Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 15–16

A MODIFIED LYZENGA'S MODEL FOR MULTISPECTRAL BATHYMETRY USING TIKHONOV REGULARIZATION

ISABEL NARRA FIGUEIREDO, LUÍS PINTO AND GIL GONÇALVES

ABSTRACT: The derivation of shallow-water bathymetry from multispectral satellite images has become a highly active field of research in recent years. Nowadays, as satellite images become more and more freely available worldwide and easily accessible, this type of technique is a cost-effective surrogate for the derivation of bathymetric information, about even the most remote areas. In fact, traditional bathymetric methods, such as acoustic and LIDAR (LIght Detection And Ranging) systems, are still very expensive and difficult to operate. Among all the models that have been presented in the literature for multispectral bathymetry, the log-linear inversion model proposed by Lyzenga is still the most popular one, due to its simplicity and physically intuitive nature. But it is well known that it has a relatively low accuracy and the optical uniformity assumption is unrealistic. We propose a modified Lyzenga's model that can account for spatial heterogeneity. This is particularly important when the imaged area corresponds to heterogeneous bottom types and varying water quality. The estimation of the bathymetric parameters is performed by solving an inverse problem with a Tikhonov-like regularization term. We test the proposed model with satellite *Landsat 8* multispectral images and *in-situ* depth measurements of a shallow water site. The results obtained indicate that the new model is more accurate, with negligible extra complexity.

KEYWORDS: Bathymetry, *Landsat 8*, Lyzenga's method, Tikhonov regularization.

Reliable and constantly updated shallow waters bathymetric information is extremely important for a wide range of applications including maritime and river transport, management of the traditionally densely populated coastal areas, and protection of vulnerable ecosystems like coral reefs and lagoons [1, 2, 4]. Classical methods for shallow waters bathymetric mapping include ship mounted acoustic-based sensors, e.g. multibeam echo sounders [5], and the LIDAR airborne remote sensing technique, which is based on laser light sensors [8]. With the recent advances in these technologies, it is now possible to perform bathymetric surveys with high level of accuracy. For instance, under

Received April 15, 2015.
This work was partially supported by the project PTDC/MATNAN/0593/2012 funded by FCT (Portuguese national funding agency for science, research and technology), and also by CMUC (Center for Mathematics, University of Coimbra) and FCT, through European program COMPETE/ FEDER and projects PEst-C/MAT/UI0324/2013 and PEst-OE/EEI/UI308/2014. We would like to acknowledge the *Instituto Hidrográfico de Portugal* for providing the depth data.

optimal clear water and weather conditions, depth measurements of up to 50 m and 15 cm accuracy are commonly reported for LIDAR systems [6]. The main limitations associated with LIDAR are the expensive operational cost, the time consuming and complex procedures, and limitation to relatively small areas, due to the airborne nature. Furthermore, another issue occurs with extremely shallow waters of less than 2 m. In fact, because of the physical processes governing LIDAR, measurements for this depth range are problematic and often not possible [15]. In this scenario, ship-acoustic techniques, are not a viable alternative, because of the obvious navigation problems. More generally, acoustic-based systems have essential the same strengths and weaknesses of LIDAR systems, namely, they are accurate, but costly and complex to deploy [3].

Since the early development of satellite technology in the 60's, the production of shallow water bathymetric charts from multispectral satellite data has been presented as a possible alternative to the conventional methods. Nowadays, the increasing availability of inexpensive satellite imagery, in parallel with the development of high resolution multispectral and hyperspectral sensors, has led to a renewed interest in this topic. The potential of this technique is now well established and documented in the research literature. Namely, satellite bathymetry is highly attractive as a fast and cost-effective alternative when compared with LIDAR or acoustic sensors. The non-intrusive nature and worldwide availability of satellite imagery make it suitable for sensitive and remote ecosystems, and coastal areas with intensive maritime activities.

Among the methods proposed, the one that has received the most attention was the one popularized by Lyzenga [12, 13]. Basically, this physically based method, relies in a number of "true depth" measurements, obtained for instance by LIDAR, to calibrate the model parameters, and afterwards, estimate the water depth for pixels with unknown depths using satellite radiance data. This method has been applied by many authors for different types of environments and using data from several different satellites including *Landsat* [12], *QuickBird* [9], *IKONOS* [7, 16], and *WorldView* [10]. Although all the research, some problems still remain, such as lower accuracy compared to traditional bathymetric methods, dependence on "true depth" data, and the impact of water and atmospheric conditions.

