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LAX ORTHOGONAL FACTORISATION SYSTEMS

MARIA MANUEL CLEMENTINO AND IGNACIO LÓPEZ FRANCO

Abstract: This paper introduces *lax orthogonal algebraic weak factorisation systems* on 2-categories and describes a method of constructing them. The related notions of *KZ lifting operation*, *lax natural lifting operation* and *lax orthogonality* between morphisms are studied, and a number of examples provided. These examples rest in the notion of *simple 2-monad*, that is a generalisation of the simple reflections studied by Cassidy, Hébert and Kelly. Each simple 2-monad on a finitely complete 2-category gives rise to a lax orthogonal algebraic weak factorisation system. Examples of simple 2-monads are: completion under a class of colimits, the filter monad on topological spaces and Cauchy completion on Lawvere's metric spaces.

Keywords: Lax idempotent algebraic weak factorisation system, algebraic weak factorisation system, weak factorisation system, lax idempotent 2-monad, simple reflection.
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1. Introduction

This paper contains four main contributions: the introduction of *lax orthogonal algebraic weak factorisation systems* (AWFSs); the introduction of the concept of *KZ diagonal fillers* and the study of their relationship to lax orthogonal AWFSs; the introduction of *simple 2-monads*, and the proof that each such induces an AWFSs; the proof that some well-known 2-monads, on topological spaces, on categories and on Lawvere metric spaces, are simple, and a description of the corresponding induced factorisations.

Weak factorisation systems form the basic ingredient of *Quillen model structures* [20], and, as the name indicates, are a weakening of the ubiquitous orthogonal factorisation systems. A weak factorisation system (WFS) on a category consists of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ satisfying two

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properties: every morphism can be written as a composition of a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$, and for any commutative square, with vertical morphisms in $\mathcal{L}$ and $\mathcal{R}$ as depicted, there exists a diagonal filler. One says that r has the right lifting property with respect to ℓ and that ℓ has the left lifting property with respect to r.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \cdot \arrow[d, "r\in\mathcal{R}"] \\ \cdot \arrow[r] \arrow[ur, dotted, "\exists"] & \cdot \end{tikzcd} \tag{1.1}$$

When r is an identity 1_C, one usually says that C is injective with respect to ℓ.

In order to unify the study of injectivity with respect to different classes of continuous maps between $T0$ topological spaces, Escardó [8] employed lax idempotent 2-monads, also known as KZ monads, on poset-enriched categories – these are the same as 2-categories whose hom-categories are posets. For example, if T is such a lax idempotent 2-monad, the T-algebras can be described as the objects A that are injective to all the morphisms ℓ such that $T\ell$ is a coretract left adjoint – a T-embedding. A central point is that not only each morphism $\mathrm{dom}(\ell) \to A$ has an extension along ℓ, but moreover it has a *least* extension: one that is smallest amongst all extensions.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & A \\ \cdot \arrow[ur, dotted] \arrow[ur, bend right=50] \arrow[ur, phantom, shift right=2, "\leqslant"] & \end{tikzcd}$$

The assignment that sends a morphism to its least extension can be described in terms of the 2-monad T, so one no longer has the *property* of the existence of at least one extension, but the *algebraic structure* that constructs the extension. If one wants to describe WFSs in this context, instead of just injectivity, in the sense that r may not be an identity, one is led to consider algebraic weak factorisation systems (AWFSs), to which we shall return in this introduction.

1.1. Injective continuous maps. One of the basic examples that fit in the framework of [8] is that of the filter monad on the category of $T0$ spaces, that assigns to each space its space of filters of open sets. It was shown in [6] that the algebras for this monad are the topological spaces that arise as continuous lattices with the Scott topology. These spaces were known to be precisely those injective with respect to subspace embeddings [22]. In [3] this and other related results are generalised, characterising those continuous maps of $T0$ spaces that have the right lifting property with respect to different classes of embeddings, and exhibiting for each a WFS in the category of $T0$ spaces. A morphism $f\colon X \to Y$ is factorised through the subspace $Kf \subseteq TX \times Y$ of those (φ, y) such that $Tf(\varphi) \leqslant \{U \in \mathcal{O}(Y) : y \in U\}$. The monad T can be the filter monad or a variant of it.

$$
\begin{tikzcd}
X \arrow[dr, "\lambda_f"] \arrow[drr, bend left, "i_X"] \arrow[ddr, bend right, "f"'] & & \\
& Kf \arrow[r] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\geqslant"] & TX \arrow[d, "Tf"] \\
& Y \arrow[r] & TY
\end{tikzcd}
\tag{1.2}
$$

Central to the arguments in [3] is the fact that the monad $f \mapsto \rho_f$ is lax idempotent or KZ. This property is intimately linked with the fact that $T\lambda_f$ is always an embedding of the appropriate variant – ie λ_f is a T-embedding.

The construction of the factorisation of maps just described resembles the classical case of simple reflections [4]. One of the aims of the present paper is to show that both constructions are particular instances of a more general one.

1.2. Algebraic weak factorisation systems. Algebraic weak factorisation systems (AWFSs) were introduced in [11] with the name *natural weak factorisation systems*, with a distributive axiom later added in [10]. Many of the factorisations systems that occur in practice provide a construction for the factorisation of an arbitrary morphism. Such a structure on a category $\mathcal{C}$ is called a *functorial factorisation* and can be described in more than one way: as a functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ that is compatible with domain and codomain; as a codomain-preserving – ie with identity codomain component – pointed endofunctor $\Lambda\colon 1 \Rightarrow R$ of $\mathcal{C}^{\mathbf{2}}$; as a domain-preserving copointed endofunctor $\Phi\colon L \Rightarrow 1$ of $\mathcal{C}^{\mathbf{2}}$. Then, a morphism f factors as $f = Rf \cdot Lf$. Any such functorial factorisation has an underlying WFS $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ consists of those morphisms that admit an (L, Φ)-coalgebra structure and $\mathcal{R}$ of those that admit an (R, Λ)-algebra structure. One wants, however, to guarantee that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$, for which one requires extra data in the form of a comultiplication that makes (L, Φ) into a comonad L and a multiplication that makes (R, Λ) into a monad R. The pair (L, R), together with an extra distributivity condition, is an AWFS.

The underlying WFS of an AWFS (L,R) is an orthogonal factorisation system precisely when L and R are idempotent [11]; it is enough if either is idempotent [2].

All the above constructions can be performed on 2-categories instead of categories. Two morphisms $\ell\colon A \to B$ and $r\colon C \to D$ in a 2-category $\mathscr{K}$ are *lax orthogonal* when the comparison morphism

$$\mathscr{K}(B,C) \to \mathscr{K}(A,C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B,D)$$

has a left adjoint coretract. – Compare with the usual definition of weak orthogonality and orthogonality where it must be an epimorphism and, respectively, an isomorphism. – Such a left adjoint provides diagonal fillers that moreover satisfy a universal property with respect to 2-cells. A choice of diagonal fillers like these that is in addition natural with respect to ℓ and r we call a *KZ lifting operation.*

When the 2-category $\mathscr{K}$ is locally a preorder, the lax orthogonality of ℓ and r reduces to the statement, encountered before in this introduction, that for each commutative square (1.1) there exists a least diagonal filler.

One could then ask what is the property on an AWFS that corresponds to the existence of KZ diagonal fillers. The answer is that both the 2-comonad and the 2-monad of the AWFS must be *lax idempotent* – Theorem 6.6. Equivalently, either the 2-comonad or the 2-monad must be lax idempotent – Section 4. This last statement mirrors the case of AWFSs whose underlying WFS is orthogonal, for which, as mentioned earlier, it is enough that either the comonad or the monad be idempotent.

A basic example of a lax idempotent AWFS is the one that factors a functor $f\colon A \to B$ as a left adjoint coretract $A \to f \downarrow B$ followed by the split opfibration $f \downarrow B \to B$. We refer to this AWFS as the coreflection–opfibration AWFS.

1.3. Simple reflections. The paper [4] studies the relationship between orthogonal factorisation systems, abbreviated OFSs, and reflections. Every OFS $(\mathcal{E},\mathcal{M})$ in a category $\mathcal{C}$ induces a reflection on $\mathcal{C}$ as long as $\mathcal{C}$ has a terminal object 1; the reflective subcategory is $\mathcal{M}/1$, the full subcategory of those objects X such that $X \to 1$ belongs to $\mathcal{M}$. Under certain hypotheses, a reflection T on $\mathcal{C}$ induces an OFS. One of the possible hypotheses is that T be *simple*, which means that for any morphism f the dashed morphism into the pullback depicted below is inverted by T. The factorisation of f is then given by $f = \rho_f \cdot \lambda_f$, and the left class of morphisms consists of those which are inverted by T.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\lambda_f"] \arrow[drr, bend left, "\rho_f"] \arrow[ddr, bend right, "f"'] & & \\
& Kf \arrow[r] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\text{p.b.}"] & TA \arrow[d, "Tf"] \\
& B \arrow[r] & TB
\end{tikzcd}$$

One way of expressing the construction of the OFS from T is the following. On any category $\mathcal{A}$ we have the OFS $(\mathrm{Iso}, \mathrm{Mor})$, with left class the isomorphisms and right class all morphisms. Isomorphisms are the coalgebras for the idempotent comonad L' on $\mathcal{A}^{\mathbf{2}}$ given by $L'(f) = 1_{\mathrm{dom}(f)}$. If $F \dashv U \colon \mathcal{A} \hookrightarrow \mathcal{C}$ is the adjunction induced by the reflection T, the copointed endofunctor (L, Φ) defined by pullback along the unit of the adjunction satisfies the property that the rectangle on the right hand side below is a pullback. In other words, (L, Φ)-coalgebras are those morphisms that are inverted by F, equivalently by T. Any morphism that is inverted by T is orthogonal to Tf and therefore to its pullback ρ_f; in particular, λ_f satisfies this if the reflection is simple. Therefore, we obtain an OFS when T is simple, with left class those morphisms that are inverted by T.

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\Phi"'] & U^{\mathbf{2}}L'F^{\mathbf{2}} \arrow[d, "\Phi'"] \\
1 \arrow[r] & U^{\mathbf{2}}F^{\mathbf{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
(L,\Phi)\text{-Coalg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & (L',\Phi')\text{-Coalg} \arrow[d] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"] & \mathcal{A}^{\mathbf{2}}
\end{tikzcd}
\tag{1.3}$$

An alternative way to prove that we obtain an OFS is to show that (L, Φ) has an extension to an idempotent comonad. The comultiplication $\Sigma \colon L \Rightarrow L^2$ is the morphism that corresponds to the pair of morphisms $\Sigma_0 \colon L \Rightarrow U^{\mathbf{2}}L'F^{\mathbf{2}}L$ and $1 \colon L \to L$, where Σ_0 is the transpose of the transformation $F^{\mathbf{2}}L \Rightarrow L'F^{\mathbf{2}}L$ with component at f the (L', Φ')-coalgebra structure of $F\lambda_f$, ie $(1, (F\lambda_f)^{-1}) \colon 1_{\mathrm{dom}(f)} \longrightarrow F\lambda_f$. The pointed endofunctor (R, Λ) given by $f \mapsto \rho_f$ underlies a monad by construction.

The above analysis can be adapted to the case where categories are substituted by 2-categories and OFSs by lax orthogonal AWFSs. Reflections are substituted by lax idempotent 2-monads, idempotent (co)monads by lax idempotent 2-(co)monads, the simple reflections by appropriately defined simple 2-adjunctions or simple 2-monads. The reflective subcategory Iso of the arrow category is substituted by the lax idempotent 2-comonad whose algebras are coretract left adjoints, while Mor is substituted by the free split opfibration 2-monad.

1.4. Simple 2-monads. Generalising the construction on topological spaces described in Section 1.1 above, one can consider a factorisation of f as depicted in (1.2) but where the inequality is a general 2-cell of a comma object, and ask when does $f = \rho_f \cdot \lambda_f$ arise from a lax idempotent AWFS. This is analogous to the construction of OFSs from simple reflections discussed in the previous section.

A 2-monad T is *simple* if $T\lambda_f$ has a certain right adjoint retract. If T is moreover a lax idempotent 2-monad on $\mathscr{K}$, and $(\mathsf{L}', \mathsf{R}')$ is the coreflection–opfibration AWFS on $\mathsf{T}\text{-Alg}_s$, the copointed endofunctor (L, Φ) constructed as in (1.3), and given by $f \mapsto \lambda_f$, can be extended to a 2-comonad, and the pointed endofunctor $f \mapsto \rho_f$ can be extended to a 2-monad that combine into a lax idempotent AWFS on $\mathscr{K}$.

The filter 2-monad on topological spaces is lax idempotent, where the category of topological spaces is made into a 2-category by the opposite of the specialisation order. We show that it is simple, therefore inducing a lax orthogonal AWFS on the category of topological spaces.

Given a class of colimits, there exists a 2-monad on **Cat** whose algebras are categories with chosen colimits of that class. We show that these 2-monads are simple, giving rise to AWFSs on **Cat**.

Another example of simple 2-monad is the one given by Cauchy completion on the 2-category of Lawvere metric spaces – categories enriched in the extended non-negative real numbers. To give an idea about the lax orthogonal AWFS within the space constraints of this introduction, one can look at maps between *metric* spaces. Left maps between metric spaces are dense isometries. Right maps $f\colon A \to B$ between metric spaces are those distance decreasing maps with the property that each Cauchy sequence in A, such that its image under f converges to a point $b \in B$, converges to a point of A over b.

When all the 2-categories involved are in fact categories, lax idempotent 2-monads reduce to reflections and our concept of simple 2-monad to the one of simple reflection. Therefore, we know that there are lax idempotent 2-monads that are not simple, as [4] gives examples of reflections that are not simple.

1.5. Description of sections. Section 2 can be regarded as a fairly self-contained recount of the basic definitions and properties of algebraic weak factorisations systems. The approach to diagonal fillers via modules or profunctors appears to be novel.

One of our main tools will be the lax idempotent 2-(co)monads, facts about which we put together at the beginning of Section 3, before introducing lax orthogonal AWFS, our main subject of study.

Section 4 proves that in order for an AWFS to be lax orthogonal it suffices that either the 2-monad or the 2-comonad be lax idempotent.

In a 2-category one can consider the usual lifting operations, but also lax natural ones. We define lax natural and KZ diagonal fillers in Section 5 and prove that lax orthogonal AWFSs give rise to KZ diagonal fillers. Lax orthogonal functorial factorisations are briefly considered.

In Section 6 we characterise lax orthogonal AWFSs as those AWFS (L, R) for which R-algebras are algebraically KZ injective to all L-coalgebras, or equivalently, for which natural KZ diagonal fillers exist for squares from L-coalgebras to R-algebras.

Sections 7 to 9 are perhaps more technical, and give conditions that allow the coreflection–opfibration lax orthogonal AWFS on a 2-category $\mathscr{A}$ to be transferred along a left 2-adjoint $\mathscr{B} \to \mathscr{A}$ to a lax idempotent AWFS on $\mathscr{B}$. Section 10 studies the case when the left 2-adjoint is the free algebra 2-functor of a special kind of 2-monad, that we call *simple* 2-monad, as it generalises the notion of simple reflection [4]. Conditions that guarantee that a lax idempotent 2-monad is simple are provided.

Section 11 looks at the case of locally preordered 2-categories, or categories enriched in the category of preorders, where various simplifications take place, especially in the case where the morphisms in the right part of the factorisation system are *fibrewise posetal*; this is the case in some of our examples.

The example of the filter monad on topological spaces is spelled out in Section 12, recovering a WFS considered in [3].

Section 13 studies the example of completion under colimits. We show that for a class of colimits Φ, the 2-monad on **Cat** whose algebras are categories with *chosen* colimits of that class is simple, whence inducing a lax orthogonal AWFS (L, R). We prove in Section 13.2 that R-algebras are always split opfibrations with fibrewise chosen Φ-colimits, but the converse does not always hold, as shown by Section 13.3.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\lambda_f"'] & C \arrow[d, "\lambda_g"] \\ Kf \arrow[r, "K(h,k)"] \arrow[d, "\rho_f"'] & Kg \arrow[d, "\rho_g"] \\ B \arrow[r, "k"] & C \end{tikzcd}$$

FIGURE 1. A functorial factorisation.

Lastly, a lax orthogonal AWFS in generalised or Lawvere metric spaces – categories enriched in the category of extended non-negative real numbers [18, 19] – is described in Section 14 as the AWFS induced by Cauchy completion.

2. Background on algebraic weak factorisation systems

In this section we recall notions related to algebraic weak factorisation systems [11], following [10] for the most part.

2.1. Basic definitions. Given a category $\mathcal{C}$ consider the functors d_0, d_1, $d_2\colon \mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$ that send a pair of composable morphisms $(f\colon A \to B, g\colon B \to C)$ in $\mathcal{C}$ to: $d_0(f,g) = f$, $d_1(f,g) = g \cdot f$, $d_2(f,g) = g$.

A *functorial factorisation* in $\mathcal{C}$ is a functor $K\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ such that $d_1 K = 1$. This means that for each morphism $(h,k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ we have a factorisation, functorial in (h,k), depicted in Figure 1.

A functorial factorisation as above induces a pointed endofunctor $\Lambda\colon 1 \Rightarrow R$ and a copointed endofunctor $\Phi\colon L \Rightarrow 1$ on $\mathcal{C}^{\mathbf{2}}$. The endofunctor L is given by $Lf = \lambda_f$, and the component of the copoint Φ at the object f is depicted on the left hand side of (2.1). Similarly, $Rf = \rho_f$, and the component of the point Λ at the object f is depicted on the right hand side of (2.1). We note that $\mathrm{dom}L = \mathrm{dom}$ and $\mathrm{cod}R = \mathrm{cod}$, as functors $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\lambda_f"'] & A \arrow[d, "f"] \\ Kf \arrow[r, "\rho_f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\lambda_f"] \arrow[d, "f"'] & Kf \arrow[d, "\rho_f"] \\ B \arrow[r, equal] & B \end{tikzcd} \tag{2.1}$$

An *algebraic weak factorisation system* [11, 10] is a functorial factorisation where the copointed endofunctor $\Phi\colon L \Rightarrow 1$ is equipped with a comultiplication $\Sigma\colon L \Rightarrow L^2$, making it into a comonad L, and the pointed endofunctor $\Lambda\colon 1 \Rightarrow R$ is equipped with a multiplication $\Pi\colon R^2 \Rightarrow R$, making it into a monad R, plus a distributivity condition. The components of this comultiplication and multiplication will be denoted by as follows.

$$\Sigma_f = \begin{tikzcd} A \arrow[r, equal] \arrow[d, "\lambda_f"'] & A \arrow[d, "\lambda_{\lambda_f}"] \\ Kf \arrow[r, "\sigma_f"] & K\lambda_f \end{tikzcd} \qquad \Pi_f = \begin{tikzcd} K\rho_f \arrow[r, "\pi_f"] \arrow[d, "\rho_{\rho_f}"'] & Kf \arrow[d, "\rho_f"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

One of the ideas behind this definition is that the L-coalgebras have the left lifting property with respect to the R-algebras, as explained below.

An L-coalgebra structure on a morphism $f\colon A \to B$, respectively, an R-algebra structure on f, is given by morphisms in $\mathcal{C}^{\mathbf{2}}$ of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d, "\lambda_f"] \\ B \arrow[r, "s"'] & Kf \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Kf \arrow[r, "p"] \arrow[d, "\rho_f"'] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

The horizontal identity morphisms are such as a consequence of the counit axiom of the comonad L, respectively unit axiom of the monad R. These axioms also imply $\rho_f \cdot s = 1_B$ and $p \cdot \lambda_f = 1_A$.

Continuing, given a morphism (h, k) in $\mathcal{C}^{\mathbf{2}}$ as in Figure 1, we get a diagonal filler as depicted.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "\lambda_f"] \arrow[dd, "f"'] & & C \arrow[r, equal] \arrow[d, "\lambda_g"'] & C \arrow[dd, "g"] \\ & Kf \arrow[r, "K(h,k)"] \arrow[d, "\rho_f"] & Kg \arrow[ur, "p"'] \arrow[dr, "\rho_g"] & \\ B \arrow[ur, "s"] \arrow[r, equal] & B \arrow[rr, "k"'] & & B \end{tikzcd} \tag{2.2}$$

The distributivity condition introduced in [10] asserts that the natural transformation $\Delta\colon LR \to RL$ with components

$$\Delta_f = \begin{tikzcd} \cdot \arrow[r, "\sigma_f"] \arrow[d, "\lambda_{\rho_f}"'] \arrow[dr, "1"] & \cdot \arrow[d, "\rho_{\lambda_f}"] \\ \cdot \arrow[r, "\pi_f"'] & \cdot \end{tikzcd} \tag{2.3}$$

is a distributive law, ie that the diagrams in Figure 2 commute. In fact, the two triangles automatically commute as a consequence of the comonad and monad axioms for L and R.

$$\begin{tikzcd}
LR \arrow[rr, "\Delta"] \arrow[dr, "\Phi R"'] & & RL \arrow[dl, "R\Phi"] \\
& R &
\end{tikzcd}
\qquad
\begin{tikzcd}
& L \arrow[dl, "L\Lambda"'] \arrow[dr, "\Lambda L"] & \\
LR \arrow[rr, "\Delta"] & & RL
\end{tikzcd}$$

$$\begin{tikzcd}
LR \arrow[rr, "\Delta"] \arrow[d, "\Sigma R"'] & & RL \arrow[d, "R\Sigma"] \\
L^2R \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta L"] & RL^2
\end{tikzcd}
\qquad
\begin{tikzcd}
LR^2 \arrow[r, "\Delta R"] \arrow[d, "L\Pi"'] & RLR \arrow[r, "R\Delta"] & R^2L \arrow[d, "\Pi L"] \\
LR \arrow[rr, "\Delta"] & & RL
\end{tikzcd}$$

FIGURE 2. Distributivity axioms, of which the two triangles are automatically satisfied.

Every AWFS (L,R) has an underlying WFS $(\mathcal{L},\mathcal{R})$, where $\mathcal{L}$ consists of the coalgebras for the copointed endofunctor (L,Φ) and $\mathcal{R}$ consists of the algebras for the pointed endofunctor (R,Λ).

We continue with some more background, in this case, the characterisation of orthogonal factorisation systems in terms of the associated AWFS. Clearly, any orthogonal factorisation system $(\mathcal{E},\mathcal{M})$ in a category $\mathcal{C}$ induces an AWFS. This is a consequence of the uniqueness of the factorisations. One can easily characterise the AWFS obtained in this way.

Proposition 2.1 ([11, Thm 3.2]). *The following are equivalent for an AWFS* (L,R)*: the comonad* L *and the monad* R *are idempotent; the underlying WFS is an OFS.*

In fact, if R is idempotent, then so is L, a proof of which the second author learned from Richard Garner. During the preparation of this manuscript a full proof of this fact appeared in [2].

2.2. Digression into modules. As a preamble to next section, let us briefly remind the reader about the language of modules or profunctors, which will be heavily used henceforth. A *module* or *profunctor* ϕ from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted by $\phi\colon \mathcal{A} \twoheadrightarrow \mathcal{B}$, is a functor $\mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to$ **Set**, and a module morphism is a natural transformation. Given another module $\psi\colon \mathcal{B} \twoheadrightarrow \mathcal{C}$, the composition $\psi \cdot \phi$ is defined by the coend formula $(\psi\cdot\phi)(C,A) = \int^B \psi(C,B)\times\phi(B,A)$; the identity $1_{\mathcal{A}}$ for this composition is given by $1_{\mathcal{A}}(A,A') = \mathcal{A}(A,A')$. In this way we obtain a bicategory **Mod**.

There is a pseudofunctor $(-)_*\colon \mathbf{Cat} \to \mathbf{Mod}$ that is the identity on objects and sends a functor $F\colon \mathcal{A} \to \mathcal{B}$ to the module F_* given by $F_*(B,A) = \mathcal{B}(B,FA)$. There is an adjunction $F_* \dashv F^*$, where $F^*\colon \mathcal{B} \twoheadrightarrow \mathcal{A}$ is given by

$F^*(A,B) = \mathcal{B}(FA,B)$, with unit and counit

$$\mathcal{A}(A,A') \xrightarrow{F} \mathcal{B}(FA,FA') \cong F^* \cdot F_*(A,A')$$

$$F_* \cdot F^*(B,B') = \int^A \mathcal{B}(B,FA) \times \mathcal{B}(FA,B') \xrightarrow{\text{comp}} \mathcal{B}(B,B').$$

Precisely the same description applies to enriched categories and enriched modules, by substituting **Set** by another symmetric monoidal closed category. We will be interested in **Cat**-enriched modules later.

The following easy lemma will be useful in the next section.

Lemma 2.2. *Let (G,e) be a copointed endofunctor on the category $\mathcal{X}$, and U the corresponding forgetful functor from (G,e)-*Coalg*. The module morphism*

$$\big(U^* \xrightarrow{U^* \cdot \eta} U^* \cdot G^* \cdot G_* \xrightarrow{\cong} (G \cdot U)^* \cdot G_* \xrightarrow{s^* \cdot G_*} U^* \cdot G_*\big) \tag{2.4}$$

is a right inverse of

$$\big(U^* \cdot G \xrightarrow{U^* \cdot e_*} U^*\big) \tag{2.5}$$

where $\eta\colon 1 \to G^* \cdot G_*$ is the unit of $G_* \dashv G^*$ and $s\colon U \to G \cdot U$ is the G-coalgebra structure of U.

Proof: The module morphisms (2.4) and (2.5) have respective components

$$\mathcal{X}(U(A),X) \xrightarrow{G} \mathcal{X}(GU(A),G(X)) \xrightarrow{\mathcal{X}(s_A,1)} \mathcal{X}(U(A),G(X)) \tag{2.6}$$

$$\mathcal{X}(GU(A),G(X)) \xrightarrow{\mathcal{X}(1,e_X)} \mathcal{X}(GU(A),X). \tag{2.7}$$

If $x\colon UA \to X$ is a morphism, apply (2.6) and then (2.7) to obtain $e_X \cdot G(x) \cdot s_A$, which is equal to $x \cdot e_{UA} \cdot s_A$ naturality, and therefore to x by $e_{UA} \cdot s_A = 1$. ∎

2.3. Natural diagonal fillers. Fix a category $\mathcal{C}$. Recall that there are retract adjunctions

$$\mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom}\colon \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C} \tag{2.8}$$

Define a module (profunctor) $\mathfrak{D}_{\mathcal{C}}\colon \mathcal{C}^{\mathbf{2}} \nrightarrow \mathcal{C}^{\mathbf{2}}$ in the following way. Given two morphisms f, g in $\mathcal{C}$, $\mathfrak{D}_{\mathcal{C}}(f,g)$ is the set of commutative diagrams

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] \arrow[ur, "d"] & D \end{tikzcd} \tag{2.9}$$

The action of $\mathcal{C}^{\mathbf{2}}$ on either side is simply by pasting the appropriate commutative square.

Lemma 2.3. *There are isomorphisms of modules* $\mathfrak{D}_{\mathcal{C}} \cong \mathrm{id}_* \cdot \mathrm{id}^* \cong (\mathrm{id} \cdot \mathrm{dom})_* \cong \mathrm{cod}^* \cdot \mathrm{dom}_* \cong (\mathrm{id} \cdot \mathrm{cod})^*$.

The second isomorphism is the one induced by the fact that $\mathrm{id}^* \cong \mathrm{dom}_*$ – see (2.8). The isomorphism $\mathfrak{D}_{\mathcal{C}}(f,g) \cong (\mathrm{id} \cdot \mathrm{dom})_*(f,g) = \mathcal{C}^{\mathbf{2}}(f, 1_{\mathrm{dom}\, g})$ is given by

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"'] \arrow[ur, "d"] & \cdot \end{tikzcd} \longmapsto \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "1_{\mathrm{dom}\, g}"] \\ \cdot \arrow[r, "d"'] & \cdot \end{tikzcd}$$

The counit of $\mathrm{id}_* \dashv \mathrm{id}^*$ is a module morphism

$$\mathfrak{D}_{\mathcal{C}} \longrightarrow 1_{\mathcal{C}^{\mathbf{2}}} \tag{2.10}$$

whose component at (f,g) sends the element (2.9) to the outer commutative square. It corresponds, under $\mathfrak{D}_{\mathcal{C}} \cong (\mathrm{id} \cdot \mathrm{dom})_*$, to the module morphism induced by the natural transformation with f-component

$$\mathrm{id} \cdot \mathrm{dom} \Longrightarrow 1_{\mathcal{C}^{\mathbf{2}}} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "1_{\mathrm{dom}\, f}"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd} \tag{2.11}$$

Definition 2.4. Let $(\mathcal{A}, U)$, $(\mathcal{B}, V)$ be two objects of $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$, and define a module

$$\mathfrak{D}(U,V) \colon \mathcal{B} \xrightarrow{V_*} \mathcal{C}^{\mathbf{2}} \xrightarrow{\mathfrak{D}_{\mathcal{C}}} \mathcal{C}^{\mathbf{2}} \xrightarrow{U^*} \mathcal{A}. \tag{2.12}$$

The module morphism $\mathfrak{D}_{\mathcal{C}} \to 1$ induces another $\mathfrak{D}(U,V) \to U^* \cdot V_*$. A *lifting operation* for U, V is a section for this module morphism, and amounts to a choice, for each square in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "Ua"'] & C \arrow[d, "Vb"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \qquad a \in \mathcal{A},\ b \in \mathcal{B} \tag{2.13}$$

of a diagonal filler, in such a way that it is natural with respect to composition on either side.

Example 2.5. A functorial factorisation system, with associated copointed endofunctor (L, Φ) and pointed endofunctor (R, Λ), gives rise to a lifting operation for the forgetful functors $U\colon (L,\Phi)\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$ and $V\colon (R,\Lambda)\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$.

Recall from Remarks 2.17 and 2.18 the transformations $(1,\lambda)\colon \mathrm{id}\,\mathrm{dom} \Rightarrow L$ and $(1,p)\colon LV \Rightarrow \mathrm{id}\,\mathrm{dom}\,V$, satisfying $(1,p)\cdot((1,\lambda)V) = 1$. Denote by $\eta\colon 1 \Rightarrow L^*\cdot L_*$ the unit of $L_* \dashv L^*$, and by $s\colon U \to L\cdot U$ the (L,Φ)-coalgebra structure of U.

We claim that the following module morphism is a section to $\mathfrak{D}(U,V) \to U^*\cdot V_*$.

$$U^*\cdot V_* \xrightarrow{U^*\cdot\eta\cdot V_*} U^*\cdot L^*\cdot L_*\cdot V_* \xrightarrow{\cong} (L\cdot U)^*\cdot(L\cdot V)_* \xrightarrow{s^*\cdot 1} U^*\cdot(L\cdot V)_* \to \\ \xrightarrow{1\cdot(1,p)_*} U^*\cdot(\mathrm{id}\cdot\,\mathrm{dom}\cdot V)_* \xrightarrow{\cong} \mathfrak{D}(U,V) \quad (2.14)$$

Indeed, $(s^*\cdot(L\cdot V)_*)\cdot(U^*\cdot\eta\cdot V_*)$ has right inverse $U^*\cdot(\Phi\cdot V)_*$, by Lemma 2.2. Thus, the composition (2.14) has right inverse

$$\mathfrak{D}(U,V) \xrightarrow{\cong} U^*(\mathrm{id}\cdot\,\mathrm{dom}\cdot V)_* \xrightarrow{U^*\cdot((1,\lambda)\cdot V)_*} U^*\cdot(L\cdot V)_* \xrightarrow{U^*\cdot(\Phi\cdot V)_*} U^*\cdot V_*. \quad (2.15)$$

The composition $\Phi\cdot(1,\lambda)\colon \mathrm{id}\cdot\,\mathrm{dom} \Rightarrow L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ is precisely the transformation described in (2.11) – see Remark 2.17 for a description of the components of $(1,\lambda)$ – therefore (2.15) is (2.10), as required.

Remark 2.6. It can be instructive to describe explicitly the component of (2.14) at an object (f,g) of $(L,\Phi)\text{-Coalg} \times (R,\Lambda)\text{-Alg}$. If $(1,s)\colon f \to Lf$ is the coalgebra structure of f and $(p,1)\colon Rg \to g$ the algebra structure of g, the component is given by

$$\mathcal{C}^{\mathbf{2}}(f,g) \xrightarrow{L} \mathcal{C}^{\mathbf{2}}(Lf,Lg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(s,1)} \mathcal{C}^{\mathbf{2}}(f,Lg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(1,(1,p))} \mathcal{C}^{\mathbf{2}}(f,1_{\mathrm{dom}(g)}) \qquad (2.16)$$

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] & & \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "k"'] & \cdot & \longmapsto & \cdot \arrow[r, "p\cdot K(h,k)\cdot s"'] & \cdot \end{tikzcd}$$

Composing (2.16) with the isomorphism $\mathcal{C}(f,1_{\mathrm{dom}(g)}) \cong \mathcal{C}(1_{\mathrm{cod}(f)},g)$, we get a module morphism with components

$$\mathcal{C}^{\mathbf{2}}(f,g) \xrightarrow{R} \mathcal{C}^{\mathbf{2}}(Rf,Rg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(1,p)} \mathcal{C}^{\mathbf{2}}(Rf,g) \xrightarrow{\mathcal{C}^{\mathbf{2}}((s,1),1)} \mathcal{C}^{\mathbf{2}}(1_{\mathrm{cod}(f)},g)$$

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd} \longmapsto \begin{tikzcd} \cdot \arrow[r, "p\cdot K(h,k)\cdot s"] \arrow[d, "1"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$$

Example 2.7. A functorial factorisation corresponds to an orthogonal factorisation system when $\mathfrak{D}(U,V) \to U^* \cdot V_*$ is invertible.

