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A PROOF OF THE $C^{p'}$-REGULARITY CONJECTURE IN THE PLANE

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ABSTRACT: We establish a new oscillation estimate for solutions of nonlinear partial differential equations of elliptic, degenerate type. This new tool yields a precise control on the growth rate of solutions near their set of critical points, where ellipticity degenerates. As a consequence, we are able to prove the planar counterpart of the longstanding conjecture that solutions of the degenerate p-Poisson equation with a bounded source are locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$; this regularity is optimal.

KEYWORDS: Nonlinear pdes, regularity theory, sharp estimates.
AMS SUBJECT CLASSIFICATION (2010): Primary 35B65. Secondary 35J60, 35J70.

1. Introduction

In this paper we investigate sharp $C^{1,\alpha}$-regularity estimates for solutions of the degenerate elliptic equation, with a bounded source,

$$-\Delta_p u = f(x) \in L^\infty(B_1), \qquad p > 2. \tag{1.1}$$

Establishing optimal regularity estimates is quite often a delicate matter and, in particular, $f(x) \in L^\infty$ is known to be a borderline condition for regularity. In the linear, uniformly elliptic case $p = 2$, solutions of

$$-\Delta u = f(x) \in L^\infty(B_1)$$

are locally in $C^{1,\alpha}$, for every $\alpha \in (0, 1)$, but may fail to be in $C^{1,1}$. Obtaining such an estimate in specific situations, like free boundary problems, often involves a deep and fine analysis.

In the degenerate setting $p > 2$, the smoothing effects of the operator are far less efficient. Nonetheless, it is well established, see for instance [3, 16],

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that a weak solution to (1.1) is locally of class $C^{1,\beta}$, for some exponent $\beta > 0$ depending on dimension and p. If p' denotes the conjugate of p, i.e.,

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

the radial symmetric example

$$-\Delta_p \left(c_p |x|^{p'} \right) = 1$$

sets the limits to the optimal regularity and gives rise to the following well known open problem among experts in the field.

Conjecture ($C^{p'}$-regularity conjecture). *Solutions to* (1.1) *are locally of class* $C^{1,\frac{1}{p-1}} = C^{p'}$.

This problem touches very subtle issues in regularity theory. As mentioned above, the conjecture is not true in the linear, uniformly elliptic setting, $p = 2$, where merely $C^{1,\mathrm{LogLip}}$ estimates are possible. Notice further that a positive answer implies that $|x|^{p'}$ – a function whose p-laplacian is constant (real analytic) – is the least regular among all functions whose p-laplacian is bounded. This is, at first sight, counterintuitive.

While this question still seems out of reach in the d-dimensional space, when restricted to the plane, more is known about the underlying regularity theory for this class of problems. By means of a hodograph transformation, Iwaniec and Manfredi in [6] give explicit estimates for the Hölder continuity exponent of the gradient of p-harmonic functions. In a somewhat related issue, let us mention that, yet in the plane, Evans and Savin proved in [5] (see also [9]) that infinity harmonic functions, i.e., viscosity solutions of

$$\Delta_\infty u := u_{x_i x_j} u_{x_i} u_{x_j} = 0,$$

are locally of class $C^{1,\gamma}$ for some $0 < \gamma \ll 1$. Whether infinity harmonic functions are of class C^1 in higher dimensions is still a major open problem in the field.

Concerning the $C^{p'}$-regularity conjecture, Lindgren and Lindqvist [8], see also [7], have recently proved an asymptotic version, namely that solutions to (1.1), in the plane, are locally of class $C^{p'-\epsilon}$, for $\epsilon > 0$. However, passing from such an asymptotic result to the sharp, full conjecture requires new insights and a novel approach. Our main result is the following.

Theorem 1.1. *Let $B_1 \subset \mathbb{R}^2$, and let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f(x)$, with $f \in L^\infty(B_1)$. Then $u \in C^{p'}(B_{1/2})$ and*

$$\|u\|_{C^{p'}(B_{1/2})} \leq C_p \left(\|f\|_{L^\infty(B_1)}^{\frac{1}{p-1}} + \|u\|_{L^p(B_1)} \right).$$

The proof is based on a new oscillation estimate (Theorem 4.3), which is interesting on its own. It gives a precise control on the oscillation of a solution to (1.1) in terms of the magnitude of its gradient. The insight to obtain such a refined control comes from the striking results in [12], where *improved* regularity estimates are obtained for degenerate equations precisely along the set of critical points, $\{\nabla u = 0\}$.

