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CONVERGENCE RATES FOR WEIGHTED SUMS OF ASSOCIATED RANDOM VARIABLES AND A MARCINKIEWICZ-ZYGMUND LAW

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Abstract: We study the almost sure convergence and rates of weighted sums of associated random variables under the classical assumption of existence of Laplace transforms. This assumption implies the existence of every moment, so we address the same problem assuming a suitable decrease rate on tail joint probabilities which only implies the existence of finitely many moments, proving the analogous characterizations of convergence and rates. Still relaxing further the assumptions on moment existence, we also prove a Marcinkiewicz-Zygmund for associated variables without means, complementing existing results for this dependence structure.

Keywords: almost sure convergence; association; convergence rates; exponential inequalities; Marcinkiewicz-Zygmund law.
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1. Introduction

Many linear statistics are written as weighted sums of random variables, raising thus the interest in the characterization of the asymptotics of such sums conveniently normalized. Since Baum and Katz [2] proved an almost sure result for constant weights with a normalization sequence $n^{-1/p}$, where p describes the moment condition on the variables, many authors studied this problem. Chow [5] and Cuzick [6] obtained conditions for the convergence for weighted sums with independent variables, later extended by Cheng [4], Bai and Cheng [1], or Sung [13] relaxing the moment assumption. This convergence has also been considered for dependent variables. Louhichi [9] obtained sufficient conditions for the convergence with constant weights but requiring only the existence of low order, less than 2, moments. These results were, more recently, extended for weighted sums in Oliveira [11] and Çağın and Oliveira [3], using an approach similar to Louhichi's [9]. Here we follow

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the method used in Ioannides and Roussas [8] and Oliveira [10] for the proof of exponential inequalities, to prove conditions for the almost sure convergence and of its rate. These conditions depend on the covariances, thus require the existence of moments of order at least 2, and link p with the behaviour of the weighting coefficients. We try to avoid the classical assumption of existence of Laplace transforms, as this implies the finiteness of moments of every order, replacing this by a polynomial decrease rate on the tail joint probabilities, which implies only the existence of finitely many moments.

Finally, for variables without means, we prove a Marcinkiewicz-Zygmund strong law that complements earlier results by Louhichi [9], Oliveira [11] and Çağın and Oliveira [3]. Again, assuming a suitable decrease rate on the tail joint probabilities, we find a simple version of this strong law.

The paper is organized as follows. Section 2 describes the framework and useful preliminary results, Section 3 describes the conditions for the almost sure convergence of weighted sums and its rates for bounded variables. This is a tool for the results considered in Section 4 and 5 that study the cases of variables with infinitely or only finitely many moments, extending the characterizations of almost sure convergence and its rates. Finally, Section 6 proves a Marcinkiewicz-Zygmund law for associated variables without means.

2. Definitions and preliminary results

Let us assume that the X_n, $n \geq 1$, are centered and associated random variables and denote $S_n = X_1 + \cdots + X_n$. Let $a_{n,i}$, $i = 1, \ldots, n$, $n \geq 1$, be nonnegative real numbers and define, for some $\alpha > 1$, $A_{n,\alpha}^{\alpha} = n^{-1} \sum_{i=1}^{n} |a_{n,i}|^{\alpha}$. Except in Section 6, where a simpler weighting will be considered, we will be interested in the convergence of $T_n = \sum_{i=}^{n} a_{n,i} X_i$ assuming that

$$A_{\alpha} = \sup_n A_{n,\alpha} < \infty. \tag{1}$$

This relaxes the assumption on the weights when compared to Oliveira [11] or Çağın and Oliveira [3]. However, in Section 5 we will need to strengthen this assumption on the weights, as done in [11, 3]. Remark that, due to the nonnegativity of the weights, the variables T_n, $n \geq 1$, are associated. Define the usual Cox-Grimmett coefficients

$$u(n) = \sup_{k \geq 1} \sum_{j:|k-j| \geq n} \mathrm{Cov}(X_j, X_k). \tag{2}$$

If the random variables are stationary, then $u(n) = 2\sum_{j=n+1}^{\infty} \mathrm{Cov}(X_1, X_j)$. As moments and, especially, covariances will play a significant role throughout the paper, let us define, for each $i, j \geq 1$,

$$\Delta_{i,j}(u,v) = \mathrm{P}(X_1 > u, X_j > v) - \mathrm{P}(X_1 > u)\mathrm{P}(X_j > v), \quad u, v \in \mathbb{R}, \tag{3}$$

and, for $x\,y \geq 0$,

$$G_{i,j}(x,y) = \int_{-x}^{x}\int_{-y}^{y} \Delta_{i,j}(u,v)\, dudv.$$

It is obvious that $\mathrm{Cov}(X_i, X_j) = G_{i,j}(+\infty, +\infty)$. Moreover, remark that due to the association $\Delta_{i,j}$ is nonnegative, which means that $G_{i,j}$ is nondecreasing in each variable.

Consider p_n a sequence of natural numbers such that $p_n < \frac{n}{2}$, r_n the largest integer less or equal to $\frac{n}{2p_n}$, and define the variables

$$Y_{n,j} = \sum_{i=(j-1)p_n+1}^{jp_n} a_{n,i} X_i, \qquad j = 1, \ldots, 2r_n.$$

These random variables are associated, due to the nonnegativity of the weights. Moreover, if the variables X_n are uniformly bounded by $c > 0$, then it is obvious that $|Y_{n,j}| \leq c A_\alpha n^{1/\alpha} p_n$. Finally, put

$$T_{n,od} = \sum_{j=1}^{r_n} Y_{n,2j-1} \quad \text{and} \quad T_{n,ev} = \sum_{j=1}^{r_n} Y_{n,2j}.$$

We prove first an easy but useful upper bound.

Lemma 2.1. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary, bounded (by $c > 0$) and $u(0) < \infty$. Then $\mathrm{E}S_n^2 \leq 2c^* n$, where $c^* = c^2 + u(0)$.*

Proof: As the variables are stationary, it follows easily that $\mathrm{E}S_n^2 = n\mathrm{Var}(X_1) + 2\sum_{j=1}^{n-1}(n-j)\mathrm{Cov}(X_1, X_{j+1}) \leq 2nc^2 + 2nu(0)$. ■

The next result is an extension of Lemma 3.1 in Oliveira [10].

Lemma 2.2. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary, bounded (by $c > 0$), $u(0) < \infty$ and the nonnegative weights satisfy (1). If $d_n \geq 1$ and $0 < \lambda < \frac{d_n-1}{d_n}\frac{1}{cA_\alpha n^{1/\alpha}p_n}$, then*

$$\prod_{j=1}^{r_n} \mathrm{E}e^{\lambda Y_{n,2j-1}} \leq \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n\right)$$

and

$$\prod_{j=1}^{r_n} \mathrm{E}e^{\lambda Y_{n,2j}} \leq \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n\right).$$

Proof: As remarked above, the variables X_n being bounded, we have $|Y_{n,j}| \leq cA_\alpha n^{1/\alpha}p_n$. So, using a Taylor expansion it follows that

$$\mathrm{E}e^{\lambda Y_{n,2j-1}} \leq 1 + \lambda^2 \mathrm{E}Y_{n,2j-1}^2 \sum_{k=2}^{+\infty} (cA_\alpha \lambda n^{1/\alpha} p_n)^{k-2}.$$

Now, $\mathrm{E}Y_{n,2j-1}^2 = \sum_{\ell,\ell'} a_{n,\ell}a_{n,\ell'}\mathrm{Cov}(X_\ell, X_{\ell'}) \leq n^{2/\alpha}A_\alpha^2 \mathrm{E}S_{p_n}^2$, due to the stationarity and the nonnegativity of the weights and covariances. So, applying Lemma 2.1, it follows that

$$\mathrm{E}e^{\lambda Y_{n,2j-1}} \leq 1 + \frac{2\lambda^2 c^* A_\alpha^2 n^{2/\alpha} p_n}{1 - cA_\alpha \lambda n^{1/\alpha} p_n} \leq \exp\left(2\lambda^2 c^* A_\alpha^2 n^{2/\alpha} p_n d_n\right).$$

To conclude, multiply the upper bounds and remember that $2r_n p_n \leq n$. ∎

A basic tool for the analysis of the convergence and its rates is the following inequality due to Dewan and Prakasa Rao [7].