Remark 2.8. Let us now assume that in Definition 2.4 U has a right adjoint G. Then, the module (2.12) is isomorphic to $(G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V)_*$, $U^* \cdot V_*$ is isomorphic to $(G \cdot V)_*$, and the module morphism $U^* \cdot \mathfrak{D}_{\mathcal{C}} \cdot V_* \to U^* \cdot V_*$ corresponds to the natural transformation

$$G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow G \cdot V \tag{2.17}$$

induced by the counit of the adjunction id $\dashv$ dom, ie the transformation with component at $b \in \mathcal{B}$

$$G(1, Vb) \colon G1_{\mathrm{dom}(Vb)} \longrightarrow GVb.$$

Now suppose that the functor U is the forgetful functor $U \colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$, for a comonad L, and still denote by G its right adjoint. Denote by $F_{\mathsf{L}} \colon \mathcal{C}^{\mathbf{2}} \to \mathrm{Kl}(\mathsf{L})$ the Kleisli construction of L. The natural transformation (2.17), belonging to the full image of G, can be described as a morphism in $[\mathcal{B}, \mathrm{Kl}(\mathsf{L})]$

$$F_{\mathsf{L}} \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow F_{\mathsf{L}} \cdot V. \tag{2.18}$$

Proposition 2.9. *Given a comonad* L *on* $\mathcal{C}^{\mathbf{2}}$*, lifting operations for the functors* $U \colon \mathsf{L}\text{-Colag} \to \mathcal{C}^{\mathbf{2}}$ *and* $V : \mathcal{B} \to \mathcal{C}^{\mathbf{2}}$ *are in bijective correspondence with sections of the natural transformation* (2.18)*.*

Example 2.10. An AWFS (L, R) induces a lifting operation – Example 2.5 – which corresponds to a section of (2.18), by Proposition 2.9. In this example we explicitly describe this section in terms of the AWFS.

Consider the transformation $(1, p) \colon L \cdot V \Rightarrow \mathrm{id} \cdot \mathrm{dom} \cdot V$ as in Remark 2.18, and denote by $\theta : V \to \mathrm{id} \cdot \mathrm{dom} \cdot V$ the associated morphism in $[\mathsf{R}\text{-Alg}, \mathrm{Kl}(\mathsf{L})]$. It is easy to check that θ is the required section: $F_{\mathsf{L}}(1, g) \cdot \theta_g$ is, as a morphism in $\mathcal{C}^{\mathbf{2}}$,

$$(1, g) \cdot (1, p) = (1, \rho_g) = \Phi_g.$$

2.4. The universal category with chosen diagonal fillers. Given a functor $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$, [10] defined a category $\mathcal{A}^{\pitchfork}$ and a functor $U^{\pitchfork}\colon \mathcal{A}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ as follows. The objects of $\mathcal{A}^{\pitchfork}$ are pairs (g, ϕ^g), where $g \in \mathcal{C}^{\mathbf{2}}$ and ϕ^g is an assignment of a diagonal filler for each square

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi^g(a,h,k)" description] & \cdot \end{tikzcd}$$

which are compatible with morphisms $U\alpha\colon Ua' \to Ua$, in the sense that

$$\phi^g(a,h,k) \cdot \operatorname{cod}(U\alpha) = \phi^g(a', h \cdot \operatorname{dom}(U\alpha), k \cdot \operatorname{cod}(U\alpha)).$$

A morphism $(g, \phi^g) \to (e, \phi^e)$ is a morphism $(u,v)\colon g \to e$ in $\mathcal{C}^{\mathbf{2}}$ such that $u \cdot \phi^g(a,h,k) = \phi^e(a, u \cdot h, v \cdot k)$, for all (h,k).

The functor $U^{\pitchfork}$ is the universal one equipped with a lifting operation for U, $U^{\pitchfork}$, ie a section of $U^* \cdot \mathfrak{D}_{\mathcal{C}} \cdot U^{\pitchfork}_* \to U^* \cdot U^{\pitchfork}_*$. It can be constructed as a certain limit in **Cat**, of the form

$$\begin{tikzcd} & \mathcal{C}^{\mathbf{2}} \arrow[r, "Y"] \arrow[d, phantom, "\Downarrow"] & \mathcal{P}(\mathcal{C}^{\mathbf{2}}) \arrow[dr, "\mathcal{P}(U^*)"] & \\ \mathcal{A}^{\pitchfork} \arrow[ur, "U^{\pitchfork}"] \arrow[dr, "U^{\pitchfork}"'] & & & \mathcal{P}(\mathcal{A}) \\ & \mathcal{C}^{\mathbf{2}} \arrow[r, "\hat{\mathfrak{D}}_{\mathcal{C}}"'] & \mathcal{P}(\mathcal{C}^{\mathbf{2}}) \arrow[ur, "\mathcal{P}(U^*)"'] & \end{tikzcd}$$

where $\mathcal{P}(\mathcal{X})$ denotes the presheaf category on $\mathcal{X}$, and $\hat{\mathfrak{D}}_{\mathcal{C}}$ is the functor associated to $\mathfrak{D}_{\mathcal{C}}$. Equally well, $U^{\pitchfork}$ is a certain enhanced limit, in the sense of [17].

We continue with some further observations from [10]. The universal property of $U^{\pitchfork}$ implies that lifting operations for the pair of functors $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}} \leftarrow \mathcal{B} : V$ are in bijection with functors $\mathcal{B} \to \mathcal{A}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$ – [10]. In particular, each AWFS (L, R) in $\mathcal{C}$ gives rise to a canonical functor R-Alg $\to$ L-Coalg$^{\pitchfork}$. Furthermore, this functor is fully faithful, as shown in the subsequent paragraph.

Let $(p,1)\colon Rg \to g$ and $(p',1)\colon Rg' \to g'$ be two R-algebra structures, and $(u,v)\colon g \to g'$ a morphism in L-Coalg$^{\pitchfork}$. We know that the chosen diagonal filler of the square $(1, \rho_g)\colon \lambda_g \to g$ is p, and similarly for g' and p', so we have $u \cdot p = p' \cdot K(u,v)$. Hence, $(p',1) \cdot R(u,v) = (u,v) \cdot (p,1)$, so (u,v) is a morphism of R-algebras.

Lemma 2.11. *Given a functor $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$, an adjunction $U \dashv G$, and $g \in \mathcal{C}^{\mathbf{2}}$, there is a bijection between structures of an object $(g, \phi^g) \in \mathcal{A}^{\pitchfork}$ and sections s of $G(1, g)\colon G(1_{\mathrm{dom}\, g}) \to Gg$ in $\mathcal{A}$. If $(f, \phi^f) \in \mathcal{A}^{\pitchfork}$ is another object, with associated section t, morphisms $(g, \phi^g) \to (f, \phi^f)$ in $\mathcal{A}^{\pitchfork}$ are in bijection with morphisms $(h, k)\colon g \to f$ in $\mathcal{C}^{\mathbf{2}}$ such that $G(h, k) \cdot s = t \cdot G(h, k)$.*

Proof: See discussion before Proposition 2.9. ∎

The previous lemma can be reinterpreted in the following way. If we denote by $\epsilon\colon UG \Rightarrow 1$ the counit of the adjunction, a structure of an object of $\mathcal{A}^{\pitchfork}$ on g is given by a diagonal filler d^g:

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{dom}(\epsilon_g)"] \arrow[d, "UGg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "\mathrm{cod}(\epsilon_g)"'] \arrow[ur, "d^g"] & \cdot \end{tikzcd}$$

A morphism in $\mathcal{A}^{\pitchfork}$ is one that preserves these diagonal fillers, in an obvious sense.

Lemma 2.12. *Assume the conditions of Lemma 2.11. Then, for any full subcategory $\mathcal{F} \subset \mathcal{A}$ containing the full image of G, the functor $\mathcal{A}^{\pitchfork} \to \mathcal{F}^{\pitchfork}$ induced by the inclusion is an isomorphism.*

Proof: Denote by $J\colon \mathcal{F} \hookrightarrow \mathcal{A}$ the fully faithful inclusion functor, and by H the right adjoint to UJ, observing that $JH = G$. An object of $\mathcal{F}^{\pitchfork}$ is a lifting operation for the functors UJ and $g\colon \mathbf{1} \to \mathcal{C}^{\mathbf{2}}$, ie a section to the module morphism $(UJ)^* \cdot \mathcal{D}_{\mathcal{C}} \cdot g_* \to (UJ)^* \cdot g_*$. The same data can be equally given by a section to the morphism $H(1, g)\colon H(1_{\mathrm{dom}(g)}) \to Hg$ in $\mathcal{F}$; or a section to the image of this morphism under the fully faithful J. But $JH = G$, so we simply have a section of $G(1, g)$, precisely an object of $\mathcal{A}^{\pitchfork}$. This shows that $\mathcal{A}^{\pitchfork} \to \mathcal{F}^{\pitchfork}$ is bijective on objects. The proof that it is fully faithful is along the same lines, and is left to the reader. ∎

Corollary 2.13. *The category L-Coalg$^{\pitchfork}$ can be described, for a domain-preserving comonad L on $\mathcal{C}^{\mathbf{2}}$, as having objects (g, d^g)*

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "\lambda_g"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "\rho_g"'] \arrow[ur, "d^g"] & \cdot \end{tikzcd}$$

and morphisms $(g, d^g) \to (e, d^e)$ morphisms $(h, k) \colon g \to e$ in $\mathcal{C}^{\mathbf{2}}$ such that $h \cdot d^g = d^e \cdot K(h, k)$. If $\mathcal{F} \subset$ L-Coalg is a full subcategory containing the cofree L-coalgebras, the induced functor $\mathsf{L}\text{-Coalg}^{\pitchfork} \longrightarrow \mathcal{F}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$ is an isomorphism.

Proof: Combine Lemma 2.11, the comments that follow it and Lemma 2.12. ∎

2.5. Double-categorical aspects. We continue this section with remarks on double categories and AWFSs, due to R Garner. The standard category object in $\mathbf{Cat}^{\mathrm{op}}$ displayed on the left below induces a category object in $\mathbf{Cat}$, that is, a double category, displayed in the centre, that we may call the double category of squares and denote by $\mathbb{S}\mathrm{q}(\mathcal{C})$. Objects of $\mathbb{S}\mathrm{q}(\mathcal{C})$ are those of $\mathcal{C}$, vertical morphisms are morphisms of $\mathcal{C}$, as are horizontal morphisms, while 2-cells in $\mathbb{S}\mathrm{q}(\mathcal{C})$ are commutative squares in $\mathcal{C}$.

$$
\begin{tikzcd}
\mathbf{3} & \mathbf{2} \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] & \mathbf{1} \arrow[l, shift right=2] \arrow[l, shift left=2]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\mathbf{3}} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{C}^{\mathbf{2}} \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & \mathcal{C} \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{R}\text{-Alg} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "V"'] & \mathcal{C} \arrow[l] \arrow[d, equal] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, shift left=2, "\mathrm{cod}"] \arrow[r, shift right=2, "\mathrm{dom}"'] & \mathcal{C} \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\tag{2.19}
$$

If (L, R) is an AWFS on $\mathcal{C}$, R-algebras can be composed, in the sense that if $f \colon A \to B$ and $g \colon B \to C$ are R-algebras, then an R-algebra structure for $g \cdot f$ can be constructed from the AWFS. The forgetful functor from R-algebras forms part of a double functor, depicted on the right hand side above.

Interestingly, the converse also holds. If R is a codomain-preserving monad on $\mathcal{C}$, there is a bijection between AWFSs with monad R and extensions of the diagram on the right hand side of (2.19) to a double functor, by which we mean extensions of the reflexive graph $\mathsf{R}\text{-Alg} \rightrightarrows \mathcal{C}$ to a category object that makes (2.19) into a functor internal to $\mathbf{Cat}$ – a double functor into $\mathbb{S}\mathrm{q}(\mathcal{C})$.

If (f, p_f) and (g, p_g) are R-algebras with $\mathrm{cod}(f) = \mathrm{dom}(g)$, the double category structure provides for a vertical composition $(g, p_g) \bullet (f, p_f) = (g \cdot f, p_g \bullet p_f)$ with underlying morphism $g \cdot f$. The identities for the vertical composition are the R-algebras $(1, \rho_1)$. Morphisms of R-algebras can be vertically composed too: given such morphisms $(h, k) \colon f \to g$ and $(k, \ell) \colon f' \to g'$, where $\mathrm{cod}(f) = \mathrm{dom}(f')$ and $\mathrm{cod}(g) = \mathrm{dom}(g')$, then (h, ℓ) is a morphism $f' \bullet f \to g' \bullet g$.

Later on we will use the following construction of the comonad L from the double category structure. The codomain preserving pointed endofunctor (R, Λ) already gives us a domain preserving copointed endofunctor (L, Φ).

The only datum that remains to give is the comultiplication $\Sigma_f = (1, \sigma_f)\colon Lf \to L^2 f$. Define a morphism of R-algebras $(\sigma_f, 1)\colon Rf \to Rf \bullet RLf$ as the one corresponding under free R-algebra adjunction to $(\lambda_{\lambda_f}, 1)\colon f \to Rf \cdot RLf$.

$$\begin{tikzcd}
Kf \arrow[r, "\sigma_f"] \arrow[dd, "\rho_f"'] & K\lambda_f \arrow[d, "\rho_{\lambda_f}"] \\
 & Kf \arrow[d, "\rho_f"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\qquad \sigma_f \cdot \lambda_f = \lambda_{\lambda_f}$$

The components σ_f form a comultiplication for L. More details can be found in [21, Thm 2.24].

Given a double functor $U\colon \mathbb{A} \to \mathbb{S}\mathrm{q}(\mathcal{C})$, define a category $\mathbb{A}^{\pitchfork\!\!\!\pitchfork}$ in the following manner. An object is a morphism $g \in \mathcal{C}^{\mathbf{2}}$ equipped with a section ϕ of $\mathcal{C}(\mathrm{cod}(U-), \mathrm{dom}(g)) \to \mathcal{C}^{\mathbf{2}}(U-, g)$, that can be depicted by the diagram on the left below, that are natural with respect to squares in $\mathbb{A}$, and that satisfies a further condition: for any pair of vertical morphisms a, a' in $\mathbb{A}$, and a square $(h, k)\colon Ua' \cdot Ua \to g$,

$$\phi\big(a', \phi(a, h, k \cdot Ua'), k\big) = \phi\big(a' \bullet a, h, k\big).$$

$$\begin{tikzcd}
\cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "k"'] \arrow[ur, dotted, "{\phi(a,h,k)}" description] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[rr, "h"] \arrow[d, "Ua"'] & & \cdot \arrow[dd, "g"] \\
\cdot \arrow[urr, dotted, "{\phi(a,h,k\cdot Ua')}" description] \arrow[d, "Ua'"'] & & \\
\cdot \arrow[rr, "k"'] \arrow[uurr, dotted, "{\phi(a',\phi(a,h,k\cdot Ua'),k)}" description] & & \cdot
\end{tikzcd}$$

So objects of $\mathbb{A}^{\pitchfork\!\!\!\pitchfork}$ are objects of the category $(\mathbb{A}_h)^{\pitchfork}$ induced by the horizontal part $U_h\colon \mathbb{A}_h \to \mathbb{S}\mathrm{q}(\mathcal{C})_h = \mathcal{C}^{\mathbf{2}}$ that satisfy an additional property. We define $\mathbb{A}^{\pitchfork\!\!\!\pitchfork}$ as the full sub category of $\mathcal{A}^{\pitchfork}$ consisting of the objects just described. A morphism $(g, \phi) \to (g', \phi')$ in $\mathbb{A}^{\pitchfork\!\!\!\pitchfork}$ is, thus, a morphism $g \to g'$ in $\mathcal{C}^{\mathbf{2}}$ that is compatible with ϕ and ϕ'.

Each double functor $F\colon (\mathbb{B}, V) \to (\mathbb{A}, U)$ over $\mathbb{S}\mathrm{q}(\mathcal{C})$ induces a functor $F^{\pitchfork\!\!\!\pitchfork}\colon \mathbb{A}^{\pitchfork\!\!\!\pitchfork} \to \mathbb{B}^{\pitchfork\!\!\!\pitchfork}$ that is simply given by $F_h^{\pitchfork}\colon \mathbb{A}_h^{\pitchfork} \to \mathbb{B}_h^{\pitchfork}$. In other words, $F^{\pitchfork\!\!\!\pitchfork}(g, \phi)(b, h, k) = \phi(Fb, h, k)$.

Theorem 2.14. *Given an AWFS (L, R) on $\mathcal{C}$, there is an isomorphism*

$$\mathsf{R}\text{-}\mathrm{Alg}_s \cong \mathsf{L}\text{-}\mathbb{C}\mathrm{oalg}_s^{\pitchfork\!\!\!\pitchfork}.$$

The result above is included in [2], which appeared during the preparation of this manuscript, and for this reason we have the freedom of omitting the proof. We limit ourselves to point out that the R-algebra structure on g associated to (g,ϕ) is $\phi(Lg, 1_{\mathrm{dom}(g)}, \rho_g)$.

Lemma 2.15. *For any double functor $U\colon \mathbb{A} \to \mathbb{S}\mathrm{q}(\mathcal{C})$ the resulting category $\mathbb{A}^{\pitchfork}$ is a fibration over $\mathcal{C}$ in such a way that the functor $U^{\pitchfork}\colon \mathbb{A}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ is a strictly cartesian functor.*

Corollary 2.16. *For any AWFS (L,R) the codomain functor exhibits $\mathsf{R}\text{-Alg}_s \to \mathcal{C}$ as a fibration over $\mathcal{C}$, such that the forgetful functor $\mathsf{R}\text{-Alg}_s \to \mathcal{C}^{\mathbf{2}}$ is a cartesian functor.*

2.6. Miscellaneous results. Below we collect a number of observations that will be of use in later sections.

Remark 2.17. Recall that there are retract adjunctions $\mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom}\colon \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}$. Suppose given a functorial factorisation, with associated copointed endofunctor (L,Φ) and pointed endofunctor (R,Λ). The identity natural transformation $1_{\mathcal{C}} = \mathrm{dom}\cdot L\cdot \mathrm{id}$ corresponds under $\mathrm{id} \dashv \mathrm{dom}$ to a natural transformation $(1,\lambda)$ with f-component equal to the morphism depicted on the right hand side below.

$$\begin{tikzcd} \mathcal{C}^2 \arrow[r, bend left, "\mathrm{id}\cdot\mathrm{dom}"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow(1,\lambda)"] & \mathcal{C}^2 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{dom}\, f \arrow[r, equal] \arrow[d, "1"'] & \mathrm{dom}\, f \arrow[d, "\lambda_f"] \\ \mathrm{dom}\, f \arrow[r, "\lambda_f"'] & Kf \end{tikzcd}$$

Remark 2.18. Given a functorial factorisation given by (L,Φ) and (R,Λ) in $\mathcal{C}$, denote by $V\colon (R,\Lambda)\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$ the corresponding forgetful functor. Define a natural transformation

$$\begin{tikzcd} (R,\Lambda)\text{-Alg} \arrow[r, "V"] \arrow[d, "V"'] \arrow[dr, phantom, "(1,p)\Nearrow"] & \mathcal{C}^{\mathbf{2}} \arrow[d, "\mathrm{id}\cdot\mathrm{dom}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "L"'] & \mathcal{C}^{\mathbf{2}} \end{tikzcd} \tag{2.20}$$

in the following way. Start with the identity natural transformation $\mathrm{cod}\cdot L = \mathrm{dom}\cdot R$, which corresponds under $\mathrm{cod} \dashv \mathrm{id}$ to a transformation $L \Rightarrow \mathrm{id}\,\cdot$

dom $\cdot R$. Now, (2.20) is the composition

$$(1, p)\colon L \cdot V \Longrightarrow \mathrm{id} \cdot \mathrm{dom} \cdot R \cdot V \Longrightarrow \mathrm{id} \cdot \mathrm{dom} \cdot V$$

where the first arrow is the transformation just defined and the second is the application of the (R, Λ)-algebra structure of $R \cdot V \Rightarrow V$. Explicitly, the component of $(1, p)$ on an (R, Λ)-algebra (f, p_f) is

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\lambda_f"'] & A \arrow[d, "1"] \\ Kf \arrow[r, "p_f"'] & A \end{tikzcd}$$

Remark 2.19. The (unique possible) pasting of the transformation $(1, \lambda)$ of Remark 2.17 with the transformation $(1, p)$ of Remark 2.18 is the identity. Indeed, for any (R, Λ)-algebra (f, p_f), its unit axiom says precisely this, ie $p_f \cdot \lambda_f = 1$.

3. Lax orthogonal AWFSs

This section introduces the fundamental definition of this work, lax orthogonal AWFSs, and describes the most basic 2-categorical example. Before all that, we shall recall some facts about lax idempotent 2-monads.

3.1. Lax idempotent 2-monads. We begin the section by introducing some space-saving terminology. Suppose given an adjunction $f \dashv g$ in a 2-category, with unit $\eta\colon 1 \Rightarrow g \cdot f$ and counit $\varepsilon\colon f \cdot g \Rightarrow 1$. We say that $f \dashv g$ is a *retract (coretract)* adjunction when the counit (unit) is an identity 2-cell.

Recall that a 2-monad $\mathsf{T} = (T, i, m)$ on a 2-category $\mathscr{K}$ is *lax idempotent*, or *Kock-Zöberlein*, or simply *KZ*, if any of the following equivalent conditions hold. (i) $Ti \dashv m$ with identity unit. (ii) $m \dashv iT$ with identity counit. (iii) Each T-algebra structure $a\colon TA \to A$ on an object A is part of an adjunction $a \dashv i_A$ with identity counit. (iv) There is a modification $\delta\colon Ti \Rightarrow iT$ satisfying $\delta \cdot i = 1$ and $m \cdot \delta = 1$. (v) The forgetful 2-functor $U_\ell\colon \mathsf{T}\text{-}\mathrm{Alg}_\ell \to \mathscr{K}$ is fully faithful. (vi) For any pair of T-algebras A, B, every morphism $f\colon UA \to UB$ in $\mathscr{K}$ admits a unique structure of a lax morphism of T-algebras. (vii) For any morphism $f\colon X \to A$ into a T-algebra (A, a), the identity 2-cell exhibits $a \cdot Tf$ as a left extension of f along i_X.

It may be useful to say a few words about how to obtain a left extension from the modification δ. If $f\colon X \to A$ and $g\colon TX \to A$ are morphisms into a T-algebra (A, a), and $\alpha\colon f \Rightarrow g \cdot i_X$ a 2-cell, then the corresponding 2-cell $a \cdot Tf \Rightarrow g$ is constructed as $Tg \cdot \delta_X$.

A 2-comonad $\mathsf{G} = (G, e, d)$ on $\mathscr{K}$ is *lax idempotent*, or *KZ*, if the 2-monad $(G^{\mathrm{op}}, e^{\mathrm{op}}, d^{\mathrm{op}})$ on $\mathscr{K}^{\mathrm{op}}$ is lax idempotent. This means that we have conditions dual to the ones spelled out above for 2-monads; eg adjunctions $eG \dashv d \dashv Ge$, a modification $\delta\colon Ge \Rightarrow eG$, etc. We state one of the conditions in full: given a morphism $f\colon A \to X$ from a G-coalgebra (A, s), the identity 2-cell exhibits $Gf \cdot s\colon A \to GX$ as a left lifting of f through e_X.

Remark 3.1. Given a lax idempotent 2-monad T on $\mathscr{K}$, the right adjoint of its Kleisli construction $U_{\mathsf{T}}\colon \mathrm{Kl}(\mathsf{T}) \to \mathscr{K}$ is locally fully faithful. This is easily verified, since U_{T} is the composition of the full and faithful comparison 2-functor $\mathrm{Kl}(\mathsf{T}) \to \mathsf{T}\text{-}\mathrm{Alg}_s$ with the forgetful 2-functor from the 2-category of T-algebras.

3.2. Definition and basic properties of AWFSs.

Definition 3.2. An AWFS on a 2-category $\mathscr{K}$ consists of a pair (L, R) formed by a 2-comonad and a 2-monad on $\mathscr{K}^{\mathbf{2}}$ satisfying the same properties as AWFSs on categories; ie the 2-comonad must be domain-preserving and the 2-comonad codomain-preserving, the copointed endofunctor of L and the pointed endofunctor of R must give rise to the same functorial factorisation of morphisms in $\mathscr{K}$, and Δ (2.3) must be a distributive law.

Definition 3.3. An AWFS (L, R) in a 2-category $\mathscr{K}$ is said *lax orthogonal* if the 2-comonad L and the 2-monad R are lax idempotent.

We will later see in Section 4 that it is enough to require that L *or* R be lax idempotent.

Remark 3.4. R is lax idempotent precisely when each morphism $(h, k)\colon g \to g'$ between R-algebras has a unique structure of a lax morphism. It is useful to make the point that such a structure $(\bar{h}, \bar{k})\colon (p', 1) \cdot R(h, k) \Rightarrow (h, k) \cdot (p, 1)$ necessarily satisfies $\bar{k} = 1$ – where $(p, 1)$ and $(p', 1)$ are the R-algebra structures of g and g'. This is a consequence of the unit axiom for lax morphisms: $(\bar{h}, \bar{k}) \cdot (\lambda_g, 1) = 1$. This axiom also implies $g' \cdot \bar{h} = 1$.

Remark 3.5. It was observed in Remark 2.19 that the transformation $(1, p)\colon L \cdot V \to \mathrm{id} \cdot \mathrm{dom} \cdot V$ of Remark 2.18 has as right inverse $(1, \lambda) \cdot V$, where $(1, \lambda)$ is the transformation of Remark 2.17. We claim that, when the 2-monad R is lax idempotent, we also have a retract adjunction in the 2-category

2-Cat(R-Alg$_s$, $\mathscr{K}^{\mathbf{2}}$) of 2-functors, 2-natural transformations and modifications

$$(1,p) \dashv (1,\lambda)\cdot V \colon \mathrm{id}\cdot \mathrm{dom}\cdot V \Longrightarrow L\cdot V\cdot$$

The counit of this adjunction is the identity modification, and the unit has components

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\lambda_f"'] & A \arrow[r, equal] \arrow[d, "1"'] & A \arrow[d, "\lambda_f"] \\ Kf \arrow[r, "p_f"] \arrow[rr, bend right, "1"'] & A \arrow[r, "\lambda_f"] \arrow[d, phantom, "\eta_f\Uparrow"] & Kf \\ & & \end{tikzcd}$$

for $f \in \mathsf{R}$-Alg$_s$, where η_f is the domain component of the unit of the adjunction $(p_f, 1) \dashv (\lambda_f, 1)$ provided by the fact that R is lax idempotent. The fact that this defines a modification with components $(1, \eta_f)$ follows, and clearly satisfies the triangular identities.

Remark 3.6. Further to Remark 3.5, $(1,p)$ extends to an oplax natural transformation $L\cdot V_\ell \Rightarrow \mathrm{id}\cdot \mathrm{dom}\cdot V_\ell$, where $V_\ell \colon \mathsf{R}\text{-Alg}_\ell \to \mathscr{K}^{\mathbf{2}}$ is the forgetful 2-functor from the 2-category of R-algebras and lax morphisms.

To see this, recall that, for a general 2-monad T on a 2-category $\mathscr{L}$, the T-algebras structures $a\colon TA \to A$ form an oplax natural transformation as depicted on the left hand side, with 2-dimensional component at the lax morphism $(f,\bar f)$ from (A,a) to (B,b) given by the 2-cell on the right hand side.

$$\begin{tikzcd} T\text{-Alg}_\ell \arrow[r, "U_\ell"] \arrow[dr, "U_\ell"'] \arrow[dr, phantom, shift left=3, "\Swarrow"] & \mathscr{L} \arrow[d, "T"] \\ & \mathscr{L} \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\bar f \nearrow"] & A \arrow[d, "f"] \\ TB \arrow[r, "b"'] & B \end{tikzcd}$$

In the case of Remark 3.5, we obtain an oplax natural transformation $R\cdot V_\ell \to V_\ell$, which provide us, for each morphism $(h,k)\colon f \to g$ in R-Alg$_s$, with 2-cells $p_g \cdot K(h,k) \Rightarrow h\cdot p_f$, where $(p_f, 1)\colon Rf \to f$ is the R-algebra structure. One can now verify that these same 2-cells endow the morphisms $(1,p)$ with the required oplax transformation structure.

Remark 3.7. The oplax natural transformation $L\cdot V_\ell \Rightarrow \mathrm{id}\cdot \mathrm{dom}\cdot V_\ell$ defined in Remark 3.6, when composed with the counit 2-natural transformation $\mathrm{id}\cdot\mathrm{dom}\cdot V_\ell \Rightarrow V_\ell$, equals the counit $\Phi\cdot V_\ell$; in particular, this composition is 2-natural. Both transformations have trivial domain components, and $(1,g)\cdot$

$(1, p_g) = (1, \rho_g) = \Phi_g$, so we must only verify that the domain component $p_g \cdot K(h, k) \Rightarrow h \cdot p_f$ of the lax morphism structure of (h, k), composed with g, is the identity 2-cell. This is clear, since the lax morphism structure in question is a 2-cell in $\mathscr{K}^{\mathbf{2}}$.

3.3. A basic example. Every functor $f\colon A \to B$ factors as $\lambda_f\colon A \to Kf = (f \downarrow B)$ followed by $\rho_f\colon Kf \to B$, where $\lambda_f(a) = (a, 1\colon f(a) \to f(a), a)$, and $\rho_f(a, \beta\colon f(a) \to b, b) = b$. The associated pointed endofunctor R on $\mathbf{Cat}^{\mathbf{2}}$ given by $f \mapsto \rho_f$ underlies the free split opfibration monad R. Precisely the same factorisation can be constructed in any 2-category $\mathscr{K}$ with the necessary comma objects. At this point one could deduce that there is an AWFS (L, R) by observing that split opfibrations compose and the results cited in Section 2.5, and furthermore, one could use the results of Section 4 to prove that the AWFS is lax orthogonal. Instead, we shall give an explicit description of the comonad and its coalgebras, as they will become important in later sections. First, let us say a few words about R. The free split opfibration on f is given by a comma object as depicted on the left hand side. The unit of R has components $\Lambda_f = (\lambda_f, 1)$, where $\lambda_f\colon A \to Kf$ is the unique morphism such that $\rho_f \cdot \lambda_f = f$, $q_f \cdot \lambda_f = 1$ and $\nu_f \cdot \lambda_f = 1$. The multiplication $\Pi_f = (\pi_f, 1)$ is given by the unique morphism $\pi_f\colon K\rho_f \to Kf$ satisfying the three equalities depicted on the right hand side.

$$\begin{tikzcd} Kf \arrow[r, "q_f"] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\swarrow \nu_f"] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \end{tikzcd} \qquad q_f \cdot \pi_f = q_f \cdot q_{\rho_f} \qquad \rho_f \cdot \pi_f = \rho_{\rho_f} \qquad \nu_f \cdot \pi_f = \nu_{\rho_f}(\nu_f \cdot q_{\rho_f}) \tag{3.1}$$

Observe that λ_f comes equipped with an adjunction $\lambda_f \dashv q_f$ with identity unit, where q_f is the projection. The counit $\omega_{\lambda_f}\colon \lambda_f \cdot q_f \Rightarrow 1$ is the one induced by the universal property of comma objects and the pair of 2-cells $q_f \cdot \omega_{\lambda_f} = 1\colon q_f \Rightarrow q_f$ and $\rho_f \cdot \omega_{\lambda_f} = \nu_f\colon \rho_f \cdot \lambda_f \cdot q_f = f \cdot q_f \Rightarrow \rho_f$.

The copointed endofunctor L underlies a comonad with comultiplication $\Sigma\colon L \Rightarrow L^2$, defined by the following equality and the universal property of comma objects.

$$\begin{tikzcd} Kf \arrow[r, "\sigma_f"] & K\lambda_f \arrow[r, "q_{\lambda_f}"] \arrow[d, "\rho_{\lambda_f}"'] \arrow[dr, phantom, "\swarrow \nu_{\lambda_f}"] & A \arrow[d, "\lambda_f"] \\ & Kf \arrow[r, equal] & Kf \end{tikzcd} = \begin{tikzcd} Kf \arrow[r, "q_f"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow \omega_f"] & A \arrow[d, "\lambda_f"] \\ Kf \arrow[r, equal] & Kf \end{tikzcd} \tag{3.2}$$

The 2-monad R is well-known to be lax idempotent. To see that the comonad L is lax idempotent, one can exhibit an adjunction $\Phi_{Lf} \dashv \Sigma_f$ with identity counit.

The existence of an adjunction $\rho_{\lambda_f} \dashv \sigma_f$, with identity counit, follows from Remark 3.8 below. The fact that this adjunction provides another $\Phi_{Lf} \dashv \Sigma_f$ can be readily checked. We leave the verification of the distributivity law between the 2-comonad and 2-monad to the reader.

