multidimensional generalisations of the Veselova problem of a non-holonomic rigid body motion. The authors consider the so called *LR systems* — non-holonomic systems with a left-invariant metric and right-invariant non-holonomic constraints. A particular case of these investigations is the Jacobi problem for geodesics on an ellipsoid.

A good review of related problems can be found in [2]. The authors consider several isomorphisms between Jacobi geodesic problem and integrable cases from rigid body dynamics: Moser, Brun, Chaplygin and the Neumann system. The isomorphism of the Brun and Neumann problems is described. Isomorphism between the Chaplygin ball problem (the rolling of a dynamically asymmetrical balanced ball without slipping on a plane) and the Neumann system.

In this paper we derive the integrals of motion for an ellipsoid rolling along geodesic upon its (affine) tangent space. The notion of rolling maps for Riemannian manifolds is revisited in Section 2. We present some results concerning geometric properties of an ellipsoid in Section 2.2. An elementary proof of the formula for the Gaussian curvature of an ellipsoid is given in Proposition 4. A remarkable simple formula for the angular velocity in the body coordinate system in terms of the rolling curve is given in Corollary 6. Section 3 formulates the Jacobi and the Neumann problems. In Section 3.1 we give a brief description of yet another point of view of the Neumann problem through a Lie algebra with semi-direct product. Our main result, Theorem 11, relates the Uhlenbeck algebraic integrals to the rolling ellipsoid. Section 4 concludes this paper.

2. Background

Consider an n-dimensional Riemannian manifold $\mathbf{M}$ isometrically embedded in $\mathbb{R}^m$, with $1 < n < m$. The embedding defines a Riemannian metric g on $\mathbf{M}$, inherited from the ambient Euclidean metric in $\mathbb{R}^m$. Let $\mathfrak{X}(\mathbf{M})$ be the space of smooth vector fields on $\mathbf{M}$. Then $\nabla_X Y \colon \mathfrak{X}(\mathbf{M}) \times \mathfrak{X}(\mathbf{M}) \to \mathfrak{X}(\mathbf{M})$ is the Riemannian (or Levi-Civita) connection of g. For any smooth curve $\gamma\colon I \to \mathbf{M}$ parameterised by t, where $I \subset \mathbb{R}$ is an interval, let $\mathcal{D}_t V$ denote the covariant derivative of a vector field V along γ, *cf.* [15, page 57].

We now recall a definition and some results concerning rolling maps.

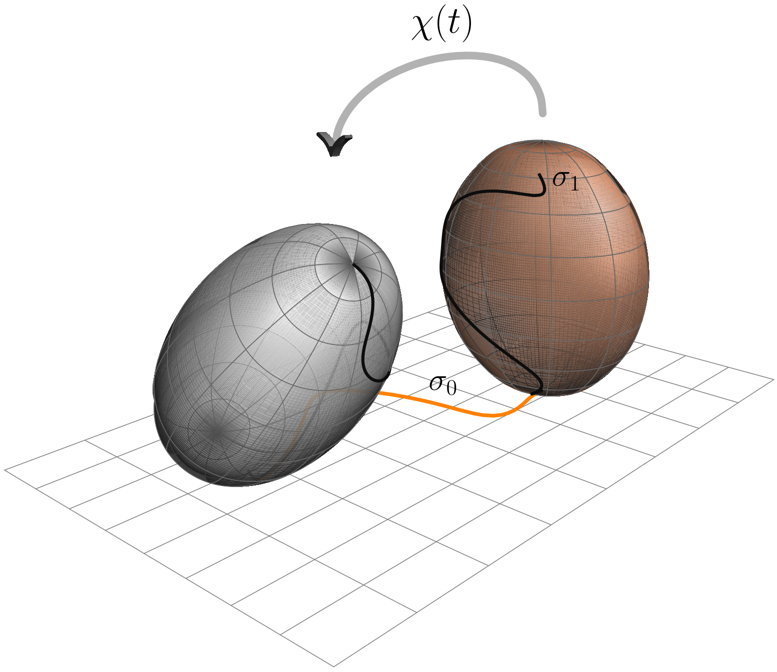

FIGURE 1. The rolling map $\boldsymbol{\chi}$ of an ellipsoid $\boldsymbol{\mathcal{E}}^2$ on a plane along a rolling curve σ_1.

2.1. Rolling maps

We follow Sharpe's definition [19, Appendix B] of a *rolling map* in Euclidean space extended to Riemannian manifolds, as in [6].

Definition 1. *Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be two n-manifolds isometrically embedded in an m-dimensional Riemannian manifold $\mathbf{M}$ and let $\sigma_1 \colon I \to \mathbf{M}_1$ be a piecewise smooth curve in $\mathbf{M}_1$, where $1 < n < m$. A* rolling map *of $\mathbf{M}_1$ on $\mathbf{M}_0$ along σ_1, without slipping or twisting, is a piecewise smooth map $\boldsymbol{\chi} \colon I \to \mathsf{Isom}(\mathbf{M})$, a Lie group of isometries of $\mathbf{M}$, satisfying the following conditions:*

Rolling: *for all $t \in I$,*
(a) $\boldsymbol{\chi}(t)(\sigma_1(t)) \in \mathbf{M}_0$*;*
(b) $\mathbf{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}(\boldsymbol{\chi}(t)(\mathbf{M}_1)) = \mathbf{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}\mathbf{M}_0$.
The curve $\sigma_0 \colon I \to \mathbf{M}_0$ defined by $\sigma_0(t) := \boldsymbol{\chi}(t)(\sigma_1(t))$ is called the development curve *of σ_1.*

No-slip: $\dot\sigma_0(t) = \boldsymbol{\chi}_*(t)(\dot\sigma_1(t))$*, for almost all $t \in I$, where $\boldsymbol{\chi}_*$ is the push-forward of $\boldsymbol{\chi}$.*

No-twist: *two complementary conditions, for almost all $t \in I$,*
tangential: $(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t))_*(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0) \subset \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0^\perp$,

***normal*:** $(\dot{\boldsymbol{\chi}}(t)\circ\boldsymbol{\chi}^{-1}(t))_*(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0^\perp) \subset \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$, *where* $\mathbf{T}_p\mathbf{M}_0^\perp$ *denotes the* normal *space at* $p \in \mathbf{M}_0$.

