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WHAT IS AN IDEAL A ZERO-CLASS OF?

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ABSTRACT: We characterise, in pointed regular categories, the ideals as the zero-classes of surjective relations. Moreover, we study a variation of the *Smith is Huq* condition: two surjective left split relations commute as soon as their zero-classes commute.

1. Introduction

The description of congruences, and of more general compatible relations, in terms of their zero-classes is a very classical topic in universal algebra. It led to the study of different notions of subalgebras in pointed varieties; let us mention here the ones of ideal [7, 11, 19] and clot [1].

Later these notions have been considered in a categorical context [9, 10, 12]. Clots were characterised as as zero-classes of internal reflexive relations, and ideals were characterised as regular images of clots. However, a characterisation of ideals as zero-classes of suitable relations was still missing, both in categorical and in universal algebra.

The aim of the present paper is to fill this gap. We prove that, in every pointed regular category, the ideals are the zero-classes of what we call *surjective relations*. Such is any relation from an object X to an object Y where the projection on Y is a regular epimorphism. In fact, we can always choose a *left split* surjective relation to represent a given ideal, which means that moreover the projection on X is a split epimorphism. We also show that, in general, it is not possible to describe ideals by means of endorelations on an object X. The

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table at the end of the introduction gives an overview of the description of all the notions mentioned above in terms of zero-classes.

This study naturally led us to consider a variation of the so-called *Smith is Huq* condition, which says that two equivalence relations on the same object commute in the Smith–Pedicchio sense [18, 17] if and only if their zero-classes commute in the Huq sense [8]. Our condition is then the following: two semi-split surjective relations commute if and only if their zero-classes (their associated ideals) commute. This provides a conceptual interpretation of the *admissibility* condition introduced in [13] and further explored in [6, 15]. We consider some equivalent and some stronger conditions, and we compare them with the standard *Smith is Huq* condition.

The paper is organised as follows. In Section 2 we recall the notions of ideal and clot, both from the categorical and the universal-algebraic points of view, and we prove some stability properties of ideals. In Section 3 we prove that ideals are exactly zero-classes of surjective relations (or, equivalently, of semi-split surjective relations) and we consider some concrete examples. In Section 4 we study the above-mentioned variations of the *Smith is Huq* condition.

any (left split) relation	surjective (left split) relation	reflexive relation	equivalence relation	effective equivalence relation
monomorphism	ideal	clot	normal monomorphism	kernel

TABLE 1. Several types of monomorphisms in pointed regular categories

2. Ideals and clots

The notion of *ideal* was introduced in [7] in the context of groups with multiple operators (also called Ω-groups), and then extended in [11]—and further studied in [19] and in subsequent papers—to varieties of algebras with a constant 0. We recall here the definition in the case of **pointed** varieties: those with a unique constant 0.

Definition 2.1. A term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in a pointed variety $\mathscr{C}$ is said to be an **ideal term** in $y_1, \ldots, y_n$ if $t(x_1, \ldots, x_m, 0, \ldots, 0) = 0$ is an identity in $\mathscr{C}$. A subalgebra I of an algebra A in $\mathscr{C}$ is an **ideal** of A if $t(x_1, \ldots, x_m, i_1, \ldots, i_n)$ belongs to I for all $x_1, \ldots, x_m \in A$, all $i_1, \ldots, i_n \in I$ and every ideal term t.

Later, as an alternative, in the paper [1] the concept of *clot* was introduced:

Definition 2.2. A subalgebra K of A in $\mathscr{C}$ is called a **clot** in A if

$$t(a_1, \ldots, a_m, 0, \ldots, 0) = 0$$

and $k_1, \ldots, k_n \in K$ imply $t(a_1, \ldots, a_m, k_1, \ldots, k_n) \in K$ for all $a_1, \ldots, a_m$, $k_1, \ldots, k_n$ in A and every $(m+n)$-ary term function t of A.

It was shown in [1] that clots are exactly 0-classes of semi-congruences, that is, of those reflexive relations which are compatible with all the operations in the variety. Thus, for any algebra A in any variety $\mathscr{C}$ there is an inclusion

$$\mathrm{N}(A) \subseteq \mathrm{Cl}(A) \subseteq \mathrm{I}(A),$$

where $\mathrm{N}(A)$ is the set of normal subalgebras of A (that are the 0-classes of the congruences on A), $\mathrm{Cl}(A)$ is the set of clots of A and $\mathrm{I}(A)$ is the set of ideals.

