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THE SIGNIFICANCE OF LIE THEORY IN THE GEOMETRY OF ROLLING MOTIONS

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ABSTRACT: The main goal of this paper is to present a unifying theory to describe the pure rolling motions of Riemannian symmetric spaces. We make a clear connection between the structure of the kinematic equations of rolling and the natural decomposition of the Lie algebra associated to the symmetric space. This emphasizes the relevance of Lie theory in the geometry of rolling manifolds. It becomes clear why many particular examples scattered through the existing literature always show a common pattern.

1. Introduction

Riemannian symmetric spaces play an important role in many areas that are interrelated to information geometry, such as image processing, machine learning and data analysis.

Examples of symmetric spaces that became popular in these areas are, for instance: the Graßmann manifold, each point of which is associated to a set of images; the Essential manifold, which parameterizes the epipolar constraint encoding the relation between correspondences across two images of the same scene taken from two different locations; the manifold of special orthogonal matrices which plays an important role in biomedical applications, ranging from feature and object detection tasks to image enhancement and image restoration techniques.

Image interpolation is one of the most elementary image processing tasks. Many image interpolation techniques have been proposed in the literature. When the data is represented on some manifold, one approach that is quite effective is based on the rolling motions of a manifold over another, subject to nonholonomic constraints of *"no-slip"* and *"no-twist"*. The main idea behind these algorithms is to use rolling motions to project the data from

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the manifold to a simpler space. Then classical methods can be applied and thereafter one rolls back the solution in order to solve the initial problem on the manifold.

In this paper, we concentrate on rolling motions of symmetric homogeneous spaces over the affine tangent space at a point. Our main goal is to find a unifying theory that incorporates all the existing scattered results and explains the common pattern that is observed in the kinematic equations of many rolling motions. A preliminary and shorter study of this subject is presented in [9].

The structure of the paper is the following. Section 2 introduces the necessary background, including the definition of rolling subject to the constraints of *"no-slip"* and *"no-twist"*, and the fundamentals of homogeneous symmetric spaces. Our results are given in Section 3. The main result is stated in Theorem 6, where a strong relationship between rolling maps and the structure of the Lie algebra associated to the symmetric space is revealed. Theorem 7 shows how to generate left-invariant parallel vector fields on symmetric spaces from rolling maps. These results are illustrated by several examples. The first three examples concern manifolds embedded in Euclidean space. Example 1 shows how the Lie algebra forces the structure of the kinematic equations for the rolling sphere and this is also illustrated with the Graßmann manifold in Example 2 and the Essential manifold in Example 3. The next two examples deal with non-Euclidean manifolds of co-dimension one, namely the Lorentzian sphere, which is a pseudo-Riemannian manifold, and the ellipsoid equipped with a left-invariant metric. Finally, the last Example 6 contains the pseudo-orthogonal groups and shows that the theory developed in this paper can be extended to manifolds that are not sub-manifolds of Euclidean space neither have co-dimension one. We finish with a few concluding remarks.

2. Preliminaries

We are interested in submanifolds of a Riemannian manifold $\widetilde{\mathbf{M}}$. Typically, $\widetilde{\mathbf{M}}$ will be the Euclidean space $\mathbb{R}^m$.

2.1. Rolling Maps. In this section we introduce a rolling map of submanifolds isometrically embedded in a Riemannian manifold. The definition of rolling is a generalization of that given in [12, Appendix B] applicable to a general situation, where the embedding space $\mathbb{R}^m$ is replaced by an orientable

Riemannian manifold $\widetilde{\mathbf{M}}$, *cf.* [5]. We assume here and in the remainder of this paper that all manifolds are connected and orientable.

Let $\widetilde{\mathbf{M}}$ be a Riemannian complete m-dimensional manifold and let $\widetilde{\mathfrak{G}}$ be the group of isometries on $\widetilde{\mathbf{M}}$. Let $I \subset \mathbb{R}$ be a closed interval. From now on, we closely follow the notations used in [5].