During the years, some modifications and improvements have been made to the original method. We refer for instance to the non-linear model proposed by Stumpf *et al.* [16] for multispectral data, and the 3 extensions developed by Kanno *et al.* in [9]. Usually, these modifications try to address unrealistic assumptions of the original model with regard to the uniformity of the optical properties of the area of interest. With respect to the non-linear model of Stumpf *et al.*, although some improvements were reported in accuracy, the results are not consistent. For example, in [7], the authors found virtually no difference between the models and conclude that is still uncertain whether or not this model is superior to the conventional one. Moreover, the calibration of the non-linear model is somehow more involved. On the other hand, the models of Kanno *et al.* [9] have yet to be properly tested, the number of results is still very limited. Possibly, this lack of experimental studies, is related to the difficulties in implementation of these methods, as mentioned by the authors.

We propose a new method for bathymetry estimation from multispectral satellite images. It is based on a simple and straightforward modification of the original method of Lyzenga, and it arises with the introduction of spatial dependent coefficients. We expect this modification to make the suggested model less vulnerable to optical heterogeneities (of bottom material and water) and even atmospheric and sunglint perturbations. After this introduction, in Section 1, we start by briefly describing the method of Lyzenga, then we present our model as well as the rationale behind it. In Section 2, using *Landsat 8* data collected over the coastal region of Lisbon, Portugal, and "true depth" measurements, we report and discuss the performance of the proposed model. A comparison between the results of both models is also given. Finally, in Section 3, we draw some conclusions and outline some future research directions.

1. Proposed Model

The bathymetric inversion model derived by Lyzenga [13, 12] for two or multiple spectral bands is given by:

$$z = a_0 + \sum_{i=1}^{N} a_i \ln[L(\lambda_i) - L_\infty(\lambda_i)], \tag{1}$$

where z is the depth, a_0 and a_i are constant coefficients, N is the number of spectral bands, λ_i is the ith spectral band, $L(\lambda_i)$ is the top-of-atmosphere spectral radiance (after atmospheric and sunglint corrections) for band λ_i, and $L_\infty(\lambda_i)$ is the deep water radiance for band λ_i. Model (1), to which we refer as the Lyzenga's model, is a physically based method that can be derived from a basic radiative transfer model for shallow waters. Essentially, it is based on Beer's law for the absorption of light in an optical medium. This law states that the intensity of light passing through a medium decreases exponentially with the thickness of the medium. In this context, assuming that the intensity of light reflected by a water column of thickness z, as measured by the satellite, is given by $L(\lambda_i) - L_\infty(\lambda_i)$, we can write the Beer's law as:

$$\ln[L(\lambda_i) - L_\infty(\lambda_i)] = -g(\lambda_i)z + c(\lambda_i). \tag{2}$$

Here, $g(\lambda_i)$ is the effective absorption coefficient of the ith band and $c(\lambda_i)$ is a constant that takes into account several transmittance, reflectance, and irradiance effects, associated for instance with the state of the atmosphere, and the surface and bottom water type [13, 10, 11, 7]. Following [12, 13], the Lyzenga's model (1) can be obtained from (2) assuming that

$$\sum_{i=0}^{N} a_i g(\lambda_i) = -1 \quad \text{and} \quad a_0 + \sum_{i=1}^{N} a_i c(\lambda_i) = 0. \tag{3}$$

However, in practice, it is not feasible to obtain from (3) the model parameters a_i, $i = 0, \ldots, N$. Therefore, in general a least-squares method is applied to estimate the values of the unknown coefficients a_i using pixels with known depth. This is an inverse problem, that is formulated as follows:

$$\min_{a_0,a_1,a_2} \sum_{m=1}^{M} \left[a_0 + \sum_{i=1}^{N} a_i \ln[L(\lambda_i)_m - L_\infty(\lambda_i)] - z_m \right]^2 . \tag{4}$$

Here $L(\lambda_i)_m$ is the spectral value at pixel m, z_m is the measured depth at pixel m and M is the total number of pixels with measured depth. After estimating the values of a_0, a_1, and a_2, by solving (4), the unknown depths in the area under study can be predicted from (1).