All these notions were then studied in a categorical context (see [9, 10, 12]). Before recalling the categorical counterparts of the definitions above, we need to introduce some terminology. The context that we consider is the one of pointed regular categories.

Definition 2.3. Given a span

$$\begin{tikzcd} & R \arrow[dl, "d"'] \arrow[dr, "c"] & \\ X & & Y \end{tikzcd} \tag{A}$$

A **zero-class** of it is the arrow $i\colon I \to Y$ in the pullback

$$\begin{tikzcd} I \arrow[r, "l"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "{(d,c)}"] \\ Y \arrow[r, "{(0,1_Y)}"'] & X \times Y. \end{tikzcd} \tag{B}$$

Clearly if (d, c) is a relation—when d and c are jointly monomorphic—then its zero-class is a monomorphism, since pullbacks preserve monomorphisms.

Definition 2.4. A **normalisation** of $(\mathbf{A})$ is the composite $ck\colon K \to X$, where $k\colon K \to R$ is a kernel of d.

Again, (d, c) being a relation implies that the normalisation is a monomorphism. Of course the zero-class and the normalisation of a span are unique up to isomorphism, so (abusing terminology) we may talk about "the" zero-class and "the" normalisation. In fact, the two procedures give the same result:

Proposition 2.5. *For any span (d, c) its zero-class coincides with its normalisation.*

Proof: It is easily seen that the morphism l in the diagram $(\mathbf{B})$ is a kernel of d. As a consequence, $i = cl$. On the other hand, any square such as $(\mathbf{B})$ in which $l = \ker(d)$ and $i = cl$ is a pullback. ■

Definition 2.6. A **normal subobject** of an object A is the zero-class of an equivalence relation on A.

We observe that this notion is a generalisation of the notion of **kernel** of a morphism: indeed, kernels are exactly zero-classes of effective equivalence relations. It is also easy to see that, in the pointed case, the definition above is equivalent to the one introduced by Bourn in [3]: see [12] and Example 3.2.4, Proposition 3.2.12 in [2].

Definition 2.7. A **clot** of A is the zero-class of a reflexive relation on A.

The original categorical definition of clot, given in [9], was different: roughly speaking, a clot of an object A was defined as a subobject which is invariant under the *conjugation action* on A. However, the two definitions are equivalent, as already observed in [9].

The following categorical definition of ideal was proposed in [10]. It was observed in [9] that, in the varietal case, it coincides with Definition 2.1 above.

Definition 2.8. A monomorphism $i \colon I \to Y$ is an **ideal** if there exists a commutative square

$$\begin{tikzcd} K \arrow[r, two heads, "q"] \arrow[d, tail, "k"'] & I \arrow[d, tail, "i"] \\ X \arrow[r, two heads, "p"'] & Y \end{tikzcd} \qquad (\mathbf{C})$$

in which p and q are regular epimorphisms and k is a clot.

The following fact was already observed in [10, Corollary 3.1]:

Proposition 2.9. *Every ideal is the regular image of a kernel along a regular epimorphism.*

Proof: Proposition 2.5 tells us that the morphism k in Diagram $(\mathbf{C})$ is of the form cl for some kernel l. The claim now follows, since a composite of two regular epimorphisms in a regular category is still a regular epimorphism. ■

The first aim of this paper is to characterise the ideals as the zero-classes of suitable relations. Before doing that, we prove some stability properties of ideals.

Proposition 2.10. *Ideals are stable under:*

(a) *direct images;*
(b) *pullbacks;*
(c) *intersections;*
(d) *compositions with product inclusions.*

Proof: (a) is immediate from the definition and (b) holds because, in a pointed regular category, pullbacks preserve regular epimorphisms, normalisations and reflexive relations.

For the proof of (c), first suppose i is an ideal as in $(\mathbf{C})$ and $l\colon L \to Y$ is a clot. We consider the commutative cube

$$\begin{tikzcd}
K \cap L' \arrow[rr, dotted, two heads] \arrow[dr, tail] \arrow[dd, tail] & & I \cap L \arrow[dr] \arrow[dd, tail] & \\
& L' \arrow[rr, two heads, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & L \arrow[dd, tail, "l"] \\
K \arrow[rr, two heads, "q", near start] \arrow[dr, tail, "k"'] & & I \arrow[dr, tail, "i"] & \\
& X \arrow[rr, two heads, "p"'] \arrow[from=uu, tail, crossing over, "p^*l"'] & & Y
\end{tikzcd}$$

in which the front, left and right squares are pullbacks by construction. Hence the back square is also a pullback, so that the dotted arrow is a regular epimorphism. Since the intersection $K \cap L'$ is still a clot, this proves that $I \cap L$ is an ideal (we are using the fact that clots are stable under intersections and pullbacks).