Theorem 2.3. *Assume $X_1, \ldots, X_n$ are centered, associated and uniformly bounded (by $c > 0$). Then, for every $\lambda > 0$,*

$$\left| \mathrm{E}e^{\lambda \sum_j X_j} - \prod_j \mathrm{E}e^{\lambda X_j} \right| \leq \frac{1}{2}\lambda^2 e^{c\lambda n} \sum_{j \neq k} \mathrm{Cov}(X_j, X_k). \tag{4}$$

3. The case of uniformly bounded variables

We assume first that there exists some $c > 0$ such that, with probability 1, $|X_n| \leq c$, for every $n \geq 1$. This allows for a direct use of the results proved above. We start by deriving an upper bound for the tail probabilities for $T_{n,od}$ and $T_{n,ev}$.

Lemma 3.1. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary, bounded (by $c > 0$) and $u(0) < \infty$. If the nonnegative weights satisfy (1), $d_n \geq 1$ and $0 < \lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$, then, for every $\varepsilon > 0$ and n large enough,*

$$\begin{aligned} \mathrm{P}(T_{n,od} > n^{1/p}\varepsilon) \ \leq \ & \frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon\right) u(p_n) \\ & + \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right). \end{aligned} \tag{5}$$

An analogous inequality for $\mathrm{P}(T_{n,ev} > n^{1/p}\varepsilon)$ also holds.

Proof: If we apply (4) to $T_{n,od}$ we find

$$\left| \mathrm{E}e^{\lambda T_{n,od}} - \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}} \right| \leq \frac{1}{2}\lambda^2 \exp\left(cA_\alpha r_n p_n n^{1/\alpha}\lambda\right) \sum_{j \neq j'} \mathrm{Cov}(Y_{n,j}, Y_{n,j'}). \tag{6}$$

Now, as, for $n \geq 1$ and $i \leq n$, it holds that $0 \leq a_{n,i} \leq n^{1/\alpha} A_{n,\alpha}$, we have

$$\mathrm{Cov}(Y_{n,j}, Y_{n,j'}) \leq \sum_{\ell,\ell'} a_{n,\ell} a_{n,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}) \leq n^{2/\alpha} A_\alpha^2 \sum_{\ell,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}).$$

Put $Y_{n,j}^* = \sum_{\ell=(j-1)p_n+1}^{jp_n} X_\ell$, $j = 1, \ldots, r_n$. Then, the previous inequality rewrites as

$$\mathrm{Cov}(Y_{n,j}, Y_{n,j'}) \leq n^{2/\alpha} A_\alpha^2 \mathrm{Cov}(Y_{n,j}^*, Y_{n,j'}^*).$$

Using twice the stationarity of the random variables we obtain

$$\sum_{j \neq j'} \mathrm{Cov}(Y_{n,j}^*, Y_{n,j'}^*) = 2 \sum_{j=1}^{r_n - 1} (r_n - j) \mathrm{Cov}(Y_{n,1}^*, Y_{n,2j-1}^*)$$

and

$$\begin{aligned} & \mathrm{Cov}(Y_{n,1}^*, Y_{n,2j-1}^*) \\ & \quad \leq \sum_{\ell=0}^{p_n - 1} (p_n - \ell) \mathrm{Cov}(X_1, X_{2jp_n + \ell + 1}) + \sum_{\ell=1}^{p_n - 1} (p_n - \ell) \mathrm{Cov}(X_\ell, X_{2jp_n + 1}) \\ & \quad \leq p_n \sum_{\ell=(2j-1)p_n + 2}^{(2j+1)p_n} \mathrm{Cov}(X_1, X_\ell). \end{aligned}$$

Inserting these inequalities in (6) we find

$$\begin{aligned}
&\left| \mathrm{E} e^{\lambda T_{n,od}} - \prod_j \mathrm{E} e^{\lambda Y_{n,2j-1}} \right| \\
&\qquad \leq \frac{1}{2} \lambda^2 n^{2/\alpha} A_\alpha^2 r_n p_n \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda\right) \sum_{\ell=p_n+2}^{2r_n-1} \mathrm{Cov}(X_1, X_\ell) \\
&\qquad \leq \frac{1}{4} \lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda\right) u(p_n+2).
\end{aligned}$$

We can now use this together with Markov's inequality to find that, for every $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{P}(T_{n,od} > n^{1/p}\varepsilon) &\leq e^{-\lambda n^{1/p}\varepsilon} \left| \mathrm{E} e^{\lambda T_{n,od}} - \prod_j \mathrm{E} e^{\lambda Y_{n,2j-1}} \right| + e^{-\lambda n^{1/p}\varepsilon} \prod_j \mathrm{E} e^{\lambda Y_{n,2j-1}} \\
&\leq \frac{1}{4} \lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon\right) u(p_n+2) \\
&\qquad + \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right),
\end{aligned}$$

and remember that $u(p_n+2) \leq u(p_n)$, due to the nonnegativity of the covariances. ∎

We may now prove the almost sure convergence of $n^{-1/p} T_n$, assuming a convenient decrease rate on the Cox-Grimmett coefficients.

Theorem 3.2. *Assume the random variables X_n, $n \geq 1$, are centered, associated, strictly stationary and bounded (by $c > 0$). Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} = \frac{1}{2} + \xi$, for some $\xi > 0$, and $u(n) \sim \rho^n$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p} T_n \longrightarrow 0$.*

Proof: Consider the decomposition of T_n into the blocks $Y_{n,j}$ defined previously, taking $p_n \sim n^\theta$, for some $\max(0, \frac{1}{2} - \xi) < \theta < \min(1, \frac{1}{2} + \xi)$. It is obviously enough to prove that both $n^{-1/p} T_{n,od}$ and $n^{-1/p} T_{n,ev}$ converge almost surely to 0. As these terms are analogous we will concentrate on the former, starting from (5). A minimization of the exponent on the second term of the upper bound in (5) leads to the choice

$$\lambda = \frac{\varepsilon}{2c^* A_\alpha^2} \frac{n^{1/p-1-2/\alpha}}{d_n}, \tag{7}$$

meaning that

$$\exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right) = \exp\left(-\frac{\varepsilon^2 n^{2\xi}}{4c^* A_\alpha^2 d_n}\right).$$

Assume that, for some $\beta > 1$,

$$\frac{\varepsilon^2 n^{2\xi}}{4c^* A_\alpha^2 d_n} = \beta \log n \quad \Leftrightarrow \quad d_n = \frac{\varepsilon^2}{4c^* A_\alpha^2 \beta} \frac{n^{2\xi}}{\log n}. \tag{8}$$

As $\xi > 0$, it follows that, for n large enough, we have $d_n > 1$ as required by Lemma 2.2. In order to use this lemma we also need to verify that the condition on λ is satisfied: $\lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$. Replacing (7) and remembering d_n is larger than 1, the assumption on Lemma 2.2 translates into

$$\frac{\varepsilon}{2c^* A_\alpha^2} \frac{n^{1/p-1-2/\alpha}}{d_n} < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n} < \frac{1}{cA_\alpha n^{1/\alpha} p_n}.$$

Using now (8) to replace d_n, the previous inequality is equivalent to

$$\frac{\varepsilon n^{1/p-1-2/\alpha}}{2c^* A_\alpha^2} < \frac{1}{cA_\alpha n^{1/\alpha+\theta}} \frac{\varepsilon^2}{4c^* A_\alpha^2 \beta} \frac{n^{2\xi}}{\log n} \quad \Leftrightarrow \quad \varepsilon^{-1} \leq \frac{1}{2cA_\alpha \beta} \frac{n^{1/2+\xi}}{n^\theta \log n}. \tag{9}$$

As $\theta < \frac{1}{2} + \xi$ this upper bound grows to infinity, so this inequality is satisfied, at least for n large enough.

