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LETTER TO THE EDITOR
on replicas of known results

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The aim of this letter is to point out some questionable issues raised by a paper of R. Fernandes published in 2009 [2]. This paper announces refinements of Cauchy's interlacing inequalities for the eigenvalues of special Hermitian matrices, but no refinement is given; moreover, referencing is defective when overlooking G. Godsil's book [4], and crediting to [3] the versions of the Christoffel-Darboux identities developed in [4].

We shall use the notations, page numbers and result numbers of [2]. The main result of the paper is theorem 5.1, which may be briefly described as follows: the full power of Cauchy's interlacing theorem for arbitrary Hermitian matrices is used to prove the interlacing theorem for Hermitian matrices of a special kind.

Let us state the Cauchy interlacing theorem in the following form: if A is an $n \times n$ Hermitian matrix and Π is a principal submatrix of order $n-1$, with eigenvalues $\pi_1 \leqslant \cdots \leqslant \pi_{n-1}$, then *the i-th eigenvalue of A lies in the interval* $[\pi_{i-1}, \pi_i]$, for $i = 1, \ldots, n$, with the convention $\pi_0 = -\infty$ and $\pi_n = +\infty$.

In theorem 5.1 the matrices A and Π have special structures which do not really matter (the graph of A has a cut edge, etc.) and the conclusion is

it is possible to choose exactly one distinct eigenvalue of A in each interval $[\pi_{i-1}, \pi_i]$.

This sentence seems to be taken from [1], where variants of it are lingo to express Cauchy interlacing often with open intervals to express strict interlacing. However, if all intervals are closed, there is no improvement on Cauchy interlacing, as example 5.2 clearly illustrates. Note that theorem 5.1 is a particular case of [1, corollary 5.5] for Hermitian matrices; moreover, theorems 5.3 and 5.4 are simple consequences of the general Cauchy interlacing theorem (as well as almost all section 5 of [1]).

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A feature of [2] is a systematic reference to [3], a critical example being section 4 on the Christoffel-Darboux identities. The history of these identities is told in C. Godsil's book [4, chapter 4], with an account of their most general form, due to Godsil himself, and valid for arbitrary complex matrices [4, p. 61]. To present Godsil's version of the identities, we let A be an n-square matrix, and denote by G the weighted directed graph associated with A, with weight function $\omega(i,j) = a_{ij}$. The set of all (oriented) paths from i to j is denoted $\mathcal{P}_{ij}$ ($\mathcal{P}_{ii} = \{i\}$); if P is a path in G, $A(P)$ denotes the principal submatrix of A obtained by deleting A's rows and columns having indices in P; $\omega(P)$, the *weight* of path P, is $\prod_{(i,j)\in P} a_{ij}$ (if $P = i$, $\omega(P) = 1$); $\phi(A, x)$ denotes the characteristic polynomial of A, and $\phi_{ij}(A, x)$ is the ij-entry of the adjugate of $xI - A$. Godsil formulas in [4, pp. 56,60] are:

$$\phi_{ij}(A,x)\phi(A,y) - \phi_{ij}(A,y)\phi(A,x) = (y-x)\sum_{k=1}^{n} \phi_{ik}(A,x)\phi_{kj}(A,y) \tag{1}$$

$$\phi_{rs}(A,x) = \sum_{P\in\mathcal{P}_{rs}} \omega(P)\phi(A(P),x). \tag{2}$$

The first is the proper Christoffel-Darboux identity, and the second may be used to replace some or all occurrences of the ϕ_{rs} in (1) as desired; we get, for example:

$$\begin{aligned}\phi_{ij}(A,x)\phi(A,y) - \phi_{ij}(A,y)\phi(A,x) &= \\ = (y-x)\sum_{k=1}^{n}\sum_{P\in\mathcal{P}_{ik}}\sum_{Q\in\mathcal{P}_{kj}} &\omega(P)\omega(Q)\ \phi(A(P),x)\ \phi(A(Q),y).\end{aligned} \tag{3}$$

When A is Hermitian and G is a tree, $\mathcal{P}_{ik}$ is a singleton, say $\mathcal{P}_{ik} = \{P_{ik}\}$, and $\omega(P_{ik}) = \overline{\omega(P_{ki})}$. So theorem 4.1 follows by trivial specialization of Godsil's formula (3). And the same can be said of corollaries 4.3 and 4.5, which are the direct outcome of formulas in [4, pp. 60, 70]. The author unduely attributed these results to [3]; an extenuating circumstance is the fact that the author of [3] states and proves these particular cases of results of [4] with no attribution to [4].

When A is Hermitian and G is the cycle C_n, we get theorem 4.2 by mere replacement in Godsil's formula (3) and term regrouping. We may draw the same kind of conclusions upon corollaries 4.4 and 4.6 as compared to the much more general formulas found in Godsil [4, pp. 60, 70].

Finally, note the complexity of the Christoffel-Darboux-Godsil replicated formulas and proofs of [2] and [3]. In contrast, Godsil makes it really candid and simple, and it is instructive to quote the very beginning of [4, §4]:

> *The identity we are about to derive looks somewhat complicated at first, and its proof is so short that it is difficult to believe that it can have any content.*

References

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