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ON PERMUTABLE PAIRS OF QUASI-UNIFORMITIES

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ABSTRACT: As it is well known, the concepts of normality and extremal disconnectedness of a topological space are dual to each other in some sense. This is nicely illustrated by several pairs of famous results in classical topology. A recent paper by E. P. de Jager and H.-P. A. Künzi provides some interesting pairs of results of the kind in the asymmetric setting of quasi-uniform spaces. The aim of this paper is to shed a more unifying light on these results. Besides extending them to a setting determined by more general fixed classes of subspaces of the underlying space, encompassing some weak variants of normality, we determine sufficient conditions on the fixed class of subspaces that enable us to unify each pair of results under the same proof.

KEYWORDS: Entourage, quasi-uniformity, quasi-uniform space, lattice of quasi-uniformities, permutable quasi-uniformities, Pervin quasi-uniformity, normal space, extremally disconnected space.

AMS SUBJECT CLASSIFICATION (2010): 54E15, 54E55, 54D15, 54G05.

1. Introduction

Normality is one of the most important topological separation properties. There is a large literature devoted to it and the most recent one is fraught with all kinds of (weak) variants of it (see, for instance, [2, 3, 4, 5, 14, 15, 21, 22, 24, 26, 25, 27]). Let us recall that a topological space X is *normal* provided that any two disjoint closed sets in X can be separated by open sets. In other words, X is normal if and only if for every open subsets A and B of X,

$$A \cup B = X \;\Rightarrow\; \exists \text{ open } U, V\colon\; U \cap V = \emptyset,\, A \cup U = X = B \cup V.$$

On the other hand, a topological space X is said to be *extremally disconnected* if every open set in X has an open closure. Equivalently, any two disjoint open subsets of X have disjoint closures, that is, X is extremally disconnected if and only if for every open subsets A and B of X,

$$A \cap B = \emptyset \;\Rightarrow\; \exists \text{ open } U, V\colon\; U \cup V = X,\, A \cap U = \emptyset = B \cap V.$$

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Hence, the property of extremal disconnectedness is, in lattice-theoretical terms, dual to normality (cf. [16, p. 301]). This nice observation was first pointed out by T. Kubiak in [17, 18]. This duality is revealed in some famous pairs of theorems like Urysohn and Gillman-Jerison separation type lemmas, Tietze and Stone extension type theorems, Katětov-Tong and Stone insertion type theorems and Hausdorff mapping invariance type theorems (see Table 1 in [11] for more information). But most interestingly, the duality is not completely symmetric in the sense that not every result in each pair is directly obtainable from its dual one (simply because in some cases the conditions required for it are not exactly the duals of the conditions required for the dual result).

Recently, E. P. de Jager and H.-P. A. Künzi [13] proved the following result in the realm of quasi-uniform spaces:

Theorem 1. *Let $\mathcal{P}$ be the Pervin quasi-uniformity on a topological space X. Then:*

(1) $\mathcal{P} \circ \mathcal{P}^{-1}$ *is a (quasi-)uniformity if and only if X is normal.*
(2) $\mathcal{P}^{-1} \circ \mathcal{P}$ *is a (quasi-)uniformity if and only if X is extremally disconnected.*
(3) $\mathcal{P}$ *and* $\mathcal{P}^{-1}$ *permute if and only if X is normal and extremally disconnected.*

The motivation for this paper arose from a conversation of the third author with Prof. H.-P. A. Künzi about this result, in particular, and the nature of the normality/extremal disconnectedness duality, in general. Our primary goal with it is to investigate whether it is possible to formulate Theorem 1 in a "two for the price of one" setting so that the proof of assertion (2) (and hence of (3)) is a direct consequence of (1) by some kind of dualization process. Concurrently, the extended setting should allow for the formulation and unification of several weak variants of the notion of normality. Our approach follows the idea introduced in [11] that by selecting different classes $\mathscr{A}$ of subspaces of the underlying space of the (quasi-)uniform space $(X, \mathcal{U})$, one can deal with relative notions of normality and extremal disconnectedness, unifying the different variants. This development enables us to obtain the sufficient conditions on $\mathscr{A}$ and $\mathcal{U}$ that allow to extend the proofs of E. P. de Jager and H.-P. A. Künzi [13].

We will conclude that the dualization of part of Theorem 1(1) yields precisely the desired result in the disconnectedness side (2) while the other part does not (just because in this case, the conditions on the class $\mathscr{A}$ are required for arbitrary joins, not only the finite ones). The interesting aspect of this work is that it reveals precisely whether it is possible to get each dual result for free.

We point out that all definitions and results in the paper are written in a way to be easily extendable to the point-free setting of frames and locales with the help of the tools introduced in [7, 8]. We keep everything in the point-set classical setting just to make the connections with the results in [13] more apparent.

We now give an overview of the contents of the paper. The paper begins with some background material on quasi-uniform spaces in Section 2. The relations amongst the several versions of the notions of normality and extremal disconnectedness collected from the literature together with the relative general notions that unify them are given in Section 3. The corresponding relative notions of a compatible quasi-uniformity and the Pervin quasi-uniformity are presented in Section 4. The proofs of our two main theorems and their corollaries are provided in Sections 5 and 6, the core sections of the paper.

2. Background on quasi-uniformities

Let X be a set. A filter $\mathcal{U}$ on $X \times X$ such that each $U \in \mathcal{U}$ is a reflexive relation and for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$ is called a *quasi-uniformity* on X and the pair $(X, \mathcal{U})$ is a *quasi-uniform space*. Note that for any quasi-uniformity $\mathcal{U}$ the filter $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$, where $U^{-1} = \{(y, x) \in X \times X \mid (x, y) \in \mathcal{U}\}$, is also a quasi-uniformity on X, the *conjugate* of $\mathcal{U}$. A quasi-uniformity $\mathcal{U}$ satisfying $\mathcal{U} = \mathcal{U}^{-1}$ is called a *uniformity*. For each $A \subseteq X$ and each $x \in A$, let

$$U(x) = \{y \in X \mid (x, y) \in U\}$$

and

$$U(A) = \bigcup_{x \in A} U(x).$$

The topology $\tau(\mathcal{U})$ *induced by* $\mathcal{U}$ *on* X consists of all $A \subseteq X$ such that for each $a \in A$ there is some $U \in \mathcal{U}$ satisfying $U(a) \subseteq A$. Then, obviously,

$$\forall U \in \mathcal{U},\ \forall A \subseteq X,\ A \subseteq \mathrm{int}_{\tau(\mathcal{U})}(U(A)). \tag{QU1}$$

Moreover, for any base $\mathcal{B}$ of $\mathcal{U}$ and any $A \subseteq X$,

$$\mathrm{cl}_{\tau(\mathcal{U})}(A) = \bigcap\{U^{-1}(A) \mid U \in \mathcal{B}\} \quad [9, \text{Prop. } 1.8]. \tag{QU2}$$

Although $U(x)$ may not be in $\tau(\mathcal{U})$, there is a base $\mathcal{B}$ for $\mathcal{U}$ such that

$$\forall B \in \mathcal{B},\ \forall x \in X,\ \forall S \subseteq X,\ B(x), B(S) \in \tau(\mathcal{U}).$$

A quasi-uniformity $\mathcal{U}$ on X induces the bitopological space $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$. The pairwise completely regular bispaces are precisely the bispaces that are induced by some quasi-uniformity.