The paper is organized as follows. To render the paper reasonably self-contained, we gather in section 2 a few tools and well known results that will be used in the proof of Theorem 1.1. In section 3 we introduce C^1- small correctors that link the regularity theory for (1.1) to that of p-harmonic functions. The key, new oscillation estimate is delivered in section 4, and in section 5 we conclude the proof of the main theorem.

2. Warming up

In this section we revisit the $C^{1,\alpha}$ regularity theory for p-harmonic functions, i.e., solutions to the homogeneous equation

$$-\Delta_p u = 0. \tag{2.1}$$

That p-harmonic functions are locally of class $C^{1,\alpha(d,p)}$, for some exponent $0 < \alpha(d,p) < 1$, that depends dimension d and power exponent p, is known since the late 60's (see [17]). Away from the set of critical points

$$\mathcal{S}(u) := \{x \mid \nabla u(x) = 0\},$$

p-harmonic functions are C^∞-smooth; however $C^{1,\alpha(d,p)}_{\text{loc}}$ is in fact the best possible regularity class since, along $\mathcal{S}(u)$, the Hessian of a p-harmonic function may become unbounded.

Very little, if anything, is known concerning the value of the sharp Hölder exponent $\alpha(d,p)$ when $d \geq 3$. In the plane, $d = 2$, a remarkable result, due to Iwaniec and Manfredi in [6], assures that

$$\alpha(2,p) = \frac{1}{6}\left(\frac{p}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}} \right). \tag{2.2}$$

Note, for $p > 2$, the following strict inequality

$$\alpha(2,p) > \frac{1}{p-1} \tag{2.3}$$

holds true. Let us write the above conclusions as a proposition for future reference.

Proposition 2.1 (Iwaniec and Manfredi). *For any $p > 2$, there exists a $0 < \tau_0 < \frac{p-2}{p-1}$ such that p-harmonic functions in $B_1 \subset \mathbb{R}^2$ are locally of class $C^{p'+\tau_0}$.*

It is also well known that functions whose p-laplacian is bounded are locally of class $C^{1,\beta}$, for some β depending on d and p. A proof of this fact may be found for instance in [3, 16].

Proposition 2.2 ($C^{1,\beta}$-estimates). *Let $B_1 \subset \mathbb{R}^d$ and $u \in W^{1,p}(B_1)$ be a weak solution to* (1.1). *There exists a constant $\beta(d,p)$ such that $u \in C^{1,\beta(d,p)}(B_{1/2})$ and*

$$\|u\|_{C^{1,\beta(d,p)}(B_{1/2})} \leq C(d,p,\|u\|_{L^p(B_1)},\|f\|_{L^\infty(B_1)}).$$

Obviously, since a p-harmonic function has a bounded p-laplacian,

$$\beta(d,p) \leq \alpha(d,p). \tag{2.4}$$

Moreover, due to the explicit solution $\Delta_p(c_p|x|^{p'}) = 1$, it is clear that

$$\beta(d,p) \leq \frac{1}{p-1}.$$

Hence, in view of (2.3), strict inequality occurs in (2.4) when $d = 2$. Whether the strict inequality is true in higher dimensions is a tantalizing question.

In the next three sections we will deliver a proof of the implication

$$\alpha(d,p) > \frac{1}{p-1} \quad \Longrightarrow \quad \beta(d,p) = \frac{1}{p-1}. \tag{2.5}$$

The proof holds for any dimension and, from the above discussion, yields the $C^{p'}$-conjecture in the plane. What prevents us from proving the conjecture in higher dimensions is simply not knowing if the hypothesis in (2.5) holds for $d \geq 3$.

3. Existence of C^1-small correctors

In this section, we show that if u is a normalized solution of

$$-\Delta_p u = f(x),$$

and $\|f\|_\infty \ll 1$, then we can find a C^1 corrector ξ, with $\|\xi\|_{C^1} \ll 1$, such that $u + \xi$ is p-harmonic. This will allow us to frame the $C^{p'}$ conjecture into the formalism of the so called geometric tangential analysis, e.g. [4], [1, 2] and [10, 11, 12, 13, 14, 15]. Here is the precise statement.