We consider now the first term, involving the Cox-Grimmett coefficients, in (5). The exponent in this term is

$$\frac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon = \frac{c\varepsilon}{4c^* A_\alpha} \frac{n^{1/2+\xi}}{d_n} - \frac{\varepsilon^2}{2c^* A_\alpha^2} \frac{n^{2\xi}}{d_n}.$$

The second term above is, up to multiplication by 2, the exponent that was found after the optimization with respect to λ of the exponent on the second term of (5). So, to control the upper bound in (5) we can factor this part of the exponential, leaving to control, after substitution the expressions for ε and d_n,

$$\frac{1}{4} \lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\frac{c\beta}{\varepsilon} n^{1/2-\xi} \log n\right) u(p_n). \tag{10}$$

As the term that we factored defines a convergent series, it is enough to verify that (10) is bounded. Further, the polynomial part in (10) is clearly dominated by the exponential, thus we may drop it, verifying only that there exists some $c_0 > 0$,

$$\exp\left(\frac{c\beta}{\varepsilon} n^{1/2-\xi}\log n\right) u(p_n) \leq c_0 \quad \Leftrightarrow \quad \frac{c\beta}{\varepsilon} n^{1/2-\xi}\log n + n^{\theta}\log\rho \leq \log c_0. \tag{11}$$

But this a consequence of $\theta > \frac{1}{2} - \xi$ and $\rho \in (0,1)$. Thus, given the above choices we have that $\mathrm{P}\left(|T_{n,od}|\, n^{1/p}\varepsilon\right) \leq (c_0+1)n^{-\beta}$, where $\beta > 1$, so $n^{-1/p}T_{n,od} \longrightarrow 0$ with probability 1. Reasoning analogously we obtain the same result on what respects $n^{-1/p}T_{n,ev}$, so the proof is completed. ∎

A small modification of the previous arguments allows for the identification of a convergence rate for the almost sure convergence just proved.

Theorem 3.3. *Assume the random variables X_n, $n \geq 1$, are centered, associated, strictly stationary and bounded (by $c > 0$). Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} = \frac{1}{2} + \xi$, for some $\xi > 0$, and $u(n) \sim \rho^n$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$ with convergence rate $\frac{\log n}{n^{\xi-\delta}}$, for arbitrarily small $\delta > 0$.*

Proof: We start again as in the proof of Theorem 3.2 choosing $\theta = \frac{1}{2} + \delta$, with $0 < \delta < \min(\frac{1}{2}, \xi)$ and $p_n \sim n^{\theta}$. Now, on (8), allow ε to depend on n:

$$\varepsilon_n^2 = \frac{4\beta c^* A_\alpha^2 d_n \log n}{n^{2\xi}}.$$

The verification of the assumptions of Lemma 2.2, given above by (9), becomes now:

$$\frac{n^{\xi}}{2(\beta c^*)^{1/2} A_\alpha d_n^{1/2} (\log n)^{1/2}} \leq \frac{1}{2c\beta A_\alpha} \frac{n^{\frac{1}{2}+\xi}}{n^{\frac{1}{2}+\delta}\log n},$$

which is equivalent to $d_n \geq \frac{c^2\beta}{c^*} n^{2\delta}\log n$. Thus, as we are interested in a slow growing sequence, we choose $d_n \sim n^{2\delta}\log n$. As a consequence, $\varepsilon_n^2 \sim n^{2(\delta-\xi)}(\log n)^2 \longrightarrow 0$, given the choice for δ. To complete the proof, it is enough to bound $\exp(\frac{1}{2}cn^{1+1/\alpha}\lambda)u(p_n)$. It is easily verified that $n^{1+1/\alpha}\lambda = \frac{1}{c}n^{1/2-\delta}$, so,

$$\frac{1}{2}cn^{1+1/\alpha}\lambda + \log u(p_n) \sim n^{1/2-\delta} + n^{\theta}\log\rho = n^{1/2-\delta} + n^{1/2+\delta}\log\rho.$$

But, the boundedness of this term is an immediate consequence of $\rho \in (0,1)$ and $\delta > 0$, so the proof is concluded. ∎

4. General random variables with infinitely many moments

We begin by treating the case of general random variables assuming the existence of Laplace transforms in some neighbourhood of the origin. This is a strong assumption, as it implies the existence of moments of every order. But it is in this case that we can prove results that fully extend the ones we found in the previous section. Let us assume the random variables X_n, $n \geq 1$, are centered, strictly stationary and

$$\exists M>0, \eta>0,\ \sup_{|t|\leq\eta} \mathrm{E}e^{tX_1} \leq M < +\infty. \tag{12}$$

For the present framework we can not use directly Lemma 3.1, as this result depends on the boundedness of the variables. To circumvent this difficulty we introduce a truncation on the random variables, allowing to treat these truncated variables using the results in Section 3, and then control the remaining tails. Define, for each $c>0$, the nondecreasing functions $g_c(u) = \max(\min(u,c),-c)$, $u \in \mathbb{R}$, which perform a truncation at level c. Let c_n, $n \geq 1$, be a sequence of nonnegative real numbers such that $c_n \longrightarrow +\infty$ and define, for each $i,n \geq 1$,

$$\begin{aligned}
X_{1,i,n} &= g_{c_n}(X_i) = -c_n \mathbb{I}_{(-\infty,-c_n)}(X_i) + X_i \mathbb{I}_{[-c_n,c_n]}(X_i) + c_n \mathbb{I}_{(c_n,+\infty)}(X_i), \\
X_{2,i,n} &= (X_i - c_n)\mathbb{I}_{(c_n,+\infty)}(X_i), \qquad X_{3,i,n} = (X_i + c_n)\mathbb{I}_{(-\infty,-c_n)}(X_i),
\end{aligned} \tag{13}$$

where $\mathbb{I}_A$ represents the characteristic function of the set A. Notice that the above transformations are monotonous, so these new families of variables are still associated. Morevoer, it is obvious that, for each $n \geq 1$ fixed, the variables $X_{1,1,n}, \ldots, X_{1,n,n}$ are uniformly bounded. Consider, as before, a sequence of natural numbers p_n such that, for each $n \geq 1$, $p_n < \frac{n}{2}$ and define r_n as the largest integer less or equal to $\frac{n}{2p_n}$. For $q=1,2,3$, and $j=1,\ldots,2r_n$, define

$$Y_{q,j,n} = \sum_{i=(j-1)p_n+1}^{jp_n} a_{n,i}\left(X_{q,i,n} - \mathrm{E}X_{q,i,n}\right), \tag{14}$$

and

$$T_{q,n,od} = \sum_{j=1}^{r_n} Y_{q,2j-1,n}, \qquad T_{q,n,ev} = \sum_{j=1}^{r_n} Y_{q,2j,n}, \tag{15}$$

For $q=2,3$, as we have assumed that the variables are strictly stationary, we find

$$\begin{aligned}
&\mathrm{P}\left(\left|\sum_{i=1}^{n} a_{n,i}\left(X_{q,i,n}-\mathrm{E}X_{q,i,n}\right)\right|>n^{1/p}\varepsilon\right) \\
&\quad\leq n\mathrm{P}\left(\left|X_{q,1,n}-\mathrm{E}X_{q,1,n}\right|>\frac{n^{1/p-1}\varepsilon}{A_\alpha}\right) \\
&\quad\leq \frac{n^{3-2/p}A_\alpha^2}{\varepsilon^2}\mathrm{Var}(X_{q,1,n}) \leq \frac{n^{3-2/p}A_\alpha^2}{\varepsilon^2}\mathrm{E}X_{q,1,n}^2.
\end{aligned}$$

The following result is an easy extension of Lemma 4.1 in [10].