For more information about quasi-uniform spaces we refer the reader to [9, 19]. Here we just recall the specific notions and facts that are relevant to our discussion.

Throughout the paper we denote the lattice of open sets (resp. closed sets) of a topological space X by $\mathfrak{O}(X)$ (resp. $\mathfrak{C}(X)$). A quasi-uniformity $\mathcal{U}$ on a space X is *compatible* with the topology of X if $\tau(\mathcal{U})$ coincides with the given topology $\mathfrak{O}(X)$. Clearly, this is equivalent to say that the following two conditions hold:

(C1) $\forall U \in \mathcal{U}$, $\forall A \subseteq X$, $\exists B \in \mathfrak{O}(X)\colon A \subseteq B \subseteq U(A)$.
(C2) $\forall a \in A \in \mathfrak{O}(X)$, $\exists U \in \mathcal{U}\colon U(a) \subseteq A$.

Moreover, notice from (QU2) that

(C3) $\forall U \in \mathcal{U}^{-1}$, $\forall A \subseteq X$, $\exists F \in \mathfrak{C}(X)\colon A \subseteq F \subseteq U(A)$.

For each $A \subseteq X$,

$$S_A = [(X \smallsetminus A) \times X] \cup [X \times A]$$

is a transitive entourage of X. Then the set of entourages $\{S_A \mid A \in \mathfrak{O}(X)\}$ is a subbase for a totally bounded transitive quasi-uniformity on X, compatible with $\mathfrak{O}(X)$. This is the well-known *Pervin quasi-uniformity* $\mathcal{U}_P$ on X. Since $S_{X \smallsetminus A} = S_A^{-1}$, it follows that the quasi-uniformity $(\mathcal{U}_P)^{-1}$ is generated by $\{S_F \mid F \in \mathfrak{C}(X)\}$.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two quasi-uniformities on a set X and $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then $\mathcal{U}_1$ is said to be *coarser* than $\mathcal{U}_2$ or that $\mathcal{U}_2$ is *finer* than $\mathcal{U}_1$. Let $\{\mathcal{U}_i\}_{i \in I}$ be a family of quasi-uniformities on X. The *supremum* of $\{\mathcal{U}_i\}_{i \in I}$ is the coarsest quasi-uniformity on X that is finer than every $\mathcal{U}_i$. The supremum always exists and it is the filter on $X \times X$ generated by the subbase $\bigcup_{i \in I} \mathcal{U}_i$. Of course, the set $q(X)$ of all quasi-uniformities on X equipped with the set-theoretic inclusion $\subseteq$ is a complete lattice (see, for instance, [12]).

The *infimum* of $\{\mathcal{U}_i\}_{i \in I}$, that is, the finest quasi-uniformity that is coarser than every $\mathcal{U}_i$, is then the supremum of the family of all quasi-uniformities on X that are coarser than every $\mathcal{U}_i$.

The operation of conjugation of quasi-uniformities commutes with the supremum and the infimum operations. Indeed, suppose that $\mathcal{V}$ (resp. $\mathcal{W}$) is the infimum of a family $\{\mathcal{U}_i\}_{i\in I}$ of quasi-uniformities on X (resp. the family of conjugate quasi-uniformities $\{\mathcal{U}_i^{-1}\}_{i\in I}$). Then $\mathcal{W}^{-1}$ is a lower bound of $\{\mathcal{U}_i\}_{i\in I}$ and thus $\mathcal{W}^{-1} \subseteq \mathcal{V}$. Similarly $\mathcal{V}^{-1} \subseteq \mathcal{W}$ by the analogous conjugate argument, and thus $\mathcal{V} = \mathcal{W}^{-1}$ (a similar proof for the statement about suprema can be given).

In particular, the supremum and infimum of an arbitrary family of uniformities in $(q(X), \subseteq)$ is a uniformity and for any quasi-uniformity $\mathcal{U}$, both $\mathcal{U} \vee \mathcal{U}^{-1}$ and $\mathcal{U} \wedge \mathcal{U}^{-1}$ are uniformities.

3. Relative normality and relative extremal disconnectedness

Throughout the present paper no separation axiom is assumed. Let X be a topological space and let $A \subseteq X$. The closure of A will be denoted by $\overline{A}$ or $\operatorname{cl} A$ and the interior by $\operatorname{int} A$. Recall that a set $A \subseteq X$ is said to be *regularly open* if $A = \operatorname{int} \overline{A}$. The complement of a regularly open set is called *regularly closed*. Clearly, the intersection of any two regularly open sets is regularly open. A finite (resp. arbitrary) union of regularly open sets is called a *π-open* (resp. *δ-open*) set. The complement of a π-open (resp. δ-open) set is called *π-closed* (resp. *δ-closed*). Of course, δ-open sets form a topology (the *semiregularization topology* in X, that is, the topology generated by regularly open sets). Hence:

$$\text{clopen} \;\Rightarrow\; \text{regularly open} \;\Rightarrow\; \pi\text{-open} \;\Rightarrow\; \delta\text{-open} \;\Rightarrow\; \text{open}. \tag{3.1.1}$$

A set $A \subseteq X$ is called a *regular F_σ-set* if it is a countable union of open sets whose closures are contained in A, i.e., if $A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \overline{A_n}$, where each A_n is an open subset of X. The complement of a regular F_σ-set is called a *regular G_δ-set*. Recall also that a set $A \subseteq X$ is a *zero-set* if there exists a continuous real-valued function f on X such that $A = f^{-1}(\{0\})$. The complement of a zero-set is a *cozero-set*. It is clear that in any space X,

$$\text{clopen} \;\Rightarrow\; \text{cozero-set} \;\Rightarrow\; \text{regular } F_\sigma\text{-set} \;\Rightarrow\; \text{open}. \tag{3.1.2}$$

Definitions 3.1. A topological space X is said to be

(i) *almost normal* if any two disjoint closed sets, one of which is regularly closed, can be separated by open sets [26];

(ii) *mildly normal* if any two disjoint regularly closed sets can be separated by open sets [27];
(iii) π-*normal* if any two disjoint closed sets, one of which is π-closed, can be separated by open sets [14];
(iv) *quasi-normal* if any two disjoint π-closed sets can be separated by open sets [14];
(v) Δ-*normal* if any two disjoint closed sets, one of which is δ-closed, can be separated by open sets [3];
(vi) *weakly* Δ-*normal* if any two disjoint δ-closed sets can be separated by open sets [3];
(vii) δ-*normal* if any two disjoint closed sets, one of which is a regular G_δ-set, can be separated by open sets [22];
(viii) *weakly* δ-*normal* if any two disjoint regular G_δ-sets can be separated by open sets [15];
(ix) *lightly normal* if any two disjoint closed sets, one of which is a zero-set, can be separated by open sets [25];
(x) *weakly lightly normal* if any two disjoint closed sets, one of which is regularly closed and the other a zero-set, can be separated by open sets [15].

The diagram in Table 1 depicts the relations between these classes of spaces (none of these implications is reversible, see [3, 4, 15]).

In view of the definitions above it appears natural to introduce the following generalization of the topological notion of normality.

Given a space X, let $\mathscr{A}, \mathscr{B} \subseteq \mathcal{P}(X)$ be two fixed classes of open subspaces of X. We call them *open subspace selections* on X and denote by $\mathscr{A}^{\mathsf{c}}$ the class $\{X \smallsetminus A \mid A \in \mathscr{A}\}$ of all complements of elements of $\mathscr{A}$.