It has been proved in [13] that Lyzenga's model can account, to some extent, for variations on optical properties of water and bottom surface, but in fact, the assumption that optical properties do not vary spatially in the area under analysis, is still essential for deriving this model. Since this strong assumption is not valid in many cases, we can expect significant errors in the model predictions. Water turbidity, suspended or dissolved materials in water, and complex and heterogeneous water bottom, are common factors which cause the discrepancy between model predictions and real depth measurements.

Being aware of this severe limitation, we propose herein a novel method for the remote sensing bathymetry problem based on Lyzenga's model and Tikhonov-regularization theory.

Firstly, in order to overcome the homogeneity assumption, we consider spatially dependent coefficients in (1). The modified model is then defined by

$$z = A_0 + \sum_{i=1}^{N} A_i \ln[L(\lambda_i) - L_\infty(\lambda_i)], \tag{5}$$

where A_i, $i = 0, \dots, N$ are matrices (whose components are constants) and with the same size of the image bands. We expect these new matrix coefficients A_i to be able to capture the spatial variation of the optical properties.

Secondly, we estimated the unknown parameters A_i by solving the following inverse problem with Tikhonov-regularization [17] (compare with (4))

$$\min_{A_0^M,\dots,A_N^M} \left[F(A_0^M, \dots, A_N^M) + \frac{1}{2}\sum_{i=0}^{N} \alpha_i J(A_i^M)^2 \right], \tag{6}$$

where the fitting term $F(A_0^M, \dots, A_N^M)$ is defined by

$$\sum_{m=1}^{M} \left[A_{0,m}^M + \sum_{i=1}^{N} A_{i,m}^M \ln[L(\lambda_i)_m - L_\infty(\lambda_i)] - z_m \right]^2,$$

and $J(\cdot)$ is the l_2-regularization term, defined by

$$J(A_i^M) := \left[\sum_{i=1}^{N} \sum_{m=1}^{M} (A_{i,m}^M)^2 \right]^{1/2}. \tag{7}$$

Moreover, α_i is a positive constant, that balances the influence of the similarity and regularity terms in the cost functional of the optimization problem (6).

We remark that notation $A_{i,m}^M$ represents the value of the component of matrix A_i at pixel m, while the superscript M is used to emphasize that A_i^M is a submatrix of A_i with M components. In general M is much smaller than the size of A_i, thus after solving (6) there are still unknown coefficients; that is, from (6), we can only estimate the M components of A_i, corresponding to the location were the depth is known and measured. The remaining coefficients of A_i have to be obtained by interpolation. In this paper, we have used linear interpolation, whenever required.

The regularization term (7) has been commonly used in least squares model fitting. The motivation for this term is to avoid over-fitting and improve the model's prediction accuracy at the cost of introducing some bias [17]. The l_2-regularization term penalizes large coefficients and shrinks the fitted coefficients A_i^M towards zero as the regularize parameter α_i increases.

We note that in order to reduce the complexity of model (5), we can consider there only one matrix coefficient and assume that the remaining ones are scalars. In fact, in the experiments we have done, this modification did not seem to affect the overall quality and performance of the model. This simplification can be justified by the hypothesis that only one spatially varying coefficient is sufficient to capture the optical heterogeneities. Moreover, in the experiments described in Section 2, only bands λ_1 (Coastal Aerosol band) and λ_2 (Blue band) were used. The remaining bands were not considered because their contribution was found to be not significant. This is an expected behavior, since these bands have wavelengths longer than 500 nm which attenuate rapidly in water. This same explanation, can be applied to account for the slightly worse results when A_2 is consider to be a matrix, while A_1 is considered a scalar. In fact, the short-wavelength of band λ_1 (430 to 450 nm), allows a deeper penetration depth into water, making it the optimal spectral band for bathymetric measurement.