Definition 1. *Let $\mathbf{M}$ and $\mathbf{M}_0$ be two n-manifolds isometrically embedded in an m-dimensional Riemannian manifold $\widetilde{\mathbf{M}}$. Then a* rolling *of $\mathbf{M}$ on $\mathbf{M}_0$ without slipping or twisting is a map $\chi \colon I \to \widetilde{\mathfrak{G}}$ satisfying the following conditions.*

Rolling: *There is a piecewise smooth rolling curve on $\mathbf{M}$ given by $\sigma \colon I \to \mathbf{M}$ such that:*
(a) $\chi(t) \cdot \sigma(t) \in \mathbf{M}_0$, and
(b) $\mathbf{T}_{\chi(t)\cdot\sigma(t)}(\chi(t)(\mathbf{M})) = \mathbf{T}_{\chi(t)\cdot\sigma(t)}\mathbf{M}_0$, for all $t \in I$.
These properties imply that at each point of contact, both manifolds, $\mathbf{M}_0$ and $\chi(t)(\mathbf{M})$, have the same tangent space. This is identified as a subspace of the tangent space of $\widetilde{\mathbf{M}}$ at the considered point. The curve $\sigma_0 \colon I \to \mathbf{M}_0$ defined by $\sigma_0(t) := \chi(t) \cdot \sigma(t)$ is called the development curve *of σ.*

No-slip: *$\dot{\sigma}_0(t) = \chi(t)_* \cdot \dot{\sigma}(t)$, for almost all $t \in I$. This condition expresses the fact that the two curves have the same velocity at the point of contact.*

No-twist: *the two complementary conditions:*
***tangential*:** $(\dot{\chi}(t)\ \chi(t)^{-1})_*(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0) \subset \mathbf{T}^{\perp}_{\sigma_0(t)}\mathbf{M}_0$, *and*
***normal*:** $(\dot{\chi}(t)\ \chi(t)^{-1})_*(\mathbf{T}^{\perp}_{\sigma_0(t)}\mathbf{M}_0) \subset \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$, *for almost all $t \in I$.*

Figure 1 illustrates the rolling motion of $\mathbf{S}^2$ upon another two dimensional manifold along development curve σ_0.

We conclude this part by a crucial observation about the operator $(\dot{\chi}\ \chi^{-1})_*$ made by Sharpe in [12, page 379], when $\widetilde{\mathbf{M}}$ is the Euclidean space, and in [5] in a more general setting. If χ is a rolling map of $\mathbf{M}$ upon $\mathbf{M}_0$, then in suitable coordinates in a neighbourhood of $p \in \mathbf{M}_0$ we may choose orthonormal basis in $\mathbf{T}_p\widetilde{\mathbf{M}} = \mathbf{T}_p\mathbf{M}_0 \oplus \mathbf{T}^{\perp}_p\mathbf{M}_0$ so that the operator $(\dot{\chi}\ \chi^{-1})_*$ has the matrix form