Definitions 3.2. We say that X is $(\mathscr{A}, \mathscr{B})$-*normal* if for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$,

$$A \cup B = X \;\Rightarrow\; \exists U \in \mathscr{A}, \exists V \in \mathscr{B}\colon\; U \cap V = \emptyset,\, A \cup U = X = B \cup V.$$

Dually, we say that X is $(\mathscr{A}, \mathscr{B})$-*disconnected* if for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$,

$$A \cap B = \emptyset \;\Rightarrow\; \exists U \in \mathscr{A}, \exists V \in \mathscr{B}\colon\; U \cup V = X,\, A \cap U = \emptyset = B \cap V.$$

In the case where $\mathscr{B} = \mathscr{A}$ we simply say that X is $\mathscr{A}$-normal or $\mathscr{A}$-disconnected.

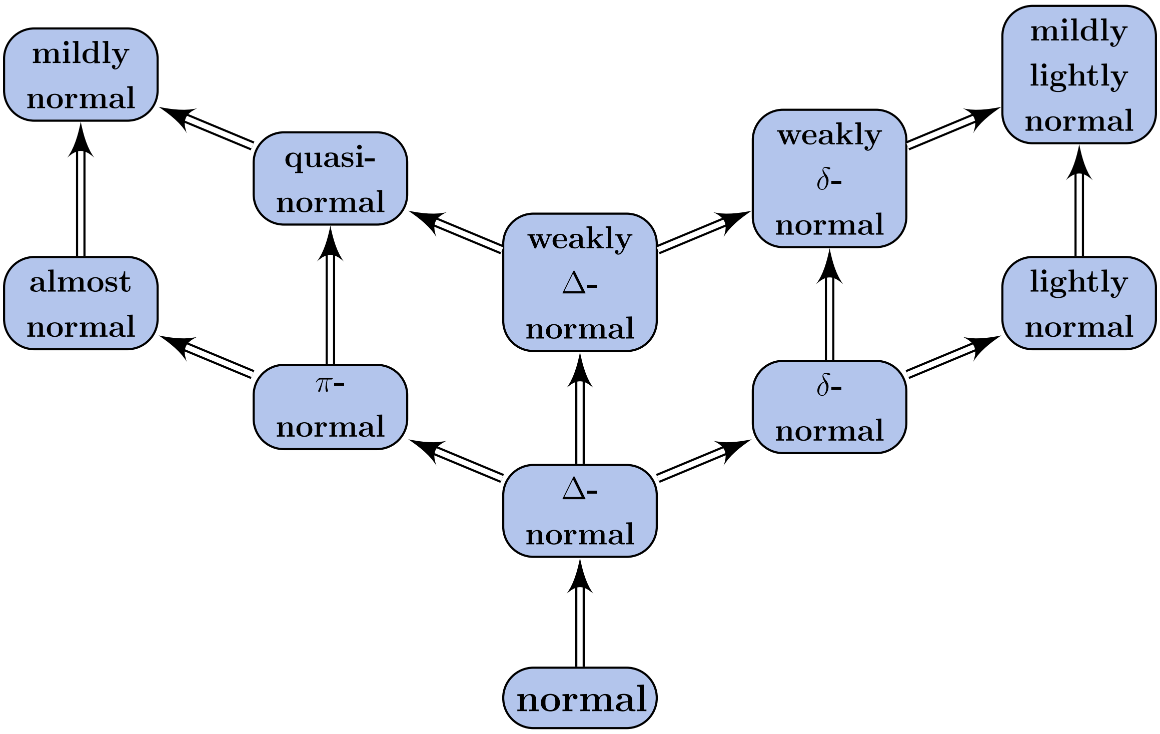

TABLE 1. Variants of normality.

Of course, the particular case where $\mathscr{A} = \mathscr{B} = \mathfrak{O}(X)$ yields the usual notions of normality and extremal disconnectedness, and for any space X,

$$X \text{ is } (\mathscr{A},\mathscr{B})\text{-disconnected iff it is } (\mathscr{A}^{\mathsf{c}},\mathscr{B}^{\mathsf{c}})\text{-normal.} \tag{3.2.1}$$

This explicitly shows that these two notions are dual to each other. The following lemma shows that this duality is not symmetric: the duals of many of the variants of normality presented above collapse into extremally disconnected spaces.

Lemma 3.3. *Let $\mathscr{A},\mathscr{B}$ be two open subspace selections on a space X containing all regularly open sets. Then:*

(a) *X is $(\mathscr{A},\mathscr{B})$-normal if and only if for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$,*

$$A \cup B = X \;\Rightarrow\; \exists\, U, V \in \mathfrak{O}(X):\; U \cap V = \emptyset,\; A \cup U = X = B \cup V. \tag{3.3.1}$$

(b) *X is $(\mathscr{A},\mathscr{B})$-disconnected if and only if it is extremally disconnected.*

Proof: (a) Clearly, $(\mathscr{A},\mathscr{B})$-normality implies that any pair (A, B) in $\mathscr{A} \times \mathscr{B}$ satisfies (3.3.1). Conversely, given an open set U, let U^* denote the regularly open set $\operatorname{int}(X \smallsetminus U) = X \smallsetminus \operatorname{cl}(U)$. It is easy to check that $U^{**} \supseteq U$ and

that $U \cap V = \emptyset$ implies $U^{**} \cap V^{**} = \emptyset$. Hence, given the open sets U and V provided by (3.3.1), it suffices to consider the regularly open sets U^{**} and V^{**} which are in $\mathscr{A}$ and $\mathscr{B}$ by assumption.
(b) The implication "$\Leftarrow$" follows easily, in a way similar to the preceding proof, from the properties $U \subseteq U^{**}$ and

$$A \cap U = \emptyset \Rightarrow A \cap U^{**} = \emptyset.$$

Conversely, consider $A, B \in \mathfrak{O}(X)$ such that $A \cap B = \emptyset$. Then $A^{**} \cap B^{**} = \emptyset$. Once $A^{**} \in \mathscr{A}$ and $B^{**} \in \mathscr{B}$, there exist $U \in \mathscr{A} \subseteq \mathfrak{O}(X)$ and $V \in \mathscr{B} \subseteq \mathfrak{O}(X)$ such that $U \cup V = X$ and $A \cap U \subseteq A^{**} \cap U = \emptyset = B^{**} \cap V \supseteq B \cap V$. ■

Examples 3.4. Consider the following selections of open sets for $\mathscr{A}$ and $\mathscr{B}$:

(1) open sets, (2) regularly open sets, (3) π-open sets,
(4) δ-open sets, (5) cozero-sets, (6) regular F_σ-sets.

Note that selections (1), (3), (4), (5), (6) are clearly sublattices of $\mathfrak{O}(X)$ while (2) is only closed under finite meets. They yield the classes of spaces listed in Table 2 below. Let us explain each one in detail.

	$\mathscr{A}$	$\mathscr{B}$	**$(\mathscr{A},\mathscr{B})$-normal spaces**	**$(\mathscr{A},\mathscr{B})$-disconnected spaces**
1:	(1)	(1)	normal	extremally disconnected
2:	(2)	(2)	mildly normal	extremally disconnected
3:	(1)	(2)	almost normal	extremally disconnected
4:	(3)	(3)	quasi-normal	extremally disconnected
5:	(1)	(3)	π-normal	extremally disconnected
6:	(4)	(4)	wΔ-normal	extremally disconnected
7:	(1)	(4)	Δ-normal	extremally disconnected
8:	(5)	(5)	all spaces	F-spaces
9:	(1)	(5)	lightly normal*	basically disconnected
10:	(2)	(5)	weakly lightly normal*	basically disconnected
11:	(6)	(6)	δ-normal*	extremally δ-disconnected
12:	(1)	(6)	weakly δ-normal*	extremally δ-disconnected

TABLE 2. Examples of $(\mathscr{A},\mathscr{B})$-normal and $(\mathscr{A},\mathscr{B})$-disconnected spaces.