Lemma 4.1. *Let X_n, $n\geq 1$, be strictly stationary random variables satisfying (12). Then, for $t\in(0,\eta]$,*

$$\mathrm{P}\left(\left|\sum_{i=1}^{n} a_{n,i}\left(X_{q,i,n}-\mathrm{E}X_{q,i,n}\right)\right|>n^{1/p}\varepsilon\right)\leq\frac{2MA_\alpha^2 n^{3-2/p}e^{-tc_n}}{t^2\varepsilon^2},\ q=2,3. \quad (16)$$

We may now prove the extensions of the results proved for uniformly bounded sequences of random variables. The main argument in the proofs in Section 3 was the control of the exponent in the exponential upper bounds found. The bound obtained in (16) is, essentially, of the same form, depending on the choice of the truncating sequence. So, we will obtain the same characterizations for the almost sure convergence and for its rate, as in the case of uniformly bounded sequences of random variables. Remark that, due to the association of the variables,

$$\mathrm{Cov}(X_{1,1,n},X_{1,j,n})=G_{1,j}(c_n,c_n)\leq G_{1,j}(+\infty,+\infty)=\mathrm{Cov}(X_1,X_j).$$

Obviously, this inequality holds even if $\mathrm{Cov}(X_1,X_j)$ is not finite.

Theorem 4.2. *Assume the random variables X_n, $n\geq 1$, are centered, associated, strictly stationary and satisfy (12) with $\eta>5$. Assume that $\frac{2}{\eta-4}\leq p<2$ and $\alpha>1$ are such that $\frac{1}{p}-\frac{1}{\alpha}=\frac{1}{2}+\xi$, for some $\xi>0$, and $u(n)\sim\rho^n$, for some $\rho\in(0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n\longrightarrow 0$.*

Proof: To control the tail terms, that is, $T_{q,n,od}$ and $T_{q,n,ev}$, for $q=2,3$, choose the truncating sequence $c_n=\log n$ and $t=\beta>4+\frac{2}{p}$. Thus according to Lemma 4.1, the probabilities depending on these tail terms are bounded above by $n^{-\beta-3+2/p}$, which defines a convergent series. Concerning the remaining term, we follow the proof of Theorem 3.2 keeping in mind that c and c^* now depend on n. Taking into account Lemma 2.1, we have $c_n^* = c_n^2 + u(0) \sim (\log n)^2$. Thus, instead of (7), we find the choice

$$\lambda = \frac{n^{1/p-1-2/\alpha}\varepsilon}{2c_n^* A_\alpha^2 d_n} \sim \frac{n^{1/p-1-2/\alpha}\varepsilon}{d_n(\log n)^2},$$

and

$$\frac{n^{2\xi}\varepsilon^2}{4A_\alpha^2 d_n(\log n)^2} = \beta \log n \quad \Leftrightarrow \quad d_n = \frac{\varepsilon^2}{4\beta A_\alpha^2}\frac{n^{2\xi}}{(\log n)^3}.$$

As $\xi > 0$, this means that d_n will, for n large enough, be larger that 1, as required by Lemma 2.2. To define the size if the blocks used to decompose the summations, choose $p_n \sim n^\theta$, for some $\max(0, \frac{1}{2} - \xi) < \theta < \min(1, \frac{1}{2} + \xi)$. The condition on λ required by Lemma 2.2 translates now into

$$\varepsilon^{-1} \leq \frac{n^{1/2+\xi}}{2c_n\beta A_\alpha n^\theta \log n} \sim \frac{n^{1/2+\xi-\theta}}{(\log n)^2},$$

thus, is verified, at least for n large enough. We still have to control the behaviour of the term corresponding to (11). The exponent in this expression takes now the form $c_n n^{1+1/\alpha}\lambda \sim n^{1-1/p+1/\alpha}(\log n)^2$, that is, the same we found in course of proof of Theorem 3.2 multiplied by a logarithmic factor that, as is easily verified, does not affect the remaining argument of that proof. ∎

We state next the description of the convergence rate. The proof follows easily along the arguments used to prove Theorem 3.3 with adaptations similar to the ones used in the previous proof.

Theorem 4.3. *Assume the random variables X_n, $n \geq 1$, are centered, associated, strictly stationary and satisfy (12) with $\eta > 5$. Assume that $\frac{2}{\eta-4} \leq p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} = \frac{1}{2} + \xi$, for some $\xi > 0$, and $u(n) \sim \rho^{-n}$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$ with convergence rate $\frac{(\log n)^2}{n^{\xi-\delta}}$, for arbitrarily small $\delta > 0$.*

The above statements includes an assumption on the Cox-Grimmett coefficients of the original untruncated variables. In fact, this assumption may be relaxed, as we only need the coefficients corresponding to the truncated variables defined as, assuming already the stationarity of the variables,

$$\begin{aligned} u^*(n) &= 2\sum_{j=n+1}^{+\infty} \mathrm{Cov}(X_{1,1,n}, X_{1,j,n}) \\ &= 2\sum_{j=n+1}^{+\infty} G_{1,j}(c_n, c_n) \leq 2\sum_{j=n+1}^{+\infty} G_{1,j}(+\infty, +\infty) = u(n). \end{aligned}$$

5. General random variables with finitely many moments

Assumption (12) used in the previous section is a rather strong one, as it implies the existence of every moment. Moreover, Lemma 4.1 does not use the dependence structure of the random variables to control the tail terms. In this section we will relax the assumptions on moments, thus requiring a different control on the tail terms. Instead of (12), we will assume a decrease rate on the tail joint probabilities:

$$\sup_{i,j\geq 1} \Delta_{i,j}(x,y) = O\left(\max(|x|, |y|)^{-a}\right), \qquad \text{as } \max(|x|, |y|) \longrightarrow +\infty. \tag{17}$$

It is easily seen that this tail behaviour only implies the existence of moments of order $k < a$. Besides, under (17), for $q = 2, 3$, and $i, j \geq 1$,

$$\mathrm{Cov}(X_{q,i,n}, X_{q,j,n}) \leq c_1 \int_{c_n}^{+\infty}\int_{c_n}^{+\infty} \max(|x|, |y|)^{-a}\, dxdy = \frac{8c_1}{a-2} c_n^{2-a}, \tag{18}$$

if $a > 2$, where $c_1 > 0$ is a generic constant independent from i, j and n.

The control of the tail terms will be achieved through a maximal inequality on weighted partial sums. Corresponding to the variables defined in (13), introduce the partial sums $T_{q,n} = \sum_{i=1}^n a_{n,i}(X_{q,i,n} - \mathrm{E}X_{q,i,n})$, $n \geq 1$, $q = 1, 2, 3$. The following is an adapted version of Lemma 2.1 in Oliveira [11].

Lemma 5.1. *Let X_n, $n \geq 1$, be centered and associated random variables. Assume the weights are such that*

$$a_{n,i} \geq 0, \qquad \text{and} \qquad a_{n,i} \geq a_{n-1,i}, \quad i < n,\ n \geq 1. \tag{19}$$

Then, for $q = 1, 2, 3$, $\mathrm{E}\left(\max_{k\leq n} T_{q,k}^2\right) \leq \mathrm{E}T_{q,n}^2$.

An immediate consequence is the following upper bound for the tail terms considered previously.

Lemma 5.2. *Let X_n, $n \geq 1$, be centered and associated random variables satisfying (17) with $a > 2$. Assume $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} =$*

$\frac{1}{2}+\xi$, *for some* $\xi > 0$, *and the weights satisfy (1) and (19). Then there exists a generic constant* $c_1 > 0$ *such that, for* $q = 2, 3$,

$$\mathrm{P}\left(|T_{q,n}| > n^{1/p}\varepsilon\right) \leq \frac{8c_1}{(a-2)\varepsilon^2}\frac{1}{n^{2\xi-1}c_n^{a-2}}. \tag{20}$$

Proof: This is a straightforward consequence of Lemma 5.1, $a_{n,i} \leq n^{1/\alpha}A_\alpha$ and (18). ■

We may now state sufficient conditions for the almost sure convergence of $n^{-1/p}T_n$.

Theorem 5.3. *Assume the random variables* X_n, $n \geq 1$, *are centered, associated, strictly stationary satisfying (17) with* $a > 2$. *Assume that* $p < 2$ *and* $\alpha > 1$ *are such that* $\frac{1}{p} - \frac{1}{\alpha} = \frac{1}{2} + \xi$, *for some* $\xi \in \left(\frac{2}{a}, 1\right)$, *and* $u(n) \sim \rho^n$, *for some* $\rho \in (0, 1)$. *If the nonnegative weights satisfy (1) and (19), then, with probability 1,* $n^{-1/p}T_n \longrightarrow 0$.