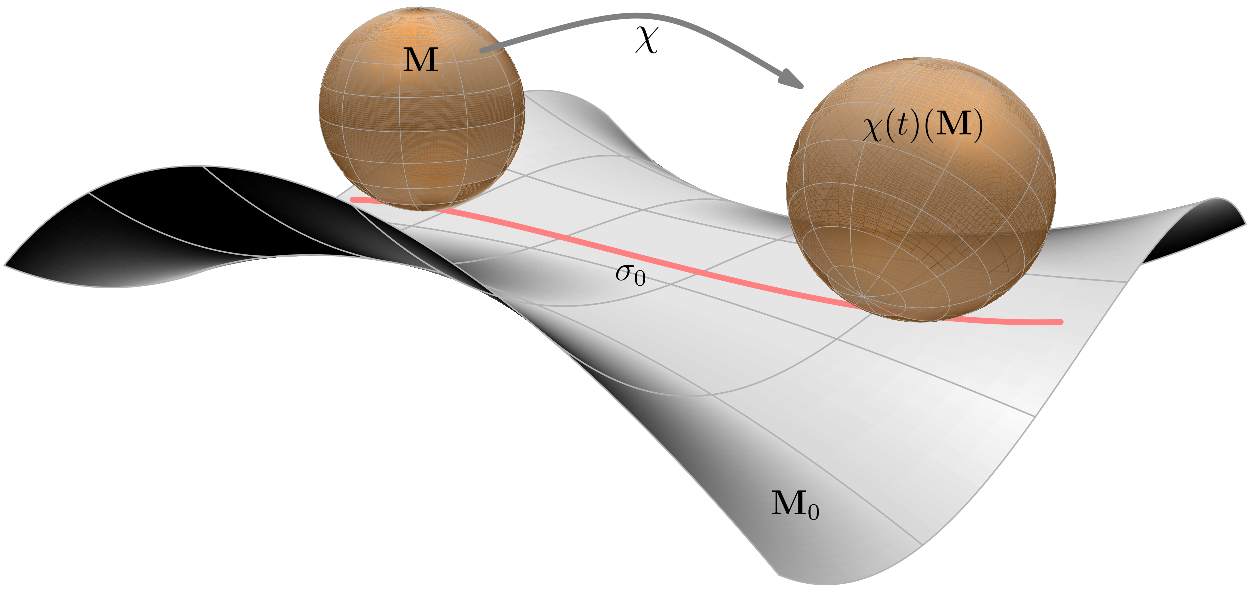

FIGURE 1. A sphere $\mathbf{M}$ is rolling upon $\mathbf{M}_0$ along the development curve σ_0 without slipping or twisting

$(m = n + r)$

$$(\dot{\chi}(t)\,\chi(t)^{-1})_* = \begin{array}{cc} \left[\begin{array}{c|c} 0 & X_{n\times r} \\ \hline -X_{n\times r}^{\mathrm{T}} & 0 \end{array}\right] & \begin{array}{c} \mathbf{T}_p\mathbf{M}_0 \\ \mathbf{T}_p^{\perp}\mathbf{M}_0 \end{array} \\ \begin{array}{cc} \mathbf{T}_p\mathbf{M}_0 & \mathbf{T}_p^{\perp}\mathbf{M}_0 \end{array} & \end{array} . \tag{1}$$

In essence, our main result, Theorem 6 captures the structure of $(\dot{\chi}\ \chi^{-1})_*$ expressed in (1), that is carried from the Lie algebra of the symmetry acting transitively on $\mathbf{M}$.

2.2. Symmetric Riemannian Homogeneous Spaces. This section gives a very brief introduction to symmetric Riemannian homogeneous spaces. For more details we refer to [4].

Let $\mathfrak{G}$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{G}$ acts transitively on a Riemannian manifold $\mathbf{M}$, i.e., there is a smooth map $\mathfrak{G}\times\mathbf{M}\to\mathbf{M}$, denoted by $(a,p)\mapsto a\cdot p$, such that, for any $p\in\mathbf{M}$: $a\cdot(b\cdot p) = (ab)\cdot p$, for any $a,b\in\mathfrak{G}$; $e\cdot p = p$, where e is the identity element of $\mathfrak{G}$; for any $q\in\mathbf{M}$ there exists an element $a\in\mathfrak{G}$ such that $q = a\cdot p$. For an arbitrary fixed point $p_0\in\mathbf{M}$ the closed subgroup

$$H := \{\, a\in\mathfrak{G}\ :\ a\cdot p_0 = p_0 \,\}$$

is an isotropy group of $\mathfrak{G}$ at p_0. Then $\mathbf{M}$ is diffeomorphic to the space $\mathfrak{G}/H$ of left cosets aH, with $p\mapsto aH$, where $a\in\mathfrak{G}$ is such that $p = a\,\cdot$

p_0. Let the metric on $\mathbf{M}$ be invariant under $\mathfrak{G}$, i.e., for any $x \in \mathfrak{G}$ the mapping $\tau(x)\colon aH \mapsto xaH$ of $\mathfrak{G}/H$ onto $\mathfrak{G}/H$ is an isometry. We will assume further that the homogeneous space $\mathfrak{G}/H$ is *reductive*, i.e., there exists a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, invariant under $\mathrm{Ad}(H)$. The natural projection $\pi\colon \mathfrak{G} \to \mathbf{M} \cong \mathfrak{G}/H$ induces the linear surjection $\pi_*\colon \mathbf{T}_e\mathfrak{G} \to \mathbf{T}_{p_0}\mathbf{M}$ and we have the following isomorphisms

$$\mathbf{T}_{p_0}\mathbf{M} \cong \mathbf{T}_e\mathfrak{G}/\ker \pi_* \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{p}.$$

The space $\mathbf{M} \cong \mathfrak{G}/H$ is called a *symmetric Riemannian homogeneous space* (*symmetric space* for short) if the above vector subspace $\mathfrak{p}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$. For such spaces we have the following relations

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}, \quad [\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{p} \quad \text{and} \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}. \tag{2}$$

This decomposition is also known as a Cartan type decomposition.