(a) $(\mathscr{A},\mathscr{B})$-normality. In each example, the condition of $(\mathscr{A},\mathscr{B})$-normality implies the corresponding property listed in the table since $\mathscr{A},\mathscr{B} \subseteq \mathfrak{O}(X)$. Regarding the converses, we have:

Example 1 is obvious. Examples 2-7 follow from Lemma 3.3(a) and relations in Definitions 3.1.1. Example 8 is a consequence of the result of Mandelker in [23] that the lattice of all cozero-sets of any space is a normal lattice. Regarding Examples 9-12, they are in general subclasses (that we distinguish by adding an asterisk to the name) of the classes of normal-like spaces in Definitions 3.1 (vii), (viii), (ix) and (x) respectively. But according to e.g. the terminology schema for F-spaces and F'-spaces (see [6]), they should be denoted the other way round: the stronger variants should get the name, not the weaker ones.

Anyway, in each case, both classes coincide whenever the space is an Oz space (Blair [1] calls a Tychonoff space X an *Oz space* if every open set of X is z-embedded). Indeed, a useful characterisation is that X is an Oz space if and only if every regularly open subset of X is a cozero-set (i.e. Oz spaces are the perfectly mildly normal spaces [20, Theorem 1.1]) and thus Lemma 3.3 applies.

(b) $(\mathscr{A},\mathscr{B})$-disconnectedness. Example 1 is obvious while Examples 2-7 follow from Lemma 3.3(b) and relations in Definitions 3.1.1. Example 8 is also easy: recall that a topological space is an *F-space* if disjoint cozero-sets are contained in disjoint zero-sets [6] and notice that being contained in disjoint zero-sets, that is being *completely separated*, is the same as saying that they are contained in disjoint cozero-sets [10, p. 17]. In Examples 11 and 12 we cannot find those classes of spaces in the literature. They are clearly the same class and we name them *extremally δ-disconnected spaces*. Finally, for Examples 9 and 10 we need the following result:

Proposition 3.5. *Let $\mathscr{A} \subseteq \mathfrak{O}(X)$ contain all regularly open sets and let $\mathscr{B}$ be the class of all cozero-sets of X. Then X is $(\mathscr{A},\mathscr{B})$-disconnected if and only if X is basically disconnected.*

Proof: Recall that a space is *basically disconnected* if every cozero-set has an open closure. This can be interpreted as saying that $A \cap B = \emptyset$, with A an arbitrary open set and B a cozero-set, implies $A^* \cup B^* = X$. So consider an open set A and a cozero-set B such that $A \cap B = \emptyset$. Then A^{**} is a regularly open set disjoint from B. Therefore by the hypothesis there exists a $U \in \mathscr{A}$

and a cozero-set V such that $U \cup V = X$ and $A^{**} \cap U = \emptyset = B \cap V$, from which it follows that $A^* \cup B^* \supseteq U \cup V = X$.

Conversely, let $A \cap B = \emptyset$ with $A \in \mathscr{A}$ and B a cozero-set. By basic disconnectedness, $A^* \cup B^* = X$. Of course, $U = A^*$ is regularly open thus belongs to $\mathscr{A}$. Further, $\overline{B}$ is clopen and therefore a zero-set. Hence $V = B^* = X \smallsetminus \overline{B}$ is a cozero-set. ■

4. Relative compatibility of a quasi-uniform structure

Let $\mathcal{U}$ be a quasi-uniformity on a space X and $\mathscr{A} \subseteq \mathfrak{O}(X)$. We say that $\mathcal{U}$ is compatible with $\mathscr{A}$ (or simply *$\mathscr{A}$-compatible*) whenever $\mathscr{A}$ is a subbase for the induced topology $\tau(\mathcal{U})$. Note that the particular case where $\mathscr{A} = \mathfrak{O}(X)$ is precisely the usual notion of a compatible quasi-uniformity on X.

Lemma 4.1. *Let $\mathcal{U}$ be a quasi-uniformity on a space X, $\mathscr{A} \subseteq \mathfrak{O}(X)$ and let $\tau_{\mathscr{A}}$ be the topology on X generated by $\mathscr{A}$. Then $\mathcal{U}$ is $\mathscr{A}$-compatible iff the following conditions hold:*
(C1) $\forall U \in \mathcal{U}, \forall S \subseteq X, \exists A \in \tau_{\mathscr{A}} \colon S \subseteq A \subseteq U(S)$.
(C2) $\forall a \in A \in \tau_{\mathscr{A}}, \exists U \in \mathcal{U} \colon U(a) \subseteq A$.

Proof: $\Rightarrow$: Suppose that $\mathscr{A}$ is a subbase for $\tau(\mathcal{U})$. Then $\tau(\mathcal{U}) = \tau_{\mathscr{A}}$ and therefore condition (C1) follows from (QU1) while (C2) follows from the definition of $\tau(\mathcal{U})$.
$\Leftarrow$: The inclusion $\tau_{\mathscr{A}} \subseteq \tau(\mathcal{U})$ follows from (C2). On the other hand, for each $A \in \tau(\mathcal{U})$ and any $a \in A$ there is some $U_a \in \mathcal{U}$ such that $a \in U(a) \subseteq A$. Consequently, by (C1), there is some $B_a \in \tau_{\mathscr{A}}$ satisfying $\{a\} \subseteq B_a \subseteq U(a) \subseteq A$. Hence $A = \bigcup_{a \in A} B_a \in \tau_{\mathscr{A}}$. ■

Lemma 4.2. *Let $\mathcal{U}$ be a quasi-uniformity on a space X, $\mathscr{A} \subseteq \mathfrak{O}(X)$ and let $\tau_{\mathscr{A}}$ be the topology on X generated by $\mathscr{A}$. If $\mathcal{U}$ is $\mathscr{A}$-compatible, then*
(C3) $\forall U \in \mathcal{U}^{-1}, \forall S \subseteq X, \exists F \in \tau_{\mathscr{A}}^{\mathsf{c}} \colon S \subseteq F \subseteq U(S)$.

Proof: The result follows from (QU2) and the fact that $\mathrm{cl}_{\tau(\mathcal{U})}(S) = \mathrm{cl}_{\tau_{\mathscr{A}}}(S) \in \tau_{\mathscr{A}}^{\mathsf{c}}$. ■

Let X be a topological space and $\mathscr{A} \subseteq \mathfrak{O}(X)$. The sets of the form

$$S_A = [(X \smallsetminus A) \times X] \cup [X \times A] \quad (A \in \mathscr{A})$$

are entourages of X that generate an $\mathscr{A}$-compatible quasi-uniformity $\mathcal{U}_{P(\mathscr{A})}$ on X:

Lemma 4.3. *The set of entourages $\{S_A \mid A \in \mathscr{A}\}$ is a subbase for a transitive totally bounded $\mathscr{A}$-compatible quasi-uniformity on X.*

Proof: Since each S_A is a reflexive and transitive relation and $\{A, X \smallsetminus A\}$ is a finite cover of X with $A\times A, X\smallsetminus A\times X\smallsetminus A \subseteq S_A$, it follows that $\{S_A \mid A \in \mathscr{A}\}$ is always a subbase for a transitive totally bounded quasi-uniformity on X.