Now suppose that both i and l are ideals. Repeating the above, through (b) we see that $K \cap L'$ is an ideal as the intersection of the clot k with the ideal p^*l. The result now follows from (a).

For the proof of (d), recall that kernels compose with product inclusions: if $k\colon K \to X$ is the kernel of $f\colon X \to X'$, then $(1_W, 0)k = (k, 0)\colon K \to X \times W$ is the kernel of $f \times 1_W\colon X \times W \to X' \times W$. If now i is an ideal as in (**C**), then $(1_W, 0)i = (i, 0)\colon I \to Y \times W$ is the direct image of $(k, 0)$ along the regular epimorphism $f \times 1_W$. ∎

3. Ideals and semi-split surjective relations

In order to characterise ideals as zero-classes, we shall be interested in spans where one of the legs is a regular or even a split epimorphism.

Definition 3.1. A **left split span** from X to Y is a diagram

$$\begin{tikzcd} & R \arrow[dl, shift right, "d"'] \arrow[dr, "c"] & \\ X \arrow[ur, shift right, "e"'] & & Y \end{tikzcd} \qquad (\mathbf{D})$$

where $de = 1_X$. A left split span (d, c, e) is called a **left split relation** when the span (d, c) is jointly monomorphic.

Proposition 3.2. *A morphism $i\colon I \to Y$ is a monomorphism if and only if it is the zero-class of a left split relation if and only if it is a zero-class of a relation on Y.*

Proof: For the first equivalence it suffices to take $X = 0$, and for the second we consider the relation $(0, i)$ on Y. ∎

Definition 3.3. A **surjective span** from X to Y is a diagram

$$\begin{tikzcd} & R \arrow[dl, "d"'] \arrow[dr, two heads, "c"] & \\ X & & Y \end{tikzcd}$$

where c is a regular epimorphism. A surjective span (d, c) is called a **surjective relation** when the span (d, c) is jointly monomorphic.

Sometimes we consider both conditions together and talk about **surjective left split** spans or relations.

We are now ready to prove our main result.

Theorem 3.4. *In any pointed regular category, for any morphism* $i\colon I \to Y$*, the following are equivalent:*

(i) *i is an ideal;*
(ii) *i is the zero-class of a surjective left split relation;*
(iii) *i is the zero-class of a surjective relation.*

Proof: To prove (i) $\Rightarrow$ (ii), suppose that i is an ideal as in ($\mathbf{C}$) above, where k is the zero-class of a reflexive relation (R, d, c, e). We consider the commutative cube

$$\begin{tikzcd}
K \arrow[rr, two heads, "q"] \arrow[dr] \arrow[dd, tail, "k"'] & & I \arrow[dd, tail, "i"' near start] \arrow[dr, dotted] & \\
& R \arrow[rr, dotted, two heads, "q'" near start] \arrow[dd, tail, "{(d,c)}"' near start] & & S \arrow[dd, dotted, tail, "{(d',c')}"] \\
X \arrow[rr, two heads, "p" near start] \arrow[dr, "{(0,1_X)}"'] & & Y \arrow[dr, "{(0,1_Y)}"'] & \\
& X \times X \arrow[rr, two heads, "1_X \times p"'] & & X \times Y
\end{tikzcd}$$

in which S is the direct image of R along $1_X \times p$ and $I \to S$ is induced by functoriality of image factorisations. We have to show that the square on the right is a pullback. Let the square on the left

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail] \\
Y \arrow[r, "{(0,1_Y)}"'] & X \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[rr, dotted] \arrow[dr] \arrow[dd, tail, "k"'] & & P \arrow[dd, tail] \arrow[dr] & \\
& R \arrow[rr, two heads, "q'" near start] \arrow[dd, tail, "{(d,c)}"' near start] & & S \arrow[dd, tail, "{(d',c')}"] \\
X \arrow[rr, two heads, "p" near start] \arrow[dr, "{(0,1_X)}"'] & & Y \arrow[dr, "{(0,1_Y)}"'] & \\
& X \times X \arrow[rr, two heads, "1_X \times p"'] & & X \times Y
\end{tikzcd}$$

be the pullback in question; then the induced arrow $I \to P$ is an isomorphism, because is both a monomorphism (since i is) and a regular epimorphism (since the dotted arrow $K \to P$ is, as a pullback of the regular epimorphism q', the bottom and left squares in the cube being pullbacks). Note that d' is split by $q'e$ and c' is a regular epimorphism because $pc = c'q'$ is.