Proof: Choose $c_n^{a-2} = n^{2-2\xi}(\log n)^b$, for some $b > 1$. As $\xi < 1$, the truncating sequence c_n does converge to $+\infty$. Replacing this expression in (18), it follows that, for $q = 2, 3$,

$$\sum_{n=1}^{+\infty}\mathrm{P}\left(|T_{q,n}| > n^{1/p}\varepsilon\right) \leq \frac{8c_1}{(a-2)\varepsilon^2}\sum_{n=1}^{+\infty}\frac{1}{n(\log n)^b} < \infty.$$

Thus, it remains to prove that $\sum_n \mathrm{P}\left(|T_{1,n}| > n^{1/p}\varepsilon\right) < \infty$. For this purpose, we will go along the arguments for the proof of Theorem 3.2. Choose $p_n \sim n^\theta$ with $\frac{1}{2} - \xi + \frac{2-2\xi}{a-2} < \theta < \frac{1}{2} + \xi - \frac{2-2\xi}{a-2}$ (it is easily verified that, as $2 < a\xi < 1$, this interval is nonempty). The minimization of the exponent in (5) leads to $\lambda = \frac{\varepsilon}{2A_\alpha^2}\frac{n^{1/p-1-2/\alpha}}{c_n^2 d_n}$, which gives raise to the term $\exp\left(-\frac{\varepsilon^2 n^{2\xi}}{4A_\alpha^2 c_n^2 d_n}\right)$ on the upper bound. Thus, we will be interested in choosing the sequences such that, for some $\beta > 1$, $\frac{\varepsilon^2 n^{2\xi}}{4c_n^2 A_\alpha^2 d_n} = \beta \log n$, that is

$$d_n = \frac{\varepsilon^2}{4A_\alpha^2\beta}\frac{n^{2\xi}}{c_n^2\log n} = \frac{\varepsilon^2}{4A_\alpha^2\beta}n^{2\xi-\frac{4(1-\xi)}{a-2}}(\log n)^{-(1+2b/(a-2))}. \tag{21}$$

As $a\xi > 2$ it follows that $\xi - 2\frac{1-\xi}{a-2} > 0$, so d_n chosen as above converges to $+\infty$, becoming, for n large enough, larger than 1, as required by Lemma 2.2. This lemma also requires the verification of a condition on λ. Reasoning as in the proof of Theorem 3.2, this means that instead of (9) we need to verify that

$$\varepsilon^{-1} \leq \frac{1}{2A_\alpha\beta}\frac{n^{1/2+\xi}}{n^\theta c_n \log n} = \frac{1}{2A_\alpha\beta} n^{\frac{1}{2}+\xi-\theta-\frac{2-2\xi}{a-2}} (\log n)^{-(1+b/(a-2))}.$$

As $\theta < \frac{1}{2}+\xi-\frac{2-2\xi}{a-2}$ the exponent of n in the previous expression is nonnegative, so this condition will be met, at least for n large enough. To conclude the proof, we still have to bound the term

$$\frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\frac{\beta}{\varepsilon} c_n n^{1/2-\xi} \log n\right) u(p_n),$$

where, as before, we may drop the polynomial term outside the exponential. After taking logarithms, the boundedness of this term is equivalent to finding an upper bound for

$$c_n n^{1/2-\xi} \log n + n^\theta \log \rho = n^{\frac{1}{2}-\xi+\frac{2-2\xi}{a-2}} (\log n)^{1+\frac{b}{a-2}} + n^\theta \log \rho.$$

But, taking into account that $\rho \in (0,1)$ and the choice for θ, the expression above is indeed bounded. ■

We may now adapt the arguments to characterize a convergence rate.

Theorem 5.4. *Assume the random variables X_n, $n \geq 1$, are centered, associated, strictly stationary satisfying (17) with $a > 2$. Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} = \frac{1}{2} + \xi$, for some $\xi > \frac{2}{a}$, and $u(n) \sim \rho^n$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (1) and (19), then, with probability 1, $n^{-1/p} T_n \longrightarrow 0$, with convergence rate $\frac{(\log n)^{1+\eta_1}}{n^{\xi-2/a-\eta_2}}$, for arbitrarily small $\eta_1, \eta_2 > 0$.*

Proof: For the block decomposition choose $p_n \sim n^\theta$, where $\theta = \frac{1}{2} + \delta$, with $0 < \delta < \min\left(\frac{1}{2}, \frac{a\xi-2}{a-2}\right)$. We will control the probabilities of $T_{q,n}$ using the arguments of Theorem 3.2 for $q = 1$ and the previous theorem for $q = 2, 3$. From the case $q = 1$, it follows that we want to choose

$$\varepsilon_n^2 = \frac{4A_\alpha^2 \beta c_n^2 d_n \log n}{n^{2\xi}},$$

for some $\beta > 1$, where $d_n > 1$ and $c_n \longrightarrow +\infty$. As in the proof of Theorem 3.3, we need to choose $d_n = \beta n^{2\delta} \log n$ to fulfill the assumptions of Lemma 2.2. Now, to define the truncating sequence c_n we use inequality (20), the representation for ε_n above and, as done in the proof of Theorem 3.3, $d_n = \beta n^{2\delta}\log n$, to find

$$\begin{aligned}\sum_{n=1}^{+\infty} \mathrm{P}\left(|T_{1,n}| > n^{1/p}\varepsilon\right) &\leq \frac{8c_1}{(a-2)}\sum_{n=1}^{+\infty}\frac{1}{\varepsilon_n^2 n^{2\xi-1}c_n^{a-2}} \\ &= \frac{8c_1}{a-2}\sum_{n=1}^{+\infty}\frac{n}{d_n c_n^a \log n} \\ &= \frac{8c_1}{a-2}\sum_{n=1}^{+\infty}\frac{1}{n^{2\delta-1}c_n^a(\log n)^2}.\end{aligned}$$

Choose $c_n^a = n^{2(1-\delta)}(\log n)^b$, for some $b>0$, so the series above is convergent. With this choice we have

$$\varepsilon_n^2 = 4A_\alpha^2\beta^2\frac{(\log n)^{2+2b/a}}{n^{2(\xi-\delta)-4(1-\delta)/a}},$$

which identifies the convergence rate stated by taking $\eta_1 = \frac{b}{a}$ and $\eta_2 = \delta(1+\frac{2}{a})$. To conclude we still need to control the term $\exp\left(\frac{A_\alpha}{2}c_n n^{1+1/\alpha}\lambda\right)u(p_n)$. Now, $\frac{A_\alpha}{2}c_n n^{1+1/\alpha}\lambda = \frac{\beta^{1/2}}{2}n^{1/2-\xi}$, so

$$\frac{A_\alpha}{2}c_n n^{1+1/\alpha}\lambda + n^{1/2+\delta}\log\rho = \frac{\beta^{1/2}}{2}n^{1/2-\xi} + n^{1/2+\delta}\log\rho,$$

is bounded as $\rho\in(0,1)$. ■

Corollary 3.5 in Çağın and Oliveira [3] is similar to our Theorems 4.2 and 5.3. The result in [3], besides always assuming the weights satisfy (1), (19) and $p\in(1,2)$ assumed

$$\sum_{n=1}^{+\infty}\int_{(n+1)^{(\alpha-2p)/(\alpha p)}}^{+\infty} v^{\frac{(\alpha-2)p}{\alpha-2p}-3}G_{1,n}(v,v)\,dv < \infty.$$

It is easily verified that if $\mathrm{Cov}(X_1,X_n)=\rho^n$, for some $\rho\in(0,1)$, the above assumption is satisfied whenever $\alpha > \frac{2p}{2-p}$, but this is equivalent to $\frac{1}{p}-\frac{1}{\alpha} > \frac{1}{2}$. So, our Theorems 4.2 and 5.3 complement Corollary 3.5 in Çağın and Oliveira [3], strengthening the moment assumptions and enlarging the choice for the weights and the variability of p.