Regarding compatibility, we need to show that $\tau_{\mathscr{A}} = \tau(\mathcal{U}_{P(\mathscr{A})})$.
$\subseteq$: Let $A \in \mathscr{A}$ and $a \in A$. Since $S_A \in \mathcal{U}_{P(\mathscr{A})}$ and $S_A(a) = A$, it follows that $A \in \tau(\mathcal{U}_{P(\mathscr{A})})$ and $\tau_{\mathscr{A}} \subseteq \tau(\mathcal{U}_{P(\mathscr{A})})$.
$\supseteq$: Since each $U(x)$ such that $U \in \mathcal{U}_{P(\mathscr{A})}$ is a nhood of x in $\tau(\mathcal{U}_{P(\mathscr{A})})$ for every $x \in X$, it suffices to check that each $U(x)$ is a nhood of x in $\tau_{\mathscr{A}}$. To this end, take $U \in \mathcal{U}_{P(\mathscr{A})}$. Then $\bigcap_{i=1}^n S_{A_i} \subseteq U$ for some $A_1, \ldots, A_n \in \mathscr{A}$. If $x \notin \bigcup_{i=1}^n A_i$ (that is, $x \in X \smallsetminus A_i$ for every i), then $(x, y) \in \bigcap_{i=1}^n S_{A_i}$ for every $y \in X$, that is,

$$\left[\bigcap_{i=1}^n S_{A_i}\right](x) = X \in \tau_{\mathscr{A}}$$

is contained in $U(x)$. Otherwise, if $x \in \bigcup_{i=1}^n A_i$, then

$$\left[\bigcap_{i=1}^n S_{A_i}\right](x) = \bigcap\{A_i \mid x \in A_i\} \in \tau_{\mathscr{A}}. \quad \blacksquare$$

We call $\mathcal{U}_{P(\mathscr{A})}$ the *Pervin quasi-uniformity* induced by $\mathscr{A}$ in X.

Remarks 4.4. (1) Note that $S_A^{-1} = S_{X\smallsetminus A}$ and thus $\mathcal{U}_{P(\mathscr{A}^c)} = \mathcal{U}^{-1}_{P(\mathscr{A})}$. This implies that $\mathcal{U}$ is a quasi-uniformity finer than $\mathcal{U}_{P(\mathscr{A})}$ if and only if $\mathcal{U}^{-1}$ is a quasi-uniformity finer than $\mathcal{U}_{P(\mathscr{A}^c)}$.
(2) Of course, the case where $\mathscr{A} = \mathfrak{O}(X)$ yields precisely the standard Pervin quasi-uniformity of X.

5. Quasi-uniformities that permute with their conjugate

Following the notation in [13], given two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X, $\mathcal{U} \circ \mathcal{V}$ denotes the filter on $X \times X$ generated by the base

$$\{U \circ V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

As it is shown in [13, Lemma 1],

$$\mathcal{U} \circ \mathcal{V} \text{ is a quasi-uniformity iff } \mathcal{U} \circ \mathcal{V} = \mathcal{U} \wedge \mathcal{V} \text{ iff } \mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}. \quad (5.1.1)$$

In particular, $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity if and only if it is a uniformity.

The quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ are said to *permute* (and called *permutable*) if $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$. Hence $\mathcal{U}$ and $\mathcal{V}$ permute if and only if both $\mathcal{U} \circ \mathcal{V}$ and $\mathcal{V} \circ \mathcal{U}$ are quasi-uniformities.

Theorem 5.1. *Let $\mathcal{U}$ be a quasi-uniformity on a space X that is finer than the Pervin quasi-uniformity $\mathcal{U}_{P(\mathscr{A})}$. If $\mathcal{U}$ satisfies the condition*

$$\forall U \in \mathcal{U}, \forall A \in \mathscr{A}^{\mathsf{c}}, \exists B \in \mathscr{A} \colon A \subseteq B \subseteq U(A) \qquad (\mathscr{A}\text{-int})$$

and $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity, then X is $\mathscr{A}$-normal.

Proof: Suppose that $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity. By (5.1.1), $\mathcal{U} \circ \mathcal{U}^{-1}$ is equal to the uniformity $\mathcal{U} \wedge \mathcal{U}^{-1}$. Let $A, B \in \mathscr{A}$ with $A \cup B = X$. Set $U = S_A \cap S_B \in \mathcal{U}_{P(\mathscr{A})}$. Then $U \in \mathcal{U}$. Note that $U^{-1} = S_{X \smallsetminus A} \cap S_{X \smallsetminus B}$ by Remark 4.4. Moreover

$$U \circ U^{-1} = \bigcup_{x \in X} (U(x) \times U(x)) = (A \times A) \cup (B \times B).$$

Since $U \circ U^{-1} \in \mathcal{U} \circ \mathcal{U}^{-1} = \mathcal{U} \wedge \mathcal{U}^{-1}$ and $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a uniformity, there exists $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$ such that $V^2 \subseteq U \circ U^{-1}$. In particular, there is some $W \in \mathcal{U}$ such that $W \cup W^{-1} \subseteq V$. Hence

$$\bigcup_{x \in X} (W^{-1}(x) \times W^{-1}(x)) \subseteq W^{-1} \circ W \subseteq V^2 \subseteq U \circ U^{-1}.$$

Furthermore, let us check that $W(X \smallsetminus A) \cap W(X \smallsetminus B) = \emptyset$. Indeed, if $x \in W(X \smallsetminus A) \cap W(X \smallsetminus B)$ we would have $(\alpha, x), (\beta, x) \in W$ for some $\alpha \in X \smallsetminus A$ and $\beta \in X \smallsetminus B$ and thus

$$(\alpha, \beta) \in W^{-1}(x) \times W^{-1}(x) \subseteq U \circ U^{-1} = (A \times A) \cup (B \times B),$$

a contradiction.

Now, using ($\mathscr{A}$-int), we obtain $A', B' \in \mathscr{A}$ satisfying $X \smallsetminus A \subseteq A' \subseteq W(X \smallsetminus A)$ and $X \smallsetminus B \subseteq B' \subseteq W(X \smallsetminus B)$. Therefore $A' \cap B' = \emptyset$ and $A \cup A' = X = B \cup B'$, which shows that X is $\mathscr{A}$-normal. ∎

Corollary 5.2. *Let $\mathcal{U}$ be an $\mathscr{A}$-compatible quasi-uniformity on a space X that is finer than the Pervin quasi-uniformity $\mathcal{U}_{P(\mathscr{A})}$. In the case where $\mathscr{A}$ is a topology on X, if $\mathcal{U} \circ \mathcal{U}^{-1}$ is a quasi-uniformity then X is $\mathscr{A}$-normal.*

Proof: The result follows from the fact that Condition (C1) of Lemma 4.1 combined with the fact that $\mathscr{A}$ is a topology yields condition ($\mathscr{A}$-int). ∎

Remarks 5.3. (1) The case where $\mathscr{A} = \mathfrak{O}(X)$ in the preceding corollary is precisely Lemma 2(a) of [13].