(ii) $\Rightarrow$ (iii) is obvious. For the proof of (iii) $\Rightarrow$ (i), let $i\colon I \to Y$ be the zero-class (**B**) of a surjective relation (d, c). Consider the pullback

$$
\begin{tikzcd}
T \arrow[r, dotted] \arrow[d, tail, dotted, "{(d',c')}"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "{(d,c)}"] \\
R \times R \arrow[r, "d\times c"'] & X \times Y
\end{tikzcd}
$$

of (d, c) and $d \times c$, which defines a reflexive relation (T, d', c', e') on R where e' is $((1_R, 1_R), 1_R)$. We prove that i is the direct image of the zero-class k of T along the regular epimorphism c as in the square on the left.

$$
\begin{tikzcd}
K \arrow[r, two heads, "q"] \arrow[d, tail, "k"'] & I \arrow[d, tail, "i"] \\
R \arrow[r, two heads, "c"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[rr, dotted, two heads, "q"] \arrow[dr] \arrow[dd, tail, "k"'] & & I \arrow[dr] \arrow[dd, tail, "i"' near start] & \\
& T \arrow[rr, two heads] \arrow[dd, tail, "{(d',c')}" near start] & & R \arrow[dd, tail, "{(d,c)}"] \\
R \arrow[rr, two heads, "c" near start] \arrow[dr, "{(0,1_R)}"'] & & Y \arrow[dr, "{(0,1_Y)}"'] & \\
& R \times R \arrow[rr, two heads, "d\times c"'] & & X \times Y
\end{tikzcd}
$$

Here it suffices to consider the cube on the right, noting that q is a regular epimorphism because all vertical squares are pullbacks and c is a regular epimorphism by assumption. ∎

As the following example shows, in general it is not possible to see any ideal as a zero-class of a surjective endorelation. We are grateful to Sandra Mantovani for suggestions concerning this example.

Example 3.5. Let $\mathscr{C}$ be the variety defined by a unique constant 0 and a binary operation s satisfying just the identity $s(0, 0) = 0$. In this variety, ideal terms are all "pure": in any term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ which is an ideal term in y_1, $\ldots$, y_n, necessarily $m = 0$. Therefore all subalgebras are ideals. Consider then the three element algebra $A = \{0, 1, a\}$, with $s(a, 1) = s(1, a) = s(a, a) = a$, and $s(x, y) = 0$ otherwise. $C = \{0, 1\}$ is a subalgebra, and we have that $s(a, 0) = 0$ lies in C, but $s(a, 1) = a$ does not belong to C. Hence C is an ideal, but not a clot. Suppose that there exists a surjective relation R on A such that C is its zero-class. Then there should exist $x \in A$ such that xRa. But it cannot be $0Ra$, because $a \notin C$. $1Ra$ is impossible, too, because otherwise $0 = s(1, 1)Rs(a, a) = a$. Similarly, aRa is impossible, otherwise $0 = s(0, a)Rs(1, a) = a$. Hence such a surjective endorelation R cannot exist.

We conclude this section with the following observation. It is well known [12] that, in any pointed exact Mal'tsev category, ideals and kernels coincide. Theorem 3.4 provides us with the following quick argument.

Proposition 3.6. *In any pointed exact Mal'tsev category, ideals and kernels coincide.*

Proof: Let $i\colon I \to Y$ be the zero-class of a surjective left split relation (d, c, e) as in (**D**). Theorem 5.7 in [5] tells us that the pushout of d and c is also a pullback; as a consequence, i is the kernel of the pushout $c_*(d)$ of d along c. ∎

4. The *Smith is Huq* condition

From now on we work in a category which is pointed, regular and weakly Mal'tsev [14]. We first recall the meaning of the last condition.

Definition 4.1. A finitely complete category is weakly Mal'tsev if, for any pullback of a split epimorphism along a split epimorphism:

$$\begin{tikzcd}
A \times_B C \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \\
A \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

the morphisms $e_1 = (1_A, sf)$ and $e_2 = (rg, 1_C)$, induced by the universal property of the pullback, are jointly epimorphic.