It is known that rolling maps are *symmetric* and *transitive*. To be more precise, if $\mathbf{M}_1$ rolls upon $\mathbf{M}_0$ with rolling map $\boldsymbol{\chi}$, along σ_1, then $\mathbf{M}_0$ rolls upon $\mathbf{M}_1$ with rolling map $\boldsymbol{\chi}^{-1}$ along the development curve σ_0. If $\mathbf{M}_1$ rolls upon $\mathbf{M}_2$ with rolling map $\boldsymbol{\chi}_1$, rolling curve σ_1 and development curve σ_2, and $\mathbf{M}_2$ rolls upon $\mathbf{M}_3$ with rolling map $\boldsymbol{\chi}_2$, rolling curve σ_2 and development curve σ_3, then $\mathbf{M}_1$ rolls upon $\mathbf{M}_3$ with rolling map $\boldsymbol{\chi}_2\circ\boldsymbol{\chi}_1$, along rolling curve σ_1 and development curve σ_3. Rolling maps preserve *parallel transport* and *covariant differentiation*. Hence, the geodesic curvatures of rolling and development curves are equal. In particular, geodesics are mapped to geodesics through rolling.

Theorem 2 (Sharpe [19, Appendix B])**.** *Let* $\mathbf{M}_0, \mathbf{M}_1 \hookrightarrow \mathbb{R}^{n+r}$ *be* n*-submanifolds. Given a piecewise smooth curve* $\sigma_1 \colon I \to \mathbf{M}_1$ *there* is a unique rolling map $\boldsymbol{\chi}\colon I \to \mathbb{SE}(n+r)$, *with rolling curve* σ_1.

We have the following Proposition 3 stating differential equations for rolling maps of smooth manifolds of co-dimension 1. These equations are a direct consequence of [19, Lemma 2.3, Appendix B]. An interested reader can find a proof in [12]. First, recall the two operators of differential geometry, the *second fundamental form* $\mathbf{II}\colon \mathfrak{X}(\mathbf{M})\times\mathfrak{X}(\mathbf{M}) \to \mathfrak{X}(\mathbf{M})^\perp$ and the *Weingarten map* $\boldsymbol{\Xi}\colon \mathfrak{X}(\mathbf{M})\times\mathfrak{X}(\mathbf{M})^\perp \to \mathfrak{X}(\mathbf{M})$. Both operators are tensor fields and are related by

$$\langle X, \boldsymbol{\Xi}(Y,\Lambda)\rangle = -\langle \mathbf{II}(X,Y),\Lambda\rangle \quad \text{for all} \quad X, Y \in \mathfrak{X}(\mathbf{M}) \quad \text{and} \quad \Lambda \in \mathfrak{X}(\mathbf{M})^\perp. \tag{1}$$

Proposition 3. *In terms of angular velocity in the* body coordinate system *the rolling equations of an* n*-manifold* $\mathbf{N}$ *isometrically embedded in* $\mathbb{R}^{n+1}$ *on its affine tangent space are given by*

$$(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_*V = -\mathbf{II}^1(\dot\sigma_1, V) \quad \textit{and} \quad (\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_*\Lambda = -\boldsymbol{\Xi}^1(\dot\sigma_1,\Lambda),$$

for all $V \in \mathbf{T}_{\sigma_1(t)}\mathbf{N}$ *and* $\Lambda \in (\mathbf{T}_{\sigma_1(t)}\mathbf{N})^\perp$.

In this paper we are concerned with an n-ellipsoid embedded in $\mathbb{R}^{n+1}$, denoted $\boldsymbol{\mathcal{E}}^n \hookrightarrow \mathbb{R}^{n+1}$, *cf.* Figure 1. Here, the Euclidean group $\mathbb{SE}(n+1) = \mathbb{SO}(n+1)\ltimes\mathbb{R}^{n+1}$ is the group of isometries acting on $\boldsymbol{\mathcal{E}}^n$, and both, $(\dot{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1})_*$ and $(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_*$ are vectors in $\mathfrak{se}(n+1) = \mathbf{T}_e\mathbb{SE}(n+1)$ that can be identified with $(n+1)\times(n+1)$-skew-symmetric matrices.

Proposition 3 can be illustrated with the following (non-commuting) diagram:

$$
\begin{tikzcd}
& \mathbf{T}_p\mathbb{R}^{n+1} \arrow[dl, "\pi^{\top}"'] \arrow[dr, "\pi^{\perp}"] & \\
\mathbf{T}_p\boldsymbol{\mathcal{E}}^n \arrow[rr, leftrightarrow, "(\chi^{-1}\dot{\chi})_*" description] \arrow[rr, bend right, "-\mathbf{II}^1_{\dot{\sigma}_1}"'] & & (\mathbf{T}_p\boldsymbol{\mathcal{E}}^n)^{\perp} \arrow[ll, bend right, "-\mathbf{\Xi}^1_{\dot{\sigma}_1}"']
\end{tikzcd}
$$

2.2. Rolling ellipsoid in Euclidean space

Rolling an ellipsoid in Euclidean space has been already studied in [6]. Then in a different context in [13], whose extended version [14] include derivations of some differential geometric properties of an ellipsoid. We begin by recalling a few of them that will be needed here: the Weingarten map, the second fundamental form and the Gaussian curvature. From now on we assume that $\boldsymbol{\mathcal{E}}^n \hookrightarrow \mathbb{R}^{n+1}$, where $n > 1$.