Remark 3.8. Given a comma object as exhibited on the left below, each adjunction $\ell \dashv r$ induces a retract adjunction $p \dashv s$, where s is defined by the equality on the right hand side.

$$\begin{tikzcd} \ell\downarrow t \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\not\Downarrow\nu"] & A \arrow[d, "\ell"] \\ X \arrow[r, "t"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "s"] & \ell\downarrow t \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\not\Downarrow\nu"] & A \arrow[d, "\ell"] \\ & X \arrow[r, "t"'] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, "t"] & B \arrow[r, "r"] \arrow[dr, "1"'] \arrow[dr, phantom, bend left, "\not\Downarrow\varepsilon"] & A \arrow[d, "\ell"] \\ & & B \end{tikzcd}$$

The unit $\eta\colon 1 \Rightarrow s\cdot p$ is the unique 2-cell satisfying $p\cdot\eta = 1$ and

$$\begin{tikzcd} \ell\downarrow t \arrow[r, bend left, "1"] \arrow[r, bend right, "s\cdot p"'] \arrow[r, phantom, "\Downarrow\eta"] & \ell\downarrow t \arrow[r, "q"] & A \end{tikzcd} = \begin{tikzcd} \ell\downarrow t \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\not\Downarrow\nu"] & A \arrow[r, equal] \arrow[d, "\ell"] \arrow[dr, phantom, "\Downarrow"] & A \\ X \arrow[r, "t"'] & B \arrow[ur, "r"'] & \end{tikzcd}$$

We make a final observation that will be of use later on. Suppose that the unit of $\ell \dashv r$ is an identity and $h\colon Z \to \ell\downarrow t$ is any morphism such that $\nu\cdot h$ is an identity 2-cell. Then $\eta\cdot h$ is an identity 2-cell.

Proposition 3.9. *(1) There is an isomorphism between* (L,Φ)-Coalg *and the 2-category with*
- *Objects* (f,v,ξ) *where* $f\colon A \rightleftarrows B : v$ *and* $\xi\colon f\cdot v \Rightarrow 1$ *satisfy* $v\cdot f = 1_A$ *and* $\xi\cdot f = 1$ *– strong deformation retracts of* B.
- *Morphisms* $(f,v,\xi) \to (f',v',\xi')$, *morphisms* $(h,k)\colon f \to f'$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that* $h\cdot v = v'\cdot k$ *and* $\xi'\cdot k = k\cdot\xi$.
- *2-cells* $(h,k) \Rightarrow (\bar h,\bar k)\colon (f,v,\xi) \to (f',v',\xi')$, *2-cells* $(\alpha,\beta)\colon (h,k) \Rightarrow (\bar h,\bar k)$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that* $\alpha\cdot v = v'\cdot\beta$.

(2) There is an isomorphism between L-Alg$_s$ *and the 2-category with*
- *Objects coretract adjunctions* $f \dashv v$ *– with counit denoted by* ϵ.
- *Morphisms from* $f \dashv v$ *to* $f' \dashv v'$ *morphisms* $(h,k)\colon f \to f'$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that* $h\cdot v = v'\cdot k$ *and* $\epsilon'\cdot k = k\cdot\epsilon$.
- *2-cells those of* $\mathscr{K}^{\mathbf{2}}$.

(3) Cofree L*-coalgebras correspond to the coretract adjunctions* $\lambda_f \dashv q_f$*.*
(4) The double category structure on $\mathsf{L}\text{-Coalg}_s$ *induced by this AWFS is the one given by composition of coretract adjunctions.*

Proof: Recall the definition of Kf as a comma object (3.1). There is a bijection between morphisms $s\colon B \to Kf$ such that $\rho_f \cdot s = 1_B$ and morphisms $v\colon B \to A$ equipped with a 2-cell $\xi\colon f \cdot v \Rightarrow 1_B$; the bijection is given by composing with the comma object ν_f, ie $v = q_f \cdot s$ and $\xi = \nu_f \cdot s$. Under this bijection, the condition $s \cdot f = \lambda_f$, which means that $(1, s)$ is a morphism $f \to Lf$, translates into $\xi \cdot f = 1$. This completes the description of (L, Φ)-coalgebras.

Next we translate the condition $\sigma_f \cdot s = K(1, s) \cdot s$, that is the coassociativity axiom for that makes an (L, Φ)-coalgebra into an L-coalgebra. Denote the counit of $\lambda_f \dashv q_f$ by ω_f, and recall that σ_f is defined by (3.2). The morphism $\sigma_f \cdot s$ corresponds under the universal property of the comma object ν_{λ_f} to the 2-cell

$$\nu_{\lambda_f} \cdot \sigma_f \cdot s = \omega_f \cdot s\colon \lambda_f \cdot q_f \cdot s = \lambda_f \cdot v \Longrightarrow s \tag{3.3}$$

while $K(1, s) \cdot s$ corresponds to

$$\nu_{\lambda_f} \cdot K(1, s) \cdot s = s \cdot \nu_f \cdot s = s \cdot \xi\colon s \cdot f \cdot v = \lambda_f \cdot v \Longrightarrow s. \tag{3.4}$$

Therefore, s is a coalgebra precisely when (3.3) equals (3.4). These are both 2-cells between morphisms with codomain Kf, and as such they are equal if and only if their respective compositions with the projections ρ_f and q_f coincide. Their composition with ρ_f yield respectively

$$\rho_f \cdot \omega_f \cdot s = \nu_f \cdot s = \xi \qquad \text{and} \qquad \rho_f \cdot s \cdot \xi = \xi$$

while their composition with q_f yield respectively

$$q_f \cdot \omega_f \cdot s = 1 \qquad \text{and} \qquad q_f \cdot s \cdot \xi = v \cdot \xi.$$

It follows that s is coassociative if and only if $v \cdot \xi = 1$, completing the description of L-coalgebras as coretract adjunctions $f \dashv v$.

We now describe the morphisms of (L, Φ)-coalgebras from $(1, s)\colon f \to Lf$ to $(1, s')\colon f' \to Lf'$. Such a morphism is a morphism $(h, k)\colon f \to f'$ in $\mathscr{K}^{\mathbf{2}}$ satisfying $s' \cdot k = K(h, k) \cdot s$. Composing with the comma object $\nu_{f'}$, this equality translates into $v' \cdot k = h \cdot v$ and $\xi' \cdot k = k \cdot \xi$. A morphism of L-coalgebras is just a morphism between the underlying (L, Φ)-coalgebras.

A 2-cell between morphisms $(h,k), (\bar{h},\bar{k})\colon (f,s) \to (f',s')$ of (L,Φ)-algebras is a pair of 2-cells $\alpha\colon h \Rightarrow \bar{h}$ and $\beta\colon k \Rightarrow \bar{k}$ satisfying $K(\alpha,\beta)\cdot s = s'\cdot\beta$. This is an equality of 2-cells between 1-cells with codomain Kf', so it holds if and only if it does after composing with the projections $\rho_{f'}$ and $q_{f'}$. The composition of this equality with $\rho_{f'}$ yields $\beta = \beta$ – no information here – while its composition with $q_{f'}$ yields $\alpha\cdot v = \beta\cdot v'$. This completes the description of (L,Φ)-Coalg. When (f,s) and (f',s') are L-algebras, with associated coretract adjunctions (f,v,ξ) and (f',v',ξ'), this latter equality is void too, since its mate automatically holds. Explicitly,

$$\big(h\cdot v \xrightarrow{\alpha\cdot v} \bar{h}\cdot v\big) = \big(h\cdot v = v'\cdot k \xrightarrow{v'\cdot\beta} v'\cdot\bar{k} = \bar{h}\cdot v\big)$$

holds if and only if it does after precomposing with f and composing with the unit $1 = v\cdot f$ of $f\dashv v$:

$$\alpha = \alpha\cdot v\cdot f = \big(h = h\cdot v\cdot f = v'\cdot k\cdot f \xrightarrow{v'\cdot\beta\cdot f} v'\cdot\bar{k}\cdot f = \bar{h}\cdot v\cdot f = \bar{h}\big).$$

But this latter equality automatically holds, by $\beta\cdot f = f'\cdot\alpha$. This shows that 2-cells in L-Coalg_s are simply 2-cells in $\mathscr{K}^{\mathbf{2}}$.

Finally, we prove the fourth statement of the proposition. The 2-category of L-coalgebras is equipped with an obvious composition: that of coretract adjunctions. Any such composition corresponds to a unique multiplication $\bar{\Pi}\colon R^2 \to R$ that makes $(R,\Phi,\bar{\Pi})$ a 2-monad and satisfies distributivity – Section 2.1. We have to show that $\bar{\Pi}$ equals the multiplication Π of the free split opfibration 2-monad.

By the comments at the end of Section 2, or rather the dual version of those comments, $\bar{\Pi}_f = (\bar{\pi}_f, 1)$ is defined by the property that $(1,\bar{\pi}_f)$ is the unique morphism of L-coalgebras from $L(\rho_f)\bullet Lf$ to Lf that composed with the counit $\Phi_f = (1,\rho_f)\colon Lf \to f$ yields the morphism $(1,\rho_{\rho_f})\colon \lambda_{\rho_f}\cdot\lambda_f \to f$ in $\mathscr{K}^{\mathbf{2}}$. Recall that $\bar{\pi}_f$ has domain $K\rho_f$ and codomain Kf. By the previous parts of the present proposition, to say that $(1,\bar{\pi}_f)$ is a morphism of L-coalgebras is equivalent to saying that $q_f\cdot\bar{\pi}_f = q_f\cdot q_{\rho_f}$ and

$$\big(\pi_f\cdot\lambda_{\rho_f}\cdot\lambda_f\cdot q_f\cdot q_{\rho_f} \xrightarrow{\pi_f\cdot\lambda_{\rho_f}\cdot\omega_f\cdot q_{\rho_f}} \pi_f\cdot\lambda_{\rho_f}\cdot q_{\rho_f} \xrightarrow{\pi_f\cdot\omega_{\rho_f}} \pi_f\big) = \big(\lambda_f\cdot q_f\cdot\pi_f \xrightarrow{\omega_f\cdot\pi_f} \pi_f\big) \tag{3.5}$$

In order to deduce that $\bar{\pi}_f$ is the multiplication of the free split opfibration 2-monad, we can verify that it satisfies the three equalities of (3.1). The first of these we have already verified above, while the second holds by definition of $\bar{\pi}_f$. It only remains to verify the third equality in (3.1), for which we will use $\rho_f \cdot \omega_f = \nu_f$ – see remarks that follow the equation (3.1). If we postcompose the left hand side of (3.5) with ρ_f, we obtain $\nu_{\rho_f}(\nu_f \cdot q_{\rho_f})$, while doing the same on the right hand side we obtain $\nu_f \cdot \bar{\pi}_f$. Therefore, $\bar{\pi}_f$ does satisfy the third equality of (3.1), and necessarily $\bar{\pi}_f = \pi_f$, completing the proof. ■

In general, for a copointed endofunctor (G, ε) on a category $\mathcal{C}$, and a retraction $r\colon Y \twoheadrightarrow X$ with section s in $\mathcal{C}$, each (G,ε)-coalgebra structure $\delta\colon Y \to GY$ on Y induces another on X. This induced coalgebra structure is $(Gr)\cdot\delta\cdot s\colon X \to GX$. Later we shall need the description of this construction in the case of the copointed endo-2-functor (L, Φ) of Proposition 3.9. Let (f, v, ξ) be a coalgebra and $(r_0, r_1)\colon f \to \bar{f}$ a retraction on $\mathscr{K}^{\mathbf{2}}$ with section (s_0, s_1). The induced coalgebra structure $(\bar{f}, \bar{v}, \bar{\xi})$ is given by $\bar{v} = r_0 \cdot v \cdot s_1$ and

$$\bar{f} \cdot \bar{v} = \bar{f} \cdot r_0 \cdot v \cdot s_1 = r_1 \cdot f \cdot v \cdot s_1 \xRightarrow{r_1 \cdot \xi \cdot s_1} r_1 \cdot s_1 = 1.$$

4. The 2-comonad is lax idempotent if the 2-monad is so

In this section we show that, in order for an AWFS on a 2-category to be lax orthogonal, it suffices that *either* its 2-monad or its 2-comonad be lax idempotent. This result can be seen as a two-dimensional generalisation of the fact that an AWFS on a category is orthogonal if either its monad or its comonad is idempotent – a fact the second author learned from R Garner and is included in [2]. However, the proof, as it is to be expected, is more involved. Incidentally, our proof uses the double category structure on $\mathsf{R}\text{-Alg}_s$ mentioned in Section 2.

Theorem 4.1. *The 2-comonad of an AWFS on a 2-category is lax idempotent provided the 2-monad is lax idempotent.*

Proof: Denote the AWFS on the 2-category $\mathscr{K}$ by (L, R). We will exhibit a coretract adjunction $\Sigma_f \dashv L\Phi_f$ whose counit is a modification in $f\colon A \to B$ – the unit is an identity 2-cell. In Section 2 we mentioned that $(\sigma_f, 1_B)$ is a morphism of R-algebras $Rf \to Rf \bullet RLf$, where the codomain is the vertical composition of the R-algebras RLf and Rf. Consider the morphism

$$RLf \xrightarrow{R\Phi_f} Rf \xrightarrow{(\sigma_f,1_B)} Rf \bullet RLf$$

which is, by Section 3.1, a left extension along Λ_{Lf} of its composition with the unit Λ_{Lf}

$$(\sigma_f,1_B)\cdot R\Phi_f\cdot \Lambda_{Lf} = (\sigma_f,1_B)\cdot \Lambda_f\cdot \Phi_f = (\lambda_{\lambda_f},\rho_f)\colon Lf \to Rf\cdot RLf.$$

The morphism $(1_{Kf},\rho_f)\colon RLf \to Rf\cdot RLf$ in $\mathscr{K}^{\mathbf{2}}$ satisfies $(1_{Kf},\rho_f)\cdot\Lambda_{Lf} = (\lambda_{\lambda_f},\rho_f)$ too, therefore the universal property of left extensions gives a unique 2-cell $(\sigma_f,1_B)\cdot R\Phi_f \Rightarrow (1_{Kf},\rho_f)$ in $\mathscr{K}^{\mathbf{2}}$ whose composition with Λ_{Lf} is the identity 2-cell. This forces the 2-cell to be of the form

$$(\varepsilon_f, 1_{1_B})\colon (\sigma_f,1_B)\cdot R\Phi \Longrightarrow (1_{Kf},\rho_f) \tag{4.1}$$

for a 2-cell in $\mathscr{K}$

$$\varepsilon_f\colon \sigma_f\cdot K(1_A,\rho_f) \Longrightarrow 1_{K\lambda_f}\colon K\lambda_f \to K\lambda_f$$

since the codomain component of Λ_f is an identity. This definition makes $(\varepsilon_f,1_{1_B})$, and hence ε_f, a modification in f, a fact that can be verified by using the universal property of extensions.

We now proceed to prove that ε_f is the counit of a coretract adjunction $\sigma_f \dashv K(1_A,\rho_f)$ in $\mathscr{K}$, for which we must show three conditions:

$$\varepsilon_f\cdot\sigma_f = 1 \qquad K(1_A,\rho_f)\cdot\varepsilon_f = 1 \qquad \varepsilon_f\cdot\lambda_{\lambda_f} = 1. \tag{4.2}$$

The first two conditions are the triangular identities of the adjunction, while the last one means that ε_f is a 2-cell in $\mathscr{K}^{\mathbf{2}}$.

Consider the morphism of R-algebras

$$Rf \xrightarrow{(\sigma_f,1_B)} Rf\bullet RLf \xrightarrow{(\rho_f,1_B)\bullet R\Phi_f} 1_B\bullet Rf \xrightarrow{1_B\bullet(\sigma_f,1_B)} 1_B\bullet Rf\bullet RLf$$

depicted in Figure 3, which equals $(\sigma_f,1_B)$. The 2-cell $(\varepsilon_f,1)\colon(\sigma_f,1)\cdot R\Phi_f \Rightarrow 1$ induces a 2-cell

$$\big((\rho_f,1_B)\bullet(\varepsilon_f,1_{1_B})\big)\cdot(\sigma_f,1) = (\varepsilon_f\cdot\sigma_f, 1_{1_B}) \tag{4.3}$$

with domain and codomain equal to $(\sigma_f,1_B)\colon Rf \to Rf\bullet RLf$ – top of Figure 3. This 2-cell (4.3) precomposed with $\Lambda_f\colon f\to Rf$ equals the identity, for $\sigma_f\cdot\lambda_f = \lambda_{\lambda_f}$ and $\varepsilon_f\cdot\lambda_{\lambda_f} = 1$ by definition of ε_f. Since $(\sigma_f,1)$ is a left

$$
\begin{tikzcd}
Kf \arrow[r, "\sigma_f"] \arrow[ddd, "\rho_f"'] & K\lambda_f \arrow[r, "K(1_A,\rho_f)"'] \arrow[r, phantom, "\varepsilon_f\Uparrow"] \arrow[rr, bend left, "1"] \arrow[dd, "\rho_{\lambda_f}"'] & Kf \arrow[r, "\sigma_f"'] \arrow[dd, "\rho_f"'] & K\lambda_f \arrow[d, "\rho_{\lambda_f}"] \\
 & & & Kf \arrow[d, "\rho_f"] \\
 & Kf \arrow[r, "\rho_f"] \arrow[d, "\rho_f"'] & B \arrow[r, equal] \arrow[d, "1"'] & B \arrow[d, "1"] \\
B \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
K\lambda_f \arrow[r, "K(1_A,\rho_f)"'] \arrow[r, phantom, "\varepsilon_f\Uparrow"] \arrow[rr, bend left, "1"] \arrow[dd, "\rho_{\lambda_f}"'] & Kf \arrow[r, "\sigma_f"'] \arrow[dd, "\rho_f"'] & K\lambda_f \arrow[r, "K(1_A,\rho_f)"] \arrow[d, "\rho_{\lambda_f}"'] & Kf \arrow[d, "\rho_f"] \\
 & & Kf \arrow[r, "\rho_f"] \arrow[d, "\rho_f"'] & B \arrow[d, "1"] \\
Kf \arrow[r, "\rho_f"] & B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

FIGURE 3. Proof of Theorem 4.1.

extension of $(\sigma_f, 1) \cdot \Lambda_f$ along Λ_f, we must have $(\varepsilon_f \cdot \sigma_f, 1_{1_B}) = 1$, the first of the equalities (4.2).

Consider the morphism of R-algebras

$$RLf \xrightarrow{R\Phi_f} RL \xrightarrow{(\sigma_f, 1_B)} Rf \bullet RLf \xrightarrow{(\rho_f,1_B)\bullet R\Phi_f} 1_B \bullet Rf$$

depicted in the bottom of Figure 3, which equals $(K(1_A, \rho_f), 1)$. The 2-cell (4.1) induces an endo-2-cell

$$((\rho_f, 1_B) \bullet R\Phi_f) \cdot (\varepsilon_f, 1_{1_B}) \colon (K(1_A, \rho_f), \rho_f) \Rightarrow (K(1_A, \rho_f), \rho_f), \qquad (4.4)$$

which, by definition of (4.1), equals the identity when precomposed with Λ_{Lf}. The morphism $(K(1_A, \rho_f), \rho_f)$ is a morphism of R-algebras, and hence a left extension along Λ_{Lf}, from where we deduce that (4.4) must be the identity 2-cell. That is, $K(1_A, \rho_f) \cdot \varepsilon_f = 1$, the second equality of (4.2).

All that remains to verify is $\varepsilon_f \cdot \lambda_{\lambda_f} = 1$, but this is part of the definition of ε_f. ■

The foregoing theorem implies that a lax orthogonal AWFS can be equivalently given by a lax idempotent 2-monad R on $\mathscr{K}^{\mathbf{2}}$ and a double category structure on R-Alg_s as described in Section 2.

5. KZ lifting operations

Section 2.3 described the algebraic structure that provides a lifting operation, and the category $\mathcal{A}^{\pitchfork}$, in terms of modules. This section introduces variations of these notions that are suitable to lax orthogonal factorisations.

5.1. Lax natural lifting operations. Before introducing the main definitions of this section, let us remind the reader about some facts around **Cat**-modules. The bicategory **Cat**-Mod of **Cat**-categories, ie 2-categories, and **Cat** modules, can be constructed from the bicategory **Cat**-Mat of **Cat**-matrices, as explained, in the case of a more general enrichment, in [1]. This 2-category has sets as objects and homs from X to Y equal to the underlying category of $\mathbf{Cat}^{Y\times X}$, so a morphism from X to Y is a matrix $A(y,x)$ of categories, where $(y,x) \in Y \times X$. Composition of a matrix $A\colon X \to Y$ with another $B\colon Y \to Z$ is given by the formula for matrix multiplication: $(B \cdot A)(z,x) = \sum_y B(z,y) \times A(y,x)$. Note that the hom-category **Cat**-Mat(X,Y) is the underlying category of a 2-category, namely, $\mathbf{Cat}^{X\times Y}$, and the composition in **Cat**-Mat is not only functorial but 2-functorial.

A 2-category $\mathscr{A}$ with objects $|\mathscr{A}|$ is a monad on the object $|\mathscr{A}|$ in **Cat**-Mat. For a pair of 2-categories $\mathscr{A}$, $\mathscr{B}$, there is a 2-monad on **Cat**-Mat$(|\mathscr{A}|,|\mathscr{B}|)$, given by precomposing with the monad $\mathscr{A}$ and post-composing with the monad $\mathscr{B}$. The 2-category of Eilenberg-Moore algebras and strict morphisms $(\mathscr{B}\cdot - \cdot \mathscr{A})$-$\mathrm{Alg}_s$ is the 2-category of **Cat**-modules **Cat**-Mod$(\mathscr{A},\mathscr{B})$.

Observe that **Cat**-Mod(X,Y) is just **Cat**-Mat(X,Y) if X, Y are discrete 2-categories. Thus, a 2-category $\mathscr{B}$ can be regarded as a monad on $|\mathscr{B}|$ in **Cat**-Mod. The 2-category **Cat**-Mod$(\mathscr{A},\mathscr{B})$ is $(- \cdot \mathscr{B})$-Alg_s, for the 2-monad given by post-composing with the monad $\mathscr{B} : |\mathscr{B}| \twoheadrightarrow |\mathscr{B}|$ on **Cat**-Mod$(\mathscr{A},|\mathscr{B}|)$

We now substitute the category $\mathcal{C}$ in Section 2.3 by a 2-category $\mathscr{K}$, and make the modules into **Cat**-enriched modules. So $\mathfrak{D}_{\mathscr{K}}(f,g)$ is now the category with objects commutative squares with a diagonal filler as depicted in (2.9), with morphisms, from an object with diagonal d to another with diagonal d', given by 2-cells $d \Rightarrow d'$ in $\mathscr{K}$. Given 2-functors $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ and $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$, we define $\mathfrak{D}(U,V)$ in the same way as we did in Section 2.3

in the case of ordinary categories, with the difference that now the modules are **Cat**-enriched.

Let us, for the purposes of this section, denote by T the 2-monad induced by the monadic 2-functor **Cat**-Mod$(\mathscr{A},\mathscr{B}) \to$ **Cat**-Mat$(|\mathscr{A}|,|\mathscr{B}|)$. Thus, a lifting operation is just a section of $\mathfrak{D}(U,V) \to U^* \cdot V_*$ in $\mathsf{T}\text{-Alg}_s$ – Definition 2.4.

Definition 5.1. A *lax natural lifting operation* for U, V is a section of $\mathfrak{D}(U,V) \to U^* \cdot V_*$ in $\mathsf{T}\text{-Alg}_c$.

An object of the 2-category $\mathsf{T}\text{-Alg}_c$ is a **Cat**-module $\varphi\colon \mathscr{A} \twoheadrightarrow \mathscr{B}$, while a morphism $t\colon \varphi \to \psi$ is a morphism of the underlying matrices that is oplax with respect to the action of $\mathscr{A}$ and $\mathscr{B}$. This means that, given a morphism $f\colon a \to a'$ in $\mathscr{A}$, and $g\colon b \to b'$ in $\mathscr{B}$, there is extra data

$$\begin{tikzcd} \varphi(b,a') \arrow[r, "t(b{,}a')"] \arrow[d, "\varphi(g{,}f)"'] \arrow[dr, phantom, "\bar{t}_{f,g}\ \Nearrow"] & \psi(b,a') \arrow[d, "\psi(g{,}f)"] \\ \varphi(b',a) \arrow[r, "t(b'{,}a)"'] & \psi(b'a) \end{tikzcd}$$

satisfying coherence axioms.

Each component $U^* \cdot V_*(a,b) \to \mathfrak{D}(U,V)(a,b)$ of the section of Definition 5.1 gives a diagonal filler for each square $Ua \to Vb$ in $\mathscr{K}^{\mathbf{2}}$. The oplax morphism structure on the section can be described as follows. Suppose the morphisms $\alpha\colon a' \to a$ in $\mathscr{A}$ and $\beta\colon b \to b'$ in $\mathscr{B}$ are mapped by U and V to commutative squares in $\mathscr{K}$

$$\begin{tikzcd} A' \arrow[r, "x"] \arrow[d, "Ua'"'] & A \arrow[d, "Ua"] \\ B' \arrow[r, "y"] & B \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} C \arrow[r, "u"] \arrow[d, "Vb"'] & C' \arrow[d, "Vb'"] \\ D \arrow[r, "v"] & D' \end{tikzcd}$$

Consider the diagonal fillers given by the respective components of the section:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "Ua"'] & C \arrow[d, "Vb"] \\ B \arrow[r, "k"'] \arrow[ur, "d"] & C \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} A' \arrow[r, "u\cdot k\cdot x"] \arrow[d, "Ua'"'] & C' \arrow[d, "Vb'"] \\ B' \arrow[r, "v\cdot k\cdot y"'] \arrow[ur, "j"] & D' \end{tikzcd}$$

Then, the oplax morphism structure on $U^* \cdot V_* \to \mathfrak{D}(U,V)$ provides a 2-cell $\omega = \omega(\alpha,\beta)\colon j \Rightarrow u \cdot d \cdot y$, satisfying $(Vb') \cdot \omega = 1$, $\omega \cdot (Ua') = 1$, and coherence conditions that we proceed to describe. Suppose given an object d of $\mathfrak{D}(U,V)(a,b)$ as above, and morphisms in $\mathscr{A}$ and $\mathscr{B}$

$$a'' \xrightarrow{\alpha'} a' \xrightarrow{\alpha} a \quad \text{and} \quad b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$$

we have the following diagram, where the dashed arrows are chosen diagonal fillers.

$$\begin{tikzcd}
A'' \arrow[r, "x'"] \arrow[d, "Ua''"'] & A' \arrow[r, "x"] \arrow[d, "Ua'"'] & A \arrow[r, "h"] \arrow[d, "Ua"'] & C \arrow[r, "u"] \arrow[d, "Vb"] & C' \arrow[r, "u'"] \arrow[d, "Vb'"] & C'' \arrow[d, "Vb''"] \\
B'' \arrow[r, "y'"'] \arrow[urrrrr, dashed, "e"] & B' \arrow[r, "y"'] \arrow[urrr, dashed, "j"] & B \arrow[r, "k"'] \arrow[ur, dashed, "d"] & D \arrow[r, "v"'] & D' \arrow[r, "v'"'] & D''
\end{tikzcd}$$

The condition corresponding to the associativity axiom of the oplax morphism $U^* \cdot V_* \to \mathfrak{D}(U,V)$ says that

$$\big(e \xrightarrow{\omega(\alpha\cdot\alpha',\beta'\cdot\beta)} u' \cdot u \cdot d \cdot y \cdot y'\big) = \big(e \xrightarrow{\omega(\alpha',\beta')} u' \cdot j \cdot y' \xrightarrow{u'\cdot\omega(\alpha,\beta)\cdot y'} u' \cdot u \cdot d \cdot y \cdot y'\big)$$

The axiom corresponding to the unit axiom of the oplax morphism $U^* \cdot V_* \to \mathfrak{D}(U,V)$ says that $\omega(1,1) = 1$.

5.2. KZ lifting operations.

Definition 5.2. A *KZ lifting operation* in $\mathscr{K}$ for the 2-functors U, V is a left adjoint section to the morphism $\mathfrak{D}(U,V) \to U^* \cdot V_*$ in the 2-category **Cat**-Mod$(\mathscr{A},\mathscr{B})$.

In more explicit terms, a KZ lifting operation is given by, for each square (2.13) in $\mathscr{K}$, a diagonal filler $d(h,k)$, with the following universal property. For any $d'\colon B \to C$ and any pair of 2-cells α, β satisfying

$$\begin{tikzcd}
A \arrow[r, bend left, "h"] \arrow[r, bend right, "d'\cdot Ua"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \arrow[r, "Vb"] & D
\end{tikzcd} = \begin{tikzcd}
A \arrow[r, "Ua"] & B \arrow[r, bend left, "k"] \arrow[r, bend right, "Vb\cdot d'"'] \arrow[r, phantom, "\Downarrow\beta"] & D
\end{tikzcd}$$

there exists a unique 2-cell $\gamma\colon d(h,k) \to d'$ such that $\gamma \cdot Ua = \alpha$ and $Vb \cdot \gamma = \beta$. Moreover, these diagonal fillers must be natural in a and b.

Definition 5.3. A *lax natural KZ lifting operation* in $\mathscr{K}$ for the 2-functors U, V is a left adjoint section to the morphism $\mathfrak{D}(U,V) \to U^* \cdot V_*$ in the 2-category **Cat**-Mat$(|\mathscr{A}|,|\mathscr{B}|)$.

This means that a lax natural KZ lifting operation is given by a left adjoint section for each component

$$\mathfrak{D}(U,V)(a,b) \longrightarrow \mathscr{K}^{\mathbf{2}}(Ua,Vb) \qquad a \in \mathscr{A},\ b \in \mathscr{B}.$$

More explicitly, it is given by a choice, for each square $(h,k)\colon Ua \to Vb$, of a diagonal filler $d(h,k)\colon \operatorname{cod}(Ua) \to \operatorname{dom}(Vb)$ with the property that 2-cells $d(h,k) \Rightarrow d'\colon \operatorname{cod}(Ua) \to \operatorname{dom}(Vb)$ are in bijection with 2-cells $(h,k) \Rightarrow (d' \cdot Ua, Vb \cdot d')$. In other words, the same universal property of KZ lifting operation, except that the chosen diagonals $d(h,k)$ need not be natural in a, b.

Remark 5.4. A lax natural KZ lifting operation equates to providing, for each $a \in \mathscr{A}$ and $b \in \mathscr{B}$, with $Ua\colon A \to B$ and $Vb\colon C \to D$, a left adjoint section of the usual comparison functor

$$\mathscr{K}(B,C) \longrightarrow \mathscr{K}(A,C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B,D).$$

However, the presentation using modules and matrices effortlessly yields more, as discussed below.

Remark 5.5. It is clear that each KZ lifting operation is also a lax natural KZ lifting operation.

It is also clear that each *lax natural KZ* lifting operation is also a *lax natural* lifting operation – Definition 5.1. This is so by *doctrinal adjunction:* for a 2-monad S on a 2-category $\mathscr{X}$, a morphism f in $\mathsf{S}\text{-Alg}_c$ has a left adjoint (resp. left adjoint coretract) precisely when f is a pseudomorphism and its underlying morphism has left adjoint (resp. left adjoint coretract) in $\mathscr{X}$. Now one can apply this fact to the 2-monad on $\mathbf{Cat}\text{-Mat}(|\mathscr{A}|,|\mathscr{B}|)$ whose algebras are the $\mathbf{Cat}$-modules $\mathscr{B} \twoheadrightarrow \mathscr{A}$.