Lemma 3.1. *Let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$. Given $\epsilon > 0$, there exists $\delta = \delta(p, d, \epsilon) > 0$ such that if $\|f\|_\infty \leq \delta$ then we can find a corrector $\xi \in C^1(B_{1/2})$, with*

$$|\xi(x)| \leq \epsilon \quad \text{and} \quad |\nabla \xi(x)| \leq \epsilon, \quad \text{in } B_{1/2} \tag{3.1}$$

such that

$$-\Delta_p(u + \xi) = 0 \quad \text{in } B_{1/2}. \tag{3.2}$$

Proof: Suppose the result does not hold. We can then find $\epsilon_0 > 0$ and sequences of functions (u_j) and (f_j) in $W^{1,p}(B_1)$ and $L^\infty(B_1)$, respectively, such that

$$-\Delta_p u_j = f_j \quad \text{in } B_1; \qquad \|u_j\|_\infty \leq 1; \qquad \|f_j\|_\infty \leq 1/j$$

but, nonetheless, for every $\xi \in C^1(B_{1/2})$ such that

$$-\Delta_p(u_j + \xi) = 0 \quad \text{in } B_{1/2},$$

we have either $|\xi(x_0)| > \epsilon_0$ or $|\nabla \xi(x_0)| > \epsilon_0$, for a certain $x_0 \in B_{1/2}$.

From classical estimates for the p-Poisson equation (Proposition 2.2), we can extract a subsequence, such that, upon relabelling,

$$u_j \longrightarrow u_\infty$$

in $C^1(B_{1/2})$ as $j \to \infty$. Passing to the limit in the pde, we obtain

$$-\Delta_p u_\infty = 0 \quad \text{in } B_{1/2}, \qquad \text{with} \quad \|u_\infty\|_\infty \leq 1.$$

Now, let $\xi_j := u_\infty - u_j$. For $j_* \gg 1$, we have

$$-\Delta_p(u_{j_*} + \xi_{j_*}) = -\Delta_p u_\infty = 0 \quad \text{in } B_{1/2}$$

and

$$|\xi_{j_*}(x)| \leq \epsilon_0 \quad \text{and} \quad |\nabla \xi_{j_*}(x)| \leq \epsilon_0, \quad \forall x \in B_{1/2},$$

thus reaching a contradiction. ■

We conclude this section by commenting that in order to prove Theorem 1.1 it is enough to establish it for normalized solutions with small RHS, i.e., with $\|f\|_\infty \leq \delta_0$. Indeed, if u verifies $-\Delta_p u = f(x)$, with $f \in L^\infty$, then the function

$$v(x) := \frac{u(\theta x)}{\|u\|_\infty}$$

is obviously normalized and

$$-\Delta_p v = \frac{\theta^p}{\|u\|_\infty^{p-1}} f(\theta x).$$

Thus, choosing

$$\theta := \sqrt[p]{\frac{\delta_0 \|u\|_\infty^{p-1}}{\|f\|_\infty}},$$

v satisfies (1.1), with small RHS. Once Theorem 1.1 is proven for v, it immediately gives the corresponding estimate for u.

4. Analysis on the critical set

In this section, based on an iterative reasoning, we establish the main tool that allows us to prove the $C^{p'}$ conjecture in the plane. The following result is the first step in the iteration.

Lemma 4.1. *There exists $0 < \lambda_0 < 1/2$ and $\delta_0 > 0$ such that if $\|f\|_\infty \leq \delta_0$ and $u \in W^{1,p}(B_1)$ is a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$, then*

$$\sup_{x \in B_{\lambda_0}} \Big| u(x) - [u(0) + \nabla u(0) \cdot x] \Big| \leq {\lambda_0}^{p'}.$$

Proof: Take $\epsilon > 0$ to be fixed later, apply the previous lemma to find δ_0 and, under the smallness assumption on f, a respective corrector ξ satisfying (3.1) and (3.2). As $(u+\xi)$ is p-harmonic in $B_{\lambda_0} \subset B_{1/2}$ and, in view of Proposition 2.1, $(u+\xi) \in C^{p'+\tau_0}.B_{\lambda_0}$, we can estimate in B_{λ_0},

$$\begin{aligned} |u(x) - [u(0) + \nabla u(0) \cdot x]| &\leq |(u+\xi)(x) - [(u+\xi)(0) + \nabla(u+\xi)(0) \cdot x]| \\ &\quad + |\xi(x)| + |\xi(0)| + |\nabla \xi(0) \cdot x| \\ &\leq C {\lambda_0}^{p'+\tau_0} + 3\epsilon. \end{aligned}$$