Given a positive definite matrix $\mathsf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_{n+1}) \succ 0$ define an ellipsoid $\boldsymbol{\mathcal{E}}^n$ by

$$\boldsymbol{\mathcal{E}}^n := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \langle x, \mathsf{D}^{-2}x\rangle = 1 \right\}. \tag{2}$$

By differentiation of a unit normal vector $\Lambda = -\mathsf{D}^{-2}p/|\mathsf{D}^{-2}p| \in (\mathbf{T}_p\boldsymbol{\mathcal{E}}^n)^{\perp}$ moving along a curve in $\boldsymbol{\mathcal{E}}^n$ it is straightforward to find the Weingarten map Ξ_Λ at $p \in \boldsymbol{\mathcal{E}}^n$ to be given by

$$\Xi_\Lambda(X) = -\mathsf{D}^{-2}\left(\frac{X}{|\mathsf{D}^{-2}p|} - \frac{p}{|\mathsf{D}^{-2}p|^3}\langle \mathsf{D}^{-2}X, \mathsf{D}^{-2}p\rangle\right), \quad \text{for any} \quad X \in \mathbf{T}_p\boldsymbol{\mathcal{E}}^n.$$

The second fundamental form is a symmetric tensor that depends only on values of vector fields at a point, hence with (1) it can be verified that

$$\mathbf{II}(X,Y) = -\frac{\langle \mathsf{D}^{-1}X, \mathsf{D}^{-1}Y\rangle}{|\mathsf{D}^{-2}p|^2}\,\mathsf{D}^{-2}p, \quad \text{for any} \quad X, Y \in \mathbf{T}_p\boldsymbol{\mathcal{E}}^n. \tag{3}$$

The ellipsoid $\boldsymbol{\mathcal{E}}^n$ is a smooth Riemannian manifold with positive Gaussian curvature. Its precise value is given by the following proposition.

Proposition 4. *The Gaussian curvature $K(p)$ of $\boldsymbol{\mathcal{E}}^n$ isometrically embedded in $\mathbb{R}^{n+1}$ is given by*

$$K(p) = \frac{\det(\mathsf{D}^{-2})}{|\mathsf{D}^{-2}p|^{n+2}} > 0.$$

Proof: We use the following relations of the *scalar second fundamental form* h and the *shape operator* s, *cf.* [15, page 140]: $\mathbf{II}(X,Y) = h(X,Y)\Lambda$ and $h(X,Y) = \langle X, sY\rangle$, for any tangent vectors X, Y and a normal unit vector Λ. From the second fundamental form (3) it follows that

$$h(X,Y) = \frac{\langle \mathbf{D}^{-1}X, \mathbf{D}^{-1}Y\rangle}{|\mathbf{D}^{-2}p|} = \langle X, sY\rangle, \quad \text{hence} \quad sY = \frac{\mathbf{D}^{-2}Y}{|\mathbf{D}^{-2}p|}.$$

The shape operator is positive definite and self adjoint. Suppose that $sv_i = \lambda_i v_i$, for $i = 1, 2, \ldots, n$, where v_i are the eigenvectors and λ_i are the corresponding eigenvalues. Then with the *generalised cross product* to $\mathbb{R}^{n+1}$, *cf.* [20, page 84], one has

$$v_1 \times v_2 \times \cdots \times v_n = \alpha\Lambda, \quad \text{where} \quad \Lambda \in (\mathbf{T}_p\boldsymbol{\mathcal{E}}^n)^{\perp}.$$

On one hand

$$(sv_1) \times (sv_2) \times \cdots \times (sv_n) = (\lambda_1\lambda_2\cdots\lambda_n)\, v_1 \times v_2 \times \cdots \times v_n = (\alpha\Lambda)\prod_{i=1}^{n}\lambda_i,$$

on the other hand, by the properties of linear mapping acting on the cross product (*cf.* [14])

$$(sv_1)\times(sv_2)\times\cdots\times(sv_n) = \frac{\mathbf{D}^{-2}v_1}{|\mathbf{D}^{-2}p|}\times\frac{\mathbf{D}^{-2}v_2}{|\mathbf{D}^{-2}p|}\times\cdots\times\frac{\mathbf{D}^{-2}v_n}{|\mathbf{D}^{-2}p|} = \frac{\det(\mathbf{D}^{-2})}{|\mathbf{D}^{-2}p|^n}\cdot\mathbf{D}^2(\alpha\Lambda).$$

Since the Gaussian curvature is defined as the product of the eigenvalues of s, therefore

$$K(p) = \prod_{i=1}^{n}\lambda_i = \frac{\det(\mathbf{D}^{-2})}{|\mathbf{D}^{-2}p|^n}\cdot\langle\mathbf{D}^2\Lambda, \Lambda\rangle = \frac{\det(\mathbf{D}^{-2})}{|\mathbf{D}^{-2}p|^n}\cdot\frac{\langle p, \mathbf{D}^{-2}p\rangle}{|\mathbf{D}^{-2}p|^2} = \frac{\det(\mathbf{D}^{-2})}{|\mathbf{D}^{-2}p|^{n+2}}.$$

What was to show. ∎

Throughout this paper, we use the following convention. For any two vectors $X, Y \in \mathbb{R}^m$, with $m > 1$, let $X \otimes Y$ be an $m \times m$-matrix with entries $(X \otimes Y)_{ij} = X^iY^j$. If X, Y are represented as column vectors, then $X \otimes Y = X \cdot Y^{\mathrm{T}}$. Furthermore, let $X \wedge Y = X \otimes Y - Y \otimes X$, then $X \wedge Y$ is a skew-symmetric $m \times m$-matrix of rank zero or two.

Given a rolling curve $\sigma_1 \colon I \to \boldsymbol{\mathcal{E}}^n$, the following lemma establishes a relationship between the rolling map and σ_1.