Taking into account all these simplifications, the modified model (5) we consider hereafter is:

$$z = a_0 + A_1 \ln[L(\lambda_1) - L_\infty(\lambda_1)] + a_2 \ln[L(\lambda_2) - L_\infty(\lambda_2)], \tag{8}$$

with a_0, a_2 scalars and A_1 a matrix. This simplified model leads to the following version of (6):

$$\min_{a_0, A_1^M, a_2} F(a_0, A_1^M, a_2) + \frac{\alpha}{2} J(A_1^M)^2, \tag{9}$$

with the fitting term

$$\begin{aligned} F(a_0, A_1^M, a_2) = \sum_{m=1}^{M} \big[a_0 + A_{1,m}^M \ln[L(\lambda_1)_m - L_\infty(\lambda_1)] \\ + a_2 \ln[L(\lambda_2)_m - L_\infty(\lambda_2)] - z_m\big]^2. \end{aligned}$$

2. Application

2.1. Study Area and Data. The bathymetric models described in the previous section were applied to the coastal area of Lisbon, Portugal. The depth information was obtained from the Portuguese Hydrographic Institute (*IH-Instituto Hidrográfico de Portugal*). It was compiled in 2010 from several *IH* soundings and interpolated to a one minute grid using a TIN interpolator. For the experimental study, a standard multispectral and cloud free *Landsat 8* image, acquired in 2013-09-08, was also downloaded (scene id = LC2040332013251LGN00). The RGB true color composition image of the test site is shown in Figure 1. The current *Landsat 8* ground spatial resolution is of about 30 m. The spectral range, of the bands of interest, is 430-450 nm for band 1 (Coastal Aerosol), and 450-510 nm for band 2 (Blue).

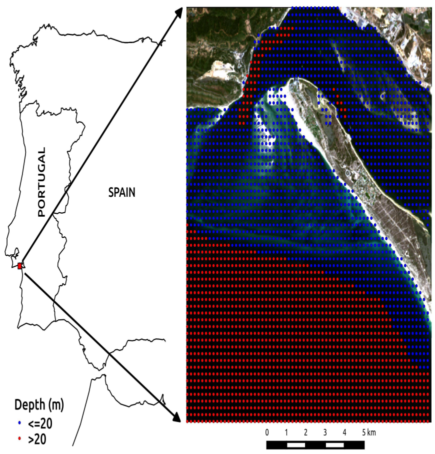

FIGURE 1. Left - location of the study area near Lisbon, Portugal. Right - *Landsat 8* true color RGB composite of the area and spatial distribution of known depth points.

In order to apply the proposed and the original Lyzenga's model, we need to estimate the values of deep water radiance $L_\infty(\lambda_i)$, $i = 1, 2$. For that, we consider the deep water pixels, i.e., the pixels with known depth bigger than 20 m, and take $L_\infty(\lambda_i)$ as the minimum value of remote sensing radiance $L(\lambda_i)$ among them. We note that in this paper we ignore the possible effects of sunglint or sun and satellite elevations. While it is recommend to apply corrections to the data to compensate these effects, it should also be recognized that this is not a mandatory procedure. For instance, when the atmosphere is homogeneous, this step can be skipped without compromising the results [14]. Furthermore, we also expect that the spatial coefficient of our model will totally or partially eliminate the need for such corrections.

Table 1. Depth statistics of the study area.

Depth [m]	Min [m]	Mean [m]	Max [m]	Number of pixels
≤ 20	0.1	6.873	19.9	1748
> 20	20.1	47.072	84.6	1474
Overall	0.1	25.263	84.6	3222

It is well known that the applicability of Lyzenga's type models, and more generally of multispectral bathymetry, is limited to shallow waters. Therefore, as displayed in Figure 1, the measured depths were separated into two groups using a depth of 20 m as the dividing line. In Table 1, we also present some statistical properties of the depth data.