We are also using the smallness of the corrector, assured by Lemma 3.1. In order to complete the proof, we now make universal choices. Initially we choose $\lambda_0 \ll 1/2$ such that

$$C{\lambda_0}^{p'+\tau_0} < \frac{1}{2}{\lambda_0}^{p'}.$$

In the sequel, we take

$$\epsilon = \frac{1}{6}{\lambda_0}^{p'},$$

which determines the smallness assumption on $\|f\|_\infty$ – constant $\delta_0 > 0$ in the statement of this current lemma – through the conclusion of Lemma 3.1. Lemma 4.1 is proven. ■

The conclusion of Lemma 4.1 does not, *per se*, allow an iteration since no obvious pde is satisfied by $u + \ell$, when ℓ is an affine function. Nonetheless, it provides the following information on the oscillation of u in B_{λ_0}.

Corollary 4.2. *Under the assumptions of the previous lemma,*

$$\sup_{x \in B_{\lambda_0}} |u(x) - u(0)| \le {\lambda_0}^{p'} + |\nabla u(0)|\lambda_0.$$

Proof: This is a immediate application of triangular inequality. ■

The idea is now to iterate Corollary 4.2 in dyadic balls, keeping a precise track on the magnitude of the influence of $|\nabla u(0)|$.

Theorem 4.3. *Under the same assumptions of Lemma 4.1, there exists a constant $C > 1$ depending only on p, such that*

$$\sup_{x \in B_r} |u(x) - u(0)| \le C r^{p'} \left(1 + |\nabla u(0)|\, r^{\frac{1}{1-p}}\right),$$

holds for all $r > 0$.

Proof: We proceed by geometric iteration. Consider the universal constants λ_0 and δ_0 obtained in the previous Lemma 4.1 and let

$$v(x) = \frac{u(\lambda_0 x) - u(0)}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0}, \qquad x \in B_1.$$

We have $\|v\|_\infty \le 1$, $v(0) = 0$, and

$$\nabla v(0) = \frac{\lambda_0}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0}\nabla u(0),$$

and

$$-\Delta_p v = \frac{\lambda_0^p}{\left({\lambda_0}^{p'} + |\nabla u(0)|\lambda_0\right)^{p-1}} f(\lambda_0 x) \le \frac{\lambda_0^p}{{\lambda_0}^{p'(p-1)}} |f(\lambda_0 x)| \le \delta_0,$$

which entitles v to Lemma 4.1. Thus

$$\sup_{x\in B_{\lambda_0}} |v(x) - v(0)| \le {\lambda_0}^{p'} + |\nabla v(0)|\lambda_0,$$

which reads

$$\sup_{x\in B_{\lambda_0}} \left| \frac{u(\lambda_0 x) - u(0)}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0} \right| \le {\lambda_0}^{p'} + \left| \frac{\lambda_0}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0} \nabla u(0) \right| \lambda_0,$$

and hence

$$\sup_{x\in B^2_{\lambda_0}} |u(x) - u(0)| \le {\lambda_0}^{p'} \left[{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0 \right] + |\nabla u(0)|\, \lambda_0^2.$$

In the sequel, we define

$$a_k := \sup_{x\in B_{\lambda_0^k}} |u(x) - u(0)|,$$

and set

$$b_k := \frac{a_k}{\lambda_0^{kp'}}.$$

Iterating the previous reasoning we obtain the recurrence law

$$a_{k+1} \le {\lambda_0}^{p'} a_k + |\nabla u(0)|\, \lambda_0^{k+1}.$$

Consequently, we estimate

$$b_{k+1} = \frac{a_{k+1}}{\lambda_0^{(k+1)p'}} \le \frac{{\lambda_0}^{p'} a_k + |\nabla u(0)|\, \lambda_0^{k+1}}{\lambda_0^{(k+1)p'}} = b_k + |\nabla u(0)|\, \lambda_0^{-(k+1)(p'-1)}.$$

Now, given $0 < r \ll \lambda_0$, let $k \in \mathbb{N}$ be such that $\lambda_0^{k+1} < r \le \lambda_0^k$. Then

$$
\begin{aligned}
\sup_{x\in B_r} \frac{|u(x)-u(0)|}{r^{p'}} &\le \sup_{x\in B_{\lambda_0^k}} \frac{|u(x)-u(0)|}{(\lambda_0^{k+1})^{p'}} = \frac{b_k}{\lambda_0^{p'}} \\
&\le \frac{b_0 + |\nabla u(0)| \displaystyle\sum_{i=1}^{k} \left[\lambda_0^{-(p'-1)}\right]^i}{\lambda_0^{p'}} \\
&= \frac{a_0 + |\nabla u(0)|\, \lambda_0^{-(p'-1)} \frac{\lambda_0^{-(p'-1)k}-1}{\lambda_0^{-(p'-1)}-1}}{\lambda_0^{p'}} \\
&\le 2 + C(\lambda_0, p')\, |\nabla u(0)|\, r^{-(p'-1)} \\
&\le C\left(1 + |\nabla u(0)|\, r^{\frac{1}{1-p}}\right),
\end{aligned}
$$