Lemma 5. *The rolling map* $\boldsymbol{\chi}$ *of an ellipsoid rolling upon its affine tangent space satisfies*

$$(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_* = -\frac{1}{|\mathbf{D}^{-2}\sigma_1|^2}\left(\mathbf{D}^{-2}\dot{\sigma}_1\right) \wedge \left(\mathbf{D}^{-2}\sigma_1\right).$$

Proof: The above equality follows directly from Proposition 3. To prove the lemma let us first rewrite the Weingarten map and the second fundamental form in a more convenient way

$$\Xi_\Lambda(X) = -\frac{1}{|\mathbf{D}^{-2}p|}\left(\mathbf{D}^{-2}X - \Lambda\langle \mathbf{D}^{-2}X, \Lambda\rangle\right) \quad \text{and}$$

$$\mathbf{II}(X, Y) = \frac{1}{|\mathbf{D}^{-2}p|}\langle \mathbf{D}^{-2}X, Y\rangle\, \Lambda.$$

Any vector $U \in \mathbf{T}_p\mathbb{R}^{n+1}$ can be uniquely written as $U = V + \alpha\Lambda$, where $V \in \mathbf{T}_p\boldsymbol{\mathcal{E}}^n$. Hence

$$\begin{aligned}
(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_* U &= (\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_*(V + \alpha\Lambda) = -\mathbf{II}(\dot{\sigma}_1, V) - \alpha\boldsymbol{\Xi}(\dot{\sigma}_1, \Lambda) \\
&= \frac{1}{|\mathbf{D}^{-2}p|}(-\langle \mathbf{D}^{-2}\dot{\sigma}_1, V\rangle\, \Lambda + \alpha\mathbf{D}^{-2}\dot{\sigma}_1 - \alpha\Lambda\langle \mathbf{D}^{-2}\dot{\sigma}_1, \Lambda\rangle) \\
&= \frac{1}{|\mathbf{D}^{-2}p|}(\langle \Lambda, U\rangle\, \mathbf{D}^{-2}\dot{\sigma}_1 - \langle \mathbf{D}^{-2}\dot{\sigma}_1, U\rangle\, \Lambda).
\end{aligned}$$

Since a covector in Euclidean space is identified with its transpose, then after expanding Λ to its full expression one arrives at

$$(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_* = -\frac{\mathbf{D}^{-2}\dot{\sigma}_1}{|\mathbf{D}^{-2}p|^2}\left(\mathbf{D}^{-2}p\right)^{\mathrm{T}} + \frac{\mathbf{D}^{-2}p}{|\mathbf{D}^{-2}p|^2}\left(\mathbf{D}^{-2}\dot{\sigma}_1\right)^{\mathrm{T}},$$

where $p = \sigma_1(t)$. The result now follows. ∎

Consider the *Gauss map*. Given a curve $\sigma_1 \colon I \to \boldsymbol{\mathcal{E}}^n$, the unit normal vector along σ_1 is given by

$$\eta := \rho \cdot \mathbf{D}^{-2}\sigma_1, \quad \text{where} \quad \rho = \frac{1}{|\mathbf{D}^{-2}\sigma_1|}. \tag{4}$$

There is the following simple formula relating a rolling curve in $\boldsymbol{\mathcal{E}}^n$ to a curve in $\mathfrak{so}(n+1)$. This follows immediately from Lemma 5.

Corollary 6. *The angular velocity in the body coordinate system of an ellipsoid rolling upon its affine tangent space satisfies*

$$(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}})_* = \eta \wedge \dot{\eta}.$$

To conclude this section we will establish a relationship between two different metrics, used in the remainder of this paper. Define the following quadratic form on the space of compatible matrices

$$\langle\!\langle \mathbf{A}, \mathbf{B} \rangle\!\rangle := \frac{1}{2}\,\mathrm{trace}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \frac{1}{2}\,\mathrm{trace}(\mathbf{A}\mathbf{B}^{\mathrm{T}}). \tag{5}$$

We will need the following property of the '$\wedge$' operator, for some later calculations, given here as Propositions 7. Simple proof is omitted.

Proposition 7. *For any $X, Y, Z, W \in \mathbb{R}^m$ there is the following relation between metric (5) and the inner (dot) Euclidean product*

$$\langle\!\langle X \wedge Y, Z \wedge W \rangle\!\rangle = \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle.$$

Remark 8. *Proposition 7 shows that $\langle\!\langle X \wedge Y, Z \wedge W \rangle\!\rangle$ has precisely the same symmetries as the Riemannian curvature tensor $\mathcal{R}m(X, Y, Z, W) := \langle \mathcal{R}(X, Y)Z, W\rangle$. In particular, for the unit sphere, the equality holds*

$$\langle\!\langle X \wedge Y, Z \wedge W \rangle\!\rangle = \mathcal{R}m(X, Y, Z, W).$$

3. Geodesics on Ellipsoid

This extremely fertile problem goes back to the XIX century and Jacobi, who first showed that the problem of geodesics on a triaxial ellipsoids can be reduced by quadrature. In 1839, in his letter to Bessel [7], C. G. J. Jacobi wrote about his discovery that the geodesic equations expressed in ellipsoidal coordinates, are separable. Surprisingly, the problem of geodesics is connected to fundamental models in classical and quantum mechanics. Our study was largely influenced by Knörrer's paper [10], where the classical C. Neumann problem is linked to geodesics on quadrics.

3.1. The Neumann problem

Consider a point on the sphere moving in a quadratic potential $\langle \gamma, \mathbf{A}\gamma\rangle$, where $\mathbf{A}$ is a symmetric matrix. The equation governing the motion of this point is

$$\ddot{\gamma} = -\mathbf{A}\gamma + u\gamma, \quad \text{where} \quad u = \langle \gamma, \mathbf{A}\gamma\rangle - \langle \dot{\gamma}, \dot{\gamma}\rangle. \tag{6}$$

If it is supposed that $\mathbf{A} = \mathrm{diag}(a_0, a_1, \dots, a_n)$ then the components γ^i of γ are eigenfunctions of the one-dimensional Schrödinger operator $(-d^2/dt^2 + u)$

with the eigenvalue a_i. Knörrer attributes finding algebraic integrals of the Neumann problem (6) to Karen Uhlenbeck's informal preprint [21], whose copy eluded the authors. Let $\mathbb{C}(z)$ denotes the set of complex rational functions that are not everywhere zero. For $x, y \in \mathbb{R}^{n+1}$ let $\Phi_z(x, y) \in \mathbb{C}(z)$ be the rational function

$$\Phi_z(x,y) = \langle x, (\mathbf{A} - z\,\mathbb{I})^{-1} y\rangle^2 + (1 - \langle x, (\mathbf{A} - z\,\mathbb{I})^{-1} x\rangle)\, \langle y, (\mathbf{A} - z\,\mathbb{I})^{-1} y\rangle, \quad (7)$$

where $\mathbb{I}$ is the identity matrix. If $\xi(t)$ is a solution to the Neumann problem (6) then the coefficients of $\Phi_z(\dot{\xi}, \xi) \in \mathbb{C}(z)$ are independent of t.