6. A Marcinkiewicz-Zygmund law for random variables without means

In the previous sections we considered random variables with at least a finite moment of order 2. We now lower this requirement to prove some Marcinkiewicz-Zygmund strong laws of large numbers. The method of approach is different, based on Shen, Wang, Yang, Hu [12] where these authors were interested in negatively associated random variables. The dependence structure studied in [12] meant that the control of variances of sums is easier than in the present framework, as these variances are smaller than the ones we find for sums of independent random variables. Thus some extra care is required to control the moments below. This methodology does not allow for doubly indexed weights as in the previous sections, so we will obtain results with a somewhat more limited scope with respect to Oliveira [11] or Çağın, Oliveira [3]. However, we will complement a Marcinkiewicz-Zygmund strong law proved in Louhichi [9] by allowing normalizations of the form $n^{1/p}$ with $p<1$ for random variables that do not have means. We start by some simple general results on almost sure convergence.

Lemma 6.1. *Let X_n, $n \geq 1$, be square integrable and associated random variables. If $\sum_{i,j=1}^{\infty} \mathrm{Cov}(X_i, X_j) < \infty$ then $\sum_{i,j=n+1}^{\infty} \mathrm{Cov}(X_i, X_j) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof: Indeed, given the association, it follows that, when $n \longrightarrow +\infty$,

$$\sum_{i,j=n+1}^{+\infty} \mathrm{Cov}(X_i, X_j) \leq \sum_{i,j=1}^{+\infty} \mathrm{Cov}(X_i, X_i) - \sum_{i,j=1}^{n} \mathrm{Cov}(X_i, X_j) \longrightarrow 0.$$

■

We may now state a general convergence result, extending classical characterizations for series of independent random variables.

Theorem 6.2. *Let X_n, $n \geq 1$, be centered, square integrable and associated random variables. If $\sum_{i,j=1}^{\infty} \mathrm{Cov}(X_i, X_j) \leq \infty$ then S_n is almost surely convergent (or, alternatively, $\sum_n X_n < \infty$ almost surely)*

Proof: We will prove that S_n is almost surely Cauchy, from what the convergence follows. For every $\varepsilon > 0$ we have that

$$\begin{aligned} &\mathrm{P}\left(\sup_{k,m\geq n} |S_k - S_m| > \varepsilon\right) \\ &\quad \leq \mathrm{P}\left(\sup_{k\geq n} |S_k - S_n| > \frac{\varepsilon}{2}\right) + \mathrm{P}\left(\sup_{m\geq n} |S_m - S_n| > \frac{\varepsilon}{2}\right). \end{aligned}$$

Both terms on the right hand side have the same form, so it is enough to treat one of them. It is obvious that

$$\mathrm{P}\left(\sup_{k\geq n} |S_k - S_n| > \frac{\varepsilon}{2}\right) = \lim_{N\to\infty} \mathrm{P}\left(\max_{n\leq k\leq N} |S_k - S_n| > \frac{\varepsilon}{2}\right).$$

So, using Markov inequality and Lemma 5.1 (define, for this purpose, all the weights equal to 1), and applying Lemma 6.1,

$$\begin{aligned} &\mathrm{P}\left(\sup_{k\geq n} |S_k - S_n| > \frac{\varepsilon}{2}\right) \\ &\quad \leq \lim_{N\to\infty} \frac{4}{\varepsilon^2} E\left(\max_{n\leq k\leq N} |S_k - S_n|^2\right) \leq \lim_{N\to\infty} \frac{8}{\varepsilon^2} \mathrm{E}\,|S_N - S_n|^2 \\ &\quad = \lim_{N\to\infty} \frac{8}{\varepsilon^2} \sum_{i,j=n+1}^{N} \mathrm{E}(X_i X_j) \leq \frac{8}{\varepsilon^2} \sum_{i,j=n+1}^{+\infty} \mathrm{E}(X_i X_j) \\ &\quad = \frac{8}{\varepsilon^2} \sum_{i,j=n+1}^{+\infty} \mathrm{Cov}(X_i, X_j) \longrightarrow 0. \end{aligned}$$

Thus, for each $\varepsilon > 0$, we have $\mathrm{P}\left(\sup_{k,m\geq n} |S_k - S_m| > \varepsilon\right) \longrightarrow 0$, that is, the sequence S_n is Cauchy in probability. Now, as $\sup_{k,m\geq n} |S_k - S_m|$ is decreasing as n increases, it follows that S_n is indeed Cauchy with probability one. ■

This result gives way to prove a version of the Three Series Theorem for associated sequences. Recall the functions $g_c(u) = \max(\min(u, c), -c)$, $u \in \mathbb{R}$, where $c > 0$ is fixed, and remember that if the original variables are associated then, as g_c is nondecreasing, the sequence $g_c(X_n)$ is also associated.

Theorem 6.3. *Let X_n, $n \geq 1$, be centered and associated random variables. Assume that for some $c > 0$ we have*

$$\sum_{n=1}^{+\infty} \mathrm{P}\left(|X_n| > c\right) < \infty, \quad \sum_{n=1}^{+\infty} \mathrm{E} g_c(X_n) < \infty, \quad \sum_{i,j=1}^{+\infty} \mathrm{Cov}\left(g_c(X_i), g_c(X_j)\right) < \infty. \tag{22}$$

Then S_n converges almost surely (or, alternatively, $\sum_n X_n < \infty$ almost surely).

Proof: We follow the classical arguments to extend the well known result for independent variables. As the truncated variables $g_c(X_n)$, $n \geq 1$, are associated, it follows from Theorem 6.2 that $\sum_{i=1}^n \left(g_c(X_i) - \mathrm{E} g_c(X_i)\right)$ converges almost surely, thus the same holds for $\sum_{i=1}^n g_c(X_i)$. Moreover, from the assumptions, we have $\sum_n \mathrm{P}\left(X_n \neq g_c(X_n)\right) = \sum_n \mathrm{P}\left(|X_n| > c\right) < \infty$, thus the Borel-Cantelli Lemma implies that $\mathrm{P}\left(X_n \neq g_c(X_n)\, i.o.\right) = 0$. Hence, the series $\sum_n X_n$ converges whenever $\sum_n g_c(X_n)$ does, so the result follows. ■

As for independent variables, the previous result enables the control of weighted sums of the form $\sum_n \frac{X_n}{a_n}$. We will assume the weights $a_n > 0$, for every $n \geq 1$, so that the quotients $\frac{X_n}{a_n}$, $n \geq 1$, are still associated. Introduce now a new sequence of associated random variables $Z_n = g_1(\frac{X_n}{a_n})$, $n \geq 1$. Note that the truncation at level 1 means no less of generality as truncation at other level may be achieved replacing a_n by ca_n. In order to prepare for the convergence result, we need some auxiliary inequalities.

Lemma 6.4. *Let $h(\cdot)$ be an even function that is nondecreasing for $x > 0$ and such that $\frac{x}{h(x)}$ is also nondecreasing for $x > 0$. Then $|\mathrm{E} Z_n| \leq \mathrm{E}\left(\frac{h(X_n)}{h(a_n)}\right)$ and $\mathrm{E} Z_n^2 \leq \mathrm{E}\left(\frac{h(X_n)}{h(a_n)}\right)$.*

Proof: Write $\mathrm{E} Z_n = \mathrm{E}\left(\frac{X_n}{a_n} \mathbb{I}_{|X_n| \leq a_n}\right) + \mathrm{E}\left(\mathbb{I}_{|X_n| > a_n}\right)$. As h is even and nondecreasing for $x > 0$, $a_n < |X_n|$ implies $h(a_n) \leq h(X_n)$ thus $\mathrm{E}\left(\mathbb{I}_{|X_n| > a_n}\right) \leq \mathrm{E}\left(\frac{h(X_n)}{h(a_n)} \mathbb{I}_{|X_n| > a_n}\right)$. Now, as $\frac{x}{h(x)}$ is nondecreasing for $x > 0$,

$$\begin{aligned} |X_n| \leq a_n \quad &\Rightarrow \quad \frac{|X_n|}{h(X_n)} \leq \frac{a_n}{h(a_n)} \\ &\Leftrightarrow \quad \frac{h(a_n)}{h(X_n)} \leq \frac{a_n}{|X_n|} \quad \Leftrightarrow \quad \frac{|X_n|}{a_n} \leq \frac{h(X_n)}{h(a_n)}. \end{aligned}$$