2.2. Results and Discussion. We start the discussion of our model with a synthetic example. With this example, we intend to show the denoising effect of the regularization term in (9) as well as a comparison with Lyzenga's model (1) with two spectral bands. For that, we generate a synthetic depth function z using equation (8), by assigning prescribed values to all the parameters in the right hand side of (8). To simulate the errors in remote sensing images, that can occur, *e.g.*, as a result of atmospheric variations, we add Gaussian noise to both $L(\lambda_1)$ and $L(\lambda_2)$. The results, obtained by using 10% of the noise-free depth points to solve the corresponding minimization problem (9), are shown in Figure 2. Not only the visual appearance of the

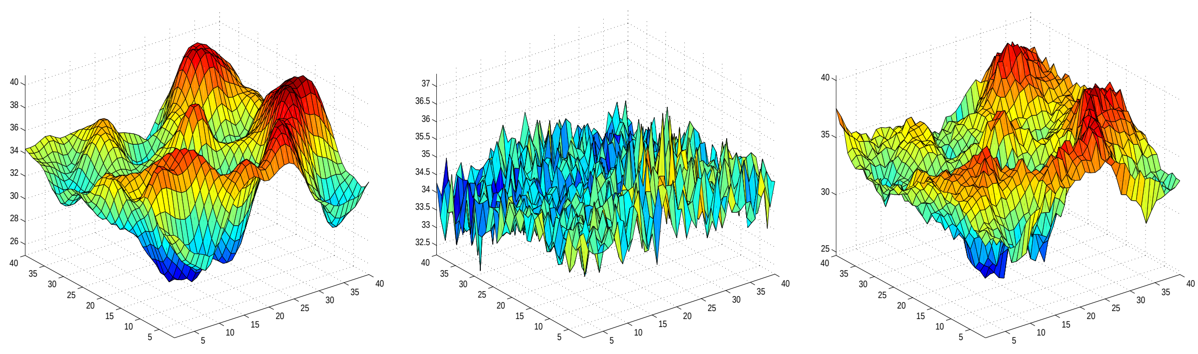

Figure 2. From left to right: synthetic noise-free depth z, recovered depth z Lyzenga's model (1), and recovered depth z with proposed model ($\alpha = 1$).

recovered depth z but also the Euclidian norm error are 125.943 and 39.981 for (1) and (9), respectively, confirm the advantage of the new model.

In order to further test the efficiency of the proposed model, we performed a cross-validation experiment using as data set the 1748 points of shallow water of the area exhibited in Figure 1. In each round, we randomly selected 10% of depth measured points, as training data, and used the remaining 90% as testing data. This cross-validation procedure was repeated 500 times. For each run, we evaluated the model prediction accuracy using the root mean square error (RMSE), defined as

$$\mathrm{RMSE} = \left[\frac{1}{N_p} \sum_{n=1}^{N_p} (\bar{z}_n - z_n)^2 \right]^{1/2}, \tag{10}$$

where N_p is the number of points in the testing set, z_n the measured depth at point n, and $\bar{z}_n$ the depth predicted by the model. One example of training and testing data used in the cross-validation is depicted in Figure 3. All the minimization problems, were solved using the Optimization Toolbox of the software MATLAB® R2013b (The Mathworks, Inc.)

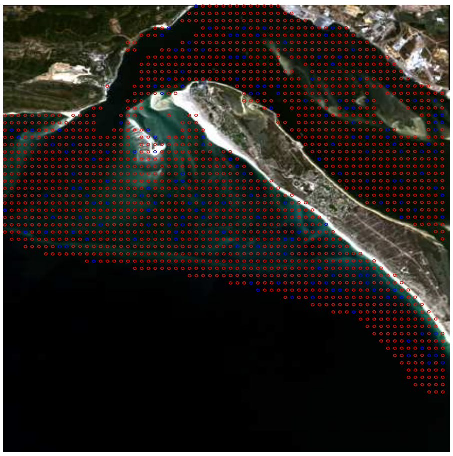

FIGURE 3. Typical distribution of the testing data (red circles) and the training data (blue circles) used in the cross-validation procedure.

We start our discussion of the results by examining the behavior of the regularization parameter α. In order to do so, we carried out experiments varying α from 0 (no regularization) to 20. The results are given in Figure 4, where each point is the average of 500 repetitions. We also present, for comparison, in the same figure, the RMSE obtained with the original Lyzenga's model. The small fluctuations observed in the RMSE of this model are only the consequence of the random nature of the cross-validation.