The following geometric interpretation of (7) can be found in [17]. Let Q_z be a bi-linear symmetric form on $\mathbb{R}^{n+1}$ given by $Q_z(x, y) := \langle x, (\mathbf{A} - z\,\mathbb{I})^{-1} y\rangle$ and let $Q_z(x)$ denote $Q_z(x, x)$. Then $Q_z(x) = 1$ generates a family of quadrics. Assume for a moment that a_i are ordered $a_0 < a_1 < \cdots < a_n$ then the quadrics are ellipsoids, when $z < a_0$, and hyperboloids, for all $a_{i-1} < z < a_i$, $i = 1, 2, \ldots, n$, *cf.* Figure 2. For any $x \in \mathbb{R}^{n+1}$ the equation $Q_z(x) = 1$ has exactly $n + 1$ solutions, lying in the above intervals, each. Therefore, through every point $x \in \mathbb{R}^{n+1}$, such that $x^j \neq 0$, $j = 0, \ldots, n$, pass exactly $n + 1$ quadrics. Because

$$\Phi_z(x, y) = Q_z^2(x, y) + (1 - Q_z(x))\, Q_z(y) = Q_z(y) - (Q_z(x)\, Q_z(y) - Q_z^2(x, y))$$

then $\Phi_z(x, y) = 0$ if and only if the line $x + ty$ is tangent to the quadric $Q_z = 1$ at point x, under the condition that $Q_z(y) \neq 0$.

Knörrer [10] shows that if a geodesic γ on a quadric Q, defined by $q(x) = 0$, is parameterised by $\ddot{\gamma}(t) = -\varepsilon\, dq(\gamma(t)) + v(t) \cdot \dot{\gamma}(t)$, with a constant $\varepsilon = \pm 1$, then the corresponding unit normal vector $\xi(t)$ at the point $\gamma(t)$ is a solution of the Neumann problem

$$\ddot{\xi} = -\varepsilon \mathbf{A}\xi + u\xi, \quad \text{where} \quad \mathbf{A} = \operatorname{Hess}(q) = \left(\frac{\partial^2 q}{\partial x_i\, \partial x_j}\right) \quad \text{and} \quad u := \frac{1}{4}v^2 - \frac{1}{2}\dot{v}.$$

Specifically, in the case of the ellipsoid (2) one considers the Gauss map η, as in (4), along a reparametrised geodesic. By Knörrer's result, the Gauss map solves the Neumann problem, therefore the Uhlenbeck's algebraic integrals (7) hold, in this case.

The Neumann problem can be also analysed through actions of Lie groups. Consider $\boldsymbol{\vartheta} \in \mathfrak{so}(n+1)$ and $\boldsymbol{\zeta} \in \mathfrak{sym}(n+1)$ given by

$$\boldsymbol{\vartheta} := \dot{\eta} \wedge \eta \quad \text{and} \quad \boldsymbol{\zeta} := \eta \otimes \eta, \quad \text{where} \quad \langle \eta, \eta\rangle = 1 \quad \text{and} \quad \langle \dot{\eta}, \eta\rangle = 0.$$

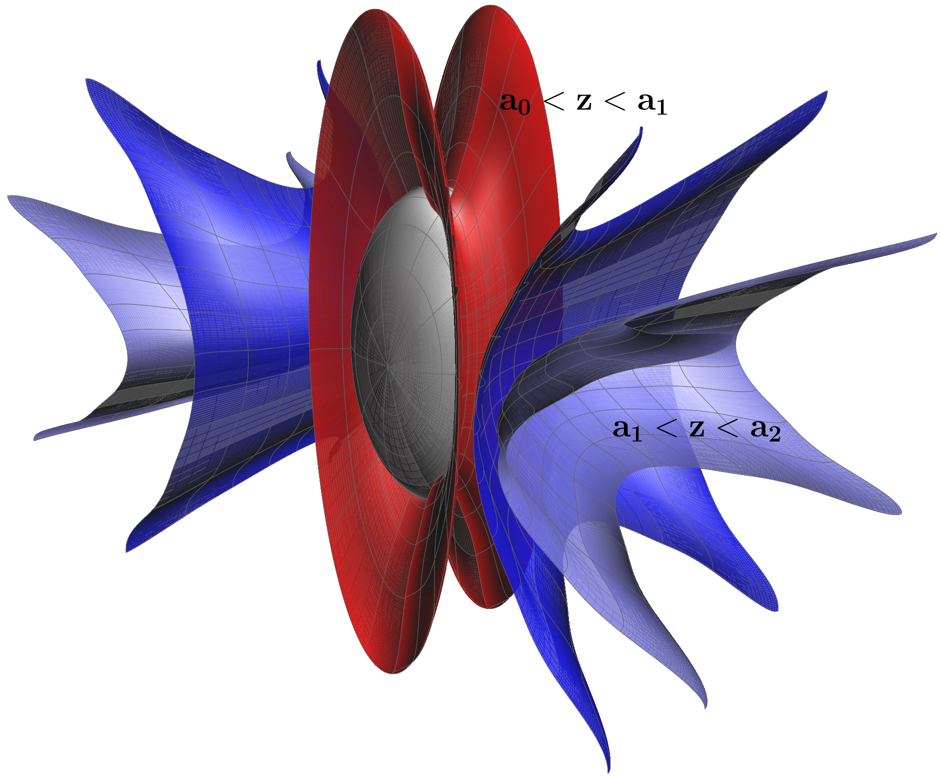

FIGURE 2. A family of confocal quadrics: ellipsoids, when $z < a_0$ (grey), hyperboloids of one sheet, for $a_0 < z < a_1$ (red) and two-sheeted elliptic hyperboloids, when $a_1 < z < z_2$ (blue).