Hence $\left|\mathrm{E}\left(\frac{X_n}{a_n}\mathbb{I}_{|X_n|\leq a_n}\right)\right| \leq \mathrm{E}\left(\frac{h(X_n)}{h(a_n)}\mathbb{I}_{|X_n|\leq a_n}\right)$, so summing the two upper bounds, the first inequality is proved. On what concerns the second inequality, write $\mathrm{E}Z_n^2 = \mathrm{E}\left(\frac{X_n^2}{a_n^2}\mathbb{I}_{|X_n|\leq a_n}\right) + \mathrm{E}\left(\mathbb{I}_{|X_n|>a_n}\right)$ and repeat the arguments above noticing that, when $|X_n| \leq a_n$ we have, as h is nondecreasing for $x > 0$, $0 \leq \frac{h(X_n)}{h(a_n)} \leq 1$, so $\frac{X_n^2}{a_n^2} \leq \frac{(h(X_n))^2}{(h(a_n))^2} \leq \frac{h(X_n)}{h(a_n)}$. ■

Lemma 6.5. *Assume the same conditions as in Lemma 6.4. Then, for every* $i \neq j$, $|\mathrm{E}\,(Z_iZ_j)| \leq \mathrm{E}\left(\frac{h(X_i)}{h(a_i)}\frac{h(X_j)}{h(a_j)}\right)$.

Proof: Decompose, analogously as before, $\mathrm{E}\,(Z_iZ_j) = \mathrm{E}\left(\frac{X_i}{a_i}\frac{X_j}{a_j}\mathbb{I}_{|X_i|\leq a_i}\mathbb{I}_{|X_j|\leq a_j}\right) + \mathrm{E}\left(\frac{X_i}{a_i}\mathbb{I}_{|X_i|\leq a_i}\mathbb{I}_{|X_j|>a_j}\right) + \mathrm{E}\left(\frac{X_j}{a_j}\mathbb{I}_{|X_i|>a_i}\mathbb{I}_{|X_j|\leq a_j}\right) + \mathrm{E}\left(\mathbb{I}_{|X_i|>a_i}\mathbb{I}_{|X_j|>a_j}\right)$. Reasoning as in the proof of Lemma 6.4 we can get the following inequalities:

$$\left|\mathrm{E}\left(\frac{X_i}{a_i}\frac{X_j}{a_j}\mathbb{I}_{|X_i|\leq a_i}\mathbb{I}_{|X_j|\leq a_j}\right)\right| \leq \mathrm{E}\left(\frac{h(X_i)}{h(a_i)}\frac{h(X_j)}{h(a_j)}\mathbb{I}_{|X_i|\leq a_i}\mathbb{I}_{|X_j|\leq a_j}\right),$$
$$\left|\mathrm{E}\left(\frac{X_i}{a_i}\mathbb{I}_{|X_i|\leq a_i}\mathbb{I}_{|X_j|>a_j}\right)\right| \leq \mathrm{E}\left(\frac{h(X_i)}{h(a_i)}\frac{h(X_j)}{h(a_j)}\mathbb{I}_{|X_i|\leq a_i}\mathbb{I}_{|X_j|>a_j}\right),$$
$$\left|\mathrm{E}\left(\frac{X_j}{a_j}\mathbb{I}_{|X_i|>a_i}\mathbb{I}_{|X_j|\leq a_j}\right)\right| \leq \mathrm{E}\left(\frac{h(X_i)}{h(a_i)}\frac{h(X_j)}{h(a_j)}\mathbb{I}_{|X_i|>a_i}\mathbb{I}_{|X_j|\leq a_j}\right),$$
$$\left|\mathrm{E}\left(\mathbb{I}_{|X_i|>a_i}\mathbb{I}_{|X_j|>a_j}\right)\right| \leq \mathrm{E}\left(\frac{h(X_i)}{h(a_i)}\frac{h(X_j)}{h(a_j)}\mathbb{I}_{|X_i|>a_i}\mathbb{I}_{|X_j|>a_j}\right).$$

Summing up these inequalities the conclusion of the lemma follows. ■

We may now state a general convergence result.

Theorem 6.6. *Let $h(\cdot)$ be an even function that is nondecreasing for $x > 0$ and such that $\frac{x}{h(x)}$ is also nondecreasing for $x > 0$. Assume that*

$$\mathrm{E}\left(\sum_{n=1}^{+\infty}\frac{h(X_n)}{h(a_n)}\right)^2 < \infty. \tag{23}$$

Then $\sum_n \frac{X_n}{a_n}$ is almost surely convergent.

Proof: We will verify that the Z_n random variables satisfy the assumptions of Theorem 6.3. As what regards the first assumption, uisng Markov's inequality,

$$\begin{aligned}\sum_{n=1}^{+\infty} \mathrm{P}\left(|Z_n|>1\right) &= \sum_{n=1}^{+\infty} \mathrm{P}\left(h(X_n)>h(a_n)\right) \\ &\leq \sum_{n=1}^{+\infty} \frac{\mathrm{E}(h(X_n))}{h(a_n)} = \mathrm{E}\left(\sum_{n=1}^{+\infty} \frac{h(X_n)}{h(a_n)}\right) < \infty,\end{aligned}$$

as this summation is assumed to be square integrable. The second summation in the assumptions of Theorem 6.3 is controlled applying twice the first upper bound proved in Lemma 6.4:

$$\sum_{n=1}^{+\infty} |\mathrm{E}Z_n| \leq 2\sum_{n=1}^{+\infty} \mathrm{E}\left(\frac{h(X_n)}{h(a_n)}\right) < \infty.$$

Finally,

$$\begin{aligned}&\sum_{i,j=1}^{+\infty} \mathrm{Cov}(Z_i,Z_j) \\ &\quad\leq \sum_{i,j=1}^{+\infty} \left(|\mathrm{E}\left(Z_iZ_j\right)| + |\mathrm{E}Z_i|\,|\mathrm{E}Z_j|\right) \\ &\quad\leq \sum_{i=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)} + \sum_{i\neq j} \mathrm{E}\left(\frac{h(X_i)}{h(a_i)}\frac{h(X_j)}{h(a_j)}\right) + \sum_{i,j=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)}\frac{\mathrm{E}h(X_j)}{h(a_j)} \\ &\quad\leq \sum_{i=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)} + \mathrm{E}\left(\sum_{i=1}^{+\infty} \frac{h(X_i)}{h(a_i)}\right)^2 + \left(\sum_{i=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)}\right)^2 < \infty\,.\end{aligned}$$

Applying now Theorem 6.3, the present result follows. ■

Corollary 6.7. *Let $h(\cdot)$ be an even function that is nondecreasing for $x>0$ and such that $\frac{x}{h(x)}$ is also nondecreasing for $x>0$. Assume that (23) holds and the weights $a_n \nearrow +\infty$. Then, $a_n^{-1}S_n \longrightarrow 0$ almost surely.*

Proof: Apply Kronecker's Lemma to the conclusion of Theorem 6.6. ■

We can find a condition for the convergence of $\sum_{i=1}^n \frac{X_i}{a_i}$ that is a somewhat weaker than (23) or, at least, may be written in a weaker form. It follows from the previous bounds that, for $i \neq j$,

$$\begin{aligned}\mathrm{Cov}(Z_i, Z_j) &\leq |\mathrm{E}(Z_i Z_j)| + |\mathrm{E}Z_i|\,|\mathrm{E}Z_j| \\ &= \mathrm{Cov}\left(\frac{h(X_i)}{h(a_i)}, \frac{h(X_j)}{h(a_j)}\right) + 2\frac{\mathrm{E}h(X_i)}{h(a_i)}\frac{\mathrm{E}h(X_j)}{h(a_j)}.\end{aligned}$$

Hence

$$\sum_{i,j=1}^{+\infty} \mathrm{Cov}(Z_i, Z_j) \leq \sum_{i=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)} + \sum_{i,j=1}^{+\infty} \mathrm{Cov}\left(\frac{h(X_i)}{h(a_i)}, \frac{h(X_j)}{h(a_j)}\right) + 2\left(\sum_{i=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)}\right)^2.$$

We thus have the following alternative result.