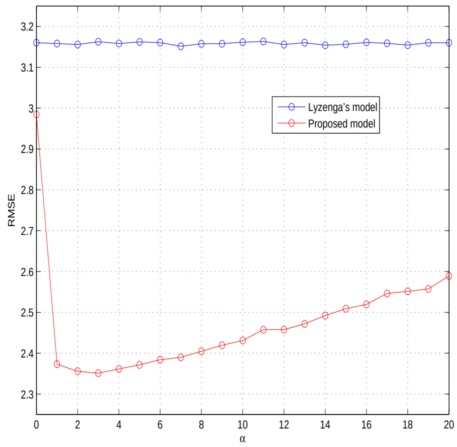

FIGURE 4. In red, impact of the regularization parameter α on the accuracy of the proposed model. In blue, results obtained with Lyzenga's model.

The analysis of Figure 4 reveals that without regularization the RMSE is much larger, which is a typical consequence of using an overparameterized model. An overparameterized model tends to overfit the training data, fitting noise and outliers, and thus is more likely to give poor prediction. It is also observed that as the value of the regularization parameter increases, the accuracy of the proposed model decreases. This is caused by the fact that when α is too large (over-regularization) the parameter A_1 tends to 0, preventing the model from predicting the data satisfactorily. According to Figure 4, we can say that the optimal value for the regularization parameter is between 1 and 7. For this range of values the RMSE fluctuates smoothly between 2.373 m and 2.390 m. In particular, the minimum value of RMSE is achieved when $\alpha = 3$ and corresponds to 2.351m. From the results presented, it is also clear the better performance of the proposed model. In fact, we observe that we smallest RMSE for Lyzenga's model is 3.151 m. This means that the proposed model gives a reduction in RMSE of 0.8 m. Note that we also implemented an l_1 regularized version of our model, by defining $J(A_1^M) = |A_1^M|^{1/2}$ in (9), but no improvement was obtained.

To gain more insight about the performance of both models, we present in Table 2, the RMSE obtained from four different water depth levels. As before, the values reported are averages over 500 runs of the cross-validation procedure. For this experiment, the parameter α was fixed at 3. From Table 2, we can conclude that the proposed model outperforms Lyzenga's model at all the levels, as the RMSE is always considerably lower. However, this is particularly true in the range 15-20 m, where the new model improves the accuracy by almost 2 m. As expected, we can also see that prediction RMSE of both methods increases with depth.

Table 2. RMSE of Lyzenga's and proposed model for different water depth ranges.

Model	0-5m	5-10m	10-15m	15-20m	Overall
Lyzenga's	2.604	2.670	3.773	5.240	3.164
Proposed	2.005	2.377	2.503	3.373	2.356

Finally, to further test the robustness of the new method, we repeated the cross-validation procedure, but reducing the size of the training set to only 5% of the total data. In this experiment, the minimum RMSE was 3.211 m for Lyzenga's model, and 2.718 m for the new model (obtained for $\alpha = 7$). Not surprisingly, the RMSE increased for both methods, particularly for the proposed model. Nevertheless, the relative improvement of our model is still substantial, around 0.5 m. Let us remark that, as illustrated by Figure 3, such percentage of points can already be considered a very sparse sample. We also observe that the results were again similar to the ones given in Figure 4, and therefore, they confirm the importance of the regularization term and the robustness of the method in relation to the regularization parameter, since different values of α lead to almost the same RMSE.

3. Conclusion

Our results demonstrate that the modified model, with spatial variable coefficients, is clearly more effective than the original Lyzenga's model, given that the observed RMSE is significantly lower. The new model is simple and easy to apply, since the calibration of the bathymetric parameters can be done using standard least-squares fitting with a Tikhonov regularization term, and it requires exactly the same information as Lyzenga's model. That is, no additional information about optical properties are needed. Also, the proposed model is not highly sensitive to the choice of the regularization parameter, and is quite robust with respect to the training set used. Despite the promising results, these findings need to be confirmed with other data sets. In future work, we intend to address this issue, as well as to study other regularization schemes, which could improve even further the efficiency of the proposed model.

References

[1] S. Andréfouët, Kramer. P., D. Torres-Pulliza, K. E. Joyce, E. J. Hochberg, R. Garza-Pérez, Mumby. P. J., B. Riegl, H. Yamano, W. H. White, M. Zubia, J. C. Brock, S. R. Phinn, A. Naseer, B. G. Hatcher, and F. E. Muller-Karger. Multi-site evaluation of IKONOS data for classification of tropical coral reef environments. *Remote Sensing of Environment*, 88(1-2):128–143, 2013.