It has been noticed by Uhlenbeck and described by Tudor Ratiu that the Neumann problem is equivalent to the following one.

Lemma 9 (Uhlenbeck, *cf.* [18])**.** *The system (6) is equivalent to*

$$\dot{\boldsymbol{\zeta}} = [\boldsymbol{\vartheta}, \boldsymbol{\zeta}] \quad \textit{and} \quad \dot{\boldsymbol{\vartheta}} = [\boldsymbol{\zeta}, \mathbf{A}], \quad \textit{where} \quad \langle \eta, \eta \rangle = 1 \quad \textit{and} \quad \langle \dot{\eta}, \eta \rangle = 1. \tag{8}$$

By "normalising" the matrices $\mathbf{A} \mapsto \mathbf{A} - \operatorname{trace}(\mathbf{A})\, \mathbb{I}/\,(n+1)$ and $\boldsymbol{\zeta} \mapsto \boldsymbol{\zeta} - \mathbb{I}/\,(n+1)$, equations (8) remain the same. Therefore one can consider this system to be set out for $\mathbf{A}, \boldsymbol{\zeta}, \boldsymbol{\vartheta} \in \mathfrak{sl}(n+1)$.

Ratiu [18] has found the following beautiful interpretation of system (8) with the language of Lie algebras of which we give a brief account, now.

3.2. Hamiltonian formalism

Consider a Lie group $\mathcal{G}$ given by a semi-direct product $\mathcal{G} = \mathbb{SL}(n+1) \ltimes \overline{\mathfrak{sl}(n+1)}$ of $\mathbb{SL}(n+1)$, the special linear group, i.e., real matrices with determinant 1, with the Lie algebra $\overline{\mathfrak{sl}(n+1)}$ acting as a vector space. The group operation is defined by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 + \mathsf{Ad}_{g_1} h_2).$$

The identity of $\mathcal{G}$ is $(\mathsf{id}, 0)$ and the inverse $(g, h)^{-1} = (g^{-1}, -\mathsf{Ad}_{g^{-1}} h)$. Let $\widetilde{\mathfrak{sym}}(n+1) \subset \mathfrak{sl}(n+1)$ denote the vector space of all symmetric matrices of trace zero. Then the Lie group $\mathcal{N}$ defined as $\mathcal{N} = \mathbb{SO}(n+1) \ltimes \widetilde{\mathfrak{sym}}(n+1)$ is a subgroup of $\mathcal{G}$, with Lie algebra $\mathfrak{N}$. Therefore $\mathfrak{G} = \mathfrak{N} \oplus \mathfrak{K}$ is the direct sum decomposition of the Lie algebra of $\mathcal{G}$, where $\mathfrak{N}$ is a sub-algebra of $\mathfrak{G}$ and $\mathfrak{K}$ is a vector sub-space of $\mathfrak{G}$. The following diagram illustrates the above structure.

$$\begin{tikzcd}
\mathcal{N} = \mathbb{SO}(n+1) \ltimes \widetilde{\mathfrak{sym}}(n+1) \subset & \mathcal{G} = \mathbb{SL}(n+1) \ltimes \overline{\mathfrak{sl}(n+1)} & \\
& \mathfrak{G} = \mathfrak{sl}(n+1) \ltimes \overline{\mathfrak{sl}(n+1)} \arrow[u, "\exp"'] \arrow[dl, "\Pi_{\mathfrak{N}}"'] \arrow[dr, "\Pi_{\mathfrak{K}}"] & \\
\mathfrak{N} = \mathfrak{so}(n+1) \ltimes \widetilde{\mathfrak{sym}}(n+1) \arrow[uu, "\exp"] & & \mathfrak{K} = \widetilde{\mathfrak{sym}}(n+1) \times \mathfrak{so}(n+1)
\end{tikzcd}$$

The co-adjoint orbits are important in studies of Hamiltonian functions, the conserved quantities. They also lead to integrable systems. In particular one is interested in the action of $\mathcal{N}$ on $\mathfrak{N}^\star$, but since $\langle\!\langle \mathfrak{N}, \mathfrak{R} \rangle\!\rangle = 0$, this can be simplified, because the metric on $\mathcal{G}$ induces isomorphisms $\mathfrak{K}^\perp \cong \mathfrak{R}^\star$ and $\mathfrak{R}^\perp \cong \mathfrak{K}^\star$. Furthermore, the algebra $\mathfrak{G}$ has the non-degenerate, bi-invariant two-form k_s, the *semi-direct product* that is given by

$$k_s((\xi_1, \eta_1), (\xi_2, \eta_2)) := \langle\!\langle \xi_1, \eta_2 \rangle\!\rangle + \langle\!\langle \xi_2, \eta_1 \rangle\!\rangle.$$

With such product k_s there is $\mathfrak{K}^\perp = \mathfrak{K}$ and $\mathfrak{R}^\perp = \mathfrak{R}$. Therefore the co-adjoint action of $\mathcal{N}$ on $\mathfrak{N}^\star$ induces the adjoint action of $\mathcal{N}$ on $\mathfrak{K}$. The adjoint action of $\mathcal{G}$ on $\mathfrak{G}$ is given by

$$\mathsf{Ad}_{(g,h)}(\xi, \eta) = (\mathsf{Ad}_g \xi, \mathsf{Ad}_g \eta + [h, \mathsf{Ad}_g \xi]).$$

With this in mind, it is easy to calculate the required adjoint action of $(g, \mathbf{A})$ on $(\mathbf{z} \otimes \mathbf{z}, 0)$, where $\mathbf{z} = (1, 1, \ldots, 1)/\sqrt{n+1}$ is a vector in $\mathbb{R}^{n+1}$.

$$\mathsf{Ad}_g(\mathbf{z} \otimes \mathbf{z}) = \eta \otimes \eta = \boldsymbol{\zeta} \quad \text{and} \quad [\mathbf{A}, \boldsymbol{\zeta}] = \mathbf{A}(\eta \otimes \eta) - (\eta \otimes \eta)\mathbf{A} = (\mathbf{A}\eta) \wedge \eta.$$