Proposition 5.6. *KZ lifting operations for the 2-functors $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ and $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$ are, if U has a right adjoint G, in bijective correspondence with left adjoint sections of the morphism $G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \to G \cdot V$ induced by the counit of* $\mathrm{id} \dashv \mathrm{dom}$ *– with components $G(1, Vb)$ – in the 2-category $[\mathscr{B},\mathscr{A}]$ of 2-functors $\mathscr{B} \to \mathscr{A}$.*

Proof: By the same argument deployed in Remark 2.8, the $\mathbf{Cat}$-module transformation

$$U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot V_* \longrightarrow U^* \cdot V_* \tag{5.1}$$

corresponds to the 2-natural transformation of the statement

$$G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow G \cdot V \tag{5.2}$$

via the pseudofunctor $(-)_*$ from 2-**Cat** to **Cat**-**Mod**. Since the 2-functor from $[\mathscr{B}, \mathscr{A}]$ to **Cat**-**Mod**$(\mathscr{B}, \mathscr{A})$ is full and faithful as a 2-functor, (5.1) has a left adjoint coretract if and only if (5.2) does so. ■

Proposition 5.7. *Given a lax orthogonal AWFS* (L, R) *on* $\mathscr{K}$*, the 2-natural transformation* $F^{\mathsf{L}} \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Rightarrow F^{\mathsf{L}} \cdot V$ *induced by the counit of* $\mathrm{id} \dashv \mathrm{dom}$ *has a left adjoint section in* $[\mathsf{R}\text{-}\mathrm{Alg}_s, \mathsf{L}\text{-}\mathrm{Coalg}_s]$*, where* $F^{\mathsf{L}} \colon \mathscr{K}^{\mathbf{2}} \to \mathsf{L}\text{-}\mathrm{Coalg}_s$ *is the cofree coalgebra 2-functor and* V *the forgetful 2-functor from* $\mathsf{L}\text{-}\mathrm{Coalg}_s$*.*

Proof: Given an R-coalgebra structure $(p_g, 1) \colon Rg \to g$ we need to exhibit a coretract adjunction in $\mathsf{L}\text{-}\mathrm{Coalg}_s$ with right adjoint $L(1, g) \colon L1_{\mathrm{dom}(g)} \to Lg$. We know from Remark 3.5 that there is a coretract adjunction $(1, p_g) \dashv (1, \lambda_g)$, whose unit we denote by η_g; the same Remark points out that these adjunctions are 2-natural in (g, p_g). Together with the adjunction $\Sigma_g \dashv L\Phi_g$ that exhibits L as lax idempotent, we obtain

$$L(1, p_g) \cdot \Sigma_g \dashv L\Phi_g \cdot L(1, \lambda_g) = L(1, g).$$

The unit of this composition of adjunctions is

$$1 = L\Phi_g \cdot \Sigma_g \xrightarrow{L\Phi_g \cdot L(\eta_g) \cdot \Sigma_g} L\Phi_g \cdot L(1, \lambda_g) \cdot L(1, p) \cdot \Sigma_g = 1,$$

which is the identity since $\Phi_g \cdot \eta_g = 1$ – again by Remark 3.5. ■

Theorem 5.8. *Each lax orthogonal AWFS* (L, R) *on the 2-category* $\mathscr{K}$ *induces (1) A KZ lifting operation for* $\mathsf{L}\text{-}\mathrm{Coalg}_s \to \mathscr{K}^{\mathbf{2}}$ *and* $\mathsf{R}\text{-}\mathrm{Alg}_s \to \mathscr{K}^{\mathbf{2}}$*. (2) A lax natural KZ lifting operation for* $U_\ell \colon \mathsf{L}\text{-}\mathrm{Coalg}_\ell \to \mathscr{K}^{\mathbf{2}}$ *and* $V_\ell \colon \mathsf{R}\text{-}\mathrm{Alg}_\ell \to \mathscr{K}^{\mathbf{2}}$*. Moreover, the diagonal fillers are those given by the AWFS on the usual way –* (2.2)*.*

Proof: The first part is a direct consequence of Propositions 5.6 and 5.7. The second part means that there must exist a left adjoint coretract to each functor

$$\mathfrak{D}(U_\ell, V_\ell)((f, s), (g, p)) = \mathscr{K}(\mathrm{cod}(f), \mathrm{dom}(g)) \longrightarrow \mathscr{K}^{\mathbf{2}}(f, g) \tag{5.3}$$

where (f, s) is an L-coalgebra and (g, p) an R-algebra. We know that such a left adjoint coretract does exist, by the fist part of the statement, and the proof is complete. ■

Theorem 5.8 (2) can be rephrased by saying that the usual lifting operation for (L,R) is, when both L and R are lax idempotent, lax natural with respect to all morphisms in $\mathscr{K}$. This is so since every morphism in $\mathscr{K}^{\mathbf{2}}$ has a unique structure of a lax morphism of L-coalgebras, respectively, R-algebras.

Remark 5.9. It may be useful to exhibit the counit of the coretract adjunction in the proof of Theorem 5.8, in which (5.3) is the right adjoint, even though it is not necessary to prove that result. Let d be a diagonal filler for a square (h,k) from an L-coalgebra (f,s) to an R-algebra (g,p), as depicted in (2.9). The diagonal filler given by the lifting operation is $p\cdot K(h,k)\cdot s$, and the counit $p\cdot K(h,k)\cdot s\Rightarrow d$ is

$$p\cdot K(h,k)\cdot s=\rho_{1_C}\cdot K(1,p)\cdot\sigma_g\cdot K(1,\rho_g)\cdot K(1,\lambda_g)\cdot K(h,d)\cdot s\Longrightarrow$$
$$\Longrightarrow\rho_{1_C}\cdot K(1,p)\cdot K(1,\lambda_g)\cdot K(h,d)\cdot s=\rho_{1_C}\cdot K(h,d)\cdot s=d\cdot\rho_f\cdot s=d$$

where the 2-cell is the one induced by the counit $\sigma_g\cdot K(1,\rho_g)\Rightarrow 1$.

5.3. Lax orthogonal functorial factorisations. We have seen in the previous sections that the lifting operation of a lax orthogonal AWFS has the extra structure of a *KZ* lifting operation. One could ask what extra structure is inherited from a lax orthogonal AWFS to its underlying WFS. Since we work with algebraic factorisations, we have at our disposal not only mere WFSs but functorial factorisations, and it is for these that we answer the question.

Let $\mathscr{A}$, $\mathscr{B}$ be 2-categories and $\mathscr{X}$ be $\mathbf{Cat}$-Mod$(\mathscr{B},\mathscr{A})$. Denote by M the 2-monad (M,Λ^M,Π^M) on $\mathscr{X}^{\mathbf{2}}$ whose algebras are morphisms in $\mathscr{X}$ equipped with a left adjoint coretract. A dual of M has been described in Section 3.3; more precisely, if L is the 2-comonad of Proposition 3.9, whose algebras are morphisms equipped with a right adjoint retract defined on the 2-category $(\mathscr{X}^{\mathrm{op}})^{\mathbf{2}}\cong(\mathscr{X}^{\mathbf{2}})^{\mathrm{op}}$, then M is L^{op}. An algebra for the pointed endo-2-functor (M,Λ^M) is a morphism $\alpha\colon\phi\to\psi$ equipped with a coretract $\sigma\colon\psi\to\phi$ and a 2-cell $m\colon\sigma\cdot\alpha\Rightarrow 1$ such that $\sigma\cdot m=1$. This is a dual form of Proposition 3.9 (1).

Definition 5.10. Consider 2-functors U and V from $\mathscr{A}$ and $\mathscr{B}$ into $\mathscr{K}^{\mathbf{2}}$. A *lax orthogonality structure* on U, V is an (M,Λ^M)-coalgebra structure on the morphism of $\mathbf{Cat}$-modules $U^*\cdot\mathfrak{D}_{\mathscr{K}}\cdot V_*\to U^*\cdot V_*$. Consider a functorial factorisation on $\mathscr{K}$ with associated copointed endo-2-functor (L,Φ) and associated pointed endo-2-functor (R,Λ). A lax orthogonality structure on the functorial factorisation is one on U, V, for U the forgetful 2-functor from (L, Φ)-coalgebras and V the forgetful 2-functor from (R, Λ)-algebras.

Explicitly, a lax orthogonality structure as in the definition is a choice of natural diagonal fillers $D(a,b)(h,k)\colon \operatorname{cod}(Ua) \to \operatorname{dom}(Vb)$ that is functorial on squares $(h,k)\colon Ua \to Vb$, and natural on $a \in \mathscr{A}$, $b \in \mathscr{B}$. Furthermore, for any diagonal filler e of (h,k) we are given a 2-cell $\theta(a,b)(e)\colon D(a,b)(h,k) \Rightarrow e$ that is natural on e and a modification on a, b.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "Vb"] \\ \cdot \arrow[r, "k"'] \arrow[ur, bend left, "D"] \arrow[ur, bend right, "e"'] \arrow[ur, phantom, "\Downarrow\theta(e)"] & \cdot \end{tikzcd}$$

The 2-cells $\theta(a,b)(e)$ must satisfy $(Vb) \cdot \theta(a,b)(e) = 1_k$ and $\theta(a,b)(e) \cdot (Ua) = 1_h$. Naturality in e means that for each 2-cell $\epsilon\colon e \Rightarrow \bar{e}$ the equality

$$\big(\theta(a,b)(\bar{e})\big)\big(D(a,b)(\epsilon \cdot Ua, Vb \cdot \epsilon)\big) = \epsilon\theta(a,b)(e)$$

holds. The modification property for θ means that, if $\alpha\colon a' \to a$ and $\beta\colon b \to b'$ are morphisms in $\mathscr{A}$ and $\mathscr{B}$, then

$$\operatorname{dom}(V\beta) \cdot \theta(a,b)(e) \cdot \operatorname{cod}(U\alpha) = \theta(a',b')(\operatorname{dom}(V\beta) \cdot e \cdot \operatorname{cod}(U\alpha)).$$

Observe that there is no reason why θ should satisfy the extra property that the endo-2-cell $\theta(a,b)(D(a,b)(h,k))$ of $D(a,b)(h,k)$ be an identity 2-cell.

Remark 5.11. In the particular instance when $\mathscr{A} = \mathscr{B} = \mathbf{1}$, the 2-functors U and V pick out morphisms $f\colon A \to B$ and $g\colon C \to D$ in $\mathscr{K}$, and a lax orthogonality structure for f, g can be described simply as a functor D that is a section of the canonical comparison functor H into the pullback, together with a natural transformation $\theta\colon DH \Rightarrow 1$ that satisfies $H\theta = 1$. This structure can be described as a choice of a diagonal filler $D(h,k)$ for each square (h,k) and a 2-cell $\theta(e)\colon D(h,k) \Rightarrow e$ for any other diagonal filler e, that satisfies $g \cdot \theta(e) = 1$ and $\theta(e) \cdot f = 1$.

$$\mathscr{K}(B,C) \underset{D}{\overset{H}{\rightleftarrows}} \mathscr{K}(A,C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B,D)$$

Proposition 5.12. *The underlying functorial factorisation of a lax orthogonal AWFS carries a canonical lax orthogonal structure, whose diagonal fillers are those induced by the functorial factorisation in the usual way – as in Example 2.5 and Remark 2.6.*

Proof: For an AWFS (L,R), consider the forgetful functors U and V from, respectively, the 2-categories of (L,Φ)-coalgebras and (R,Λ)-algebras. Denote by $\bar{\Sigma}=(1,s)\colon U\Rightarrow LU$ the L-coalgebra structure of U, and $\bar{\Pi}=(p,1)\colon RV\Rightarrow V$ the R-algebra structure of V. The morphisms $\bar{\Sigma}^*\colon (LU)^*\to U^*$ and $\bar{\Lambda}_*\colon (RV)_*\to V_*$ induce a retraction

$$\begin{tikzcd} U^*L^*\cdot\mathfrak{D}\cdot R_*V_* \arrow[r] \arrow[d] & U^*\cdot\mathfrak{D}\cdot V_* \arrow[d] \\ U^*L^*\cdot R_*V_* \arrow[r] & U^*\cdot V_* \end{tikzcd}$$

where the object of $\mathscr{X}^{\mathbf{2}}$ on the left is a M-algebra, where the 2-category $\mathscr{X}$ and the 2-monad M are those described in the beginning of the present section. Theorem 5.8 implies that the object of $\mathscr{X}$ depicted by the leftmost vertical arrow in the diagram carries a structure of an M-algebra. Hence the object on the right hand side is a retract of a M-coalgebra, therefore it carries an (M,Λ^M)-algebra structure that makes the retraction a morphism of (M,Λ^M)-coalgebras. It remains to show that the section of $U^*\cdot\mathfrak{D}_{\mathscr{K}}\cdot V_*\to U^*\cdot V_*$ so obtained is equal to that induced by the functorial factorisation, as described in Example 2.5, for which we appeal to the comments at the end of Section 3.3. The induced section is

$$U^*V_*\xrightarrow{U^*\Phi^*\Lambda_*V_*}U^*L^*R_*V_*\to U^*L^*\mathfrak{D}_{\mathscr{K}}R_*V_*\xrightarrow{(1,s)^*\mathfrak{D}_{\mathscr{K}}(p,1)_*}U^*\mathfrak{D}_{\mathscr{K}}V_*$$

where the middle morphism is the KZ lifting operation for LU, RV. One can verify that the diagonal filler of a square $(h,k)\colon f\to g$, where (f,s) is an (L,Φ)-coalgebra and (f,p) and (R,Λ)-algebra, is $p\cdot d\cdot s$ where d is the diagonal filler of $(\lambda_g\cdot h,k\cdot\rho_f)\colon Lf\to Rg$. But $d=K(h,k)$, so $p\cdot d\cdot s$ is precisely the diagonal filler induced by the functorial factorisation. ∎

6. Algebraic KZ Injectivity

Recall from Section 2.4, and originally from [10], the definition of the free category with a lifting operation $U^{\pitchfork}\colon\mathcal{A}^{\pitchfork}\to\mathcal{C}^{\mathbf{2}}$ for $U\colon\mathcal{A}\to\mathcal{C}^{\mathbf{2}}$. If $U\colon\mathscr{A}\to\mathscr{K}^{\mathbf{2}}$ is a 2-functor instead, $\mathscr{A}^{\pitchfork}$ has objects (g,ϕ) where ϕ is a section of the 1-cell $U^*\cdot\mathfrak{D}_{\mathscr{K}}\cdot g_*\to U^*\cdot g_*$ in the 2-category $\mathbf{Cat}\text{-Mod}(\mathbf{1},\mathscr{A})$, which is isomorphic to $[\mathscr{A}^{\mathrm{op}}, \mathbf{Cat}]$. Morphisms $(g,\phi) \to (g',\phi')$ are those morphisms $(u,v)\colon g \to g'$ in $\mathscr{K}^{\mathbf{2}}$ that are compatible with the sections, while 2-cells $(u,v) \Rightarrow (\bar{u},\bar{v})$ are pairs of 2-cells $\alpha\colon u \to \bar{u}$ and $\beta\colon v \to \bar{v}$ in $\mathscr{K}$ such that the equality below holds – we omit the dots that denote composition to save space.

$$\begin{tikzcd} U^*g_* \arrow[r, "\phi"] & U^*\mathfrak{D}_{\mathscr{K}}g_* \arrow[r, bend left, "U^*\mathfrak{D}_*(u,v)_*"] \arrow[r, bend right, "U^*\mathfrak{D}_*(\bar{u},\bar{v})_*"'] \arrow[r, phantom, "\Downarrow"] & U^*\mathfrak{D}_{\mathscr{K}}g'_* \end{tikzcd} = \begin{tikzcd} U^*g_* \arrow[r, bend left, "U^*(u,v)_*"] \arrow[r, bend right, "U^*(\bar{u},\bar{v})_*"'] \arrow[r, phantom, "\Downarrow"] & U^*g'_* \arrow[r, "\phi'"] & U^*\mathfrak{D}_{\mathscr{K}}g'_* \end{tikzcd}$$

In more elementary terms, $\alpha \cdot \phi(a,h,k) = \phi'(a, \alpha \cdot h, \beta \cdot k)$, for each $a \in \mathscr{A}$ and square $(h,k)\colon Ua \to g$. The 2-functor $U^{\pitchfork}\colon \mathscr{A}^{\pitchfork} \to \mathscr{K}^{\mathbf{2}}$ is the obvious one, analogous to one constructed in the case of categories.

Next we introduce a different construction, the universal 2-category with a KZ lifting operation.

Definition 6.1. Given a 2-functor $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ define another $U^{\triangleright}\colon \mathscr{A}^{\triangleright} \to \mathscr{K}^{\mathbf{2}}$ in the following manner. Its objects are morphisms $g \in \mathscr{K}^{\mathbf{2}}$ that are *algebraically KZ injective to* U, by which we mean equipped with a KZ lifting operation for the 2-functors U, $g\colon \mathbf{1} \to \mathscr{K}^{\mathbf{2}}$; ie a left adjoint coretract to $U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot g_* \to U^* \cdot g_*$. Hence, every object is an object of $\mathscr{A}^{\pitchfork}$ equipped with the extra structure of a coretract adjunction. A morphism $g \to g'$ is a morphism (h,k) in $\mathscr{K}^{\mathbf{2}}$ such that in the diagram below not only the square formed with the right adjoints commute – this always holds – but moreover the diagram represents a morphism of adjunctions; ie it commutes with the coretracts and it is compatible with the counits.

$$\begin{tikzcd} U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot g_* \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "U^*\cdot\mathfrak{D}_{\mathscr{K}}\cdot(h,k)_*"'] & U^* \cdot g_* \arrow[l, shift left] \arrow[d, "U^*\cdot(h,k)_*"] \\ U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot g'_* \arrow[r, shift left] \arrow[r, phantom, "\perp"] & U^* \cdot g'_* \arrow[l, shift left] \end{tikzcd} \tag{6.1}$$

The 2-cells in $\mathscr{A}^{\triangleright}$ are those of $\mathscr{K}^{\mathbf{2}}$. Observe that any such 2-cell is automatically compatible with the left adjoints in (6.1) – see Proposition 3.9 (2). There are obvious forgetful 2-functors $\mathscr{A}^{\triangleright} \to \mathscr{A}^{\pitchfork}$ and $\mathscr{A}^{\triangleright} \to \mathscr{K}^{\mathbf{2}}$, the first of which is locally fully faithful by the observation of the previous paragraph.

Dually, given a 2-functor $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$ define ${}^{\triangleleft}V\colon {}^{\triangleleft}\mathscr{B} \to \mathscr{K}^{\mathbf{2}}$ by ${}^{\triangleleft}\mathscr{B} = (\mathscr{B}^{\mathrm{op}})^{\triangleright}$ and ${}^{\triangleleft}V = (V^{\mathrm{op}})^{\triangleright}$. Here we use the obvious isomorphism $(\mathscr{K}^{\mathbf{2}})^{\mathrm{op}} \cong (\mathscr{K}^{\mathrm{op}})^{\mathbf{2}}$. More explicitly, objects of ${}^{\triangleleft}\mathscr{B}$ are $f \in \mathscr{K}^{\mathbf{2}}$ equipped with a KZ lifting operation for the 2-functors $f\colon \mathbf{1} \to \mathscr{K}^{\mathbf{2}}$, V.

There is a concise way of describing $\mathscr{A}^{\triangleright}$. Let M be the 2-monad on the 2-category $\mathscr{P}(\mathscr{A})^{\mathbf{2}}$ whose algebras are right adjoint retract morphisms in $\mathscr{P}(\mathscr{A}) = [\mathscr{A}^{\mathrm{op}}, \mathbf{Cat}]$. This 2-monad can be described by performing the construction of the 2-monad of Section 3.3 starting from the 2-category $\mathscr{P}(\mathscr{A})^{\mathrm{op}}$. The $\mathbf{Cat}$-module morphism $\mathfrak{D}_{\mathscr{K}} \to 1_{\mathscr{K}^{\mathbf{2}}}$ can be equivalently described as a 2-functor

$$E \colon \mathscr{K}^{\mathbf{2}} \longrightarrow \mathscr{P}(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$$

that sends $g \in \mathscr{K}^{\mathbf{2}}$ to $\mathfrak{D}_{\mathscr{K}}(-,g) \to \mathscr{K}^{\mathbf{2}}(-,g)$. Then $\mathscr{A}^{\triangleright}$ is the pullback of the 2-category of M-algebras along $\mathscr{P}(U^*)^{\mathbf{2}}E$.

$$\begin{tikzcd}
\mathscr{A}^{\triangleright} \arrow[rr] \arrow[d] & & \mathsf{M}\text{-Alg}_s \arrow[d] \\
\mathscr{K}^{\mathbf{2}} \arrow[r, "E"] & \mathscr{P}(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}} \arrow[r, "\mathscr{P}(U^*)^{\mathbf{2}}"] & \mathscr{P}(\mathscr{A})^{\mathbf{2}}
\end{tikzcd}$$

One can express the compatibility of the morphism (h,k) in $\mathscr{A}^{\triangleright}$ with the counits required in Definition 6.1 in terms of diagonal fillers. Given a diagonal filler j as on the left hand side below, the counit provides for a 2-cell $\varepsilon_j \colon \phi(a,u,v) \Rightarrow j$. The compatibility means that $h \cdot \varepsilon_j = \varepsilon_{h \cdot j}$.

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ua"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "v"'] \arrow[ur, "j"] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ua"'] & \cdot \arrow[r, "h"] \arrow[d, "g"] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, "v"'] \arrow[ur, bend left, "\phi"] \arrow[ur, bend right, "j"'] \arrow[ur, phantom, "\Searrow"] & \cdot \arrow[r, "k"'] & \cdot
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "h \cdot u"] \arrow[d, "Ua"'] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, "k \cdot v"'] \arrow[ur, bend left] \arrow[ur, bend right, "h \cdot j"'] \arrow[ur, phantom, "\Searrow"] & \cdot
\end{tikzcd}$$

These constructions are functorial, in the sense that if $F \colon (\mathscr{A}, U) \to (\mathscr{B}, V)$ is a 2-functor over $\mathscr{K}^{\mathbf{2}}$, there is another 2-functor $F^{\triangleright} \colon (\mathscr{B}^{\triangleright}, V^{\triangleright}) \to (\mathscr{A}^{\triangleright}, U^{\triangleright})$, which sends $g \in \mathscr{K}^{\mathbf{2}}$ equipped with a KZ lifting operation for V, g to the induced choice for $U = VF, g$. A 2-functor ${}^{\triangleleft}F$ can be similarly defined.

Remark 6.2. Given $V \colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$, there is an isomorphism of categories between 2-functors $\mathscr{B} \to \mathscr{A}^{\triangleright}$ over $\mathscr{K}^{\mathbf{2}}$ and KZ lifting operations for the pair of 2-functors U, V. Similarly, there is an isomorphism of categories between 2-functors $\mathscr{A} \to {}^{\triangleleft}\mathscr{B}$ over $\mathscr{K}^{\mathbf{2}}$ and KZ lifting operations for the pair of 2-functors U, V. We hence have a natural isomorphism of *sets*

$$\mathbf{2\text{-}Cat}/\mathscr{K}^{\mathbf{2}}((\mathscr{A}, U), (\mathscr{B}^{\triangleright}, V^{\triangleright})) \cong \mathbf{2\text{-}Cat}/\mathscr{K}^{\mathbf{2}}((\mathscr{B}, V), ({}^{\triangleleft}\mathscr{A}, {}^{\triangleleft}U))$$

and an adjunction between $(-)^{\triangleright}$ and ${}^{\triangleleft}(-)$.

The unit and counit of this adjunction – or rather, both units – are 2-functors $N_U\colon \mathscr{A} \to {}^{\triangleleft}(\mathscr{A}^{\triangleright})$ and $M_U\colon \mathscr{A} \to ({}^{\triangleleft}\mathscr{A})^{\triangleright}$ commuting with the functors into $\mathscr{K}^{\mathbf{2}}$. The first one corresponds to the tautological KZ lifting operation for the pair of 2-functors $U, U^{\triangleright}$, and the second one to the tautological KZ lifting operation for ${}^{\triangleleft}U, U$.

Example 6.3. In the case when U is the 2-functor $f\colon \mathbf{1} \to \mathscr{K}^{\mathbf{2}}$ that picks out a morphism f, the objects of the 2-category $f^{\triangleright}$ are morphisms *algebraically KZ injective with respect to f*. This is a slightly abuse of language, as a morphism can be algebraically KZ injective to f in more than one way – but two such are, of course, isomorphic.

Lemma 6.4. *Given a 2-functor $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$, a 2-adjunction $U \dashv G$ and $g \in \mathscr{K}^{\mathbf{2}}$, there is a bijection of 2-categories over $\mathscr{K}^{\mathbf{2}}$ between $\mathscr{A}^{\triangleright}$ and the 2-category described by:*

- *Objects are coretract adjunctions $\ell_g \dashv G(1,g)\colon G(1_{\mathrm{dom}(g)}) \to Gg$ in $\mathscr{A}$.*
- *Morphisms from $\ell_g \dashv G(1,g)$ to $\ell_{\bar g} \dashv G(1,\bar g)$ are morphisms $(h,k)\colon g \to \bar g$ in $\mathscr{K}^{\mathbf{2}}$ such that $G(h,k)$ defines a morphism of adjunctions: $G(h,k)\cdot \ell_g = \ell_{\bar g}\cdot G(h,k)$ and $G(h,k)$ commutes with the counits.*
- *2-cells $(h,k) \Rightarrow (\bar h, \bar k)$ are 2-cells in $\mathscr{K}^{\mathbf{2}}$, with no additional conditions.*

Proof: By Proposition 5.6 there is a bijection between objects of $\mathscr{A}^{\triangleright}$ and coretract adjunctions as in the statement. The description of the morphisms and 2-cells is a direct translation from the ones of $\mathscr{A}^{\triangleright}$ – Definition 6.1. ∎

Lemma 6.5. *Assume the conditions of Lemma 6.4. Then, for any full sub-2-category $\mathscr{F} \subset \mathscr{A}$ containing the full image of G, the functor $\mathscr{A}^{\triangleright} \to \mathscr{F}^{\triangleright}$ induced by the inclusion is an isomorphism.*

Proof: If we denote by $J\colon \mathscr{F} \hookrightarrow \mathscr{A}$ the inclusion and $H = JG\colon \mathscr{A} \to \mathscr{F}$ the right adjoint of UJ, Lemma 6.4 allows us to describe $\mathscr{F}^{\triangleright}$ as the 2-category with objects coretract adjunctions $\ell_g \dashv H(1,g)\colon H(1_{\mathrm{dom}(g)}) \to Hg$ in $\mathscr{F}$. But to give this retract adjunction in $\mathscr{F}$ is equivalent to giving a retract adjunction $\ell_g \dashv G(1,g)$ in $\mathscr{A}$. The rest of the proof is similarly easy. ∎

Proposition 5.7 together with Lemma 6.4 imply that if (L,R) is a lax orthogonal AWFS, there is a 2-functor, depicted by the dashed arrow below. On objects, it sends an R-coalgebra $(p,1)\colon Rg \to g$ to the coretract adjunction $L(1,p)\cdot \Sigma_g \dashv L(1,g)$ in $\mathsf{L}\text{-Coalg}_s$, composition of the adjunctions $\Sigma_g \dashv L\Phi_g$

and $L(1,p) \dashv L(1,\lambda_g)$. On morphisms and 2-cells it is given by the identity.

$$\begin{tikzcd} \mathsf{R}\text{-Alg}_s \arrow[r, dashed] \arrow[dr] & \mathsf{L}\text{-Coalg}_s^{\triangleright} \arrow[r] \arrow[d] & \mathscr{F}^{\triangleright} \arrow[d] \\ & \mathsf{L}\text{-Coalg}_s^{\pitchfork} \arrow[r] & \mathscr{F}^{\pitchfork} \end{tikzcd} \tag{6.2}$$

Theorem 6.6. *The following are equivalent for an AWFS* (L,R) *on a 2-category. (1)* (L,R) *is a lax orthogonal AWFS. (2) There is an – essentially unique – KZ lifting operation for the forgetful 2-functors from* L*-coalgebras and from* R*-algebras. (3) There is an – essentially unique – 2-functor* $\mathsf{R}\text{-Alg}_s \to \mathsf{L}\text{-Coalg}_s^{\triangleright}$ *making* (6.2) *commutative. (4) There is an – essentially unique – 2-functor* $\mathsf{R}\text{-Alg}_s \to \mathscr{F}^{\triangleright}$ *making the outer diagram in* (6.2) *commutative, for any full sub-2-category* $\mathscr{F} \subset \mathsf{L}\text{-Coalg}_s$ *containing the cofree* L*-coalgebras.*

Proof: There is a bijection between structures in (2) and those in (3), by definition of $\mathscr{A}^{\triangleright}$, in which case both are essentially unique since KZ lifting operations are unique up to isomorphism – Remark 6.2. The equivalence of (3) and (4) follows from Lemma 6.5, while that of (1) and (3) was already explained above.

We now proceed to prove (3)⇒(1). As it has been our convention, we will denote by $\mathscr{K}$ the base 2-category, by U and V the forgetful 2-functors from the 2-categories of L-coalgebras and R-algebras, respectively.

Let (g,p) be an R-algebra. Its image in $\mathsf{L}\text{-Coalg}_s^{\pitchfork}$ can be given as in Corollary 2.13, again by (g,p). By hypothesis, (g,p) carries a structure of an object of $\mathsf{L}\text{-Coalg}_s^{\triangleright}$. By definition $p \cdot \lambda_g = 1$ and $g \cdot p = \rho_g$. Consider the diagonal

$$\begin{tikzcd} \cdot \arrow[r, "\lambda_g"] \arrow[d, "\lambda_g"'] & \cdot \arrow[d, "\rho_g"] \\ \cdot \arrow[r, "\rho_g"'] \arrow[ur, "\lambda_g \cdot p" description] & \cdot \end{tikzcd}$$

and note that $\rho_g = Rg$ is an object of $\mathsf{L}\text{-Coalg}_s^{\triangleright}$, and that the chosen diagonal filler of the outer square is the identity morphism. It follows the existence of a unique 2-cell $\eta\colon 1 \Rightarrow \lambda_g \cdot p$ such that $\eta \cdot \lambda_g = 1$ and $\rho_g \cdot \eta = 1$. The first of these two equalities is one of the triangular identities required to obtain a retract adjunction $p \dashv \lambda_g$. The second of the these equalities tells us that, if we can prove the other triangular identity, we obtain not only an adjunction in $\mathscr{K}$ but also a retract adjunction $(p,1) \dashv \Lambda_g$ in $\mathscr{K}^{\mathbf{2}}$.

We now show that $p \cdot \eta = 1$. Consider the pasting below.

$$\begin{tikzcd}
\cdot \arrow[rrr, equal] \arrow[ddd, "\lambda_g"'] & & & \cdot \arrow[ddd, "g"] \\
& & \cdot \arrow[ur, "p"] & \\
& \cdot \arrow[ur, "\lambda_g"'] \arrow[u, phantom, "\Downarrow\eta"] & & \\
\cdot \arrow[ur, "p"] \arrow[uurr, bend left, "1"] \arrow[rrr, "\rho_g"'] & & & \cdot
\end{tikzcd}$$

The chosen diagonal filler of the outer diagram is p, and $p \cdot \eta$ is an endo-2-cell of p. In addition, $g \cdot p \cdot \eta = \rho_g \cdot \eta = 1$ and $p \cdot \eta \cdot \lambda_g = 1$. By the universal property of KZ lifting operations spelled out immediately after Definition 5.2, it must be $p \cdot \eta = 1$. This finishes the proof that R-algebra structures are left adjoint retracts to the components of the unit of R, ie that R is lax idempotent.

One can show that L is lax idempotent either by appealing to Theorem 4.1 or by a duality argument. By taking opposite 2-categories, and taking into account the isomorphism $(\mathscr{K}^{\mathrm{op}})^{\mathbf{2}} \cong (\mathscr{K}^{\mathbf{2}})^{\mathrm{op}}$, the 2-functor $\mathsf{L}\text{-Coalg}_s \to {}^{\triangleleft}\mathsf{R}\text{-Alg}_s$, which exists by Remark 6.2, transforms into a 2-functor $\mathsf{L}^{\mathrm{op}}\text{-Alg}_s \to \mathsf{R}^{\mathrm{op}}\text{-Coalg}_s^{\triangleright}$ that commutes with the 2-functors into $\mathsf{R}^{\mathrm{op}}\text{-Coalg}_s^{\pitchfork}$. By the proof above we know that L^{op} is a lax idempotent 2-monad on $(\mathscr{K}^{\mathbf{2}})^{\mathrm{op}}$, which is to say that L is a lax idempotent 2-comonad. ∎

Theorem 6.6 has a dual statement of the following form: an AWFS (L, R) is lax orthogonal if and only if there exists an – essentially unique – 2-functor $\mathsf{L}\text{-Coalg}_s \to {}^{\triangleleft}\mathsf{R}\text{-Alg}_s$ commuting with the respective forgetful functors into ${}^{\pitchfork}\mathsf{R}\text{-Alg}_s$.

Remark 6.7. For a lax orthogonal AWFS (L, R), objects of $\mathsf{L}\text{-Coalg}_s^{\triangleright}$ are in bijection with normal pseudo-R-algebras. Indeed, the proof of Theorem 6.6 shows that they are in bijection with retract adjunctions $(p, 1) \dashv \Lambda_g$ in $\mathscr{K}^{\mathbf{2}}$, which are precisely normal pseudo-R-algebras

7. Extending copointed endofunctors to comonads

This section includes a series of results that provide conditions that will allow us to transfer a comonad along an adjunction. We start with the following lemma, which appears, in a slightly different form, in [12].

Lemma 7.1. *Let $F \dashv U \colon \mathcal{A} \to \mathcal{B}$ be an adjunction with unit i, (G, ε) a copointed endofunctor on $\mathcal{A}$ and (L, Φ) the copointed endofunctor on $\mathcal{B}$ defined by the pullback square on the left hand side below. Then, the square on the* *right hand side is a pullback.*

$$
\begin{tikzcd}
L \arrow[r, "\tau"] \arrow[d, "\Phi"'] & UGF \arrow[d, "U\varepsilon F"] \\
1 \arrow[r, "i"'] & UF
\end{tikzcd}
\qquad
\begin{tikzcd}
(L,\Phi)\text{-Coalg} \arrow[r] \arrow[d, "U"'] & (G,\varepsilon)\text{-Coalg} \arrow[d, "U"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{A}
\end{tikzcd}
\tag{7.1}
$$

The transformation τ has a transpose $\hat{\tau}\colon FL \Rightarrow GF$ that makes $(F, \hat{\tau})$ into a morphism of copointed endofunctors from (L, Φ) to (G, ε). This means that an (L, Φ)-coalgebra structure $s\colon X \to LX$ corresponds to the (G, ε)-coalgebra structure $\hat{s} = \hat{\tau}_X \cdot Fs\colon FX \to GFX$. Conversely, if $t\colon FX \to GFX$ is a (G, ε)-coalgebra, then the unique $s\colon X \to LX$ such that $\tau_X \cdot s = Ut \cdot i_X$ is an (L, Φ)-coalgebra.

Below we use the term coalgebra for a copointed endofunctor, or comonad, on a category $\mathcal{B}$ in the generalised sense of a functor with codomain $\mathcal{B}$ with extra structure, as opposite to the usual notion of an object of $\mathcal{B}$ with extra structure.