3. Rolling Riemannian Symmetric Spaces

In the remainder of this paper we assume that a manifold $\mathbf{M}$, isometrically embedded in the ambient space $\widetilde{\mathbf{M}}$, is rolling upon its affine tangent space at a point p_0. Let $\widetilde{\mathfrak{G}} = \mathfrak{G} \ltimes V$ be the group of isometries preserving orientation of $\widetilde{\mathbf{M}}$. For instance, if the ambient space is $\mathbb{R}^m$, its isometry group is the special Euclidean group $\mathbf{SE}(m) = \mathbf{SO}(m) \ltimes \mathbb{R}^m$. The affine tangent space to $\mathbf{M}$ at a point $p \in \mathbf{M}$ is defined as

$$\mathbf{T}_p^{\mathrm{aff}}\mathbf{M} = p + \mathbf{T}_p\mathbf{M}, \tag{3}$$

which makes sense if the ambient space is Euclidean. In a more general settings we regard the *plus sign* in (3) to be the action of an element $(e, s) \in \widetilde{\mathfrak{G}}$ on p under identification of the tangent space $\mathbf{T}_p\mathbf{M}$ with a subspace of the abelian group V.

If $\chi = (g, s)$ is a rolling map then χ acts as follows

$$\begin{array}{rcl} I \times \widetilde{\mathbf{M}} & \xrightarrow{\chi} & \widetilde{\mathbf{M}} \\ I \times \mathbf{T}_{p_0}\widetilde{\mathbf{M}} & \xrightarrow{\chi_*} & \mathbf{T}_{p_0}\widetilde{\mathbf{M}} \end{array} \qquad \begin{array}{rcl} \big(t, p\big) & \xrightarrow{\chi} & g(t) \cdot p + s(t) \\ \big(t, V\big) & \xrightarrow{\chi_*} & g(t)_* \cdot V \end{array}$$

We shall assume that $\mathbf{M}$ is the symmetric space $\mathfrak{G}/H$, that is $\mathbf{M} \cong \mathfrak{G}/H$, so that the subgroup $\mathfrak{G} \subset \widetilde{\mathfrak{G}}$ acts transitively on $\mathbf{M}$ and H is the isotropy group of $p_0 \in \mathbf{M}$. We identify elements of Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ with the vector space of linear maps from $\mathbf{T}_{p_0}\widetilde{\mathbf{M}}$ to itself. Let μ denote the group action on $\mathbf{M}$ then the above relationships can be illustrated with the following diagrams.

$$
\begin{array}{ccc}
\mathfrak{G}\times\widetilde{\mathbf{M}} & \xrightarrow{\mu} & \widetilde{\mathbf{M}} \\
\exp\big\uparrow\exp & & \big\uparrow\exp \\
\mathfrak{g}\times\mathbf{T}_{p_0}\widetilde{\mathbf{M}} & \xrightarrow{\mu_*} & \mathbf{T}_{p_0}\widetilde{\mathbf{M}}
\end{array}
\qquad
\begin{array}{ccc}
(g,p) & \xrightarrow{\mu} & g\cdot p \\
\exp\big\uparrow\exp & & \big\uparrow\exp \\
(X,V) & \xrightarrow{\mu_*} & X\cdot V
\end{array}
$$

Proposition 2. *Let $\mathfrak{h}$ be the Lie algebra of the isotropy group H of $p_0\in\mathbf{M}$. Then $\mathfrak{h}(\mathbf{T}_{p_0}\mathbf{M})\subset\mathbf{T}_{p_0}\mathbf{M}$ and $\mathfrak{h}(\mathbf{T}^\perp_{p_0}\mathbf{M})\subset\mathbf{T}^\perp_{p_0}\mathbf{M}$.*