Corollary 6.8. *Let $h(\cdot)$ be an even function that is nondecreasing for $x > 0$ and such that $\frac{x}{h(x)}$ is also nondecreasing for $x > 0$. Assume that*

$$\sum_{i,j=1}^{+\infty} \mathrm{Cov}\left(\frac{h(X_i)}{h(a_i)}, \frac{h(X_j)}{h(a_j)}\right) < \infty \qquad \text{and} \qquad \sum_{i=1}^{+\infty} \frac{\mathrm{E}h(X_i)}{h(a_i)} < \infty. \tag{24}$$

Then $\sum_n \frac{X_n}{a_n}$ is almost surely convergent. Moreover, if additionally the weights $a_n \nearrow +\infty$, then $a_n^{-1} S_n \longrightarrow 0$ almost surely.

We may now prove first a result for general weights and identically distributed variables.

Theorem 6.9. *Let X_n, $n \geq 1$, be identically distributed and associated random variables and $0 < q \leq 1$ be such that $\mathrm{E}\,|X_1|^q < \infty$. Assume the positive weights a_n are such that*

$$\sum_{n=1}^{+\infty} \frac{1}{a_n^q} < \infty \quad \text{and} \qquad \sum_{i<j} \frac{1}{a_i a_j} G_{i,j}(a_i, a_j) < \infty.$$

Then $\sum_n \frac{X_n}{a_n}$ is almost surely convergent. Moreover, if additionally the weights $a_n \nearrow +\infty$, then $a_n^{-1} S_n \longrightarrow 0$ almost surely.

Proof: Choose $h(x) = |x|^q$, fulfilling the assumptions on h of the preceding results. Then, the second assumption in (24) rewrites as

$$\sum_{n=1}^{+\infty} \frac{\mathrm{E}\,|X_1|^q}{a_n^q} < \infty.$$

As what regards the first inequality in (24), we go back to $\mathrm{Cov}(Z_i, Z_j)$. We may write, using the $\Delta_{i,j}$ functions defined in (3),

$$\begin{aligned}\mathrm{Cov}(Z_i, Z_j) &= \int_{-1}^{1}\int_{-1}^{1} \Delta_{i,j}(a_i u, a_j v)\, dudv \\ &= \frac{1}{a_i a_j}\int_{a_i}^{a_i}\int_{-a_j}^{a_j} \Delta_{i,j}(u,v)\, dudv = \frac{1}{a_i a_j} G_{i,j}(a_i, a_j).\end{aligned}$$

Finally, using the second inequality from Lemma 6.4,

$$\begin{aligned}\sum_{i,j=1}^{+\infty} \mathrm{Cov}(Z_i, Z_j) &= \sum_{i=1}^{+\infty} \mathrm{Var}(Z_i) + 2\sum_{i<j} \frac{1}{a_i a_j} G_{i,j}(a_i, a_j) \\ &\leq \sum_{i=1}^{+\infty} \mathrm{E} Z_i^2 + 2\sum_{i<j} \frac{1}{a_i a_j} G_{i,j}(a_i, a_j) \\ &\leq \mathrm{E}\,|X_1|^q \sum_{i=1}^{+\infty} \frac{1}{a_i^q} + 2\sum_{i<j} \frac{1}{a_i a_j} G_{i,j}(a_i, a_j) < \infty,\end{aligned}$$

so the proof is concluded. ■

Choosing now a suitable sequence of weights we prove a Marcinkiewicz-Zygmund strong law requiring, at most, the existence of means.

Corollary 6.10. *Let X_n, $n \geq 1$, be identically distributed and associated random variables and $0 < p < q \leq 1$ be such that $\mathrm{E}\,|X_1|^q < \infty$. Assume that*

$$\sum_{i<j} \frac{1}{i^{1/p} j^{1/p}} G_{i,j}(i^{1/p}, j^{1/p}) < \infty. \tag{25}$$

Then $n^{-1/p} S_n \longrightarrow 0$ almost surely.

Proof: With respect to the proof of Theorem 6.9 choose $a_n = n^{1/p}$. Then $\sum_n \frac{1}{a_n^q} = \sum_n \frac{1}{n^{q/p}} < \infty$, thus the assumptions of Theorem 6.9 are satisfied. ■

This result complements Theorem 1 in Louhichi [9], where it was assumed that $\mathrm{E}\,|X_1|^q < \infty$, for $q \in [1, 2)$, and a suitable increase rate, similar to (25), on the truncated covariances.

If we assume a convenient decrease rate on the joint tail probabilities we may even verify that (25) holds.

Corollary 6.11. *Let X_n, $n \geq 1$, be identically distributed and associated random variables and $0 < p < q \leq 1$ be such that* $\mathrm{E}\,|X_1|^q < \infty$. *Assume that (17) holds for some $a > 2p$. Then $n^{-1/p} S_n \longrightarrow 0$ almost surely.*

Proof: We need to verify that (25) holds. From (17) it follows that there exists some constants $u_0 > 0$ and $c_1 > 0$ such that for $|u|, |v| \geq u_0$ and every $i, j \geq 1$ we have $\Delta_{i,j}(u,v) \leq c_1 \max(|u|,|v|)^{-a}$. As, obviously, for every $u, v \in \mathbb{R}$ and $i, j \geq 1$, $\Delta_{i,j}(u,v) \leq 1$, we have

$$
\begin{aligned}
&G_{i,j}(i^{1/p}, j^{1/p}) \\
&\quad = u_0^2 + 2c_1 \int_{u_0}^{i^{1/p}} \int_{-u}^{u} dv\, |u|^{-a}\, du + 2c_1 \int_{u_0}^{i^{1/p}} \int_{-v}^{v} du\, |v|^{-a}\, dv \\
&\quad\quad +2c_1 \int_{i^{1/p}}^{j^{1/p}} \int_{-i^{1/p}}^{i^{1/p}} du\, v^{-a}\, dv \\
&\quad = \left(u_0^2 - \frac{8c_1}{2-a} u_0^{2-a} \right) + c_1 \left(\frac{8}{2-a} - \frac{4}{1-a} \right) i^{\frac{2-a}{p}} + \frac{4c_1}{1-a} i^{1/p} j^{\frac{1-a}{p}}.
\end{aligned}
$$

Replacing this in (25) and dropping the constants, we need to control:

- $\displaystyle\sum_{i<j} i^{-1/p} j^{-1/p} \leq \frac{1}{2} \sum_{i \neq j} i^{-1/p} j^{-1/p} + \frac{1}{2} \sum_i i^{-2/p} = \frac{1}{2} \left(\sum_i \frac{1}{i^{1/p}} \right)^2 < \infty$, as $p < 1$;
- $\displaystyle\sum_{i<j} i^{\frac{2-a}{p}} i^{-1/p} j^{-1/p} = \sum_{i=1}^{\infty} i^{\frac{1-a}{p}} \sum_{j=i}^{\infty} j^{-1/p} \sim \sum_{i=1}^{\infty} i^{\frac{1-a}{p}} i^{1-1/p} = \sum_{i=1}^{\infty} i^{1-a/p} < \infty$, as $a > 2p$ implies $1 - \frac{a}{p} < -1$;
- $\displaystyle\sum_{i<j} i^{-1/p} j^{-1/p} i^{1/p} j^{\frac{1-a}{p}} = \sum_{i<j} j^{-\frac{a}{p}} < \sum_{j=2}^{\infty} \sum_{i=1}^{j} j^{-a/p} = \sum_{j=2}^{\infty} j^{(1-a/p)} < \infty$, as $a > 2p$,

so the proof is concluded. ∎

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