(2) By (QU1), any quasi-uniformity $\mathfrak{U}$ such that $\tau(\mathfrak{U}) \subseteq \mathscr{A}$ satisfies condition ($\mathscr{A}$-int): just take $B = \mathrm{int}_{\tau(\mathfrak{U})}(U(A))$.

(3) Let $\mathscr{A}$ be closed under arbitrary unions and set

$$\mathrm{int}_{\mathscr{A}}(S) = \bigcup\{A \in \mathscr{A} \mid A \subseteq S\}$$

for any $S \subseteq X$. If $A \subseteq \mathrm{int}_{\mathscr{A}}(U(A))$ for any $U \in \mathfrak{U}$ and $A \in \mathscr{A}^{\mathsf{c}}$, then $\mathfrak{U}$ satisfies ($\mathscr{A}$-int).

By taking $\mathfrak{U}^{-1}$ for $\mathfrak{U}$ and $\mathscr{A}^{\mathsf{c}}$ for $\mathscr{A}$, Theorem 5.1 yields immediately the following dual result:

Let $\mathfrak{U}^{-1}$ be a quasi-uniformity on a space X that is finer than the Pervin quasi-uniformity $\mathfrak{U}_{P(\mathscr{A}^{\mathsf{c}})}$. If $\mathfrak{U}^{-1}$ satisfies the condition

$$\forall U \in \mathfrak{U}^{-1}, \forall A \in \mathscr{A}, \exists B \in \mathscr{A}^{\mathsf{c}} \colon A \subseteq B \subseteq U(A)$$

and $\mathfrak{U}^{-1} \circ \mathfrak{U}$ is a quasi-uniformity, then X is $\mathscr{A}^{\mathsf{c}}$-normal.

Using Remark 4.4(1) and (3.2.1) we then get the following:

Corollary 5.4. *Let $\mathfrak{U}$ be a quasi-uniformity on a space X that is finer than the Pervin quasi-uniformity $\mathfrak{U}_{P(\mathscr{A})}$. If $\mathfrak{U}$ satisfies the condition*

$$\forall U \in \mathfrak{U}^{-1}, \forall A \in \mathscr{A}, \exists B \in \mathscr{A}^{\mathsf{c}} \colon A \subseteq B \subseteq U(A) \tag{$\mathscr{A}$-cl}$$

and $\mathfrak{U}^{-1} \circ \mathfrak{U}$ is a quasi-uniformity, then X is $\mathscr{A}$-disconnected. ∎

Corollary 5.5. *Let $\mathfrak{U}$ be an $\mathscr{A}$-compatible quasi-uniformity on a space X that is finer than the Pervin quasi-uniformity $\mathfrak{U}_{P(\mathscr{A})}$. In the case where $\mathscr{A}$ is a topology on X, if $\mathfrak{U}^{-1} \circ \mathfrak{U}$ is a quasi-uniformity then X is $\mathscr{A}$-disconnected.*

Proof: The result follows from the fact that Condition (C3) of Lemma 4.2 combined with the fact that $\mathscr{A}$ is a topology yields condition ($\mathscr{A}$-cl). ∎

Remarks 5.6. (1) The case where $\mathscr{A} = \mathfrak{O}(X)$ in the preceding corollary is precisely Lemma 2(b) of [13].

(2) By (QU2), any quasi-uniformity $\mathfrak{U}$ such that $\tau(\mathfrak{U}) \subseteq \mathscr{A}$ satisfies condition ($\mathscr{A}$-cl): just take $B = \mathrm{cl}_{\tau(\mathfrak{U})}(U(A))$.

(3) Let $\mathscr{A}$ be closed under arbitrary unions and set

$$\mathrm{cl}_{\mathscr{A}}(S) = X \smallsetminus \mathrm{int}_{\mathscr{A}}(X \smallsetminus S) = \bigcap\{X \smallsetminus A \mid A \in \mathscr{A}, X \smallsetminus A \supseteq S\} \in \mathscr{A}^{\mathsf{c}}$$

for any $S \subseteq X$. If $\mathrm{cl}_{\mathscr{A}}(A) \subseteq U(A)$ for every $U \in \mathfrak{U}^{-1}$ and $A \in \mathscr{A}$, then $\mathfrak{U}$ satisfies ($\mathscr{A}$-cl).

6. On the converse results

Let X be a topological space and $\mathscr{A} \subseteq \mathfrak{O}(X)$. From now on we assume that $X \in \mathscr{A}$. We say that a cover $\mathscr{C}$ of X is an $\mathscr{A}$-cover if $C \in \mathscr{A}$ for all $C \in \mathscr{C}$.

Consider now the Pervin quasi-uniformity $\mathfrak{U}_{P(\mathscr{A})}$ induced by $\mathscr{A}$ in X. We have:

Lemma 6.1. *For each $U \in \mathfrak{U}_{P(\mathscr{A})}$ there is a finite $\mathscr{A}$-cover $\mathscr{C}$ of X such that $\bigcap_{C\in\mathscr{C}} S_C \subseteq U$.*

Proof: Let $U \in \mathfrak{U}_{P(\mathscr{A})}$. Then $\bigcap_{i=1}^{n} S_{A_i} \subseteq U$ for some $A_1, \ldots, A_n \in \mathscr{A}$. Since $S_X = X \times X$, it suffices to take $\mathscr{C} = \{A_1, \ldots, A_n, X\}$. ■

Moreover:

Lemma 6.2. *Let X be a $\mathscr{A}$-normal space and let $\mathscr{C} = \{A_1, A_2, \ldots, A_n\}$ be a finite $\mathscr{A}$-cover of X.*

(1) *If $\mathscr{A}$ is closed under finite unions, then for each $i \in \overline{n} = \{1, 2, \ldots, n\}$ there is some $V_i \in \mathscr{A}$ such that $\overline{V_i} \subseteq A_i$ and $\{V_i \mid i \in \overline{n}\}$ is a finite $\mathscr{A}$-cover of X.*

(2) *If $\mathscr{A}$ is closed under arbitrary unions, then for each $i \in \overline{n}$ there is some $V_i \in \mathscr{A}$ such that $\mathrm{cl}_{\mathscr{A}}(V_i) \subseteq A_i$ and $\{V_i \mid i \in \overline{n}\}$ is a finite $\mathscr{A}$-cover of X.*

Proof: (1) Since $\mathscr{A}$ is closed under finite unions we may apply $\mathscr{A}$-normality to A_1 and $A_2 \cup \cdots \cup A_n$ and conclude that there is some $U_1, V_1 \in \mathscr{A}$ such that $U_1 \cap V_1 = \emptyset$ and $U_1 \cup A_1 = X = V_1 \cup A_2 \cup \cdots \cup A_n$. Clearly, $V_1 \subseteq \overline{V_1} \subseteq A_1$. Now we may apply $\mathscr{A}$-normality to A_2 and $V_1 \cup A_3 \cup \cdots \cup A_n$ and conclude that there is some $U_2, V_2 \in \mathscr{A}$ such that $U_2 \cap V_2 = \emptyset$ and $U_2 \cup A_2 = X = V_2 \cup V_1 \cup A_3 \cup \cdots \cup A_n$. Clearly, $V_2 \subseteq \overline{V_2} \subseteq A_2$. Proceeding inductively we get, at step n, $U_n, V_n \in \mathscr{A}$ such that $U_n \cap V_n = \emptyset$ and $U_n \cup A_n = X = V_n \cup \cdots \cup V_2 \cup V_1$ from which it follows that $V_n \subseteq \overline{V_n} \subseteq A_n$.