Lemma 7.2. *In the situation of Lemma 7.1, there is a bijection between (L, Φ)-coalgebra structures $\Sigma\colon L \Rightarrow L^2$ on L and (G, ε)-coalgebra structures $\hat{\Sigma}\colon FL \Rightarrow GFL$ on FL. Moreover,*

(1) Σ is coassociative if and only if $F \cdot \Sigma\colon (FL, \hat{\Sigma}) \to (FL^2, \hat{\Sigma}L)$ is a morphism of (G, ε)-coalgebras.
(2) $(L \cdot \Phi)\Sigma = 1$ if and only if $(GF \cdot \Phi)\hat{\Sigma} = \hat{\tau}$.
(3) (L, Φ, Σ) is a 2-comonad if and only if the conditions in (1) and (2) hold.
(4) An (L, Φ)-coalgebra structure $s\colon X \to LX$ in $\mathcal{B}$ is an (L, Φ, Σ)-coalgebra structure if and only if $Fs\colon (FX, \hat{s}) \to (FLX, \hat{\Sigma}_X)$ is a morphism of (G, ε)-coalgebras.

Proof: The bijection between Σ and $\hat{\Sigma}$ was described in the paragraph previous to this lemma. The counit axiom $(\Phi \cdot L)\Sigma = 1$ comes at no cost from the fact that Σ is an (L, Φ)-coalgebra structure on L. Using the bijection, $(\Sigma \cdot L)\Sigma$ corresponds to $(\hat{\Sigma} \cdot L)(F \cdot \Sigma)$, while $(L \cdot \Sigma)\Sigma$ corresponds to $(GF \cdot \Sigma)\hat{\Sigma}$. Thus, Σ is coassociative if and only if $F \cdot \Sigma$ is a morphism of (G, ε)-coalgebras as in the statement.

To prove (2), note that $(L \cdot \Phi)\Sigma = 1$ precisely when its composition with τ equals τ and its composition with Φ equals Φ. The last fact always holds, as $\Phi(L \cdot \Phi)\Sigma = \Phi(\Phi \cdot L)\Sigma = \Phi$. By taking transposes under the adjunction

$F \dashv U$, the first fact is equivalent to $(GF \cdot \Phi)\hat{\Sigma} = \hat{\tau}$. The statement (3) is obvious.

We now prove (4). The coalgebra s is a coalgebra over the comonad if $(Ls) \cdot s = \Sigma_X \cdot s$. By the definition of L as a pullback given in Lemma 7.1, this is equivalent to $\Phi_{LX} \cdot (Ls) \cdot s = \Phi_{LX} \cdot \Sigma_X \cdot s$ and $\tau_{LX} \cdot (Ls) \cdot s = \tau_{LX} \cdot \Sigma_X \cdot s$. The first of these two equalities holds since s is compatible with Φ; in fact both sides of the equality are s. The second equality is easily seen to be equivalent to $(GFs) \cdot \hat{s} = \hat{\Sigma}_X \cdot (Fs)$. ■

Denote by $\mathsf{L}' = (L', \Phi', \Sigma')$ the 2-comonad on $\mathscr{A}^{\mathbf{2}}$ part of the AWFS that factors a morphism as a left adjoint coretract followed by a split opfibration, described in Section 3.3. In Proposition 3.9 we gave a 2-category isomorphic to that of (L', Φ')-coalgebras, henceforth we shall identify the two 2-categories. For example, (L', Φ')-coalgebras shall be identified with morphisms f in $\mathscr{A}$ with a retract v and a 2-cell $\xi\colon f \cdot v \Rightarrow 1$ such that $\xi \cdot f = 1$.

Proposition 7.3. *Let $F \dashv U\colon \mathscr{A} \to \mathscr{K}$ be a 2-adjunction, where $\mathscr{A}$ has lax limits of morphisms and $\mathscr{K}$ has pullbacks. Denote by (L, Φ) the copointed endofunctor on $\mathscr{K}^{\mathbf{2}}$ constructed from (L', Φ') in Lemma 7.1.*

(1) There is a transformation $\Omega_f\colon FKf \to F(\operatorname{dom} f)$, 2-natural in $f \in \mathscr{K}^{\mathbf{2}}$, and a modification $\omega_f\colon Ff \cdot \Omega_f \Rightarrow F\rho_f$ that induces the lifting of F to $(L, \Phi)\text{-Coalg}_s \to (L', \Phi')\text{-Coalg}_s$ in the following way: if $(1, s)\colon f \to Lf$ is an (L, Φ)-coalgebra, then $v = \Omega_f \cdot Fs\colon F(\operatorname{cod} f) \to F(\operatorname{dom} f)$ together with $\omega_f \cdot Fs\colon Ff \cdot v \Rightarrow 1$ gives an (L', Φ')-coalgebra structure on Ff as described in Proposition 3.9.

(2) Given an (L, Φ)-coalgebra structure $\Sigma = (1, \sigma)\colon L \Rightarrow L^2$, or equivalently by Lemma 7.2, a 2-natural transformation $r_f = \Omega_{Lf} \cdot F\sigma_f$ and modification with components $\alpha_f\colon F\lambda_f \cdot r_f \Rightarrow 1$ that satisfies

$$r_f \cdot F\lambda_f = 1 \qquad \alpha_f \cdot F\lambda_f = 1, \tag{7.2}$$

the following are equivalent.

(a) (L, Φ, Σ) is a 2-comonad.

(b) The following equalities hold.

$$r_{\lambda_f} \cdot F\sigma_f = r_f \qquad \alpha_{\lambda_f} \cdot F\sigma_f = F\sigma_f \cdot \alpha_f \qquad F\rho_f \cdot \alpha = \omega_f. \tag{7.3}$$

Proof: As it is our custom, we denote by K and K' the functorial factorisations associated with (L, Φ) and (L', Φ'). We begin by proving (1).

We have noted before that there is a transformation $\hat{\tau}\colon FL \Rightarrow L'F$ satisfying $(\Phi' \cdot F)\hat{\tau} = F \cdot \Phi$. Since the domain component of both Φ and Φ' are identities, the same holds for $\hat{\tau}$, which is thus given by 2-natural morphisms $\theta_f\colon FKf \to K'Ff$, satisfying $\rho'_{Ff} \cdot \theta_f = F\rho_f$. By definition of $K'Ff$ as the comma object $Ff \downarrow 1$, we obtain 2-natural morphisms $\Omega_f\colon FKf \to F(\operatorname{dom} f)$ and a modification $\omega_f\colon Ff \cdot \Omega_f \Rightarrow F\rho_f$. We know that the 2-functor from (L,Φ)-coalgebras to (L',Φ')-coalgebras is induced by $\hat{\tau}$, which in terms of θ means that if $(1,s)\colon f \to Lf$ is an (L,Φ)-coalgebra, for $s\colon \operatorname{cod} f \to Kf$, then $\theta_f{\cdot}Fs\colon \operatorname{cod}(Ff) \to K'Ff$ is an (L',Φ')-coalgebra. This coalgebra structure on Ff can be described in the terms of Proposition 3.9 as the morphism $\Omega_f \cdot Fs$ equipped with a 2-cell ω_f as in the statement.

(2) By Lemma 7.2 and Proposition 3.9, to give an (L,Φ)-coalgebra structure Σ on L is equivalent to giving an (L',Φ')-coalgebra structure on FL, ie morphisms $r_f\colon Kf \to F(\operatorname{dom} f)$, 2-natural in f, and a modification with components $\alpha_f\colon F\lambda_f \cdot r_f \Rightarrow 1$ satisfying the equalities (7.2). The condition equivalent to coassociativity of Σ given in Lemma 7.2(1) translates into conjunction of the first two equalities in (7.3). The counital condition given in Lemma 7.2(2) translates to the third equality in (7.3). ∎

In the following proposition we continue with the notations used above.

Proposition 7.4. *Assume given the 2-comonad* $\mathsf{L} = (L,\Phi,\Sigma)$ *of Proposition 7.3. The category* L-Coalg_s *is isomorphic to the 2-category with*

- *Objects* (L,Φ)*-coalgebras* $(1,s)\colon f \to Lf$ *on* $f\colon A \to B$*, with corresponding* $v\colon FKf \to F(\operatorname{dom} f)$ *and* $\xi\colon Ff{\cdot}v \Rightarrow 1$ *that satisfy* $\xi{\cdot}Ff = 1$ *as in Proposition 3.9, that in addition must satisfy*
$$r_f \cdot Fs = v \qquad Fs \cdot \xi = \alpha_f \cdot Fs. \tag{7.4}$$
- *Morphisms from* (f,v^f,ξ^f) *to* (g,v^g,ξ^g)*, morphisms* $(h,k)\colon f \to g$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that*
$$v^g \cdot Fk = Fh \cdot v^f \quad \textit{and} \quad \xi^g \cdot Fk = Fk \cdot \xi^f. \tag{7.5}$$
- *For morphisms* (h,k) *and* $(\bar{h},\bar{k})$ *between the coalgebras in* (f,v^f,ξ^f) *and* (g,v^g,ξ^g) *as above, 2-cells* $(h,k) \Rightarrow (\bar{h},\bar{k})$ *are those of* $\mathscr{K}^{\mathbf{2}}$ *that satisfy* $F\alpha \cdot v^f = v^g \cdot F\beta$.

Proof: In this proof we continue using the notation of Proposition 7.3. According to Lemma 7.2, an L-coalgebra structure on $f\colon A \to B$ is given by $(1,s)\colon f \to Lf$ such that $(1,Fs)$ is a morphism of (L',Φ')-coalgebras

$(Ff,(1,\hat{s})) \to (F\lambda_f, \hat{\Sigma}_f)$, where $\hat{s}\colon F(\operatorname{cod} f) \to K'Ff$ is the (L',Φ')-coalgebra structure on Ff induced by s. By Lemma 7.2, we must show that $(1, Fs)$ is a morphism of (L',Φ')-coalgebras from Ff to $F\lambda_f$, which happens precisely if $r_f \cdot Fs = v$ and $Fs \cdot \xi_f = \alpha_f \cdot Fs$, by Proposition 3.9.

Morphisms between L-coalgebras are those between the underlying (L,Φ)-coalgebras. By the pullback of categories depicted in (7.1), a morphism from (f, v^f, ξ^f) to (g, v^g, ξ^g) is a morphism $(h,k)\colon f \to g$ in $\mathscr{K}^{\mathbf{2}}$ such that (Fh, Fk) is a morphism of (L',Φ')-coalgebras $Ff \to Fg$, which means precisely the equalities (7.5) by the description of morphisms in Proposition 3.9.

Similarly, 2-cells between two such morphisms are 2-cells in $\mathscr{K}^{\mathbf{2}}$ whose image under F are 2-cells in (L',Φ')-Coalg_s. The compatibility with v^f and v^g given in the statement is a direct consequence of the descriptions of 2-cells in Proposition 3.9(1). ■

8. Simple 2-adjunctions and lax idempotent 2-monads

This section introduces the notion of simple 2-adjunction, which can be thought as a lax version of that of simple reflection studied in [4].

Lemma 8.1. *Let $\mathscr{A}$ be a monoidal 2-category and $\mathscr{C} \subseteq \mathscr{A}$ a coreflective 2-category, closed under the monoidal structure, and (T,i,m) a monoid in $\mathscr{A}$. If $\alpha\colon S \to T$ is the coreflection of T into $\mathscr{C}$, then S carries a structure of a monoid (S,j,n) making α a monoid morphism. Assume further that $\alpha \otimes S\colon S\otimes S \to T \otimes S$ is the coreflection of $T\otimes S$. Then S is lax idempotent if there exists a coretract adjunction*

$$\big(T \xrightarrow{T\otimes j} T\otimes S\big) \dashv \big(T\otimes S \xrightarrow{T\otimes\alpha} T\otimes T \xrightarrow{m} T\big). \tag{8.1}$$

Proof: The unit $j\colon I \to S$ and multiplication $n\colon S\otimes S \to S$ are defined by $\alpha \cdot j = i$ and $\alpha\cdot n = m\cdot(\alpha\otimes\alpha)$. We shall define a 2-cell $\delta\colon S\otimes j \Rightarrow j\otimes S\colon S \to S\otimes S$. From the fact that $\alpha\otimes S$ is a coreflection, it follows that to give δ is equally well to give a 2-cell $\delta'\colon (T\otimes j)\cdot\alpha \Rightarrow i\otimes S$, and by the adjunction (8.1), to give a 2-cell $\delta''\colon \alpha \Rightarrow m\cdot(T\otimes\alpha)\cdot(i\otimes S)$, which we choose to be the identity.

The axiom $\delta\cdot j = 1$ of a lax idempotent monoid follows from the triangular identity $\varepsilon\cdot(T\otimes j) = 1$, where ε is the counit of (8.1): we show that $\delta'\cdot j = 1$ below.

$$\delta'\cdot j = ((T\otimes j)\cdot m\cdot(T\otimes\alpha)\cdot\delta'\cdot j)(\varepsilon\cdot(j\otimes S)\cdot j) = \varepsilon\cdot(T\otimes j)\cdot i = 1.$$

It only rests to verify the axiom $n \cdot \delta = 1$. By the coreflection α, we have to show $1 = \alpha \cdot n \cdot \delta = m \cdot (\alpha \otimes \alpha) \cdot \delta = m \cdot (T \otimes \alpha) \cdot \delta' = \delta''$, which holds by our choice of δ''. ∎

Before continuing, it is convenient to introduce some notation. Each endofunctor S of $\mathcal{C}^{\mathbf{2}}$ corresponds under the isomorphism $\mathrm{End}(\mathcal{C}^{\mathbf{2}}) \cong [\mathcal{C}^{\mathbf{2}}, \mathcal{C}]^{\mathbf{2}}$ to a pair of functors S_0, $S_1 \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ with a natural transformation $S_0 \Rightarrow S_1$. We denote the component of this natural transformation at f by $Sf \colon S_0 f \to S_1 f$. A morphism $S \to T$ in $\mathrm{End}(\mathcal{C}^{\mathbf{2}})$ corresponds to a pair of natural transformations $S_0 \Rightarrow T_0$ and $S_1 \Rightarrow T_1$, compatible with $S_0 \Rightarrow S_1$ and $T_0 \Rightarrow T_1$.

The following lemma is contained in [10, Prop 4.7].

Lemma 8.2. *If $\mathcal{C}$ has pushouts, then the category of codomain-preserving pointed endofunctors $1\backslash\mathrm{End}_{\mathrm{cod}}(\mathcal{C}^{\mathbf{2}})$ is a – non full – coreflective subcategory of the category of pointed endofunctors $1\backslash\mathrm{End}(\mathcal{C}^{\mathbf{2}})$. The coreflection of a monad has a canonical structure of a codomain-preserving monad that makes the counit a monad morphism.*

Given a pointed endofunctor (T, Θ), its codomain-preserving coreflection (R, Λ) is given by the following pullback square, while the point $\Lambda \colon 1 \Rightarrow R$ is induced by the universal property. The natural transformation $R \Rightarrow T$ with components given by the pullback square is the counit of the coreflection.

$$\begin{tikzcd}
A \arrow[dr, "\Lambda_{0f}"] \arrow[drr, bend left, "\Theta_{0f}"] \arrow[ddr, bend right, "f"'] & & \\
& R_0 f \arrow[r] \arrow[d, "Rf"] & T_0 f \arrow[d, "Tf"] \\
& B \arrow[r, "\Theta_{1f}"'] & T_1 f
\end{tikzcd}$$

Remark 8.3. For future reference, we state that the coreflection $R \Rightarrow T$ of a monad T on $\mathcal{C}^{\mathbf{2}}$ into a codomain-preserving monad R is a monad morphism.

Definition 8.4. Let $F \dashv U \colon \mathscr{A} \to \mathscr{K}$ be a 2-adjunction, P a 2-monad on $\mathscr{A}^{\mathbf{2}}$, and $\alpha \colon \mathsf{S} \to U^{\mathbf{2}}\mathsf{P}F^{\mathbf{2}}$ the codomain-preserving coreflection of the 2-monad $U^{\mathbf{2}}\mathsf{P}F^{\mathbf{2}}$. The 2-adjunction is said to be *simple* with respect to P if there is a coretract adjunction in the 2-category $[\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}]$

$$\big(PF \xrightarrow{PF^{\mathbf{2}}j} PFS\big) \dashv \big(PFS \xrightarrow{PF^{\mathbf{2}}\alpha} PFUPF \xrightarrow{PeLF^{\mathbf{2}}f} PPF \xrightarrow{mF} PF\big)$$

where j the unit of S, e the counit of $F \dashv U$, and m the multiplication of P in $\mathscr{K}$.

Lemma 8.5. *Given a simple 2-adjunction as in Definition 8.4, the codomain-preserving reflection* S *is a lax idempotent 2-monad.*

Proof: Let us denote by T the 2-monad $U^{\mathbf{2}}PF^{\mathbf{2}}$. By the construction of the coreflection S as a pullback, it is clear that $\alpha T\colon SS \to TS$ is the coreflection of TS. Lemma 8.1 tells us that S will be lax idempotent if we have a coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathscr{K}^{\mathbf{2}}]$

$$\big(T \xrightarrow{Tj} TS\big) \dashv \big(TS \xrightarrow{T\alpha} TT \longrightarrow T\big).$$

Such an adjunction is obtained from the one of Definition 8.4 by applying $U^{\mathbf{2}}$. ■

Remark 8.6. Given $F \dashv U$, P and the codomain-preserving coreflection $\alpha\colon S \to U^{\mathbf{2}}PF^{\mathbf{2}}$ as in Lemma 8.5 – we do not make any assumption related to idempotency, however – since α is a 2-monad (strict) morphism, we know that

$$(SS \to S \xrightarrow{\alpha} U^{\mathbf{2}}PF^{\mathbf{2}}) = (SS \xrightarrow{\alpha\alpha} U^{\mathbf{2}}PF^{\mathbf{2}}U^{\mathbf{2}}PF^{\mathbf{2}} \to U^{\mathbf{2}}PPF^{\mathbf{2}} \to U^{\mathbf{2}}PF^{\mathbf{2}})$$

The point we make, and will use in the next section, is that this morphism factors through $U^{\mathbf{2}}mF^{\mathbf{2}}$, where m is the multiplication of the 2-monad P.

Below we describe Definition 8.4 in a particular case of interest, but before let us recall a few facts about the Kleisli construction for the free split opfibration 2-monad R' on $\mathscr{A}^{\mathbf{2}}$, for a 2-category $\mathscr{A}$ with lax limits of morphisms. This Kleisli construction can be described as the inclusion 2-functor of $\mathscr{A}^{\mathbf{2}}$ into the 2-category $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$ of 2-functors from $\mathbf{2}$ to $\mathscr{A}$ and lax transformations between them. Morphisms between free R'-algebras are in bijection with morphisms in $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$, and the bijection is given as displayed below, a fact we shall soon employ.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "R'f"'] & \cdot \arrow[d, "R'g"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \cdot \arrow[r, "\lambda_f"] \arrow[d, "f"'] & \cdot \arrow[r, "h"] \arrow[d, "R'f"'] & \cdot \arrow[r, "q_g"] \arrow[d, "R'g"'] \arrow[dr, phantom, "\swarrow_{\nu}"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \arrow[r, "k"] & \cdot \arrow[r, equal] & \cdot \end{tikzcd} \tag{8.2}$$

Proposition 8.7. *Let* $F \dashv U\colon \mathscr{A} \to \mathscr{K}$ *be a 2-adjunction, where* $\mathscr{A}$ *has comma objects and* $\mathscr{K}$ *has pullbacks, and* R' *be the free split opfibration 2-monad on* $\mathscr{A}^{\mathbf{2}}$*. Denote by* R *the codomain-preserving coreflection of* $U^{\mathbf{2}}\mathsf{R}'F^{\mathbf{2}}$.

The 2-adjunction is simple with respect to the coreflection–opfibration AWFS precisely when there are coretract adjunctions $F\lambda_f \dashv q_{Ff}\cdot e_{K'Ff}\cdot F\tau_f$ 2-natural in f, where these morphisms are those defined in (8.3).

Proof: In this proof we shall denote the unit and counit of $F \dashv U$ by i and e, respectively. The definition of simple adjunction consists of a coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}]$ between 2-natural transformations that are in the image of

$$[1, U_{\mathsf{P}}] \colon [\mathscr{K}^{\mathbf{2}}, \mathrm{Kl}(\mathsf{P})] \longrightarrow [\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}].$$

When U_{P} is locally fully faithful, such a coretract adjunction can be lifted to $[\mathscr{K}^{\mathbf{2}}, \mathrm{Kl}(\mathsf{P})]$; this is possible if P is lax idempotent by Remark 3.1.

When P is the free split opfibration 2-monad R', its Kleisli construction is isomorphic to the inclusion of $\mathscr{A}^{\mathbf{2}}$ into $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$ – see the comments before the present proposition. We can use the correspondence between morphisms between free R'-algebras and morphisms in $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$ described in (8.2) to deduce the form of the lifting to $[\mathscr{K}^{\mathbf{2}}, \mathrm{Lax}[\mathbf{2}, \mathscr{A}]]$ of the coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}]$ that exhibits $F \dashv U$ as a simple 2-adjunction. The lifting has component at $f \in \mathscr{K}^{\mathbf{2}}$ displayed below, where ν is the comma object that defines ρ'.

$$\begin{tikzcd}
FA \arrow[r, "F\lambda_f"] \arrow[d, "Ff"'] & FKf \arrow[d, "F\rho_f"] \arrow[r, phantom, "\dashv"] & FKf \arrow[r, "F\tau_f"] \arrow[d, "F\rho_f"] & FUK'Ff \arrow[r, "e"] \arrow[d, "FU\rho'_{Ff}"] & K'Ff \arrow[r, "q_{Ff}"] \arrow[d, "\rho'_{Ff}"'] \arrow[dr, phantom, "\Swarrow\nu"] & FA \arrow[d, "Ff"] \\
FB \arrow[r, equal] & FB & FB \arrow[r, "Fi_B"] & FUFB \arrow[r, "e"] & FB \arrow[r, equal] & FB
\end{tikzcd} \tag{8.3}$$

This adjunction consists of a coretract adjunction as in the statement of this proposition, plus the requirement that its counit, say α_f, is a 2-cell in $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$; ie

$$F\rho_f \cdot \alpha_f = \nu \cdot e_{K'Ff} \cdot F\tau_f. \tag{8.4}$$

Thus, the direct part of the statement is trivial, and it remains to show that if $F\lambda_f$ has a right adjoint retract *in $\mathscr{K}$* as in the statement, then (8.4) automatically holds. As a consequence of the adjunction, the 2-cell on the left hand side of (8.4) is the unique 2-cell β, with the appropriate domain and codomain, such that $\beta \cdot F\lambda_f = 1$. We must verify that the 2-cell on the right hand side of (8.4) satisfies the same property. By definition of λ_f,

$$e_{K'Ff} \cdot F\tau_f \cdot F\lambda_f = e_{K'Ff} \cdot FU\lambda'_{Ff} \cdot Fi_A = \lambda'_{Ff} \cdot e_{FA} \cdot Fi_A = \lambda'_{Ff},$$

from where it is clear that $\nu \cdot e_{K'Ff} \cdot F\tau_f \cdot F\lambda_f = \nu \cdot \lambda'_{Ff} = 1$, concluding the proof. ■

Lemma 8.5 yields:

Corollary 8.8. *Given a simple 2-adjunction as in the previous Proposition, the codomain-preserving 2-monad* R *is lax idempotent.*

Remark 8.9. There is a 2-monad morphism with components $(\tau_f, i_{\mathrm{cod}(f)})\colon Rf \to UR'Ff$, by Remark 8.3. Taking the mate along $F \dashv U$, we obtain an opmorphism of 2-monads $(\hat{\tau}_f, 1_{F\,\mathrm{cod}(f)})\colon FRf \to R'Ff$.

Recall from Corollary 2.16 that the codomain functor is a fibration from (the underlying category of) $\mathsf{R}\text{-}\mathrm{Alg}_s$ to $\mathcal{C}$. In particular the category of split opfibrations in a 2-category $\mathscr{K}$ with lax limits of morphisms is a fibration over $\mathscr{K}$.

Theorem 8.10. *Assume given a 2-adjunction* $F \dashv U\colon \mathscr{A} \to \mathscr{K}$ *between 2-categories equipped with chosen lax limits of morphisms and pullbacks, strictly preserved by* U*. If the 2-monad* R *is as in Proposition 8.7, then there is a canonical 2-functor into the category of split opfibrations in* $\mathscr{K}$ *that commutes with the forgetful functors into* $\mathscr{K}^{\mathbf{2}}$*.*

$$\mathsf{R}\text{-}\mathrm{Alg}_s \longrightarrow \mathbf{OpFib}_s(\mathscr{K})$$

Proof: Denote by $\mathsf{R}'_{\mathscr{A}}$ and $\mathsf{R}'_{\mathscr{K}}$ the free split opfibration 2-monad on $\mathscr{A}^{\mathbf{2}}$ and $\mathscr{K}^{\mathbf{2}}$ respectively. Clearly $U^{\mathbf{2}}\mathsf{R}'_{\mathscr{A}} = \mathsf{R}'_{\mathscr{K}}U^{\mathbf{2}}$, and there is a monad morphism $\mathsf{R}'_{\mathscr{K}} \to U^{\mathbf{2}}\mathsf{R}'_{\mathscr{A}}F^{\mathbf{2}}$. Since R is by definition the codomain-preserving coreflection of $U^{\mathbf{2}}\mathsf{R}'_{\mathscr{A}}F^{\mathbf{2}}$, there exists a 2-monad morphism $\mathsf{R} \to \mathsf{R}'_{\mathscr{K}}$, which induces the 2-functor of the statement. ■

9. Transferring an AWFS along a left adjoint

If $(\mathsf{L}', \mathsf{R}')$ is an AWFS on $\mathcal{A}$, and $F \dashv U\colon \mathcal{A} \to \mathcal{C}$ an adjunction, we obtain a *transferred right algebraic weak factorisation system* $((L, \Phi), \mathsf{R})$ in $\mathcal{C}$. The monad R is the codomain-preserving coreflection of the monad $U^{\mathbf{2}}\mathsf{R}'F^{\mathbf{2}}$ on $\mathcal{C}^{\mathbf{2}}$ – see Lemma 8.2. This means that Rf is given by the pullback in $\mathcal{C}$ depicted on the left hand side.

$$\begin{tikzcd} Kf \arrow[r] \arrow[d, "Rf"'] & UK'Ff \arrow[d, "UR'Ff"] \\ B \arrow[r, "i_B"] & UFB \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "\lambda_f"] \arrow[ddr, bend right, "f"'] & UFA \arrow[dr, "U\lambda'_{Ff}"] & \\ & Kf \arrow[r, "\tau_f"] \arrow[d, "\rho_f"] & UK'Ff \arrow[d, "U\rho'_{Ff}"] \\ & B \arrow[r, "i_B"] & UFB \end{tikzcd}$$

The domain-preserving copointed endofunctor (L, Φ) on $\mathcal{C}^{\mathbf{2}}$ associated to (R, Λ) is the same as the one constructed from (L', Φ') and $F \dashv U$ as Lemma 7.1. Explicitly, $f = \rho_f \cdot \lambda_f \colon A \to B$ where λ_f is as in the diagram on the right hand side above.

The above considerations hold in the case of 2-categories, 2-adjunctions, etc, which we assume for the rest of the section.

We know that there is a 2-natural transformation $\Omega_f \colon FKf \to F(\operatorname{dom} f)$ and a modification $\omega_f \colon Ff \cdot \Omega_f \Rightarrow 1$ that induce the 2-functor from (L, Φ)-coalgebras to (L', Φ')-coalgebras, as shown in Proposition 7.3. Given a coalgebra $(1, s) \colon f \to Lf$ then $v = \Omega_f \cdot Fs \colon F(\operatorname{cod} f) \to F(\operatorname{dom} f)$ together with $\omega_f \cdot Fs \colon Ff \cdot v \Rightarrow 1$ make Ff into an (L', Φ')-coalgebra.

Remark 9.1. The 2-adjunction $F \dashv U$ is simple precisely when there are coretract adjunctions $F\lambda_f \dashv \Omega_f$ with counit a modification $\alpha_f \colon F\lambda_f \cdot \Omega_f \Rightarrow 1$ that must satisfy $F\rho_f \cdot \alpha_f = \omega_f$.

Theorem 9.2. *Let $F \dashv U \colon \mathscr{A} \to \mathscr{K}$ be a simple 2-adjunction, where $\mathscr{A}$ has lax limits of morphisms and $\mathscr{K}$ has pullbacks, and denote by $(\mathsf{L}', \mathsf{R}')$ the AWFS on $\mathscr{A}$ described in Section 3.3, that factors a morphism as a left adjoint coretract followed by a split opfibration. Then the right AWFS $((L, \Phi), \mathsf{R})$ on $\mathscr{K}$ described above has an extension to an AWFS (L, R), which is, moreover, lax orthogonal. Furthermore, the outer square in Figure 4 is a pullback of 2-categories and of double categories.*

Proof: By simplicity, we have coretract adjunctions $F\lambda_f \dashv \Omega_f$, 2-natural in f, and whose counit we denote by α_f, as in Remark 9.1. Moreover, $F\rho_f \cdot \alpha_f = \omega_f$. In Proposition 7.3, set $r_f = \Omega_f$ and α the counit of the adjunction, which clearly satisfy (7.2) by the triangular identities. Therefore there is a corresponding (L, Φ)-coalgebra structure $(1, \sigma) \colon L \to L^2$ on L, which is related to Ω by $\Omega_f = \Omega_{Lf} \cdot F\sigma_f$. In order to obtain a 2-comonad

$$
\begin{tikzcd}
\mathsf{L}\text{-Coalg}_s \arrow[r] \arrow[d] & \mathsf{L}'\text{-Coalg}_s \arrow[d] \\
(L,\Phi)\text{-Coalg} \arrow[r] \arrow[d] & (L',\Phi')\text{-Coalg} \arrow[d] \\
\mathscr{K}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"] & \mathscr{A}^{\mathbf{2}}
\end{tikzcd}
$$

FIGURE 4. A pullback diagram of 2-categories and of double categories.

extension of (L, Φ) we need to verify

$$\alpha_f \cdot F\lambda_f = 1 \quad \Omega_{Lf} \cdot F\sigma_f = \Omega_f \quad \alpha_{\lambda_f} \cdot F\sigma_f = F\sigma_f \cdot \alpha_f \quad F\rho_f \cdot \alpha_f = \omega_f.$$

The first of these equalities is one of the triangular identities of the adjunction, the second we saw it is satisfied by definition of σ, and the fourth equality holds by Remark 9.1. It only remains to prove $\alpha_{\lambda_f} \cdot F\sigma_f = F\sigma_f \cdot \alpha_f$, but, by taking mates along the coretract adjunction $F\lambda_f \dashv \Omega_f$ on either side of the equality, one obtains the identity 2-cell of $F\lambda_{\lambda_f}$. – Note that in this last point we used for the first time all the strength of the adjunction $F\lambda_f \dashv \Omega_f$. – Hence the original equality holds, and therefore there is a 2-comonad L.

Next we show that the outer diagram in Figure 4 is a pullback. Suppose that $f \colon A \to B$ is sent by F to an L'-coalgebra $(1, \hat{s}) \colon f \to L'f$, ie $Ff \dashv v$, with unit ξ. By Proposition 7.4, we need to prove that the corresponding (L, Φ)-coalgebra structure $(1, s) \colon f \to L$ satisfies the second equality in equation (7.4). This is easily done by taking mates with respect to $F\lambda_f \dashv \Omega_f$ and $Ff \dashv v$, obtaining on both sides of the equality the identity 2-cell of $F\lambda_f$. Together with Lemma 7.1, this means that an (L, Φ)-coalgebra f is an L-coalgebra precisely when Ff is an L'-coalgebra; the square in the statement is a pullback, at least at the level of objects. But Lemma 7.1 tells us that the bottom square in the diagram in Figure 4 is a pullback, and since the forgetful functors from the 2-categories of coalgebras for the 2-monads to the 2-category of coalgebras for the respective copointed endofunctors are fully faithful, we deduce that the outer diagram is a pullback.

To finish the proof, observe that L-Coalg$_s$ has a double category structure, being a pullback of the double category L'-Coalg$_s$; in other words, there is a composition of L-coalgebras and F preserves it. The double category structure endows the pointed endo-2-functor (R, Λ) with a multiplication $\bar{\Pi} = (\pi, 1) \colon R^2 \Rightarrow R$. Recall from the end of Section 2 that $\bar{\pi}_f$ is defined by the requirement that $(1_A, \bar{\pi}_f)\colon \lambda_{\rho_f} \cdot \lambda_f \to \lambda_f$ be the unique morphism of L-algebras $LRf \bullet Lf \to Lf$ such that $\rho_f \cdot \bar{\pi}_f = \rho_{\rho_f}$.

