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LETTER TO THE EDITOR
on flawed mu-permanental formulas

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This letter brings into discussion an article by C. Fonseca, published in 2005 [2], where an attempt is made to prove a particular case of a conjecture of R. Bapat [1] on the μ-permanent of Hermitian positive definite matrices. Unfortunately, we show with counterexamples that all C. Fonseca's μ-permanental formulas are wrong, as well as all statements leading to the attempted proof.

We shall use the notations, page numbers and result numbers of [2]. The μ-permanent of a square matrix A is defined by

$$P_\mu(A) = \sum_{\sigma\in\mathscr{S}_n} \mu^{\ell(\sigma)} \prod_{i=1}^n a_{i\sigma(i)}\,,$$

where $\ell(\sigma)$ denotes the number of inversions of the permutation σ. So $P_\mu(A)$ is a polynomial in n^2+1 variables, the entries of A and the new variable μ. The determinant and the permanent are specializations of the μ-permanent, and the latter is a much trickier function than the two former notorious prototypes. For example, while Laplace expansions along any row [column] are valid for the determinant and the permanent, the corresponding expansions for the μ-permanent only stand for the first and the last row [column] of A. The first μ-permanental formula of C. Fonseca (namely, the second displayed in p. 227) is in fact a set of n expansion formulas, one for each value of $i \in \{1,\dots,n\}$; each such formula will be referred to as $F(n,i)$. Formula $F(n,i)$, which involves the set of cycles of the digraph of A through vertex i, is well-known for the determinant and holds as well for the permanent, but

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it fails in general for the μ-permanent. The following matrices are counterexamples to the formulas $F(3,2)$ and $F(5,1)$, respectively:

$$K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

It can be proven that $P_\mu(A^\backprime) = P_\mu(A)$, where $A^\backprime$ is the *skew-transpose* of A, *i.e.*, the matrix of entries $a^\backprime_{ij} = a_{n-j+1,n-i+1}$; moreover, if X is a counterexample to $F(n,i)$, then $X^\backprime$ is a counterexample to $F(n, n-i+1)$, and $X \oplus I_m$ is a counterexample to $F(n+m, i)$. Applying this and $(A \oplus B)^\backprime = B^\backprime \oplus A^\backprime$ to the counterexamples K, L we get counterexamples to all $F(n,i)$ except for $(n,i) \in \{(3,1),(3,3),(4,1),(4,4)\}$. As all these counterexamples are symmetric $\{0,1\}$-matrices, theorem 3.2 is false for the indicated (n,i).

For a permutation $\sigma \in \mathscr{S}_n$, let Π_σ denote the permutation matrix corresponding to σ (Π_σ has ij-entry $\delta_{i,\sigma(j)}$); then $P_\mu(\Pi_\sigma) = \mu^{\ell(\sigma)}$. Moreover, Π_σ is symmetric iff σ is a product of pairwise disjoint transpositions. For $i = 1$ (and hence, by skew-transposition, for $i = n$), corollary 3.3 holds as a particular case of a result of A. Lal [3, lemma 3.3.1]. But corollary 3.3 fails for all other i, and all trees having an edge $\{p,q\}$ such that $p < i < q$, as the counterexample Π_τ shows, for $\tau = (p\ q)$.

Auspiciously the decisive argument of page 228 does not use the damaging previous formulas; it uses the correct formula of A. Lal (the case $i = 1$ of corollary 3.3). But further errors are committed which we do not analyze here. Instead, we exhibit counterexamples to the resulting lemma 4.1. Consider the $n \times n$ matrix Π_χ, where χ is the product of two disjoint transpositions, say $\chi = \tau_1\tau_2$, where $\tau_1 = (w_1\ v_1)$ and $\tau_2 = (w_2\ v_2)$ satisfy $[w_1, v_1] \cap [w_2, v_2] \neq \varnothing$. The calculations to follow depend on the relative positions of w_1, w_2, v_1, v_2; we may assume, without loss of generality, that $w_1 < w_2 < v_1$ and $w_2 < v_2$. So we essentially have two cases: when $w_1 < w_2 < v_1 < v_2$, and when $w_1 < w_2 < v_2 < v_1$. In both cases, the reader may easily prove that the derivative formula of lemma 4.1 fails for the matrix Π_χ. So we may say that lemma 4.1 is false for every tree T having at least two disjoint edges, $\{w_1, v_1\}, \{w_2, v_2\}$, such that $[w_1, v_1] \cap [w_2, v_2]$ is nonempty.

It is a well-known elementary fact from algebraic geometry that a counterexample to an algebraic condition never comes alone if the base field is infinite, Moreover, the set of all such counterexamples is Zariski open in an appropriate affine space, and when we are working with the real or complex fields, the nonempty Zariski open sets are open and dense in the Euclidean topology. We illustrate this point with a tree T for which Fonseca's lemma 4.1 fails. Denote by $\mathcal{H}_T$ the real space of complex Hermitian matrices whose graphs are subgraphs of T. Then the set of those $A \in \mathcal{H}_T$ that do not satisfy lemma 4.1 is an open dense subset of $\mathcal{H}_T$, so almost every matrix in $\mathcal{H}_T$ is a counterexample. A similar statement holds for real symmetric matrices, and we may get similar statements for all other counterexamples found in this letter.

References

[1] R. Bapat, Interpolating the Determinantal and Permanental Hadamard Inequality, *Linear and Multilinear Algebra*, 32(1992), 335-337

[2] C. Fonseca, On a conjecture about the μ-permanent, *Linear and Multilinear Algebra*, 53(2005), 225-230.

[3] A. Lal, *Coxeter Groups and Positive Matrices*, Ph.D. Thesis, Indian Statistical Institute, 1992.

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