In conclusion, $\{V_1, V_2, \ldots, V_n\}$ is the required $\mathscr{A}$-cover.

(2) In each step of the preceding proof we have $V_i \subseteq X \smallsetminus U_i \subseteq A_i$ with $U_i \in \mathscr{A}$. If $\mathscr{A}$ is closed under arbitrary unions (precisely the condition on $\mathscr{A}$ that ensures the existence of $\mathrm{cl}_{\mathscr{A}}(-)$), then that implies immediately $V_i \subseteq \mathrm{cl}_{\mathscr{A}}(V_i) \subseteq A_i$. ■

In the next lemma, $\mathrm{st}(x,\mathscr{D})$ denotes, as usual, the union

$$\bigcup\{D\in\mathscr{D}\mid x\in D\}.$$

Lemma 6.3. *Let $\mathscr{C}$ be a finite $\mathscr{A}$-cover of X. If X is $\mathscr{A}$-normal and $\mathscr{A}$ is a topology, then there exists a finite $\mathscr{A}$-cover $\mathscr{D}$ of X such that*

$$\{\mathrm{st}(x,\mathscr{D})\mid x\in X\}\leq\left\{\Big[\bigcap_{C\in\mathscr{C}}S_C\Big](x)\mid x\in X\right\}.$$

Proof: Let $\mathscr{C}=\{A_1,A_2,\dots,A_n\}$ be a finite $\mathscr{A}$-cover of X and let

$$\mathscr{C}'=\left\{\Big[\bigcap_{C\in\mathscr{C}}S_C\Big](x)\mid x\in X\right\}.$$

As observed in the proof of Lemma 4.3,

$$\Big[\bigcap_{C\in\mathscr{C}}S_C\Big](x)=\bigcap\{C\in\mathscr{C}\mid x\in C\}\in\mathscr{A}$$

(since $\mathscr{A}$ is closed under finite intersections). Hence $\mathscr{C}'$ is a finite $\mathscr{A}$-cover of X and by Lemma 6.2(2) there is a finite $\mathscr{A}$-cover $\{V_C\mid C\in\mathscr{C}'\}$ satisfying $\mathrm{cl}_{\mathscr{A}}(V_C)\subseteq C$ for all $C\in\mathscr{C}'$.

Now, for each $\mathscr{C}''\subseteq\mathscr{C}'$ set

$$D_{\mathscr{C}''}=(\bigcap_{C\in\mathscr{C}''}C)\smallsetminus\bigcup\{\mathrm{cl}_{\mathscr{A}}(V_C)\mid C\in\mathscr{C}'\smallsetminus\mathscr{C}''\}\quad\text{ and }\quad\mathscr{D}=\{D_{\mathscr{C}''}\mid\mathscr{C}''\subseteq\mathscr{C}'\}.$$

Note that $D_{\mathscr{C}''}=\bigcap_{C'\in\mathscr{C}'\smallsetminus\mathscr{C}''}\bigcap_{C\in\mathscr{C}''}(C\cap\mathrm{int}_{\mathscr{A}}(X\smallsetminus V_{C'}))$ and thus each $D_{\mathscr{C}''}$ is in $\mathscr{A}$. Furthermore, for each $x\in X$ let $\mathscr{C}''_x=\{C\in\mathscr{C}'\mid x\in\mathrm{cl}_{\mathscr{A}}(V_C)\}$. Since

$$D_{\mathscr{C}''_x}=\bigcap_{C'\in\mathscr{C}'\smallsetminus\mathscr{C}''_x}\ \bigcap_{C\in\mathscr{C}''_x}(C\cap\mathrm{int}_{\mathscr{A}}(X\smallsetminus V_{C'})),$$

it is clear that $x\in D_{\mathscr{C}''_x}$ and therefore $\mathscr{D}$ is a finite $\mathscr{A}$-cover of X. It suffices now to show that $\{\mathrm{st}(x,\mathscr{D})\mid x\in X\}\leq\mathscr{C}'$. So we need to check that for any $\mathscr{C}''\subseteq\mathscr{C}'$ with $x\in D_{\mathscr{C}''}$ there is some $C\in\mathscr{C}'$ such that $D_{\mathscr{C}''}\subseteq C$. Any $C\in\mathscr{C}''_x$ (so that $x\in\mathrm{cl}_{\mathscr{A}}(V_C)\subseteq C$) will do the job. Indeed:

(1) From $x\in D_{\mathscr{C}''}=(\bigcap_{C\in\mathscr{C}''}C)\smallsetminus\bigcup\{\mathrm{cl}_{\mathscr{A}}(V_C)\mid C\in\mathscr{C}'\smallsetminus\mathscr{C}''\}$ it follows that $x\notin\mathrm{cl}_{\mathscr{A}}(V'_C)$ for every $C'\in\mathscr{C}'\smallsetminus\mathscr{C}''$; but since $x\in\mathrm{cl}_{\mathscr{A}}(V_C)$, then $C\in\mathscr{C}''$.

(2) Finally, it is true that $D_{\mathscr{C}''}\subseteq C$ whenever $C\in\mathscr{C}''$ and $x\in C$:

$$\begin{aligned}D_{\mathscr{C}''}&=\bigcap_{C\in\mathscr{C}'\smallsetminus\mathscr{C}''}\ \bigcap_{C'\in\mathscr{C}''}(C'\cap(X\smallsetminus\mathrm{cl}_{\mathscr{A}}(V_C)))\\&\subseteq\bigcap_{F\in\mathscr{C}'\smallsetminus\mathscr{C}''}(C\cap(X\smallsetminus\mathrm{cl}_{\mathscr{A}}(V_F)))\subseteq C.\quad\blacksquare\end{aligned}$$

Theorem 6.4. *Let X be a topological space and let $\mathscr{A} \subseteq \mathfrak{O}(X)$ be a topology. If X is $\mathscr{A}$-normal, then $\mathfrak{U}_{P(\mathscr{A})} \circ \mathfrak{U}^{-1}_{P(\mathscr{A})}$ is a uniformity.*

Proof: Let $U \in \mathfrak{U}_{P(\mathscr{A})}$. By Lemma 6.1 there is a finite $\mathscr{A}$-cover $\mathscr{C}$ of X such that

$$\bigcap_{C \in \mathscr{C}} S_C \subseteq U. \tag{6.4.1}$$

Then, by Lemma 6.3, there exists a finite $\mathscr{A}$-cover $\mathscr{D}$ of X such that

$$\{\mathrm{st}(x, \mathscr{D}) \mid x \in X\} \leq \left\{ \Big[\bigcap_{C \in \mathscr{C}} S_C\Big](x) \mid x \in X \right\}. \tag{6.4.2}$$

Let $V = \bigcap_{D \in \mathscr{D}} S_D$ which clearly belongs to $\mathfrak{U}_{P(\mathscr{A})}$. We have

$$(V \circ V^{-1})^2 = (V \circ V^{-1}) \circ (V \circ V^{-1})^{-1} = \bigcup_{x \in X} \big((V \circ V^{-1})(x) \times (V \circ V^{-1})(x)\big).$$

Thus, by (6.4.2),

$$(V \circ V^{-1})^2 \subseteq \bigcup_{x \in X} \left[\Big[\bigcap_{C \in \mathscr{C}} S_C\Big](x) \times \Big[\bigcap_{C \in \mathscr{C}} S_C\Big](x) \right] \subseteq (\bigcap_{C \in \mathscr{C}} S_C) \circ (\bigcap_{C \in \mathscr{C}} S_C)^{-1}$$

and, finally, by (6.4.1), $(V \circ V^{-1})^2 \subseteq U \circ U^{-1}$. ∎

Since the conditions for $\mathscr{A}$ are not self-dual, we only get the following corollary:

Corollary 6.5. *Let X be a topological space and let $\mathscr{A} \subseteq \mathfrak{O}(X)$ be a co-topology. If X is $\mathscr{A}$-disconnected, then $\mathfrak{U}^{-1}_{P(\mathscr{A})} \circ \mathfrak{U}_{P(\mathscr{A})}$ is a uniformity.* ∎

Nevertheless, a careful analysis of the proof of the direct result in [13, Lemma 3(b)] reveals that it is possible to conform it to our relative setting and to obtain directly the next result.