In this context, we say that two left split spans (f, α, r) and (g, γ, s) from B to D as in

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\ & D & \end{tikzcd} \tag{E}$$

centralise each other or **commute** when there exists a (necessarily unique) morphism $\varphi\colon A \times_B C \to D$, called **connector** from the pullback

$$\begin{tikzcd} & C \arrow[dl, shift left, "e_2"] \arrow[dr, shift left, "g"] \arrow[drr, bend left, "\gamma"] & & \\ A \times_B C \arrow[ur, shift left, "\pi_2"] \arrow[dr, shift left, "\pi_1"] & & B \arrow[ul, shift left, "s"] \arrow[dl, shift left, "r"] \arrow[r, dotted, "\beta"] & D \\ & A \arrow[ul, shift left, "e_1"] \arrow[ur, shift left, "f"] \arrow[urr, bend right, "\alpha"'] & & \end{tikzcd}$$

of f and g to the object D such that $\varphi e_1 = \alpha$ and $\varphi e_2 = \gamma$. Note that, when this happens, $\gamma s = \alpha r$; we call this morphism $\beta\colon B \to D$. In other words, the existence of β is a necessary condition for the given left split spans to centralise each other. Note how this explains the *admissibility* condition from [13] in conceptual terms: clearly that condition deals with a certain type of commutativity, but without the notion of left split span we could not express precisely *what* commutes.

If we take $\beta = 1_B$, we immediately recover the notion of commutativity of reflexive graphs in the Smith–Pedicchio sense:

Proposition 4.2. *Two reflexive graphs commute in the Smith–Pedicchio sense if and only if they commute in the above sense.* ■

We recall that the commutativity of equivalence relations was first introduced by Smith in [18] for Mal'tsev varieties, and then extended by Pedicchio [17] to Mal'tsev categories. However, weakly Mal'tsev categories are a suitable setting for the definition (because the connector, as defined above, is unique), and the commutativity can be defined, as above, just for reflexive graphs.

If, in the diagram (**E**), we take $B = 0$, we get the definition of commutativity of two morphisms in the Huq sense [8]: two morphisms $\alpha\colon A \to D$ and $\gamma\colon C \to D$ **commute** when there exists a (necessarily unique) morphism $\varphi\colon A \times C \to D$, called the **cooperator** of α and γ, such that

$$\varphi(1_A, 0) = \alpha \qquad \text{and} \qquad \varphi(0, 1_C) = \gamma.$$

A pointed regular weakly Mal'tsev category satisfies the **Smith is Huq** condition, shortly denoted by (SH), when a pair of equivalence relations over the same object commutes as soon as their zero-classes do. (The converse is always true). We observe that the (SH) condition has the following interesting consequence. We recall that an object A is **commutative** if its identity commutes with itself (in the Huq sense); it is **abelian** if it has an internal abelian group structure.

Proposition 4.3. *If* (SH) *is satisfied, then every commutative object is abelian.*

Proof: The identity 1_X of an object X is the normalisation of the indiscrete relation ∇_X. If X is commutative, then 1_X commutes with itself; thanks to the (SH) condition, the relation ∇_X commutes with itself, too. This situation is represented by the following diagram:

$$\begin{tikzcd}
& X \times X \times X \arrow[dl, shift right] \arrow[dr, shift left] \arrow[dd, "p"] & \\
X \times X \arrow[ur, shift right] \arrow[dr, shift left] & & X \times X \arrow[ul, shift left] \arrow[dl, shift right] \\
& X. \arrow[ul, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

The connector $p\colon X \times X \times X \to X$ is then an internal Mal'tsev operation on X. To conclude the proof it suffices to observe that, in a pointed category, an object is endowed with an internal Mal'tsev operation if and only if it is endowed with an internal abelian group structure [2, Proposition 2.3.8]. ■

Our aim is to extend the (SH) condition to surjective left split relations and ideals. In order to do so, we start by introducing some terminology. We call a morphism **ideal-proper** when its image is an ideal; we say that a cospan is **ideal-proper** when so are the morphisms of which it consists.

In a pointed finitely complete category $\mathscr{C}$, given an object B, the category $\mathsf{Pt}_B(\mathscr{C})$ of so-called **points over** B is the category whose objects are couples $(p\colon E \to B, s\colon B \to E)$ where $ps = 1_B$, so split epimorphisms with codomain B and a chosen section. A morphism $(p\colon E \to B, s) \to (p'\colon E' \to B, s')$ in $\mathsf{Pt}_B(\mathscr{C})$ is a morphism $f\colon E \to E'$ in $\mathscr{C}$ such that $p'f = p$ and $fs = s'$. We have, for any B, a functor (called the **kernel functor**) $\mathrm{Ker}_B\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathscr{C}$ associating with every split epimorphism its kernel. We can now formulate the main result of this section.