This yields

$$\mathsf{Ad}_{(g,\mathbf{A})}(\mathbf{z} \otimes \mathbf{z}, 0) = (\eta \otimes \eta, \dot{\eta} \wedge \eta) = (\boldsymbol{\zeta}, \boldsymbol{\vartheta}).$$

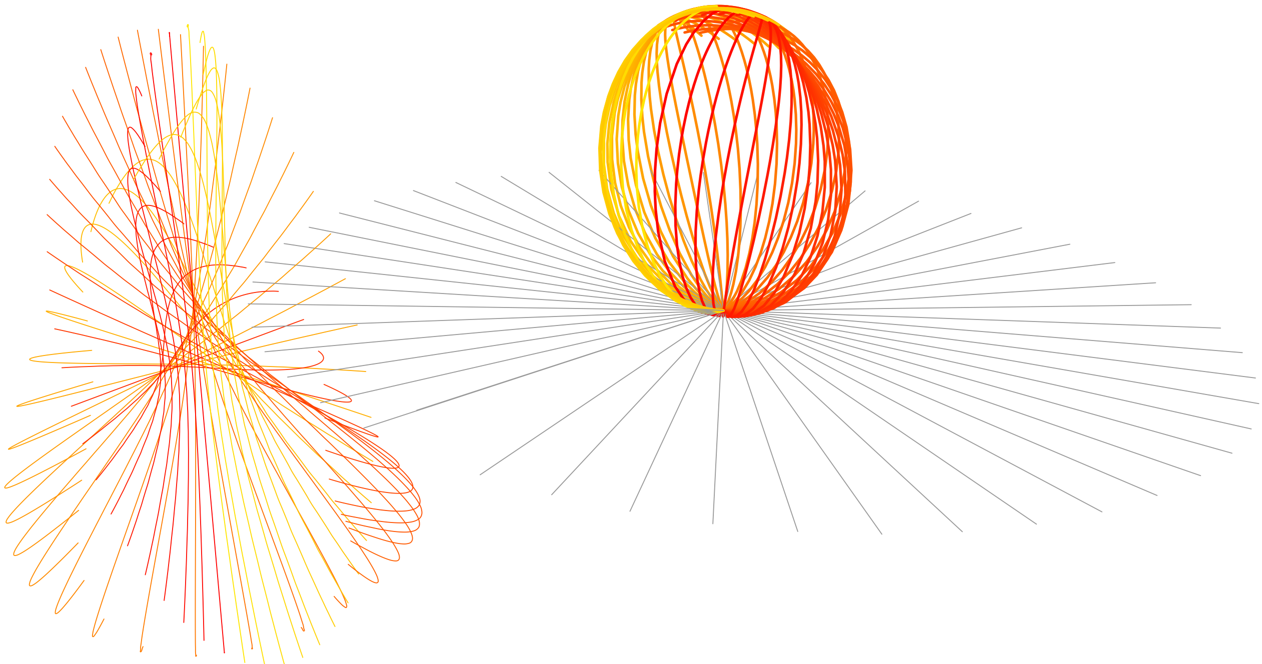

FIGURE 3. Family of geodesics γ_i on the ellipsoid with developing curves on its tangent space — straight line segments (right) and corresponding curves $\boldsymbol{\vartheta}_i$ in $\mathfrak{so}(3)$ projected onto $\mathbb{R}^3$ (left).

Since $(\mathbf{A}\eta) \wedge \eta = (\mathbf{A}\eta - \langle \mathbf{A}\eta, \eta\rangle\eta) \wedge \eta$, the equivalence with the Neumann problem can be achieved by taking $\dot{\eta} = \mathbf{A}\eta - \langle \mathbf{A}\eta, \eta\rangle\eta$. Then $\langle\!\langle \dot{\eta}, \eta \rangle\!\rangle = 0$ as needed. This Lie group construction is summarised by the following theorem.

Theorem 10 (Ratiu [18]). *The $\mathcal{N} = \mathfrak{so}(n+1) \times \widetilde{\mathfrak{sym}}(n+1)$-orbit through $(\mathbf{z}\otimes\mathbf{z} - \mathbb{I}/(n+1), 0)$ in $\mathfrak{R}^\perp = \mathfrak{R} = \widetilde{\mathfrak{sym}}(n+1)\times\mathfrak{so}(n+1)$, consists of all pairs $(\boldsymbol{\zeta}, \boldsymbol{\vartheta})$, $\boldsymbol{\zeta}_{ij} = x^i x^j$, for $i \neq j$, and $\boldsymbol{\zeta}_{ii} = x^i x^i - 1/(n+1)$, $\boldsymbol{\vartheta}_{ij} = y^i x^j - x^i y^j$, $|x| = 1$, $\langle x, y\rangle = 0$. With the Kirillov-Kostant-Souriau symplectic structure this $2n$-dimensional orbit is symplectically diffeomorphic via $(\boldsymbol{\zeta}, \boldsymbol{\vartheta}) \mapsto (x, y)$ to $\mathbf{TS}^n$ with the symplectic structure induced from $\mathbb{R}^{2n+2}$ by $-\sum_{i=1}^{n+1} dx^i \wedge dy^i$. The Hamiltonian $E(\boldsymbol{\zeta}, \boldsymbol{\vartheta}) = -\frac{1}{2}\langle\!\langle \boldsymbol{\vartheta}, \boldsymbol{\vartheta}\rangle\!\rangle + \langle\!\langle \mathbf{A}, \boldsymbol{\zeta}\rangle\!\rangle$ defines on this orbit the equations of motion of the C. Neumann problem*

$$\dot{\boldsymbol{\zeta}} = [\boldsymbol{\vartheta}, \boldsymbol{\zeta}] \quad \textit{and} \quad \dot{\boldsymbol{\vartheta}} = [\boldsymbol{\zeta}, \mathbf{A}] \quad \textit{where} \quad \langle \eta, \eta \rangle = 1 \quad \textit{and} \quad \langle \dot{\eta}, \eta \rangle = 1.$$

3.3. Algebraic integrals

Coming back to the problem of rolling ellipsoid upon its affine tangent space we derive an analogous to Uhlenbeck's algebraic integrals, for the rolling map of the ellipsoid along geodesics. From the nature of the rolling map, through the non-holonomic "no-slip" and "no-twist" constraints, in Definition 1, it is known that $\boldsymbol{\vartheta}$ is a horizontal curve for an n-dimensional distribution.