[2] O. Andrejev, T. Soomere, and K. Myrberg. The role of the spatial resolution of a three-dimensional hydrodynamic model for marine transport risk assessment. *Oceanologia*, 53(1):309–334, 2011.

[3] J. S. Bird and G. K. Mullins. Analysis of Swath Bathymetry Sonar Accuracy. *IEEE Journal of Oceanic Engineering*, 30(2):372–390, 2005.

[4] M. Davidson, M. V. Koningsveld, A. Kruif, J. Rawson, R. Holman, A. Lamberti, R. Medina, A. Kroon, and S. Aarninkhof. The CoastView project: Developing video-derived Coastal State Indicators in support of coastal zone management. *Coastal Engineering*, 54(6-7):463–475, 2007.

[5] H. K. Farr. Multibeam bathymetric sonar: sea beam and hydro chart. *Marine Geodesy*, 4(2):77–93, 1980.

[6] J. Gao. Bathymetric mapping by means of remote sensing: methods, accuracy and limitations. *Progress in Physical Geography*, 3(1):103–116, 2009.

[7] Su. H, H. Liu, and D. W. Heymana. Automated derivation of bathymetric information from multi-spectral satellite imagery using a non-linear inversion model. *Marine Geodesy*, 31(4):281–298, 2008.

[8] J. L. Irish and W. J. Lillycrop. Scanning laser mapping of the coastal zone: the SHOALS system. *ISPRS Journal of Photogrammetry and Remote Sensing*, 54(2-3):123–129, 1999.

[9] A. Kanno, Y. Koibuchi, and M. Isobe. Shallow water bathymetry from multispectral satellite images: Extensions of lyzenga's method for improving accuracy. *Coastal Engineering Journal*, 53(4):431–450, 2011.

[10] A. Kanno and Y. Tanaka. Modified Lyzenga's method for estimating generalized coefficients of satellite-based predictor of shallow water depth. *IEEE Geoscience and Remote Sensing Letters*, 9(4):715–719, 2012.

[11] A. Kanno, Y. Tanaka, A. Kurosawa, and M. Sekine. Generalized Lyzenga's predictor of shallow water depth for multispectral satellite imagery. *Marine Geodesy*, 36(4):365–376, 2013.

[12] D. R Lyzenga. Passive remote sensing techniques for mapping water depth and bottom features. *Applied Optics*, 17(3):379–383, 1978.

[13] D. R. Lyzenga, N. P. Malinas, and F. J. Tanis. Multispectral bathymetry using a simple physically based algorithm. *IEEE Transactions on Geoscience and Remote Sensing*, 44(8):2251–2259, 2006.

[14] A. Minghelli-Roman, A. Goreac, S. Mathieu, M. Spigai, and P. Gouton. Comparison of bathymetric estimation using different satellite images in coastal sea waters. *International Journal of Remote Sensing*, 30(21):5737–5750, 2009.

[15] S. Pe'eri and W. Philpot. Increasing the Existence of Very Shallow-Water LIDAR Measurements Using the Red-Channel Waveforms. *IEEE Transactions on Geoscience and Remote Sensing*, 45(5):1217–1223, 2007.
[16] Holderied K. Stumpf, R. P and M. Sinclair. Determination of water depth with high-resolution satellite imagery over variable bottom types. *Limnology and Oceanography*, 48(1):547–556, 2003.
[17] C. R. Vogel. *Computational methods for inverse problems*. SIAM, Philadelphia, USA, 2002.

Isabel Narra Figueiredo
CMUC, Dep. of Mathematics, University of Coimbra, Portugal
E-mail address: isabelf@math.uc.ut
URL: http://www.mat.uc.pt/∼isabelf/

Luís Pinto
CMUC, Dep. of Mathematics, University of Coimbra, Portugal
E-mail address: luisp@mat.uc.pt

Gil Gonçalves
INESCC and Dep. of Mathematics, University of Coimbra, Portugal
E-mail address: gil@mat.uc.pt
URL: http://www.mat.uc.pt/∼gil/