On the other hand, there is a multiplication Π given by the fact that $(\tau, i)\colon R \Rightarrow UR'F$ is the codomain-preserving coreflection of the 2-monad $UR'F$ on $\mathscr{K}^{\mathbf{2}}$ – Lemma 8.2. We will show $\Pi = \bar{\Pi}$ by proving that $(1, \pi_f)$ is a morphism of L-coalgebras $LRf \bullet Lf \to Lf$ – see previous paragraph. By the pullback square in Figure 4, we must show that $(1, F\pi_f)$ is a morphism of L'-coalgebras with domain $F(LRf \bullet Lf) = FLRf \bullet FLf$ and codomain FLf. To this end, we shall use the characterisation of Proposition 3.9. The L-coalgebra structure on $FLRf \cdot FLf$ is given by composing the relevant coretract adjunctions, ie the right adjoint is $\Omega_f \cdot \Omega_{Rf}$, and the counit is

$$FLRf \cdot FLf \cdot \Omega_f \cdot \Omega_{Rf} \xRightarrow{FLRf \cdot \alpha_f \cdot \Omega_{Rf}} FLRf \cdot \Omega_{Rf} \xRightarrow{\alpha_{Rf}} 1.$$

Then, still by Proposition 3.9, $(1, \pi_f)$ is a morphism of L-coalgebras if and only if

$$\Omega_f \cdot \Omega_{Rf} = \Omega_f \cdot F\pi_f \qquad (F\pi_f \cdot \alpha_{Rf})(F\pi_f \cdot FLRf \cdot \alpha_f \cdot \Omega_{Rf}) = \alpha_f \cdot F\pi_f. \quad (9.1)$$

Recall from Remark 8.9 that the mate of (τ, i), namely $\Theta = (\hat{\tau}, 1)\colon FR \to RF$, is an opmorphism of 2-monads when R is equipped with the multiplication Π; in particular, the following diagram commutes.

$$\begin{tikzcd} FR^2 \arrow[r, "\Theta R"] \arrow[d, "F\Pi"'] & R'FR \arrow[r, "R'\Theta"] & R'^2F \arrow[d, "\Pi' F"] \\ FR \arrow[rr, "\Theta"] & & R'F \end{tikzcd}$$

The domain component of this rectangle evaluated on $f \in \mathscr{K}^{\mathbf{2}}$ is the equation

$$\pi'_{Ff} \cdot K'(\hat{\tau}_f, 1) \cdot \hat{\tau}_{\rho_f} = \hat{\tau}_f \cdot F\pi_f. \tag{9.2}$$

Observe that $q_{Ff} \cdot \hat{\tau}_f = \Omega_f$, so composing $q_{Ff}\colon K'Ff \to FA$ with the left hand side of (9.2) we obtain:

$$\begin{aligned} q_{Ff} \cdot \pi'_{Ff} \cdot K'(\hat{\tau}_f, 1) \cdot \hat{\tau}_{Rf} &= q_{Ff} \cdot q_{R'Ff} \cdot K'(\hat{\tau}_f, 1) \cdot \hat{\tau}_f && \text{(def. of } \pi') \\ &= q_{Ff} \cdot \hat{\tau}_f \cdot q_{F\rho_f} \cdot \hat{\tau}_{Rf} && \text{(def. of } K') \\ &= \Omega_f \cdot \Omega_{Rf}. \end{aligned}$$

On the other hand, composing the right hand side of (9.2) with q_{Ff} yields $\Omega_f \cdot F\pi_f$, and we deduce the first condition of a morphism of L'-coalgebras for $(1, \pi_f)$ in (9.1). The second condition is verified by taking mates along

$$
\begin{tikzcd}
A \arrow[dr, "\lambda_f"] \arrow[drr, bend left, "i_A"] \arrow[ddr, bend right, "f"'] & & \\
& Kf \arrow[r, "t_f"] \arrow[d, "\rho_f"] \arrow[dr, phantom, "\Swarrow\mu"] & UFA \arrow[d, "UFf"] \\
& B \arrow[r, "i_B"'] & UFB
\end{tikzcd}
$$

$$
\begin{tikzcd}
Kf \arrow[r, "t_f"] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\Swarrow\mu"] & UFA \arrow[d, "UFf"] \\
B \arrow[r, "i_B"'] & UFB
\end{tikzcd}
=
\begin{tikzcd}
Kf \arrow[r, "\tau_f"] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\text{p.b.}"] & UK'Ff \arrow[r, "Uq_{Ff}"] \arrow[d, "U\rho'_{Ff}"] \arrow[dr, phantom, "\Swarrow U\nu"] & UFA \arrow[d, "UFf"] \\
B \arrow[r, "i_B"'] & UFB \arrow[r, equal] & UFB
\end{tikzcd}
$$

FIGURE 5. Factorisation of a morphism f.

$$
\begin{tikzcd}
K\rho_f \arrow[r, "\pi_f"] & Kf \arrow[r, "t_f"] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\Swarrow\mu_f"] & UFA \arrow[d, "UFf"] \\
& B \arrow[r, "i_B"'] & UFB
\end{tikzcd}
=
\begin{tikzcd}
K\rho_f \arrow[r, "t_{\rho_f}"] \arrow[d, "\rho_{\rho_f}"'] \arrow[dr, phantom, "\Swarrow\mu_{\rho_f}"] & UFKf \arrow[r, "UFt_f"] \arrow[d, "UF\rho_f"] \arrow[dr, phantom, "\Swarrow UF\mu_f"] & UFUFA \arrow[r, "Ue_{FA}"] \arrow[d, "UFUFf"] & UFA \arrow[d, "UFf"] \\
B \arrow[r, "i_B"'] & UFB \arrow[r, "UFi_B"'] & UFUFB \arrow[r, "Ue_{FB}"'] & UFA
\end{tikzcd}
$$

FIGURE 6. Characterisation of the multiplication of R.

$F\lambda_{Rf} \dashv \Omega_{LRf}$, a procedure that transforms the 2-cells at either side of the equality into the 2-cell α_f. This completes the proof that $(1, F\pi_f)$ is a morphism of L'-coalgebras, and thus the proof that $\Pi = \bar{\Pi}$.

Finally, the 2-monad $\mathsf{R} = (R, \Lambda, \Pi)$ is lax idempotent by Corollary 8.8, so the AWFS (L, R) is lax orthogonal by Theorem 4.1. ∎

The AWFS constructed in the theorem above factors a morphism $f : A \to B$ as $f = \rho_f \cdot \lambda_f$, where ρ_f is given by the comma in Figure 5, and λ_f is the unique morphism such that $\mu \cdot \lambda_f = 1$.

Now we look at the fibrant replacement 2-monad associated to the AWFS constructed.

Corollary 9.3. *Suppose that in Theorem 9.2 the 2-category $\mathscr{K}$ has a terminal object 1, and that $i_1 : 1 \to UF1$ is a right adjoint of $UF1 \to 1$. Then the restriction of R to $\mathscr{K}/1 \cong \mathscr{K}$ – the fibrant replacement 2-monad of (L, R) – is isomorphic to UF.*

Proof: Let us denote by $f\colon A \to 1$ the unique morphism into the terminal object, and by R_1 the restriction of R to $\mathscr{K}/1$. We shall show that in the comma object

$$\begin{tikzcd} Kf \arrow[r, "t_A"] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\Swarrow"] & UFA \arrow[d, "UFf"] \\ 1 \arrow[r, "i_1"'] & UF1 \end{tikzcd}$$

the projection t_A is an isomorphism. For any morphism $x\colon X \to UFA$, there exists a unique 2-cell $UFf \cdot x \Rightarrow i_1 \cdot !$, as these are in bijection, by mateship along $i_1 \dashv f$, with endo-2-cells of $X \to 1$, of which there is only one. Hence $\mathscr{K}(X, t_A)$ is an isomorphism, for each X, and thus t_A is an isomorphism. Since $t_A \cdot \lambda_f = i_A$, and the compatibility of t with the multiplication of R and UF that can be found in Figure 6, we have that t is a 2-monad isomorphism $t\colon R_1 \to UF$. ■

We conclude the section with the following lemma, which will be of use in later sections. Corollary 9.3 says that for any morphism $b\colon 1 \to B$ from the terminal object of $\mathscr{K}$ the fibre A_b of any R-algebra $g\colon A \to B$ – ie the pullback of g along b – has a structure of a T-algebra, for $T = UF$.

$$\begin{tikzcd} A_b \arrow[r, "z_b"] \arrow[d, "!"'] & A \arrow[d, "g"] \\ 1 \arrow[r, "b"'] & B \end{tikzcd}$$

Furthermore, (z_b, b) is a morphism of R-algebras.

Lemma 9.4. *Assume the conditions of Corollary 9.3, and denote by* T *the monad generated by* $F \dashv U$*. Given* $g\colon A \to B$ *and* $b\colon 1 \to B$*, the morphism*

$$(Kg)_b \xrightarrow{z_b} Kg \xrightarrow{t_g} TA$$

is a morphism of T*-algebras.*

Proof: Denote by $a\colon T(Kg)_b \to (Kg)_b$ the T-algebra structure given by Corollary 9.3. We are to show that the following rectangle commutes.

$$\begin{tikzcd} T(Kg)_b \arrow[r, "Tz_b"] \arrow[d, "a"'] & TKg \arrow[r, "Tt_b"] & T^2A \arrow[d, "m_A"] \\ (Kg)_b \arrow[r, "z_b"'] & Kg \arrow[r, "t_b"'] & TA \end{tikzcd} \tag{9.3}$$

$$
\begin{tikzcd}
T(Kg)_b \arrow[r, "a"] \arrow[d] & (Kg)_b \arrow[r, "z_b"] \arrow[d] & Kg \arrow[r, "t_g"] \arrow[d, "\rho_g"'] \arrow[dr, phantom, "\Swarrow"] & TA \arrow[d, "Tg"] \\
1 \arrow[r, equal] & 1 \arrow[r, "b"'] & B \arrow[r, "i_B"'] & TB
\end{tikzcd}
=
\begin{tikzcd}
T(Kg)_b \arrow[r, "{K(z_b,b)}"] \arrow[d] & K\rho_g \arrow[r, "\pi_g"] \arrow[d, "\rho_{\rho_g}"'] & Kg \arrow[r, "t_g"] \arrow[d, "\rho_g"'] \arrow[dr, phantom, "\Swarrow"] & TA \arrow[d, "Tg"] \\
1 \arrow[r, "b"'] & B \arrow[r, equal] & B \arrow[r, "i_B"'] & TB
\end{tikzcd}
=
$$

$$
=
\begin{tikzcd}
T(Kg)_b \arrow[r, "{K(z_b,b)}"] \arrow[d] & K\rho_g \arrow[r] \arrow[d, "\rho_{\rho_g}"'] \arrow[dr, phantom, "\Swarrow"] & TKg \arrow[r, "Tt_g"] \arrow[d, "T\rho_g"'] \arrow[dr, phantom, "\Swarrow"] & T^2A \arrow[r, "m_A"] \arrow[d, "T^2g"] & TA \arrow[d, "Tg"] \\
1 \arrow[r, "b"'] & B \arrow[r, "i_B"'] & TB \arrow[r, "Ti_B"'] & T^2B \arrow[r, "m_B"'] & TB
\end{tikzcd}
=
$$

$$
=
\begin{tikzcd}
T(Kg)_b \arrow[r, "1"] \arrow[d] \arrow[dr, phantom, "\Swarrow !"] & T(Kg)_b \arrow[r, "Tz_b"] \arrow[d] & TKg \arrow[r, "Tt_g"] \arrow[d, "T\rho_g"'] \arrow[dr, phantom, "\Swarrow"] & T^2A \arrow[r, "m_A"] \arrow[d, "T^2g"] & TA \arrow[d, "Tg"] \\
1 \arrow[r, "i_1"'] & T1 \arrow[r, "Tb"'] & TB \arrow[r, "Ti_B"'] & T^2B \arrow[r, "m_B"'] & TB
\end{tikzcd}
$$

FIGURE 7. Proof of Corollary 9.3.

In order to do so, consider the string of equalities displayed in Figure 7, the first of which holds since (z_b, b) is a morphism of R-algebras; the second holds by definition of π_g – see Figure 6; next equality reflects the definition of $K(z_b, b)$ and the fact that the restriction of R to $\mathscr{K}/1$ is T – Corollary 9.3. Now it is clear that (9.3) commutes. ∎

10. Simple 2-monads

A 2-monad T on a 2-category $\mathscr{K}$ with lax limits of morphisms is said to be *simple* if the usual Eilenberg-Moore adjunction $F \dashv U \colon \mathsf{T}\text{-}\mathrm{Alg}_s \to \mathscr{K}$ is simple with respect to the coreflection–opfibration AWFS on $\mathsf{T}\text{-}\mathrm{Alg}_s$ – in the sense of Definition 8.4. To make this definition more explicit, consider the factorisation of a morphism $f \colon A \to B$ in $\mathscr{K}$ depicted in Figure 5, and recall from Proposition 8.7 that the simplicity of $F \dashv U$ amounts to the existence of a certain coretract adjunction in $\mathsf{T}\text{-}\mathrm{Alg}_s$; namely

$$T\lambda_f \dashv m_{K'Ff} \cdot Tt_f \tag{10.1}$$

where m is the multiplication of T and the rest of the notation is as in Figure 5. This adjunction must be an adjunction in $\mathsf{T}\text{-}\mathrm{Alg}_s$ – a condition that is redundant when, for example, T is lax idempotent, as it will often be in our examples.

Remark 10.1. At this point it is useful to consider the meaning of simple 2-monads and the previous proposition when the 2-category is locally discrete, ie just a category $\mathcal{C}$. In this case comma objects are just pullbacks, and the coreflection–opfibration factorisation becomes the orthogonal factorisation $(\mathrm{Iso}, \mathrm{Mor})$ that factors a morphism f as the identity followed by f. To say that a monad T on $\mathcal{C}$ is simple is to say that the image of the comparison morphism ℓ, which goes from the naturality square of i to the pullback in (10.2), is sent to an isomorphism by the free T-algebra functor. Equivalently, one can say that $T\ell$ is an isomorphism. Observe that when T is a reflection, this gives the definition of *simple reflection* in the sense of [4].

$$\begin{tikzcd} A \arrow[dr, "\ell"] \arrow[drr, bend left, "i_A"] \arrow[ddr, bend right, "f"'] & & \\ & B \times_{TB} TA \arrow[r] \arrow[d] & TA \arrow[d, "Tf"] \\ & B \arrow[r, "i_B"'] & TB \end{tikzcd} \tag{10.2}$$

We consider in this section some properties that guarantee that a 2-monad is simple, thus inducing a transferred AWFS. We make the blanket assumption that the 2-category $\mathscr{K}$ has pullbacks and cotensor products with $\mathbf{2}$, and therefore comma objects.

Given a cospan $f\colon A \to C \leftarrow B : g$, consider the comparison 1-cell

$$k\colon T(f \downarrow g) \longrightarrow Tf \downarrow Tg. \tag{10.3}$$

Proposition 10.2 (Simplicity criterion)**.** *A 2-monad* $\mathsf{T} = (T, i, m)$ *is simple if it is lax idempotent and composing with* (10.3) *induces a bijection between the following 2-cells, where* $f\colon A \to B$*,* u*, and* v *are arbitrary morphisms.*

$$\begin{tikzcd} X \arrow[d, "v"'] \arrow[dr, bend left, "u"] \arrow[dr, phantom, "\Downarrow"] & \\ Tf \downarrow i_B \arrow[r, "i"'] & T(Tf \downarrow i_B) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \arrow[d, "v"'] \arrow[dr, bend left, "k\cdot u"] \arrow[dr, phantom, "\Downarrow"] & \\ Tf \downarrow i_B \arrow[r, "k\cdot i"'] & TTf \downarrow Ti_B \end{tikzcd}$$

Proof: As before, denote by R' the free split opfibration 2-monad on $\mathsf{T}\text{-}\mathrm{Alg}_s^{\mathbf{2}}$ and by R the transferred 2-monad along the Eilenberg-Moore adjunction $F \dashv U$, ie the codomain-preserving coreflection of the 2-monad $U^{\mathbf{2}}\mathsf{R}'F^{\mathbf{2}}$. The right part ρ_f of the factorisation of a morphism f in $\mathscr{K}$ induced by R

is given by a comma object

$$\begin{tikzcd} Kf \arrow[r, "t_f"] \arrow[d, "\rho_f"'] \arrow[dr, phantom, "\not\Leftarrow\mu_f"] & TA \arrow[d, "Tf"] \\ B \arrow[r, "i_B"'] & TB \end{tikzcd} \tag{10.4}$$

and the left part $\lambda_f \colon A \to Kf$ is the unique morphism such that $\mu_f \cdot \lambda_f = 1$. As explained at the beginning of the present section, we must exhibit a coretract adjunction (10.1) in $\mathscr{K}$; this adjunction is automatically an adjunction in T-Alg_s, since T is lax idempotent.

In order to define a counit $\alpha \colon T\lambda_f \cdot m_{K'Ff} \cdot Tt_f \Rightarrow 1$ we can give its transpose under the free T-algebra 2-adjunction, which is a 2-cell $\bar{\alpha} \colon T\lambda_f \cdot t_f \Rightarrow i_{Kf}$ in $\mathscr{K}$.

The morphism k of (10.3) is the unique such that satisfies the equality

$$\begin{tikzcd} TKf \arrow[r, "Tt_f"] \arrow[d, "T\rho_f"'] \arrow[dr, phantom, "\not\Leftarrow T\mu_f"] & T^2A \arrow[d, "T^2f"] \\ TB \arrow[r, "Ti_B"'] & T^2B \end{tikzcd} = \begin{tikzcd} TKf \arrow[r, "k"] & T^2f \downarrow Ti_B \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\not\Leftarrow\theta"] & T^2A \arrow[d, "T^2f"] \\ & TB \arrow[r, "Ti_B"'] & T^2B \end{tikzcd}$$

To give $\bar{\alpha}$ is to equally give a pair of 2-cells, corresponding to $d_0 \cdot k \cdot \bar{\alpha}$ and $d_1 \cdot k \cdot \bar{\alpha}$:

$$\bar{\alpha}_1 \colon Tt_f \cdot T\lambda_f \cdot t_f = Ti_A \cdot t_f \Rightarrow i_{TA} \cdot t_f = Tt_f \cdot i_{Kf}$$

$$\bar{\alpha}_2 \colon T\rho_f \cdot T\lambda_f \cdot t_f = Tf \cdot t_f \Rightarrow i_B \cdot \rho_f = T\rho_f \cdot i_{Kf}$$

compatible with θ, in the sense that the following two compositions of 2-cells must be equal.

$$T^2f \cdot Ti_A \cdot t_f \xrightarrow{T^2f\cdot\bar{\alpha}_1} T^2f \cdot Tt_f \cdot i_{Kf} = T^2f \cdot d_0 \cdot k \cdot i_{Kf} \xrightarrow{\theta\cdot k\cdot i_{Kf}} Ti_B \cdot d_1 \cdot k \cdot i_{Kf}$$

$$T^2f \cdot d_0 \cdot k \cdot T\lambda_f \cdot t_f \xrightarrow{\theta\cdot k\cdot T\lambda_f\cdot t_f} Ti_B \cdot d_1 \cdot k \cdot T\lambda_f \cdot t_f = Ti_B \cdot Tf \cdot t_f \xrightarrow{Ti_B\cdot\bar{\alpha}_2} Ti_B \cdot i_B \cdot \rho_f$$

Set $\bar{\alpha}_1 = \delta_A \cdot t_f$ and $\bar{\alpha}_2 = \mu_f$, where $\delta \colon Ti \Rightarrow iT$ is the modification given by the lax idempotent structure of T. We must verify the pair of 2-cells displayed above are equal. Using that $\theta \cdot k = T\mu_f$, the verification takes the following form, where the first equality is the modification property for δ and the 2-naturality of i, the second is the interchange law in a 2-category, the third holds since $\delta \cdot i = 1$, and the last holds since $\mu_f \cdot \lambda_f = 1$.

$$\big(T\mu_f \cdot i_{Kf}\big)\big(T^2 f \cdot \delta_A \cdot t_f\big) = \big(i_{TB} \cdot \mu_f\big)\big(\delta_B \cdot Tf \cdot t_f\big) = \big(\delta_B \cdot i_B \cdot \rho_f\big)\big(Ti_B \cdot \mu_f\big) = \\ = Ti_B \cdot \mu_f = (Ti_B \cdot \mu_f)(T\mu_f \cdot T\lambda_f \cdot t_f)$$

It remains to verify the triangular identities of an adjunction. One of them is $m_A \cdot Tt_f \cdot \alpha = 1$, equivalent to $m_A \cdot Tt_f \cdot \bar{\alpha} = 1$, and by definition of $\bar{\alpha}$, equivalent to $m_A \cdot \bar{\alpha}_1 = 1$. This latter equality clearly holds, since $m_A \cdot \delta = 1$,

Up to now we have only used the hypothesis in the case when v is an identity morphism. Only now, in order to prove the other triangular identity $\alpha \cdot (T\lambda_f) = 1$ we shall need the hypothesis in its general form, more precisely, for $v = \lambda_f$. The triangular equality is equivalent to $\bar{\alpha} \cdot \lambda_f = 1$, which holds since $\delta_A \cdot t_f \cdot \lambda_f = \delta_A \cdot i_A = 1$ and $\mu_f \cdot \lambda_f = 1$, finishing the proof. ■

The proposition will be usually used in the following, less powerful form.

Corollary 10.3. *A 2-monad* $\mathsf{T} = (T, i, m)$ *is simple if it is lax idempotent and composing with* (10.3) *induces a bijection between 2-cells* $u \Rightarrow i_{f\downarrow g} \cdot v$ *and* $k \cdot u \Rightarrow k \cdot i_{f\downarrow g} \cdot v$*, where* $f \colon A \to B \leftarrow C : g$ *are arbitrary morphisms.*

Let $\mu \colon h \cdot j \Rightarrow g$ be a left extension in a 2-category with comma objects. Recall that μ is a *pointwise left extension* if, whenever pasted with a comma object as depicted on the left hand side below, the resulting 2-cell is a left extension. Recall that if a 2-monad $\mathsf{T} = (T, i, m)$ is lax idempotent then the identity 2-cell below exhibits Tf as a left extension – not necessarily a pointwise extension – of $i_B \cdot f$ along i_A – Section 3.1.

$$\begin{tikzcd} j\downarrow w \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & W \arrow[d, "w"] \\ X \arrow[r, "j"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \arrow[d, phantom, shift right=3ex, "\overset{\mu}{\Rightarrow}"] \\ & Z \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & TA \arrow[d, "Tf"] \\ B \arrow[r, "i_B"] & TB \end{tikzcd} \tag{10.5}$$

Theorem 10.4. *Suppose the lax idempotent 2-monad* T *satisfies: the identity 2-cell* (10.5) *exhibits* Tf *as a pointwise left extension of* $i_B \cdot f$ *along* i_A*, for all* f*; and the components of the unit* $i \colon 1 \to T$ *are fully faithful. Then* T *is simple.*

Proof: We will verify the hypothesis of Corollary 10.3. Given a comma object $h \downarrow g$ depicted on the left below, denote by $k \colon T(h \downarrow g) \to Th \downarrow Tg$ the comparison morphism. Given a morphism $u\colon X \to T(h \downarrow g)$, we consider the diagram on the right hand side, where the unlabelled 2-cell is a comma object. This pasting exhibits $(Td_n) \cdot u$ as a left extension, since Td_n is a pointwise left extension.

$$\begin{tikzcd} h \downarrow g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\gamma \Nearrow"] & B \arrow[d, "g"] \\ A \arrow[r, "h"'] & C \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "e_1"] \arrow[d, "e_0"'] \arrow[dr, phantom, "\Nearrow"] & X \arrow[d, "u"] \\ h \downarrow g \arrow[r, "i_{h\downarrow g}"'] \arrow[d, "d_n"'] & T(h \downarrow g) \arrow[d, "Td_n"] \\ \mathrm{cod}(d_n) \arrow[r, "i"] & T(\mathrm{cod}(d_n)) \end{tikzcd}$$

Given a morphism $v\colon X \to h \downarrow g$, we will show that 2-cells $\alpha\colon k \cdot u \Rightarrow k \cdot i_{h\downarrow g} \cdot v$ are in bijection with 2-cells $u \Rightarrow i_{h\downarrow g} \cdot v$.

We begin by observing that 2-cells α are in bijection with pairs of 2-cells

$$\alpha_0\colon (Td_0) \cdot u \Rightarrow (Td_0) \cdot i_{h\downarrow g} \cdot v \qquad \text{and} \qquad \alpha_1 : (Td_1) \cdot u \Rightarrow (Td_1) \cdot i_{h\downarrow g} \cdot v$$

compatible with $T\gamma$ in the sense that $(Tg \cdot \alpha_1)(T\gamma \cdot u) = (T\gamma \cdot i_{h\downarrow g} \cdot v)(Th \cdot \alpha_0)$ holds.

By the universal property of extensions, α_n is in bijection with 2-cells $i \cdot d_n \cdot e_0 \Rightarrow (Td_n) \cdot i_{h\downarrow g} \cdot v \cdot e_1 = i_{\mathrm{cod}(d_n)} \cdot d_n \cdot v \cdot e_1$, and since i has fully faithful components, with 2-cells $\beta_n\colon d_n \cdot e_0 \Rightarrow d_n \cdot v \cdot e_1$. The compatibility between α_0, α_1, and $T\gamma$ translates into $(g \cdot \beta_1)(\gamma \cdot e_0) = (\gamma \cdot v \cdot e_1)(h \cdot \beta_0)$. By the universal property of γ, the pair β_0, β_1 is in bijection with 2-cells $e_0 \Rightarrow v \cdot e_1$, and thus with 2-cells $i_{h\downarrow g} \cdot e_0 \Rightarrow i_{h\downarrow g} \cdot v \cdot e_1$. Finally, by the description of u as a left extension, these 2-cells are in bijection with 2-cells $u \Rightarrow i_{h\downarrow g} \cdot v$, as required. ∎

The theorem can be used to prove that, for a class of **Set**-colimits, the 2-monad on **Cat** whose algebras are categories with chosen colimits of that class is simple. Section 13 proves this fact in another way, that applies to enriched categories.

11. Locally preordered 2-categories

By a locally preordered 2-category we mean one whose hom-categories are preorders: categories with at most one morphism between any pair of objects. This type of 2-category is particularly simple and includes some interesting examples, that we will mention later. Another usual way of describe them is as categories enriched in the cartesian closed category of preorders **Preord**. A morphism $x \to y$ in a preorder will usually be denoted by $x \leqslant y$.

When $\mathscr{K}$ is a locally preordered 2-category, a 2-monad $\mathsf{T} = (T, i, m)$ is lax idempotent when any of the following equivalent conditions holds: $(Ti) \cdot m \leqslant 1_{T^2}$; $1_{T^2} \leqslant (iT) \cdot m$; $Ti \leqslant iT$.

11.1. KZ lifting operations in locally preordered 2-categories. Consider 2-functors $U \colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}} \leftarrow \mathscr{B} : V$ and assume that all three 2-categories are locally preordered – in fact, $\mathscr{A}$ and $\mathscr{B}$ could be general 2-categories, but this makes no difference to our analysis. The **Cat**-modules of Section 6, $\mathfrak{D}_{\mathscr{K}} \cong (\mathrm{id} \cdot \mathrm{dom})_* \colon \mathscr{K}^{\mathbf{2}} \twoheadrightarrow \mathscr{K}^{\mathbf{2}}$, $U^* \colon \mathscr{K}^{\mathbf{2}} \twoheadrightarrow \mathscr{A}$ and $V_* \colon \mathscr{B} \twoheadrightarrow \mathscr{K}^{\mathbf{2}}$ can be regarded as **Preord**-modules.

Theorem 11.1. *In the locally preordered case, given U and V as in the preceding paragraph, a KZ lifting operation for U, V – Definition 5.2 – is a lifting operation with the extra property that given a square $(h, k) \colon Ua \to Vb$, its chosen diagonal filler is less or equal than any other diagonal filler.*

Proof: Definition 5.2 says that a KZ lifting operation for U, V is a left adjoint coretract for $U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot V_* \to U^* \cdot V_*$. Since these are modules enriched in preorders, this amounts to a lifting operation $U^* \cdot V_* \to U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot V_*$ such that

$$\big(U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot V_* \to U^* \cdot V_* \to U^* \cdot \mathfrak{D}_{\mathscr{K}} \cdot V_*\big) \geqslant 1_{U^* \cdot V_*}.$$

The naturality is automatically satisfied. This inequality can be verified component-wise, and this means precisely that given a square $(h, k) \colon Ua \to Vb$, as depicted in (2.13), and an arbitrary diagonal filler j, then $d \leqslant j$ where d is the chosen diagonal filler. ∎

The proof above also shows that distinction between lax orthogonality structures – Definition 5.10 – and KZ lifting operations disappears, so we have:

Corollary 11.2. *In the locally preordered case, a lax orthogonality structure for U, V exists if and if a KZ lifting operation does; in this case they coincide, and this structure is unique.*

Corollary 11.3. *For a 2-functor $U \colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ between locally preordered 2-categories, the forgetful 2-functor $\mathscr{A}^{\triangleright} \to \mathscr{A}^{\pitchfork}$ is fully faithful.*

Proof: Consider Definition 6.1, where a morphism between two objects of $\mathscr{A}^{\triangleright}$ is defined as a morphism in $\mathscr{A}^{\pitchfork}$ that satisfies an extra condition of compatibility with the counits of each adjunction. In the locally preordered case, this condition is void. In the same Definition, it is already mentioned that our 2-functor is locally fully faithful, completing the proof. ■

11.2. Lax orthogonal AWFSs on locally preordered 2-categories. Let (L, R) be a lax orthogonal AWFS on the locally preordered $\mathscr{K}$, with underlying WFS $(\mathcal{L}, \mathcal{R})$. Recall that $g \in \mathcal{R}$ precisely when $\Lambda_g \colon g \to Rg$ is split monic, and, by the remarks above, when g admits an R-algebra structure. The dual statement holds for $\mathcal{L}$, so $\mathcal{L}$ consists of those morphisms that admit an L-coalgebra structure, and $\mathcal{R}$ of those that admit an R-algebra structure.

Definition 11.4. We say that a morphism $f \colon A \to B$ in a locally preordered 2-category $\mathscr{K}$ is *fibrewise posetal* if, for any isomorphism $\alpha \colon u \Rightarrow v \colon X \to A$, it holds that $\alpha = 1$ whenever $f \cdot \alpha = 1$. In other words, if for all X the fibres of the functor $\mathscr{K}(X, f) \colon \mathscr{K}(X, A) \to \mathscr{K}(X, B)$ are posets.

Observe that fibrewise posetal morphisms in locally preordered 2-categories are closed under retracts.

Lemma 11.5. *Let* R *be a lax idempotent and codomain preserving 2-monad on* $\mathscr{K}^{\mathbf{2}}$*, where* $\mathscr{K}$ *is a locally preordered 2-category. If free* R*-algebras are fibrewise posetal morphisms in* $\mathscr{K}$*, then: (1) A morphism* $g \in \mathscr{K}^{\mathbf{2}}$ *admits an* R*-algebra structure if and only if there exists a left adjoint retract of* $\Lambda_g \colon g \to Rg$*. (2)* R*-algebra structures are unique. (3)* R*-algebras are closed under retracts in* $\mathscr{K}^{\mathbf{2}}$*. An object of* $\mathscr{K}^{\mathbf{2}}$ *is an* R*-algebra if and only if it is a retract of a free* R*-algebra.*

Proof: In this proof we shall use the same notation for R we employ for the right part of an AWFS. The morphism Rg will be $\rho_g \colon Kg \to B$ if $g \colon A \to B$, and the 2-monad R will have unit $\Lambda = (\lambda, 1)$ and multiplication $\Pi = (\pi, 1)$.

We must show that any left adjoint retract of Λ_g is an R-algebra structure. A retract of $\Lambda_g = (\lambda_g, 1)$ must have identity codomain component. Suppose that $(p, 1)$ is a left adjoint retract of Λ_g, so $p \cdot \lambda_g = 1$ and $1 \to \lambda_g \cdot p$ – the adjunction between the codomain components is trivial, since these are identities, so the unit and counit must be equalities. Note that g is fibrewise posetal since Rg is so. In order to show that $(p, 1)$ is an algebra structure we must show that $p \cdot K(p, 1) = p \cdot \pi_g$. There is an isomorphism

$$p \cdot K(p, 1) \cong p \cdot \pi_g \tag{11.1}$$

that is the mate of the identity $K(\lambda_g, 1) \cdot \lambda_g = \lambda_{\rho_g} \cdot \lambda_g$ under the adjunctions $K(p, 1) \dashv K(\lambda_g, 1)$, $\pi_g \dashv \lambda_{\rho_g}$ and $p \dashv \lambda_g$. When post-composed with g, both sides of (11.1) are equal – to ρ_{ρ_g} – and therefore an identity, since g is fibrewise posetal.

We now prove the uniqueness of R-algebras. By a general fact about lax idempotent 2-monads, two algebra structures $(p, 1), (p', 1)\colon Kg \to g$ are isomorphic. The codomain component of the isomorphism must be an identity, so we have $p \cong p'$. But $g \cdot p = \rho_g = g \cdot p'$, that implies $p = p'$ because g is fibrewise posetal.