Theorem 11. *Let $\gamma\colon I \to \boldsymbol{\mathcal{E}}^n$ be a geodesic, parameterised by*

$$\ddot{\gamma}(t) = -\mathbf{D}^{-2}\gamma(t) + v(t)\cdot\dot{\gamma}(t).$$

Furthermore let $\boldsymbol{\vartheta} = \dot{\eta}\wedge\eta$, where η is the unit normal vector along γ and let $\boldsymbol{\zeta} = \eta\otimes\eta$. Then the coefficients of $\Psi_z(\boldsymbol{\zeta},\boldsymbol{\vartheta}) \in \mathbb{C}(z)$ given by

$$\Psi_z(\boldsymbol{\zeta},\boldsymbol{\vartheta}) := -\frac{1}{2}\left\langle\!\left\langle \boldsymbol{\vartheta}(\mathbf{D}^{-2} - z\,\mathbb{I})^{-1}, (\mathbf{D}^{-2} - z\,\mathbb{I})^{-1}\boldsymbol{\vartheta}\right\rangle\!\right\rangle + \left\langle\!\left\langle \boldsymbol{\zeta}, (\mathbf{D}^{-2} - z\,\mathbb{I})^{-1}\right\rangle\!\right\rangle$$

are independent of t.

Proof: For the proof one can show that under the hypothesis $\Psi_z(y\otimes y, x\wedge y) = 2\,\Phi_z(x,y)$. But perhaps it will be more instructive to prove the result by going back to the original version of the Neumann problem (6). To shorten the notation, denote $\mathbf{X} = (\mathbf{D}^{-2} - z\,\mathbb{I})^{-1}$. By Knörrer's result, η is a solution to the Neumann problem

$$\ddot{\eta} = -\mathbf{D}^{-2}\eta + u\eta = -\mathbf{X}^{-1}\eta + (u-z)\,\eta.$$

Apply '$\wedge\eta$' to both sides of the above equality, then $\dot{\boldsymbol{\vartheta}} = \ddot{\eta}\wedge\eta = -(\mathbf{X}^{-1}\eta)\wedge\eta$. To prove the theorem, it is enough to show that the derivative of $\Psi_z(\boldsymbol{\zeta},\boldsymbol{\vartheta})$ with respect to t is zero.

$$\frac{d}{dt}\langle\!\langle \boldsymbol{\vartheta}\mathbf{X}, \mathbf{X}\boldsymbol{\vartheta}\rangle\!\rangle = 2\,\langle\!\langle \mathbf{X}\dot{\boldsymbol{\vartheta}}\mathbf{X}, \boldsymbol{\vartheta}\rangle\!\rangle = -2\,\langle\!\langle \eta\wedge(\mathbf{X}\eta), \dot{\eta}\wedge\eta\rangle\!\rangle = 2\,\langle\dot{\eta}, \mathbf{X}\eta\rangle = \frac{d}{dt}\langle\eta, \mathbf{X}\eta\rangle.$$

Since $\langle\eta, \mathbf{X}\eta\rangle = \operatorname{trace}(\eta\eta^{\mathrm{T}}\mathbf{X}) = 2\,\langle\!\langle \eta\otimes\eta, \mathbf{X}\rangle\!\rangle$ the result now follows. ■

The function v of the parametrisation in Theorem 11 is given by

$$v = 2\,\frac{\langle \mathbf{D}^{-2}\eta, \mathbf{D}^{-2}\dot{\eta}\rangle}{\langle\dot{\eta}, \mathbf{D}^{-2}\dot{\eta}\rangle}.$$

In the case of the unit sphere $\mathbf{D} = \mathbb{I}$. Then $v = 0$ and Ψ_z becomes

$$\Psi_z(\boldsymbol{\zeta},\boldsymbol{\vartheta}) = -\frac{1}{2(1-z)^2}\langle\!\langle \boldsymbol{\vartheta},\boldsymbol{\vartheta}\rangle\!\rangle + \frac{1}{(1-z)}, \qquad z \neq 1.$$

Thus $\Psi_z(\boldsymbol{\zeta},\boldsymbol{\vartheta})$ is now equivalent to the energy integral, with $\langle\!\langle \boldsymbol{\vartheta},\boldsymbol{\vartheta}\rangle\!\rangle = \langle\dot{\gamma},\dot{\gamma}\rangle =$ const, which encodes the fact that geodesics have constant speed. Moreover, since $v = 0$ then clearly $\boldsymbol{\vartheta}$ is a constant curve in $\mathfrak{so}(n+1)$. Thus the rolling map for the unit sphere $\mathbf{S}^n$ rolling upon its affine tangent space assigns a single point in $\mathfrak{so}(n+1)$ to every geodesic in $\mathbf{S}^n$.

4. Conclusion and Further Questions

In this paper we have shown that rolling an ellipsoid along a geodesic has a simple expression for its integrals of motions. Our approach uses a different mathematical formalism than Hamiltonian mechanics. The integrals are based on Uhlenbeck's algebraic integrals for the C. Neumann problem and on their generalisation to geodesics on quadrics found by Knörrer [10]. We have linked some expressions appearing in Hamiltonians considered in [8, 4, 18], for example, to the angular velocity for the ellipsoid $\boldsymbol{\mathcal{E}}^n \hookrightarrow \mathbb{R}^{n+1}$ rolling, without a slip or twist, upon its affine tangent space.

A possible extension to this study is to consider the pseudo-Riemannian case. Rolling maps has been studied, for instance in [3], and in [11] and connecting this knowledge with the studies of geodesics on ellipsoids, or more general on quadrics, could bring some new insights into integrals of motions for Lorentzian metrics.

We hope to consider more general setting and a geometric interpretation of the integrals of motions in future work.

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