Theorem 4.4. *In any pointed, regular and weakly Mal'tsev category $\mathscr{C}$, the following are equivalent:*

(i) *for every object B in $\mathscr{C}$, the kernel functor* $\mathrm{Ker}_B \colon \mathsf{Pt}_B(\mathscr{C}) \to \mathscr{C}$ *reflects Huq-commutativity of ideal-proper cospans;*
(ii) *a pair of surjective left split relations over the same objects commutes as soon as their zero-classes do;*
(iii) *a pair of surjective left split spans over the same objects commutes as soon as their zero-classes do.*

Proof: The equivalence between conditions (ii) and (iii) is proved just by taking direct images. In order to prove that (i) and (ii) are equivalent, given a pair of surjective left split relations over the same object, we rewrite Diagram (**E**) in the shape

$$\begin{tikzcd}
A \arrow[r, "{(\alpha,f)}"] \arrow[dr, shift left, "f"'] & D \times B \arrow[d, shift right, "\pi_B"'] & C \arrow[l, "{(\gamma,g)}"'] \arrow[dl, shift right, "g"] \\
& B \arrow[ul, shift left, "r"] \arrow[u, shift right, "{(\beta,1_B)}"'] \arrow[ur, shift right, "s"'] &
\end{tikzcd} \tag{F}$$

and consider it as a cospan $((\alpha, f), (\gamma, g))$ in $\mathsf{Pt}_B(\mathscr{C})$. Let us prove that this cospan is ideal-proper. To do that, it suffices to notice that (α, f) is the composite of the kernel $(1_A, f) \colon A \to A \times B$ with the regular epimorphism $\alpha \times 1_B \colon A \times B \to D \times B$. Indeed, the outer square in the diagram

$$\begin{tikzcd}
A \arrow[rr, "{(1_A,f)}"] \arrow[dd, "f"'] \arrow[dr, shift left, "f"] & & A \times B \arrow[dd, "f \times 1_B"] \arrow[dl, shift left, "\pi_B"] \\
& B \arrow[ul, shift left, "r"] \arrow[ur, shift left, "{(r,1_B)}"] \arrow[dl, equal] \arrow[dr, shift left, "{(1_B,1_B)}"] & \\
B \arrow[rr, "{(1_B,1_B)}"'] & & B \times B \arrow[ul, shift left, "\pi_2"]
\end{tikzcd}$$

is a pullback in $\mathsf{Pt}_B(\mathscr{C})$. The same is true for (γ, g). To conclude the proof of the equivalence between (i) and (ii) it suffices then to observe that applying the kernel functor Ker_B to the cospan (**F**) gives the normalisations of the two surjective split relations. ∎

It is immediately seen that condition (i) above is equivalent to the condition that, for every morphism $p\colon E \to B$ in $\mathscr{C}$, the pullback functor

$$p^*\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathsf{Pt}_E(\mathscr{C})$$

—which sends every split epimorphism over B into its pullback along p—reflects Huq-commutativity of ideal-proper cospans. In the same way as for the previous theorem, it can be shown that also the following conditions, which are strictly stronger than (i)–(iii) above, are equivalent.

Proposition 4.5. *In any pointed, regular and weakly Mal'tsev category $\mathscr{C}$, the following are equivalent:*

(iv) *for every object B in $\mathscr{C}$, the kernel functor* $\mathrm{Ker}_B\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathscr{C}$ *reflects Huq-commutativity of cospans;*

(v) *a pair of left split relations over the same objects commutes as soon as their zero-classes do;*

(vi) *a pair of left split spans over the same objects commutes as soon as their zero-classes do.* ■

Again, condition (iv) can be expressed equivalently in terms of all pullback functors $p^*\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathsf{Pt}_E(\mathscr{C})$.

We conclude by observing that Theorem 4.4 and Proposition 4.5 restrict to Propositions 2.5 and 3.1 in [16] (see also Theorem 2.1 in [4]) in the pointed exact Mal'tsev case, because there ideals coincide with kernels (Proposition 3.6), and the proof is based on the same ideas. In particular, then the conditions (i)–(iii) are equivalent to (SH), while (iv)–(vi) are equivalent to the strictly stronger condition (W)—see [16].

It is an open question whether the conditions (i)–(iii) are equivalent to (SH) in pointed *weakly* Mal'tsev *regular* categories.

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