It remains to show that a retract $(r_0, r_1)\colon g \to f$ of an R-algebra $g\colon C \to D$, with section (s_0, s_1), is again an R-algebra. If $(p, 1)\colon Rg \to g$ is the algebra structure, define

$$q = \big(Kf \xrightarrow{K(s_0,s_1)} Kg \xrightarrow{p} C \xrightarrow{r_0} A\big).$$

We shall prove that $(q, 1)\colon Rf \to f$ is an algebra structure, which is to say, as we have seen in the present proof, to prove that $(q, 1)$ is a left adjoint retract of Λ_f. First, $(q, 1)$ is a morphism:

$$f \cdot q = f \cdot r_0 \cdot p \cdot K(s_0, s_1) = r_1 \cdot g \cdot p \cdot K(s_0, s_1) = r_1 \cdot \rho_g \cdot K(s_0, s_1) = r_1 \cdot s_1 \cdot \rho_f = \rho_f.$$

Next, we see that q is a retract:

$$q \cdot \lambda_f = r_0 \cdot p \cdot K(s_0, s_1) \cdot \lambda_f = r_0 \cdot p \cdot \lambda_g \cdot s_0 = r_0 \cdot s_0 = 1.$$

It remains to show that

$$\lambda_f \cdot q = \lambda_f \cdot r_0 \cdot p \cdot K(s_0, s_1) = K(r_0, r_1) \cdot \lambda_g \cdot p \cdot K(s_0, s_1) \geqslant K(r_0, r_1) \cdot K(s_0, s_1) = 1$$

which ends the verification of the existence of the retract adjunction. ∎

Lemma 11.6. *For a lax orthogonal AWFS* (L, R) *on a locally preordered 2-category, with underlying WFS* $(\mathcal{L}, \mathcal{R})$*, and with the property that* R*-algebras are fibrewise posetal morphisms, the following assertions hold: (1)* $f \in \mathcal{R}$ *if and only if* f *is an* R*-algebra.* R*-algebra structures are unique. (2)* $f \in \mathcal{L}$ *if and only if* f *is an* L*-coalgebra.* L*-coalgebra structures are unique. (3)* $\mathsf{R}\text{-Alg}_s \to \mathsf{L}\text{-Coalg}_s^{\triangleright}$ *is an isomorphism.*

Proof: The assertion (1) follows from Lemma 11.5. We now prove that a right adjoint retract of $\Phi_f = (1, \rho_f)\colon Lf \to f$ is the same as an L-coalgebra structure. Such an adjoint retract must be of the form $(1, s)$, for a morphism

$s\colon B \to Kf$. Taking mates of $K(1,\rho_f)\cdot\rho_f = \rho_{\rho_f}\cdot\rho_f$ under the adjunctions $K(1,\rho_f) \dashv K(1,s)$, $\rho_f \dashv s$ and $\rho_{\rho_f} \dashv \pi_f$, we obtain an isomorphism

$$K(1,s)\cdot s \xrightarrow{\cong} \sigma_f \cdot s \tag{11.2}$$

that when post-composed with the fibrewise posetal morphism $\rho_f\cdot\rho_{\rho_f}$ equals the identity. It follows that (11.2) is an identity, so $(1,s)$ is an L-coalgebra.

By a similar argument, L-coalgebra structures are unique. If s and s' give coalgebra structures, there is an isomorphism $s \cong s'$ that composed with ρ_f is an identity, and one deduces that $s = s'$.

In order to prove that a retract of an L-coalgebra is again a coalgebra, and thus that $\mathcal{L}$ consist of the L-coalgebras, one can now deploy the same arguments – in dual form – of the last paragraph of the proof of Lemma 11.5 to show that retracts of algebras are algebras.

Finally, we prove the assertion (3) by recalling from Remark 6.7 that objects of $\mathsf{L}\text{-Coalg}_s^{\triangleright}$ are in bijection with left adjoints retracts for the different units Λ_g, and, as we saw, these are the same as R-algebra structures. ∎

Theorem 11.7. *Let $F \dashv U\colon \mathscr{A} \to \mathscr{K}$ be a 2-adjunction, where $\mathscr{A}$ has lax limits of arrows, $\mathscr{K}$ has pullbacks, and it is locally preordered. Assume further that the categories $\mathscr{K}(X, UFY)$ are posets for each X, $Y \in \mathscr{K}$. If the 2-adjunction is simple, there exists a transferred AWFS (L,R) on $\mathscr{K}$, with underlying WFS $(\mathcal{L},\mathcal{R})$, that satisfies:*

1. *It is lax orthogonal.*
2. *The following statements are equivalent for an $f \in \mathscr{K}^{\mathbf{2}}$. (a) f admits an L-coalgebra structure. (b) Ff is a left adjoint coretract in $\mathscr{A}$. (c) $f \in \mathcal{L}$.*
3. *The following statements are equivalent for a $g \in \mathscr{K}^{\mathbf{2}}$. (a) $g \in \mathscr{K}^{\mathbf{2}}$ admits an R-algebra structure. (b) g is algebraically KZ injective with respect to all morphisms as in (2) above – see Example 6.3. (c) g is injective with respect to all morphisms as in (2). (d) $g \in \mathcal{R}$.*
4. *In the underlying WFS $(\mathcal{L},\mathcal{R})$, the left class consists of morphisms as in (2), and the right class of morphisms as in (3).*

Proof: For $g\colon C \to D$ in $\mathscr{K}$, the morphism $\rho_g\colon Kg \to D$ is fibrewise posetal; indeed, the fibre of the functor $\mathscr{K}(X,\rho_g)$ over $v\colon X \to D$ is isomorphic to $\mathscr{K}(X, UFg)/v$, which is a poset since $\mathscr{K}(X, UFC)$ is so.

Lemma 11.6.2 shows the equivalence between (2a) and (2c), while the equivalence of these with (2b) is part of Theorem 9.2, as well as (1) is. Finally, (4) is part of Lemma 11.6.

By Lemma 11.5, we know that morphisms in $\mathcal{R}$ coincide with R-algebras. This shows the equivalence between (3a) and (3d). That (3a) implies (3b) follows from Theorem 6.6, while that (3c) implies (3d) is a basic fact about WFSs. ∎

12. Example: the filter monad

In this section we exhibit an AWFS on the 2-category of topological spaces arising from a simple lax idempotent 2-monad: the filter 2-monad. This factorisation was constructed in the case of $T0$ spaces in [3]. In fact, [3] constructs four WFSs, corresponding to the 2-monads for filters and its sub-2-monads of proper filters, prime filters and completely prime filters. We shall restrict only to the first of those, as only small modifications are needed to treat the other three cases. The restriction to $T0$ spaces ensures that the 2-category of spaces is locally posetal, but it is not essential, and indeed we shall be able to work with general spaces, since the free space of filters on any space is a poset.

Each topological space X carries a preorder structure given by the order

$$x \leqslant y \text{ if and only if } y \in \overline{\{x\}}, \tag{12.1}$$

ie if and only if every open neighbourhood of y is an open neighbourhood of x – this is the opposite of what is usually called the *specialisation order*. This induces a preorder structure on each hom-set $\mathbf{Top}(X,Y)$ by defining $f \leqslant g$ if $fx \leqslant gx$, for all $x \in X$, making $\mathbf{Top}$ into a preorder-enriched category, or a locally preordered 2-category.

A comma-object $f \downarrow g$ in $\mathbf{Top}$ can be described as the subspace of $A \times B$ defined by the subset $\{(a,b) \in A \times B \colon f(a) \leqslant g(b)\}$.

$$\begin{tikzcd} f \downarrow g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Denote by $\mathsf{F}\colon \mathbf{Top} \to \mathbf{Top}$ the filter monad. If X is a space, FX is the set of filters of open sets of X, with topology generated by the subsets $U^{\#} = \{\varphi \in FX : U \in \varphi\}$, where $U \in \mathcal{O}(X)$. The (opposite of the) specialisation order on FX results in the opposite of the inclusion of filters. In particular, FX is a poset. If $f\colon X \to Y$ is continuous, then Ff is defined by $(Ff)(\varphi) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \varphi\}$. The unit of the monad has components $i_X\colon X \to FX$, given by $i_X(x) = \{U \in \mathcal{O}(X) : x \in U\}$. The multiplication of the monad has components $m_X\colon F^2X \to FX$, given by $m_X(\Theta) = \{U \in \mathcal{O}(X) : U^{\#} \in \Theta\}$.

Observe that i_X is a full (and faithful) functor between the underlying preorders, by definition of the order in X. It is an injective function precisely when X is $T0$.

The 2-monad F restricts to the full sub-2-category $\mathbf{Top}_0$ of $T0$ topological spaces. It was shown in [6] that the category of algebras for this restriction is isomorphic to the category whose objects are continuous lattices [22] and morphisms poset maps that preserve directed suprema and arbitrary infima. Our choice of the (opposite of the) specialisation order on spaces, which is the opposite of the order used in [6], grants a few comments as a way of avoiding confusion. A space $X \in \mathbf{Top}_0$ has an F-algebra structure precisely when the opposite of the poset $(X, \leqslant)$ is a continuous lattice, where $\leqslant$ is the order (12.1). The topology of the space X can be recovered as the Scott topology of the continuous lattice $(X, \leqslant)^{\mathrm{op}}$. A morphism of F-algebras $f\colon X \to Y$ is a continuous function that preserves arbitrary suprema, as a poset map $(X, \leqslant) \to (Y, \leqslant)$ [6, Thm 4.4] – continuity is equivalent to preservation of $\leqslant$-directed infima.

In fact the 2-category $\mathsf{F}\text{-}\mathrm{Alg}_s$ of general topological spaces equipped with an F-algebra structure is isomorphic to the one described in the previous paragraph. This is so because any F-algebra is a retract of a free F-algebra, ie a space of filters, and these are posets, equally, $T0$. Thus, the preorder underlying any F-algebra is a poset and the algebra is a $T0$ space.

By an *embedding* we will mean a topological embedding, in the usual sense: a continuous function that is an homeomorphism onto its image, where the latter is equipped with the subspace topology.

The filter 2-monad F was shown to be lax idempotent in [9], where it is also proved that a continuous function f between $T0$ spaces is an embedding if and only if Ff is the left adjoint in a coretract adjunction.

Recall that given a continuous map $f\colon X \to Y$ the inverse image morphism $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a left adjoint of f_* given by $f_*(U) = \cup\{V \in \mathcal{O}(Y) : f^{-1}(V) \subseteq U\}$. The unit $V \subseteq f_*f^{-1}(V)$ is the inclusion of V into $\cup\{W \in \mathcal{O}(Y) : f^{-1}(W) \subseteq f^{-1}(V)\}$. The map Ff has always a right adjoint r given by

$$r(\psi) = \{U \in \mathcal{O}(X) : f_*(U) \in \psi\}. \tag{12.2}$$

The unit of $Ff \dashv r$ is given by

$$r(Ff(\phi)) = \{U \in \mathcal{O}(X) : f_*(U) \in Ff(\phi)\} = \{U \in \mathcal{O}(X) : f^{-1}f_*(U) \in \phi\} \subseteq \phi$$

where the inclusion is induced by $f^{-1}f_*(U) \subseteq U$. The counit

$$Ff \cdot r(\psi) = \{V \in \mathcal{O}(Y) : f_*f^{-1}(V) \in \psi\} \supseteq \psi$$

is induced by $V \subseteq f_*f^{-1}(V)$.

Example 12.1. For instance $(i_X)_* = (-)^\# \colon \mathcal{O}(X) \to \mathcal{O}(FX)$. To see this, recall that $\{U^\# : U \in \mathcal{O}(X)\}$ form a basis of open sets of FX, hence $\cup\{V \in \mathcal{O}(FX) : i_X^{-1}(V) \subseteq U\}$ is equal to the union of those $W^\#$ satisfying $i_X^{-1}(W^\#) \subseteq U$; but this means $W \subseteq U$, so the union is U.

The assignment $X \mapsto \mathcal{O}(X)$ can be extended into a pair of 2-functors $\mathbf{Top}^{\mathrm{coop}} \to \mathbf{Poset}$ and $\mathbf{Top} \to \mathbf{Poset}$ into the 2-category of posets. The first of these is defined on morphisms by $f \mapsto f^{-1}$. It is not hard to show that if $f \leqslant g$ in $\mathbf{Top}$ then $f^{-1}(V) \supseteq g^{-1}(V)$ for all $V \in \mathcal{O}(Y)$. The second 2-functor is given on morphisms by $f \mapsto f_*$. Taking mates, one deduces that $f_*(U) \subseteq g_*(U)$ for all $U \in \mathcal{O}(X)$ when $f \leqslant g$.

Lemma 12.2. *For a morphism f in* $\mathbf{Top}$*, the following conditions are equivalent. (1) f_* is full – equivalently, $f^{-1}f_* = 1$. (2) Ff is the left adjoint in a coretract adjunction in the 2-category* $\mathbf{Top}$*. (3) Ff is the left adjoint in a coretract adjunction in the 2-category $\mathsf{F}\text{-}\mathrm{Alg}_s$. Furthermore, if any of these conditions hold, f is a full morphism between the underlying preorders. If f is a morphism in $\mathbf{Top}_0$, (1)–(3) are equivalent to f being an embedding.*

Proof: We start by showing that for any space X, the unit i_X satisfies (1). In Example 12.1 we saw that $(i_X)_* = (-)^\#$, and $i_X^{-1}(U^\#) = U$ is easily verified. As for any adjunction with invertible counit, the right adjoint is full.

Given $f \colon X \to Y$ in $\mathbf{Top}$, consider the naturality of i, and apply the 2-functor $(-)_*$.

$$i_Y \cdot f = Ff \cdot i_X \qquad (i_Y)_* f_* = (Ff)_*(i_X)_* \tag{12.3}$$

As $(i_Y)_*$ is full – and faithful – we have that f_* is full if and only if $(Ff)_*(i_X)_*$ is full. It is clear that (1) implies that Ff is full and faithful, so the unit of $Ff \dashv r$ must be an identity, which is the statement (2).

$$\begin{tikzcd}
A \arrow[dr, "\lambda_f"] \arrow[drr, bend left, "i_A"] \arrow[ddr, bend right, "f"'] & & \\
& Ff\downarrow i_B \arrow[r] \arrow[d, "\rho_f"] \arrow[dr, phantom, "\geqslant"] & FA \arrow[d, "Ff"] \\
& B \arrow[r, "i_B"'] & FB
\end{tikzcd}$$

FIGURE 8. Factorisation of a continuous map.

Now assume (2), ie that the unit of $Ff \dashv r$ is an identity, so the same is true of $(Ff)_* \dashv r_*$. By a standard argument, this implies that the counit of $(Ff)^{-1} \dashv (Ff)_*$ is an identity, so the counit of $i_X^{-1}(Ff)^{-1} \dashv (Ff)_*(i_X)_*$ is an identity too. Equivalently, the counit of $f^{-1}i_Y^{-1} \dashv (i_Y)_*f_*$ is an identity, but this counit is $f^{-1}i_Y^{-1}(i_Y)_*f_* = f^{-1}f_* \leqslant 1$, just the counit of $f^{-1} \dashv f$, which therefore must be an identity, completing the proof of (2)⇒(1).

It remains to prove the equivalence between (3) and (2), of which the direct direction is trivial. We only have to prove the converse, for which it is enough to prove that r is a morphism of F-algebras, since the forgetful 2-functor $\mathsf{F}\text{-Alg}_s \to \mathbf{Top}$ is locally fully faithful. We need to prove, thus, that r preserves arbitrary suprema – intersections of filters, since we are working with the opposite of the inclusion order. But this is readily verified by using (12.2).

Next we show that (2) implies that f is full on the underlying preorders. The unit i_Z is full and faithful for any Z – this is precisely the way in which the order on Z is defined. Then, (12.3) shows that f is full if Ff is so, but this is precisely what (2) guarantees.

Finally, if X is a $T0$ space, its underlying preorder is a poset. In this case, f is injective, as $f(x) = f(y)$ implies both $x \leqslant y$ and $y \leqslant x$ in the X, so $x = y$. Together with $f^{-1}f_* = 1$, we have that f is an embedding. ∎

We shall show that F is simple by verifying the hypotheses of Theorem 10.4, but first we summarise the consequences of this fact. There is a lax orthogonal AWFS on **Top** that factors a morphism f as $f = \rho_f \cdot \lambda_f$, as depicted in Figure 8. We can now employ Theorem 11.7 to recover one of the WFSs considered in [3].

Proposition 12.3. *The underlying WFS of the lax orthogonal AWFS* (L, R) *on* **Top** *transferred from the coreflection–opfibration AWFS on* $\mathsf{F}\text{-Alg}_s$ *has the* *following properties. (1) $\mathcal{L}$ consists of those maps that satisfy the equivalent conditions of Lemma 12.2. (2) Maps in $\mathcal{R}$ are not just injective but also algebraically KZ injective with respect to each map in $\mathcal{L}$. If we restrict to $T0$ spaces, $\mathcal{L}$ is the class of topological embeddings.*

Theorem 12.4. *The 2-monad* F *is simple.*

Proof: The proof uses Proposition 10.2. Let $h\colon A \to B \leftarrow C : g$ be two continuous maps, and $h \downarrow g$ its comma object in **Top**, with first and second projections denoted d_0 and d_1. The comparison morphism $k\colon F(h \downarrow g) \to Fh \downarrow Fg \subset FA \times FB$ sends a filter φ on $h \downarrow g$ to the pair of filters (ψ_0, ψ_1): $\psi_0 = \{U \in \mathcal{O}(A) : d_0^{-1}(U) \in \varphi\}$, $\psi_1 = \{V \in \mathcal{O}(B) : d_1^{-1}(V) \in \varphi\}$. Given $(a, b) \in h \downarrow g$, recall that its image under the unit is $i_{h\downarrow g}(a, b) = \{W \in \mathcal{O}(h \downarrow g) : (a, b) \in W\}$. We have $(Fd_0)i_{h\downarrow g}(a, b) = i_A d_0(a, b) = i_A(a)$, and similarly, $(Fd_1)i_{h\downarrow g}(a, b) = i_B(b)$.

We claim that given $\varphi \in F(h \downarrow g)$, $(a, b) \in h \downarrow g$ as above, if $\psi_1 \leqslant i_A(a)$ and $\psi_2 \leqslant i_B(b)$ then $\varphi \leqslant i_{h\downarrow g}(a, b)$. In other words, we shall show that if the following two inclusions hold

$$\{U \in \mathcal{O}(A) : d_0^{-1}(U) \in \varphi\} \supseteq \{U \in \mathcal{O}(A) : a \in U\}$$

$$\{V \in \mathcal{O}(B) : d_1^{-1}(V) \in \varphi\} \supseteq \{V \in \mathcal{O}(B) : b \in V\}$$

then $\varphi \supseteq \{W \in \mathcal{O}(h \downarrow g) : (a, b) \in W\}$. Given $a \in U \in \mathcal{O}(A)$, $b \in V \in \mathcal{O}(B)$, then

$$(U \times V) \cap (h \downarrow g) = d_0^{-1}(U) \cap d_1^{-1}(V) \in \varphi.$$

But any neighbourhood W of (a, b) contains another of the form $(U \times V) \cap (h \downarrow g)$, so $W \in \varphi$, completing the proof of the claim. ∎

The fibrant replacement 2-monad on **Top** of this AWFS is F, as a consequence of Corollary 9.3. To prove this, first observe that the topological space $F1$ of filters of open sets of 1 is $\{\{1\}, \{\varnothing, 1\}\}$, where $\bot := \{\varnothing, 1\}$ is an open point, and $\top := \{1\}$ is not. So $F1$ is the Sierpinski space, and $\bot \leqslant \top$ with our choice of (the opposite of the specialisation) order. The unit $1 \to F1$ is the continuous function that picks out $\top \in F1$, and hence a right adjoint to $F1 \to 1$.

Example 12.5. Let P be the 2-monad on **Top** that assigns to each space the space PX of upper closed subsets of $\mathcal{O}(X)$, with topology generated by the subsets $U^{\#} = \{\varphi \in PX : U \in \varphi\}$. This 2-monad is lax idempotent by the same considerations that apply to the filter monad.

One can prove that, for a map $f\colon X \to Y$, $f_*\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ is full if and only if Pf is a left adjoint coretract; an identical proof to the case of the filter monad works just as well. This implies that P is *not* simple. If it were, we would have an induced lax idempotent AWFS on **Top**, with fibrant replacement 2-monad isomorphic to P, by the same arguments given in the paragraph immediately previous to the present example. Whence, P-algebras would be those spaces injective with respect to full maps, which are the F-algebras, yielding the contradiction that F is isomorphic to P.

13. Example: completion of $\mathbb{V}$-categories under a class of colimits

This section is divided in three parts. In the first we prove that the 2-monad whose algebras are $\mathbb{V}$-categories with chosen colimits of a class is simple, therefore inducing a lax orthogonal AWFS (L, R) on $\mathbb{V}$-**Cat**. The middle part shows that the algebras for R are, at least when $\mathbb{V} = \mathbf{Set}$, split opfibrations whose fibres are equipped with chosen colimits and whose push forward functors strictly preserve them. Intuitively, this type of split opfibration should coincide with the R-algebras, but in general they do not, and this is the subject of the last part of the section.

13.1. Simplicity of completion under a class colimits. Let Φ be a small class of colimits, and T_Φ the 2-monad on $\mathbb{V}$-**Cat** whose algebras are small $\mathbb{V}$-categories with chosen colimits of the class Φ. This 2-monad, whose existence was proven in [15], is lax idempotent. Earlier, less general, versions of this monad appeared, for example, in [16]. In this section we prove:

Theorem 13.1. *The 2-monads T_Φ are simple – in the sense of Section 10 – therefore inducing a lax orthogonal AWFS $(\mathsf{L}_\Phi, \mathsf{R}_\Phi)$ on $\mathbb{V}$-**Cat**.*

Because an explicit description of $\mathsf{T}_\Phi A$ is only possible in particular instances, we will use the usual presheaf pseudomonad and its relationship to T_Φ. The free completion of a $\mathbb{V}$-category C under small colimits can be constructed as the $\mathbb{V}$-category $\mathscr{P}C$ with objects small presheaves – ie $\mathbb{V}$-functors $C^{\mathrm{op}} \to \mathbb{V}$ that are a left Kan extension of its own restriction to a small subcategory of C^{op} – and enriched homs given by $\mathscr{P}C(\phi, \psi) = \int_c [\phi c, \psi c]$. This extends to a pseudomonad on $\mathbb{V}$-**Cat**, whose unit has components the Yoneda embedding $y_C \colon C \to \mathscr{P}C$, and whose multiplication we denote by m^{Φ}. A number of properties of $\mathscr{P}C$, in particular its completeness, are studied in [7].

The free completion of C under colimits of the class Φ – or Φ-colimits – can be constructed as the smallest full sub-$\mathbb{V}$-category of $\mathscr{P}C$ that is closed under Φ-colimits and contains the representable presheaves. One obtains a pseudomonad Φ together with a pseudomonad morphism $\Phi \to \mathscr{P}$ that has fully faithful components.

Lemma 13.2. *Given a $\mathbb{V}$-functor $f \colon C \to D$ between small $\mathbb{V}$-categories, the following statements are equivalent. (1) $\Phi f \colon \Phi C \to \Phi D$ has a right adjoint. (2) Whenever $\psi \in \Phi D$, $\psi f^{\mathrm{op}} \colon C^{\mathrm{op}} \to \mathbb{V}$ belongs to ΦC. (3) $D(f-, d) \in \Phi C$ for all $d \in D$.*

Before we prove the theorem, it is convenient to recall some useful aspects of the construction of T_{Φ} [15]. Part of this construction is an equivalence $t_A \colon T_{\Phi} \to \Phi A$ for each $\mathbb{V}$-category A, which form a pseudonatural transformation $\mathsf{T} \to \Phi$, and moreover, a pseudomonad morphism.

Lemma 13.3. *Denote by Φ a small class of $\mathbb{V}$-enriched colimits and the associated pseudomonad on $\mathbb{V}$-**Cat**. Let $f \colon A \to B$ be a $\mathbb{V}$-functor into a Φ-cocomplete $\mathbb{V}$-category, and denote by $\tilde{f} \colon \Phi(A) \to B$ a left Kan extension of f along the corestricted Yoneda embedding y_A. Then the morphisms induced by $\tilde{f}$*

$$\Phi(A)(\phi, y_A(a)) \longrightarrow B(\tilde{f}(\phi), f(a)) \tag{13.1}$$

are isomorphisms for all $\phi \in \Phi(A)$ and $a \in A$.

Proof: The morphism (13.1) can be written as the composition of $\Phi(A)(\phi, f)$ from $\Phi(A)(\phi, y_A(a))$ to $\Phi(A)(\phi, B(f-, f(a)))$ and the isomorphism between the latter and $B(\mathrm{col}(\phi, f), f(a))$. The result is an isomorphism since f is full and faithful. ∎

Proof of Theorem 13.1: Let the 2-monad T in Corollary 10.3 be the 2-monad on $\mathbb{V}$-**Cat** whose algebras are $\mathbb{V}$-categories with chosen Φ-colimits. Assume we are given morphisms f and g as in the statement of the said corollary. The $\mathbb{V}$-category $Tf \downarrow Tg$ is Φ-cocomplete, as the forgetful 2-functor from T-algebras creates comma objects. The comparison morphism of the corollary is the left Kan extension of the $\mathbb{V}$-functor $h \colon f \downarrow g \to Tf \downarrow Tg$ induced by i_A and i_B. Since h is full and faithful, then k is full and faithful on homs of the form $T(f \downarrow g)(u, i_{f\downarrow g}(v))$ by Lemma 13.3, so we have indeed the bijection of 2-cells required in Corollary 10.3. ■

13.2. Monadicity of split opfibrations with fibrewise chosen Φ-colimits. Given a small class of **Set**-colimits Φ, denote by $\mathbf{OpFib}_s$-Φ-$\mathbf{Colim}_s$ the 2-category with objects split opfibrations in **Cat** whose fibres are small categories with chosen colimits of the class Φ and whose push-forward functors strictly preserve these. Morphisms from $p\colon E \to B$ to $p'\colon E' \to B'$ are strict morphisms $(h, k)\colon p \to p'$ of split fibrations such that the restriction of h to fibres strictly preserves the chosen Φ-colimits, while 2-cells are those of $\mathbf{Cat}^{\mathbf{2}}$.

The codomain functor $\mathbf{OpFib}_s$-Φ-$\mathbf{Colim}_s \to \mathbf{Cat}$ is a fibration and the forgetful 2-functor $\mathbf{OpFib}_s$-Φ-$\mathbf{Colim}_s \longrightarrow \mathbf{Cat}^{\mathbf{2}}$ is strictly cartesian. This means that given a split opfibration $A \to B$ whose fibres have chosen Φ-colimits and the push-forward functors strictly preserve them, its pullback along any functor $B' \to B$ carries the same structure. The fibration of 2-categories $\mathbf{OpFib}_s$-Φ-$\mathbf{Colim}_s \to \mathbf{Cat}$ is equivalent to that whose fibre over B is $[B^{\mathrm{op}}, \Phi\text{-}\mathbf{Colim}_s]$ and whose pullback along $f\colon B' \to B$ is given by precomposition.

Lemma 13.4. *The forgetful 2-functor* $U\colon \mathbf{OpFib}_s$-Φ-$\mathbf{Colim}_s \to \mathbf{Cat}^{\mathbf{2}}$ *is monadic.*

Proof: We first show that the strictly cartesian functor U has a fibred left adjoint. One can write U as the composition of the forgetful 2-functors V from **OpFib**-Φ-$\mathbf{Colim}_s$ to $\mathbf{OpFib}_s$ and $\mathbf{OpFib}_s \to \mathbf{Cat}^{\mathbf{2}}$, and since the latter is monadic, it will suffice to show that V has a left adjoint. Observe that V is a strictly cartesian 2-functor between fibrations over **Cat**, so it suffices to prove that each restriction V^B to the fibre over $B \in \mathbf{Cat}$ has a left adjoint, say F^B, and that for any functor $f\colon B' \to B$, the natural transformation $F^{B'}f^* \Rightarrow f^*F^B$ is invertible. This is a version for 2-categories fibred over a 2-category of the corresponding classical result for fibred categories, but there is no real difference in the proof. The 2-functor V^B is equivalently described as $[B, \Phi\text{-}\mathbf{Colim}_s] \to [B, \mathbf{Cat}]$ given by composing with the right adjoint forgetful 2-functor $U_\Phi\colon \Phi\text{-}\mathbf{Colim}_s \to \mathbf{Cat}$. Therefore, V^B has a left adjoint, given by composing with the left adjoint, say F_Φ, of U_Φ. Given a functor f as above, the Beck-Chevalley condition says that, given $\phi\colon B \to \mathbf{Cat}$, we need the following transformation to be invertible, $F_\Phi \cdot \phi \cdot f \Rightarrow F_\Phi \cdot U_\Phi \cdot F_\Phi \cdot \phi \cdot f \Rightarrow$

$F_\Phi \cdot \phi \cdot f$, where the arrows are induced, respectively, by the unit and counit of $F_\Phi \dashv U_\Phi$. This is obviously true, so we obtain a left adjoint $F \dashv V$.

In order to show that U is monadic it remains to show that it creates coequalisers of U-split pairs. Suppose (u_0, u_1), $(v_0, v_1)\colon k \to h$ are a parallel pair and $(q_0, q_1)\colon h \to g$ a morphism in **OpFib-Φ-Colim**$_s$, with a splitting in **OpFib**$_s$ given by $(s_0, s_1)\colon g \to h$ and $(t_0, t_1)\colon h \to k$. This means $q \cdot s = 1$, $u \cdot t = 1$ and $v \cdot t = s \cdot q$. We have to show that fibres of g can be equipped with chosen Φ-colimits in a way that the restriction of q to fibres strictly preserves them, and the push-forward functors of the split opfibration g strictly preserves them too.

$$\begin{tikzcd} C \arrow[r, shift left=2, "u_0"] \arrow[r, "v_0"'] \arrow[d, "k"'] & E \arrow[l, shift left=2, "t_0"] \arrow[r, shift left, "q_0"] \arrow[d, "h"'] & A \arrow[l, shift left, "s_0"] \arrow[d, "g"] \\ D \arrow[r, shift left=2, "u_1"] \arrow[r, "v_1"'] & F \arrow[l, shift left=2, "t_1"] \arrow[r, shift left, "q_1"] & B \arrow[l, shift left, "s_1"] \end{tikzcd}$$

Given $b \in B$, the top row of the diagram restricts to the fibres $C_{t_1 s_1(b)}$, $E_{s_1(b)}$ and A_b, yielding a split diagram of categories. The first two categories are equipped with chosen Φ-colimits, and the parallel pair between them strictly preserve them, so we have that A_b has a unique choice of Φ-colimits that make $q\colon E_{s_1(b)} \to A_b$ strictly preserve them, by monadicity of categories with chosen colimits [15]. It remains to prove that each $q_0\colon E_f \to A_{q_1(f)}$ strictly preserves Φ-colimits, for any $f \in F$. Since u_0 exhibits E_f as a retract of $C_{t_1(f)}$, it suffices to prove that $q_0 \cdot u_0\colon C_{t_1(f)} \to A_{q_1(f)}$ strictly preserves Φ-colimits, as any such colimit in E_f is the image under u_0 of another in $C_{t_1(f)}$. Thus, we need to prove that $q_0 \cdot v_0\colon C_{t_1(f)} \to A_{q_1(f)}$ strictly preserves these colimits. Note that v_0 sends the fibre over $t_1(f)$ to the fibre over $v_1 t_1(f) = s_1 q_1(f)$, so it suffices to know that q_0 restricted to $E_{s_1 q_1(f)}$ strictly preserves Φ-colimits, which is the case by construction of the colimits on the fibres of A.

It is easy to see that for any morphism $\beta\colon b \to b'$ the push-forward functor $A_\beta\colon A_b \to A_{b'}$ strictly preserves the chosen colimits, since $\beta_* \cdot q_0\colon E_b \to A_b \to A_{b'}$ equals $q_0 \cdot \beta_*\colon E_b \to E_{b'} \to A_{b'}$. ■

For later use, we record the following observations, which hold not only in the case of ordinary categories but also in that of categories enriched in a suitable monoidal category whose unit object is terminal.

Remark 13.5. Let C be a category with Φ-colimits, $H\colon C \to E$ a Φ-cocontinuous functor, e an object of E and $Q\colon H/e \to C$ the projection. For any functor $D\colon J \to H/e$ and any colimiting cylinder $\eta\colon \phi \Rightarrow C(QD-, c)$ with $\phi \in \Phi$ there exists a unique $\epsilon\colon Hc \to e$ in E and a unique colimiting cylinder $\nu\colon \phi \Rightarrow H/e(D-, (c, \epsilon))$ such that $Q(\nu) = \eta$.

Moreover, if C is equipped with chosen Φ-colimits, then there exists a unique choice of Φ-colimits on H/e that is strictly preserved by Q.

Suppose $S\colon A \to H/e$ is a functor, where A has chosen Φ-colimits. Then S strictly preserves Φ-colimits if and only if $QS\colon A \to C$ does so.

Theorem 13.6. *Given a small class of* **Set***-colimits* Φ*, there exists a 2-functor*

$$\mathsf{R}\text{-}\mathrm{Alg}_s \longrightarrow \mathbf{OpFib}\text{-}\Phi\text{-}\mathbf{Colim}_s \tag{13.2}$$

that commutes with the forgetful functors into $\mathbf{Cat^2}$*, where* R *is the right part of the AWFS on* **Cat** *induced by* Φ*.*

Proof: We shall denote the 2-category $\mathbf{OpFib}\text{-}\Phi\text{-}\mathbf{Colim}_s$ by $\mathscr{F}$ in order to save space. We are to prove that each R-algebra $A \to B$ is a split opfibration whose fibres are equipped with chosen Φ-colimits and whose push-forward functors strictly preserve these. The first assertion follows from Theorem 8.10, so we only need to concern ourselves with the chosen colimits. The rest of the proof is divided in a series of steps: (a) define on Rg a structure of an object of $\mathscr{F}$, for all $g \in \mathbf{Cat^2}$; (b) any morphism of R-algebras is a morphism in $\mathscr{F}$, so we have a 2-functor $\mathrm{Kl}(\mathsf{R}) \to \mathscr{F}$ over $\mathbf{Cat^2}$; (c) the image of any parallel pair of morphisms in $\mathrm{Kl}(\mathsf{R})$ under the 2-functor of (b) has a coequaliser in $\mathscr{F}$, provided that its image under $\mathrm{Kl}(\mathsf{R}) \to \mathbf{Cat^2}$ has an absolute coequaliser in $\mathbf{Cat^2}$; (d) this means that we can left Kan extend it to a 2-functor (13.2).