Theorem 6.6. *Let X be a topological space and let $\mathscr{A} \subseteq \mathfrak{O}(X)$ be a topology. If X is $\mathscr{A}$-disconnected, then $\mathfrak{U}^{-1}_{P(\mathscr{A})} \circ \mathfrak{U}_{P(\mathscr{A})}$ is a uniformity.*

Indeed, for that we just need the following properties of $\mathscr{A}$-disconnected spaces:

Proposition 6.7. (1) *Let $\mathscr{A} \subseteq \mathfrak{O}(X)$ be closed under arbitrary joins. The following are equivalent:*

(i) *X is $\mathscr{A}$-disconnected.*
(ii) *$\mathrm{cl}_{\mathscr{A}}(A) \in \mathscr{A}$ for every $A \in \mathscr{A}$.*
(iii) *For any $A, B \in \mathscr{A}$, $A \cap B = \emptyset \Rightarrow \mathrm{cl}_{\mathscr{A}}(A) \cap \mathrm{cl}_{\mathscr{A}}(B) = \emptyset$.*

(2) *Moreover, if $\mathscr{A}$ is also closed under finite intersections, then X is $\mathscr{A}$-disconnected if and only if* $\mathrm{cl}_{\mathscr{A}}(A \cap B) = \mathrm{cl}_{\mathscr{A}}(A) \cap \mathrm{cl}_{\mathscr{A}}(B)$ *for every* $A, B \in \mathscr{A}$.

Proof: (1) (i)⇒(ii): Let $A \in \mathscr{A}$. Since $A \cap \mathrm{int}_{\mathscr{A}}(X \smallsetminus A) = A \cap X \smallsetminus \mathrm{cl}_{\mathscr{A}}(A) = \emptyset$, $\mathscr{A}$-disconnectedness provides some $U, V \in \mathscr{A}$ satisfying $U \cup V = X$ and $A \cap U = \emptyset = V \cap X \smallsetminus \mathrm{cl}_{\mathscr{A}}(A)$. Clearly, $U \subseteq \mathrm{int}_{\mathscr{A}}(X \smallsetminus A)$ and $V \subseteq \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(A))$. Therefore $\mathrm{int}_{\mathscr{A}}(X \smallsetminus A) \cup \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(A)) = X$, that is, $\mathrm{cl}_{\mathscr{A}}(A) \subseteq \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(A))$.

(ii)⇒(iii): Since $B \subseteq X \smallsetminus A \in \mathscr{A}^c$, it follows that $\mathrm{cl}_{\mathscr{A}}(B) \subseteq X \smallsetminus A$, that is, $A \cap \mathrm{cl}_{\mathscr{A}}(B) = \emptyset$. It then follows similarly that $\mathrm{cl}_{\mathscr{A}}(A) \cap \mathrm{cl}_{\mathscr{A}}(B) = \emptyset$, since $\mathrm{cl}_{\mathscr{A}}(B) \in \mathscr{A}$ by hypothesis.

(iii)⇒(i): Let $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$. It suffices to take $U = X \smallsetminus \mathrm{cl}_{\mathscr{A}}(A)$ and $V = X \smallsetminus \mathrm{cl}_{\mathscr{A}}(B)$.

(2) Let $U \in \mathscr{A}$ such that $U \cap A \cap B = \emptyset$. Then, by the property proved in the implication (ii)⇒(iii) above, we have $U \cap A \cap \mathrm{cl}_{\mathscr{A}}(B) = \emptyset$ (note that the assumption that $\mathscr{A}$ is closed under finite intersections is needed here so that $U \cap A \in \mathscr{A}$). Hence $U \cap A \cap \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(B)) = \emptyset$. This shows that

$$A \cap \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(B)) \subseteq \mathrm{cl}_{\mathscr{A}}(A \cap B)$$

and thus that $A \cap \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(B)) \subseteq \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(A \cap B))$. Similarly,

$$\mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(A)) \cap \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(B)) \subseteq \mathrm{int}_{\mathscr{A}}(\mathrm{cl}_{\mathscr{A}}(A \cap B)).$$

The conclusion follows now by application of characterization (ii).

The converse is obvious since the hypothesis implies assertion (iii) above. ■

Corollary 6.8. *Let X be a topological space and let $\mathscr{A} \subseteq \mathfrak{O}(X)$ be a topology. Then:*

(1) $\mathcal{U}_{P(\mathscr{A})} \circ \mathcal{U}^{-1}_{P(\mathscr{A})}$ *is a uniformity if and only if X is $\mathscr{A}$-normal.*
(2) $\mathcal{U}^{-1}_{P(\mathscr{A})} \circ \mathcal{U}_{P(\mathscr{A})}$ *is a uniformity if and only if X is $\mathscr{A}$-disconnected.*
(3) $\mathcal{U}_{P(\mathscr{A})}$ *and* $\mathcal{U}^{-1}_{P(\mathscr{A})}$ *permute if and only if X is $\mathscr{A}$-normal and $\mathscr{A}$-disconnected.*

Proof: Assertion (1) follows from Corollary 5.2 and Theorem 6.4, while assertion (2) follows from Corollary 5.5 and Theorem 6.6.

(3) If $\mathcal{U}_{P(\mathscr{A})}$ and $\mathcal{U}^{-1}_{P(\mathscr{A})}$ permute, then by (5.1.1) both $\mathcal{U}_{P(\mathscr{A})} \circ \mathcal{U}^{-1}_{P(\mathscr{A})}$ and $\mathcal{U}^{-1}_{P(\mathscr{A})} \circ \mathcal{U}_{P(\mathscr{A})}$ are uniformities and so by Corollaries 5.2 and 5.5 X is $\mathscr{A}$-normal and $\mathscr{A}$-disconnected. Conversely, if X is $\mathscr{A}$-normal and $\mathscr{A}$-disconnected, then by (5.1.1) and Theorems 6.4 and 6.6 we have

$$\mathcal{U}_{P(\mathscr{A})} \circ \mathcal{U}^{-1}_{P(\mathscr{A})} = \mathcal{U}_{P(\mathscr{A})} \wedge \mathcal{U}^{-1}_{P(\mathscr{A})} = \mathcal{U}^{-1}_{P(\mathscr{A})} \circ \mathcal{U}_{P(\mathscr{A})}$$

and thus $\mathcal{U}_{P(\mathscr{A})}$ and $\mathcal{U}^{-1}_{P(\mathscr{A})}$ permute. ∎

This result yields, in particular, the part of Corollary 9 of [13] about the Pervin quasi-uniformity.

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