as desired. Observe that λ_0 is a universal constant. ■

In accordance to [12], Theorem 4.3 provides the aimed regularity along the set of critical points of u, $|\nabla u|^{-1}(0)$. In the next section we show how Theorem 4.3 can be used in its full strength to yield $C^{p'}$ regularity at any point, regardless of the value of $|\nabla u|$; it will be a softer analysis.

5. Analysis on the set of non-degenerate points

We now analyze the oscillation decay around points where the gradient is large. Recall our ultimate goal is to show that

$$
\sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| \le C\, r^{p'}, \quad \forall\, 0 < r \ll 1.
$$

The idea is that Theorem 4.3 gives the conclusion in the hard case, namely where the gradient is small. For large values of $|\nabla u|$, the operator is uniformly elliptic and hence stronger estimates are available. The formal way of performing this analysis is by splitting it in two cases. When $|\nabla u(0)| \le r^{\frac{1}{p-1}}$, then Theorem 4.3 gives

$$
\begin{aligned}
\sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| &\le \sup_{x\in B_r} |u(x)-u(0)| + |\nabla u(0)|\, r \\
&\le (C+1) r^{p'}.
\end{aligned}
$$

For the complementary case, i.e., when $|\nabla u(0)| > r^{\frac{1}{p-1}}$, we argue as follows. Define $\mu := |\nabla u(0)|^{p-1}$ and take

$$w(x) := \frac{u(\mu x) - u(0)}{\mu^{p'}}.$$

Clearly

$$w(0) = 0, \quad |\nabla w(0)| = 1 \quad \text{and } \Delta_p w = f(\mu x) \in L^\infty.$$

Moreover, from Theorem 4.3, it follows that

$$\sup_{x \in B_1} |w(x)| = \sup_{x \in B_\mu} \frac{|u(x) - u(0)|}{\mu^{p'}} \leq C,$$

since $\mu^{\frac{1}{p-1}} = |\nabla u(0)|$. From classical $C^{1,\alpha}$ regularity estimates, Proposition 2.2, there exists a radius ρ_0, depending only on the data, such that

$$|\nabla w(x)| \geq \frac{1}{2}, \quad \forall\, x \in B_{\rho_0}.$$

This implies that, in B_{ρ_0}, w solves a uniformly elliptic equation. In particular, we have

$$w \in C^{1,\beta}(B_{\rho_0}), \quad \text{for some } \frac{1}{p-1} \leq \beta < 1.$$

As an immediate consequence,

$$\sup_{x \in B_r} \Big| w(x) - \nabla w(0) \cdot x \Big| \leq C\, r^{1+\beta}, \quad \forall\, 0 < r < \frac{\rho_0}{2}$$

which, in terms of u, reads

$$\sup_{x \in B_r} \Big| \frac{u(\mu x) - u(0)}{\mu^{p'}} - \mu^{1-p'} \nabla u(0) \cdot x \Big| \leq C\, r^{1+\beta}.$$

Since $p' \leq 1 + \beta$, we conclude

$$\sup_{x \in B_r} \Big| u(x) - [u(0) + \nabla u(0) \cdot x] \Big| \leq C\, r^{p'}, \quad \forall\, 0 < r < \mu \frac{\rho_0}{2}.$$

Finally, for $\mu\frac{\rho_0}{2} \le r < \mu$, we have

$$\begin{aligned}\sup_{x\in B_r}\left|u(x)-[u(0)+\nabla u(0)\cdot x]\right| &\le \sup_{x\in B_\mu}\left|u(x)-[u(0)+\nabla u(0)\cdot x]\right| \\ &\le \sup_{x\in B_\mu}|u(x)-u(0)|+|\nabla u(0)|\,\mu \\ &\le (C+1)\mu^{p'} \\ &\le C\left(\frac{2r}{\rho_0}\right)^{p'} \\ &= Cr^{p'}.\end{aligned}$$

In view of the reduction discussed at the end of Section 3, the proof of Theorem 1.1 is complete.

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