We start by proving (a). The fibres of any R-algebra are categories with chosen Φ-colimits: any such fibre is an R-algebra over 1, by Corollary 2.16, and the restriction of R to $\mathbf{Cat}/1$ is isomorphic to T, by Corollary 9.3; note that the unit $i_1\colon 1 \to T1$ is always a right adjoint. It remains to prove that for any morphism $\beta\colon b \to b'$ in B the push-forward functor $A_b \to A_{b'}$ preserves the chosen colimits. We claim that it is enough to prove it for free

R-algebras, since there will be a split coequaliser in $\mathbf{Cat}/B$

$$\begin{tikzcd} K\rho_g \arrow[r, shift left=2, "\pi_g"] \arrow[r, "K(p,1)" description] & Kg \arrow[l, shift left=2, "\lambda_{\rho_g}"] \arrow[r, shift left, "p"] & A \arrow[l, shift left, "\lambda_g"] \end{tikzcd}$$

– where p denotes the R-algebra structure of g – which then lifts to a (non-split) coequaliser in the 2-category of split opfibrations. Taking the fibre over $b \in B$ of this split coequaliser, we obtain a coequaliser in $\mathsf{T}\text{-Alg}_s$ that splits in $\mathbf{Cat}$. In particular, for any functor d into A

$$\mathrm{col}(\phi, d) = p_b(\mathrm{col}(\phi, (\lambda_g)_b \cdot d))$$

because p strictly preserves the chosen colimits. Taking fibres over b and b', we have a commutative square in $\mathbf{Cat}$ where all the categories have chosen Φ-colimits and the horizontal functors strictly preserve them.

$$\begin{tikzcd} (Kg)_b \arrow[r, "p_b"] \arrow[d, "\beta_*"'] & A_b \arrow[d, "\beta_*"] \\ (Kg)_{b'} \arrow[r, "p_{b'}"] & A_b \end{tikzcd}$$

Therefore, if the push-forward functors of Rg preserve the chosen Φ-colimits, then so does the ones of g: by commutativity of the square, $\beta_* \cdot p_b$ strictly preserves Φ-colimits, and

$$\beta_*(\mathrm{col}(\phi, d)) = \beta_* p_b(\mathrm{col}(\phi, (\lambda_g)_b \cdot d)) = \mathrm{col}(\phi, \beta_* \cdot p_b \cdot (\lambda_g)_b \cdot d) = \mathrm{col}(\phi, \beta_* \cdot d).$$

Now we prove that the push-forward functors of a free R-algebra Rg strictly preserve chosen Φ-colimits. By the description of Kg as a comma object (10.4), its objects are triples (x, b, ξ) where $x \in TA$, $b \in B$ and $\xi\colon (Tg)(x) \to i_B(b)$ is a morphism in TB. If we denote by $z_b\colon (Kg)_b \to Kg$ the inclusion of the fibre over $b \in B$ and $t_g\colon Kg \to TA$ the projection of the comma object, we showed in Lemma 9.4 that $t_g \cdot z_b\colon (Kg)_b \to TA$ strictly preserves Φ-colimits. It is clear that the triangle on the left hand side commutes, since $\beta_*(x, b, \xi) = (x, b', i_B(\beta) \cdot \xi)$.

$$\begin{tikzcd} (Kg)_b \arrow[r, "t_g \cdot z_b"] \arrow[d, "\beta_*"'] & TA \\ (Kg)_{b'} \arrow[ur, "t_g \cdot z_{g'}"'] & \end{tikzcd} \qquad \begin{tikzcd} Tg/i_B(b) \arrow[r, "\mathrm{pr}_b"] \arrow[d, "\beta_*"'] & TA \\ Tg/i_B(b') \arrow[ur, "\mathrm{pr}_{b'}"'] & \end{tikzcd}$$

But $(Kg)_b$ is the slice category $Tg/i_B(b)$, and $t_g \cdot z_b$ is the projection into TA, so we have a commutative triangle as in the right hand side. We can now apply the observations of Remark 13.5 to deduce that, since $\mathrm{pr}_{b'} \cdot \beta_*$ strictly preserve Φ-colimits, so does β_*. This completes the proof of (a).

The next step in the proof is to verify (b), which is easy. Let $g\colon A \to B$ and $g\colon A' \to B'$ be R-algebras, $(h,k)\colon g \to g'$ a morphism of R-algebras, and $b \in B$. Taking fibres over b and $k(b)$ we obtain a commutative square in $\mathbf{Cat}^{\mathbf{2}}$ displayed on the left, where the solid arrows are in $\mathscr{F}$ and the dotted arrow is the restriction of h to fibres.

$$\begin{tikzcd} A_b \arrow[r, dotted] \arrow[d] & A'_{k(b)} \arrow[d] \\ g \arrow[r, "{(h,k)}"] & g' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[r, equal] \arrow[d, "b"'] & \mathbf{1} \arrow[d, "k(b)"] \\ B \arrow[r, "k"] & B' \end{tikzcd}$$

The codomain part of this diagram is displayed on the right hand side. We deduce that the dotted functor is a morphism in $\mathscr{F}$ from the fact that the codomain functor $\mathsf{R}\text{-Alg}_s \to \mathbf{Cat}$ is a fibration – Corollary 2.16 – and the vertical morphisms, as pullback squares, are cartesian.

From (a) and (b) together we deduce that there is a 2-functor $\mathrm{Kl}(\mathsf{R}) \to \mathscr{F}$ that commutes with the canonical 2-functors into $\mathscr{K}^{\mathbf{2}}$. If a pair of morphisms between free R-algebras has an absolute coequaliser in $\mathbf{Cat}^{\mathbf{2}}$, they form a parallel pair of morphisms in $\mathscr{F}$ with an absolute coequaliser in $\mathbf{Cat}^{\mathbf{2}}$. By monadicity of $\mathscr{F}$ – Lemma 13.4 – their coequaliser exists in $\mathscr{F}$, and it is preserved by the forgetful 2-functor into $\mathbf{Cat}^{\mathbf{2}}$. This proves (c).

Finally, (d) is the general observation that, for any 2-monad S on a 2-category $\mathscr{A}$, the 2-functor $\mathrm{Kl}(\mathsf{S}) \to \mathsf{S}\text{-Alg}_s$ is dense with a density presentation given by coequalisers of those pairs whose image under $\mathrm{Kl}(\mathsf{S}) \to \mathscr{A}$ have absolute coequalisers in $\mathscr{A}$. Therefore the left Kan extension of $\mathrm{Kl}(\mathsf{R}) \to \mathscr{F}$ to a 2-functor (13.2) exists. For background on density presentations see [13, 14, Ch 5]. ∎

In many instances, the 2-functor of the theorem is an isomorphism. For example, it is not hard to verify this when Φ is the class for initial objects $\{\varnothing \to \mathbf{Set}\}$.

13.3. Split opfibrations with fibrewise chosen colimits are not always R-algebras. In this example we show that for a certain, very simple class of colimits, the 2-functor of Theorem 13.6 is not an isomorphism. Denote by R the 2-monad of the lax orthogonal AWFS induced by a class of colimits Φ, and by S the 2-monad whose 2-category of algebras is $\mathbf{OpFib}\text{-}\Phi\text{-}\mathbf{Colim}_s$ – Lemma 13.4. In fact, we prove that the restriction of the 2-monad morphism $\mathsf{S} \to \mathsf{R}$ given by Theorem 13.6 to $\mathbf{Cat}/1+1$ is not an isomorphism.

Let R_{1+1} and S_{1+1} be the restrictions to $\mathbf{Cat}/1+1$. The latter was described in the course of the proof of Theorem 13.6, and sends $g\colon A \to 1+1$ to the split opfibration – just a functor over $1+1$ – with fibres $T(A_0)$ and $T(A_1)$ over 0 and 1 respectively. Here we are writing 0, 1 for the two objects of the discrete category $1+1$. The 2-monad morphism $\mathsf{S}_{1+1} \to \mathsf{R}_{1+1}$ of Theorem 13.6 has component at g a functor that on the fibres over 0 is the unique $T(A_0) \to (Kg)_0$ that composed with the inclusion $(Kg)_0 \hookrightarrow Kg$ corresponds to

$$\begin{tikzcd}
TA_0 \arrow[rr, "Tz_0"] \arrow[dd] \arrow[dr, "T!"'] & & TA \arrow[dd, "Tg"] \\
& T1 \arrow[d, phantom, "\Swarrow"] \arrow[dr, "T0"] & \\
\mathbf{1} \arrow[r, "0"'] & 1+1 \arrow[r, "i_{1+1}"'] & T(1+1)
\end{tikzcd}$$

where the unlabelled natural transformation is the unique that yields the identity transformation when precomposed with i_{A_0}. Note that we used that T is lax-idempotent.

Below we shall exhibit an example where the functor $T(A_0) \to (Kg)_0$ is not an isomorphism, and therefore neither is $\mathsf{S}_{1+1} \to \mathsf{R}_{1+1}$.

Consider the class of colimits $\Phi = \{\Delta\varnothing\colon \mathbf{1} \to \mathbf{Set}\}$. The colimit of a functor $v\colon 1 \to \mathbf{Set}$ that picks out a set v, weighted by $\Delta\varnothing$ – the tensor product of v by $\varnothing$ – is $\mathrm{col}(\Delta\varnothing, v) = \varnothing$. The completion of a small category A under these colimits consists of the full subcategory $\Phi A \subseteq [A^{\mathrm{op}}, \mathbf{Set}]$ consisting of the representables together with the initial object: the presheaf constant at $\varnothing$. A category A has chosen Φ-colimits if there is an assignment of an initial object $0(a) \in A$ for each object $a \in A$.

We can explicitly describe the 2-monad T associated to Φ. Let $(|A| \times \mathbb{N})_{\mathrm{ch}}$ be the codiscrete, chaotic or indiscrete category with objects $|A| \times \mathbb{N}$, and $N\colon A \to (|A| \times \mathbb{N})_{\mathrm{ch}}$ the unique functor that on objects is given by $a \mapsto (a, 0)$.

Define TA as the oplax colimit in **Cat** depicted below.

$$\begin{tikzcd} (|A| \times \mathbb{N})_{\mathrm{ch}} \arrow[dr, "j"'] & & A \arrow[ll, "N"'] \arrow[dl, "i_A"] \arrow[dll, phantom, "\Rightarrow"] \\ & TA & \end{tikzcd} \tag{13.3}$$

Then TA has objects of the form either $a \in A$ or $(a, n) \in |A| \times \mathbb{N}$, the functors j and i_A are fully faithful and $TA((a, n), a') = 1$, $TA(a, (a', n)) = \varnothing$. In particular, all the objects of the form (a, n) are initial in TA. We can equip TA with chosen Φ-colimits by defining $0(a) = (a, 0)$ and $0((a, n)) = (a, n+1)$ for $n \in \mathbb{N}$.

We next show that $i_A : A \to TA$ is the free category with chosen Φ-colimits on A. Suppose $f : A \to B$ is a functor and that B has chosen Φ-colimits, ie a choice of an initial object $0(b)$ for each $b \in B$. By the universal property of the colimit (13.3), to give a functor $\hat{f} : TA \to B$ is equivalent to giving a pair of functors $h : A \to B$ and $k : (A \times \mathbb{N})_{\mathrm{ch}} \to B$ with a natural transformation $\alpha : h \Rightarrow k \cdot N$. To say that $\hat{f}$ preserves the chosen initial objects means that $k(a, 0) = \hat{f}j(a, 0) = 0(f(a))$ and $k(a, n) = \hat{f}j(a, n) = 0(\hat{f}(a, n-1))$ for $n \geqslant 1$. Therefore, k is determined, and $h = f$ if we require $\hat{f} \cdot i_A = f$. Finally, a natural transformation α will have initial objects as the domain of its components, so it clearly exists and it is unique. This completes the proof of the existence and uniqueness of $\hat{f}$, and TA is the free category with chosen Φ-colimits on A.

Given a functor $g : A \to 1 + 1$, consider $\rho_g : Kg \to 1 + 1$. Its fibre $(Kg)_0$ over 0 has objects pairs $x \in TA$, $(Tg)x \to i_{1+1}(0)$ in $T(1 + 1)$. For any $a \in A$, $i_A(a)$ is an object of $(Kg)_0$ precisely when $a \in A_0$, since $(Tg)i(a) = i_{1+1}(ga) \to i_{1+1}(0)$ exists – and is the identity – when $ga = 0$ in $1 + 1$. On the other hand, any of the chosen initial objects $(a, n) \in TA$ will be objects of $(Kg)_0$. Indeed, $(Tg)(a, n)$ is initial in $T(1 + 1)$, and thus it has a unique morphism into $i_{1+1}(0)$. Therefore, the objects of $(Kg)_0$ can be described as $|(Kg)_0| \cong |A_0| + \{(a, n) : a \in A, n \geqslant 0\}$. On the other hand, the objects of $T(A_0)$ can be described as $|T(A_0)| \cong |A_0| + (|A_0| \times \mathbb{N})$. The comparison functor $TA_0 \to (Kg)_0$ sends $a \in A_0$ to a as an object of $(Kg)_0$; it sends (a, n), where $a \in A_0$ and $n \geqslant 0$ to (a, n), now thought as an object of $(Kg)_0$. It is clear that if $a \in A_1$, then $(a, 0)$ does not belong to the image of $TA_0 \to (Kg)_0$, and this functor cannot be an isomorphism.

14. Example: Lawvere's generalised metric spaces

This section studies the example of the 2-category of Lawvere's generalised metric spaces and a lax idempotent AWFS that arises from the Cauchy completion construction. The left and right part of the AWFS, and its underlying WFS, can be described in terms of Cauchy sequences. For instance, the AWFS factorises a distance-decreasing function f between (usual) metric spaces as a dense isometry λ_f followed by a distance-decreasing function ρ_f with the property that any Cauchy sequence in its domain converges whenever its image under ρ_f converges.

The lax idempotent AWFS described in this section is related but not a particular instance of the one of Section 13. The difference resides in that the 2-monad that induces the AWFS in the named section is that for *chosen* colimits, while in the present section it will be the 2-monad that freely adds absolute colimits, not chosen ones.

Let $\bar{\mathbb{R}}_+$ be Lawvere's symmetric monoidal closed category of non-negative real numbers [18, 19]. It has objects non-negative real numbers together with an extra object ∞, and a unique morphism $\alpha \to \beta$ if $\alpha \geqslant \beta$. This category is complete and cocomplete, with limit and colimit of a functor $f\colon J \to \bar{\mathbb{R}}_+$ given by $\mathrm{col}(f) = \inf\{f(j) : j \in J\}$ and $\lim(f) = \sup\{f(j) : j \in J\}$. These inf and sup are defined in the usual way, with the obvious extension to allow for ∞.

The tensor product of the monoidal structure is given by addition, with $\alpha+\infty = \infty+\alpha = \infty$, and unit object 0. The internal hom $[\alpha, \beta]$ is $\beta-\alpha$ if this real number is non-negative, and 0 otherwise; $[\alpha, \infty] = \infty$ and $[\infty, \alpha] = 0$ for $\alpha \neq \infty$, and $[\infty, \infty] = 0$. Observe that the unit object $0 \in \bar{\mathbb{R}}_+$ is a terminal object.

Categories enriched in $\bar{\mathbb{R}}_+$ are Lawvere's generalised metric spaces, which we will refer to simply as spaces when no confusion is possible. These structures are just as metric spaces, with the difference that the distance function is not symmetric and it can take ∞ as a value. A generalised metric space X is *symmetric* if $X(x, y) = X(y, x)$ for all $x, y \in X$. A symmetric space is the same as a pseudometric space, only that the distance can attain the value ∞. Equivalently, it is the same as a family of pseudometric spaces. Functors enriched in $\bar{\mathbb{R}}_+$ are distance-decreasing functions.

The functor $\bar{\mathbb{R}}_+(0,-)\colon \bar{\mathbb{R}}_+ \to \mathbf{Set}$ takes only two values, 1 on $0 \in \bar{\mathbb{R}}_+$ and $\varnothing$ everywhere else. It can therefore be considered as a functor $\bar{\mathbb{R}}_+ \to \mathbf{2}$, which has a left adjoint $\mathbf{2} \to \bar{\mathbb{R}}_+$ given by $\bot \mapsto \infty$ and $\top \mapsto 0$.

The underlying category 2-functor can be regarded as a 2-functor $\bar{\mathbb{R}}_+$-$\mathbf{Cat} \to \mathbf{Preord}$, where $a \leqslant a'$ when the distance from a to a' is zero. It has a left adjoint that sends a poset X to the generalised metric space with points X and $X(x,x') = 0$ if $x \leqslant x'$ and ∞ otherwise.

The completion of a space A under colimits is the space $\mathscr{P}A = [A^{\mathrm{op}}, \bar{\mathbb{R}}_+]$ of distance-decreasing $\bar{\mathbb{R}}_+$-valued functions on A^{op}.

Let $f\colon A \to B$ be a distance-decreasing function. The $\bar{\mathbb{R}}_+$-functor $\mathscr{P}f$ is explicitly given by $\mathscr{P}f(\phi)(b) = \inf_{a\in A} B(b, f(a)) + \phi(a)$ and it has a right adjoint r given by $r(\psi) = \psi(f-)$. The adjunction $\mathscr{P}f \dashv r$ is a coretract adjunction, ie the unit is an identity, precisely when $\mathscr{P}f$ if full and faithful, equivalently, when f is so, which is to say, f is an isometry.

The 2-functor from $\bar{\mathbb{R}}_+$-$\mathbf{Cat}$ to $\mathbf{Preord}$ that sends a space to its underlying preorder is a 2-monad morphism between the respective free split opfibration 2-monads. This is so because these 2-monads are given by taking a comma object along an identity morphism, and the 2-functor, as a right adjoint, preserves limits. In particular, a split opfibration in $\bar{\mathbb{R}}_+$-$\mathbf{Cat}$ is a split opfibration in $\mathbf{Preord}$.

Definition 14.1. A sequence of points (x_n) in a metric space X is *Cauchy* when it satisfies the usual definition of a Cauchy sequence. Two Cauchy sequences in X are *equivalent* when they are equivalent in the usual sense.

Every Cauchy sequence in $\bar{\mathbb{R}}_+$ converges. If the sequence takes the value ∞ only finitely many times, this is the completeness of $\mathbb{R}_+$. Otherwise, we know that there exists n_0 such that for all $n \geqslant n_0$ both $[\infty, x_n] < 1$ and $[x_n, \infty] < 1$. Of these two conditions, the first is trivial, as it amounts to $0 < 1$, while the second means $\infty < 1$ if $x_n \neq \infty$, which is a contradiction. It follows that $x_n = \infty$ for all $n \geqslant n_0$, and (x_n) converges.

In Section 2.2 we have recalled the definition of $\mathbf{Set}$-module or profunctor and in Section 5.1 that of $\mathbf{Cat}$-module. An $\bar{\mathbb{R}}_+$-module $\phi\colon X \to\!\!\!\!\!\!\!\rightarrow Y$, between spaces X and Y, is a matrix $\phi(y,x) \in \bar{\mathbb{R}}_+$, $x \in X, y \in Y$, satisfying the inequality $Y(y',y) + \phi(y,x) + X(x,x') \geqslant \phi(y',x')$. An $\bar{\mathbb{R}}_+$-module from the trivial $\bar{\mathbb{R}}_+$-category $\phi\colon I \to\!\!\!\!\!\!\!\rightarrow X$ is given simply by $\phi(x) \in \bar{\mathbb{R}}_+$ for each $x \in X$, such that $X(x',x) + \phi(x) \geqslant \phi(x')$. $\bar{\mathbb{R}}_+$-categories and $\bar{\mathbb{R}}_+$-modules form a 2-category with the obvious composition: if $\phi\colon X \to\!\!\!\!\!\!\!\rightarrow Y$ and $\psi\colon Y \to\!\!\!\!\!\!\!\rightarrow Z$

are $\bar{\mathbb{R}}_+$-modules, then $(\psi \cdot \phi)(z,x) = \inf_y \psi(z,x) + \phi(y,x)$. Left adjoint morphisms in this 2-category are called *Cauchy modules*. An $\bar{\mathbb{R}}_+$-module ϕ as before is *convergent* when it is representable by a morphism $f\colon X \to Y$; ie if $\phi(y,x) = Y(y, f(x))$.

Lemma 14.2 (Lawvere)**.** *Let X be a generalised metric space. There is a bijection between the set of Cauchy modules $I \twoheadrightarrow X$ and the set of equivalence classes of Cauchy sequences in X. Furthermore, this bijection sends a Cauchy module ϕ to the equivalence class of Cauchy sequences (x_n) such that $\phi(x_n) < 1/n$ and $\phi^*(x_n) < 1/n$, and conversely a Cauchy sequence (x_n) to the module given by $y \mapsto \lim_n X(y, x_n)$.*

Proposition 14.3. *The generalised metric space QA is isomorphic to the space with points equivalence classes of Cauchy sequences in A and distance*

$$QA([x_n],[y_n]) = \lim_n A(x_n, y_n).$$

A priori, the assignment $A \mapsto QA$ is the object part of a pseudomonad on $\bar{\mathbb{R}}_+$-**Cat**. Next we see that this pseudomonad is in fact a 2-monad. We call a generalised metric space X *posetal* if its underlying preorder is a poset, ie if $X(x,y) = 0 = X(y,x)$ implies $x = y$.

Lemma 14.4. *(1) The underlying preorder of $\mathscr{P}X$ is a poset – it satisfies the antisymmetry condition – and the same holds for QX. (2) Both $\mathscr{P}$ and Q are 2-monads. Their multiplications are characterised by being left adjoints to the respective units. (3) Any normal pseudo-Q-algebra, and in particular any strict Q-algebra, is a poset.*

Proof: Note that $\mathscr{P}X$ is a poset since $\bar{\mathbb{R}}_+$ is one. Therefore, $\mathscr{P}$ is a 2-functor, since the pseudofunctor constraints $\mathscr{P}g \cdot \mathscr{P}f \cong \mathscr{P}(g \cdot f)$ and $1 \cong \mathscr{P}1$ have components in a poset, therefore they are identities. Furthermore, $\mathscr{P}$ is a 2-monad since both the associativity and unit constraints of the pseudomonad $\mathscr{P}$ are isomorphisms between morphisms into $\mathscr{P}$, and $\mathscr{P}X$ is a poset. The same holds for Q, as there exists a pseudomonad morphism $Q \to \mathscr{P}$ whose components, the inclusions of the categories of Cauchy modules into all modules, reflect identities.

By the considerations above, both $\mathscr{P}$ and Q restrict to the full subcategory of $\bar{\mathbb{R}}_+$-**Cat** consisting of the posetal spaces, which we denote by $\bar{\mathbb{R}}_+$-$\mathbf{Cat}_{\mathrm{sep}}$. This is a locally posetal 2-category, and the multiplication of any lax idempotent 2-monad on such a 2-category is characterised by being left adjoint retract to the unit. In particular, $m_X\colon \mathscr{P}^2X \to \mathscr{P}X$ is characterised by being a left adjoint retract of the Yoneda embedding $y_{\mathscr{P}X}\colon \mathscr{P}X \to \mathscr{P}^2X$. However, the retract condition is automatically satisfied: $y_{\mathscr{P}X}$ is full and faithful, so $m_X \cdot y_{\mathscr{P}X} \cong 1$, but this isomorphism must be an identity since $\mathscr{P}X$ is a poset. The same arguments apply to Q instead of $\mathscr{P}$.

It is very easy to verify that posets are closed under retracts in the category of preorders. Then, any normal pseudo-Q-algebra X, as a retract of QX, is a poset. ■

Lemma 14.5. *Let $g\colon A \to B$ be a distance-decreasing function between generalised metric spaces. If a Cauchy sequence (x_n) in A has associated Cauchy module $\phi \in QA$, then $g(x_n)$ has associated Cauchy module $(Qg)(\phi) \in QB$.*

Definition 14.6. A Cauchy sequence (x_n) in a generalised metric space X *source converges* to x if $\lim_n X(x_n, x) = 0$; we say that x is a *source limit* of (x_n).

Observe that $\lim_n X(x_n, x) = 0$ alone need not imply that (x_n) is a Cauchy sequence, by the lack of symmetry on distance of X.

Let $\phi \in QX$ be the Cauchy module associated to a Cauchy sequence (x_n) in X. It is obvious from the definition that x is a source limit of (x_n) if and only if $\phi^*(x) = 0$.

Lemma 14.7. *Let (x_n) be a Cauchy sequence in the generalised metric space X. Its associated module is represented by x if and only if (x_n) converges to x. In this case, if (x_n) source converges to y, then $X(x, y) = 0$.*

The lemma above says that a limit of a convergent sequence is initial – in the underlying preorder – amongst the source limits of the sequence.

Proposition 14.8. *The 2-monads on $\bar{\mathbb{R}}_+$-**Cat** given by completion under a class of colimits are simple. In particular, $\mathscr{P}$ and Q are simple.*

Proof: Lemma 13.3 ensures that the hypotheses of Corollary 10.3 are satisfied. ■

Observe that the proposition above is not a particular instance of Theorem 13.1, since the 2-monad Φ on $\bar{\mathbb{R}}_+$-**Cat**, where $\Phi X \subset \mathscr{P}X$ is the completion of X under colimits of a class, may be different from the 2-monad T_Φ whose algebras are $\bar{\mathbb{R}}_+$-categories with *chosen* colimits of that class; eg one can build an example with the non-posetal space $X = \{0, 1\}$, where $X(0, 1) = 0 = X(1, 0)$.

Proposition 14.9. *The following conditions on a distance-decreasing function $f\colon A \to B$ between generalised metric spaces are equivalent.*
(1) $Qf\colon QA \to QB$ has a right adjoint retract.
(2) f is an isometry and for each $b \in B$ there exists a Cauchy sequence (x_n) in A such that $B(f-, b) = \lim_n A(-, x_n)$.

Proof: Lemma 13.2 says that Qf has a right adjoint precisely when $B(f-, b) \in \mathscr{P}A$ belongs to QA, which is equivalent to the existence of a Cauchy sequence as in (2). The unit of this adjunction is an identity if and only if f is full and faithful: an isometry. ∎

Corollary 14.10. *The equivalent conditions in Proposition 14.9 are further equivalent to requiring that f be a dense isometry if the space B is symmetric.*

Proof: If the conditions of Proposition 14.9 hold, we obtain $\lim_n B(f(x_n), b) = 0$ for all $b \in B$ and some Cauchy sequence (x_n), and f is dense. To prove the converse, given $b \in B$, there is a Cauchy sequence (x_n) in X such that both $\lim_n B(f(x_n), b)$ and $\lim_n B(b, f(x_n))$ equal 0. Therefore,

$$B(f(a), b) = \lim_n B(f(a), f(x_n)) = \lim_n A(a, x_n)$$

which is one of the equivalent conditions in Proposition 14.9. ∎

Remark 14.11. For a distance-decreasing function $g\colon A \to B$, the space Kg – the comma object $Qg \downarrow y_B$ – has points (ϕ, b) such that $(Qg)(\phi) \geqslant B(-, b)$ in QB. Equivalently, $(Qg(\phi))^*(b) = 0$. By Lemma 14.5, this is equivalent to $\lim_n B(g(x_n), b) = 0$ where (x_n) is any Cauchy sequence of the equivalence class $\mathbf{x}$ associated to ϕ. The distance of Kg is given by the formula

$$Kg((\mathbf{x}, b), (\mathbf{x}', b')) = \max\{\lim_n A(x_n, x'_n), B(b, b')\}.$$

The $\bar{\mathbb{R}}_+$-functor $\lambda_g\colon A \to Kg$ sends a to $(\mathbf{c}_a, g(a))$, where $\mathbf{c}_a$ is the equivalence class of Cauchy sequences that converge to a; ie the equivalence class of the Cauchy sequence that is constant on a. We know from the general theory, but it is also readily verified, that λ_g is an isometry.

Proposition 14.12. *A distance-decreasing function $g\colon C \to D$ is an R-algebra if and only if for each equivalence class $\mathbf{x} = [x_n]$ of Cauchy sequences and each source limit $b \in B$ of $g(x_n)$ there is an element $p(\mathbf{x}, b) \in A$ over b such that for all $a \in A$*

$$A(p(\mathbf{x}, b), a) = \max\{\lim_n A(x_n, a), B(b, g(a))\} \in \bar{\mathbb{R}}_+. \tag{14.1}$$

Proof: One implication is clear, since an R-algebra structure provides the choice of elements. For the converse, suppose that a choice $p(\mathbf{x}, b) \in A$ exists. First observe that there is a choice of elements $p'(\mathbf{x}, b)$ that satisfy the same properties and in addition $p'(\mathbf{c}_a, g(a)) = a$, where $\mathbf{c}_a$ is the sequence constant at $a \in A$. To see this, define $p'(\mathbf{x}, b) = p(\mathbf{x}, b)$ unless $(\mathbf{x}, b) = (\mathbf{c}_a, g(a))$ in which case set $p'(\mathbf{c}_a, g(a)) = a$. To verify that p' satisfies (14.1), observe that $\lim_n A((\mathbf{c}_{\bar{a}})_n, a) = A(\bar{a}, a) \geqslant B(g(\bar{a}), g(a))$. The equality (14.1) says precisely that the chosen elements can be extended to a left adjoint enriched functor $p' \dashv \lambda_g$. Furthermore, $p'(\lambda_g(a)) = a$ by definition, so the inequality $A(a, \bar{a}) \geqslant A(p'\lambda_g(a), p'\lambda_g(\bar{a}))$ is an equality; in other words, $p' \cdot \lambda_g = 1$ as $\bar{\mathbb{R}}_+$-functors. The fact that $gp'(\mathbf{x}, b) = b$ means that $(p', 1)$ is a morphism $Rg \to g$. In order to have a retract adjunction $(p', 1) \dashv (\lambda_g, 1)$, which is, by Lemma 11.5, to say that $(p', 1)$ is an R-algebra structure, it remains to prove that the unit of $p' \dashv \lambda_g$ is sent to the identity by ρ_g, but this is immediate from $\rho_g(\mathbf{x}, b) = b = \rho_g(\lambda_g(p'(\mathbf{x}, b)))$. ∎

Corollary 14.13. *A distance-decreasing function $g\colon A \to B$ into a symmetric generalised metric space B is an* R*-algebra if and only if for each equivalence class $\mathbf{x} = [x_n]$ of Cauchy sequences and each limit $b \in B$ of $g(x_n)$ there is an object $p(\mathbf{x}, b) \in A$ over b such that (x_n) converges to $p(\mathbf{x}, b)$.*

Proof: Observe that a Cauchy sequence (x_n) converges to p precisely when $A(p, a) = \lim_n A(x_n, a)$ for all $a \in A$. One way of seeing this is by applying Lemma 14.7. Together with the fact that the maximum in Proposition 14.12 equals $\lim_n A(x_n, a)$ as a consequence of the symmetry of B, we obtain the result. ∎

If in the corollary above the spaces A and B are metric spaces, g is an R-algebra if the convergent Cauchy sequences in A are precisely those for which their image under g converges in B. This is so since $\lim_n x_n$ in A always satisfy $g(\lim_n x_n) = \lim_n g(x_n)$.

Theorem 14.14. *There is a lax idempotent AWFS* (L, R) *on the 2-category of generalised metric spaces, with underlying WFS $(\mathcal{L}, \mathcal{R})$, that satisfies: (1) A morphism f is an* L*-coalgebra, equivalently $f \in \mathcal{L}$, if and only if f is as in Proposition 14.9. (2) A morphism g is an* R*-algebra, equivalently $g \in \mathcal{R}$, if and only if it is as in Proposition 14.12. (3) The fibrant replacement 2-monad is* Q*, so the fibrant objects are the Cauchy complete generalised metric spaces.*

Proof: The categories $\bar{\mathbb{R}}_+$-**Cat**(X, QY) are posets since QY is a poset. Therefore, the statements (1) and (2) follow from Theorem 11.7. The statement (3) is a consequence of Corollary 9.3. ■

Finally, the AWFS of the previous theorem restricts to the 2-category of metric spaces. The associated WFS on metric spaces has been considered in [23, Examples 4.18] while studying Lawvere completeness as outlined in [5].

Corollary 14.15. *The AWFS of Theorem 14.14 restricts to the 2-category of symmetric generalised metric spaces – pseudometric spaces. The left class consists of dense isometries. The right class consists of those $g\colon A \to B$ satisfying the following condition: for all Cauchy sequence (x_n) in A such that $(g(x_n))$ converges to b, there exists a limit of (x_n) that lies over b. The AWFS also restricts to the 2-category of metric spaces, where it becomes an orthogonal factorisation system.*

Proof: If A is symmetric so it is QA, by, for example, Proposition 14.3. For the same reason, QA is metric when A is so; QA is the classical Cauchy completion of A. Then, Kg is symmetric for any $g\colon C \to D$ between symmetric spaces; see Remark 14.11. Therefore, the factorisation of a morphism between symmetric spaces is through a symmetric space. Corollaries 14.10 and 14.13, together with the fact that the underlying preorder of a metric space is discrete, conclude the proof. ■

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Maria Manuel Clementino
CMUC, Department of Mathematics, University of Coimbra, 3001–501 Coimbra, Portugal
E-mail address: `mmc@mat.uc.pt`

Ignacio López Franco
Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB2 0WB, UK
E-mail address: `il120@cam.ac.uk`