Proof: Let $g\colon(-\varepsilon,\varepsilon)\to H$ be a differentiable curve in the isotropy group H such that $g(0)$ is the identity. Moreover, let $\gamma\colon(-\delta,\delta)\to\widetilde{\mathbf{M}}$ be a differentiable curve in the ambient manifold, with $\gamma(0)=p_0$. Then $c(t,s):=g(t)\cdot\gamma(s)$ is a smooth map from $(-\varepsilon,\varepsilon)\times(-\delta,\delta)$ to $\widetilde{\mathbf{M}}$ such that $c(t,0)=p_0$, for all $t\in(-\varepsilon,\varepsilon)$. The derivative of c with respect to s is

$$\partial_s c(t,0)=g(t)_*\cdot\dot\gamma(0),$$

therefore $g(t)_*$ is a map from $\mathbf{T}_{p_0}\widetilde{\mathbf{M}}$ to itself. Since H is also a subgroup of a Lie group $\mathfrak{G}$, that acts transitively on $\mathbf{M}$, then, by restricting γ to $\mathbf{M}$, $\partial_s c(t,0)=g(t)_*\cdot V$, where $V\in\mathbf{T}_{p_0}\mathbf{M}$, is a curve in the tangent space $\mathbf{T}_{p_0}\mathbf{M}$. Similarly $g(t)_*\cdot\Lambda$, where $\Lambda\in\mathbf{T}^\perp_{p_0}\mathbf{M}$ is a curve in the normal space $\mathbf{T}^\perp_{p_0}\mathbf{M}$, because H is an isometry. Taking derivative with respect to t, noting that $g(0)=e$, yields $\dot g(0)_*\cdot\dot\gamma(0)=\partial_t\partial_s c(0,0)$, where $\dot g(0)_*\in\mathfrak{h}$. The proof is now complete. ■

The fact that $\dot g(0)_*\colon\mathbf{T}_{p_0}\mathbf{M}\to\mathbf{T}_{p_0}\mathbf{M}$ can be also seen from the Lie algebra decomposition (2), namely from $[\mathfrak{p},\mathfrak{h}]\subset\mathfrak{p}$. From the proof of Proposition 2 we can draw yet another conclusion.

Remark 3. *If $\mathbf{M}$ is of co-dimension one, i.e., $m=n+1$, then $\mathfrak{h}$ maps $\mathbf{T}^\perp_{p_0}\mathbf{M}$ to zero (in $\mathbf{T}_{p_0}\widetilde{\mathbf{M}}$). This easily follows from the fact that since $\mathbf{T}^\perp_{p_0}\mathbf{M}$ is one dimensional and H is an isometry preserving orientation then $g(t)_*\cdot V=V$, for any $V\in\mathbf{T}^\perp_{p_0}\mathbf{M}$. Therefore its derivative with respect to t is zero.*

Proposition 4. *Assume that $\widetilde{\mathbf{M}}$ is Euclidean and let $\mathfrak{p}=\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of the isotropy group H of $p_0\in\mathbf{M}$. Then $\mathfrak{p}(\mathbf{T}_{p_0}\mathbf{M})\subset\mathbf{T}^\perp_{p_0}\mathbf{M}$ and $\mathfrak{p}(\mathbf{T}^\perp_{p_0}\mathbf{M})\subset\mathbf{T}_{p_0}\mathbf{M}$.*

Proof: We will show first that $\mathfrak{p}(\mathbf{T}_{p_0}\mathbf{M}) \subset \mathbf{T}^{\perp}_{p_0}\mathbf{M}$. For any vector $V_0 \in \mathbf{T}_{p_0}\mathbf{M}$ and $X \in \mathfrak{p}$ the curve $\gamma(t) = \pi \circ \exp(tX)$ is a geodesic in $\mathbf{M}$. Define $V(t) := \gamma(t)_* \cdot V_0$, then V is a parallel vector field along γ, *cf.* [4, page 208]. Therefore its covariant derivative $D_t V$ is zero, i.e., the tangent component of its t derivative vanishes. Hence $\dot{V}(0) = X \cdot V_0 \in \mathbf{T}^{\perp}_{p_0}\mathbf{M}$. The second claim that $\mathfrak{p}(\mathbf{T}^{\perp}_{p_0}\mathbf{M}) \subset \mathbf{T}_{p_0}\mathbf{M}$ has been proved in [2]. ■

Remark 5. *We are strongly convinced that Proposition 4 is true for general Riemannian manifolds, although we have not been able to produce a complete proof yet. Our believe is based on all the cases that we have analyzed including some of the examples that appear later. The inclusion* $\mathfrak{p}(\mathbf{T}_{p_0}\mathbf{M}) \subset \mathbf{T}^{\perp}_{p_0}\mathbf{M}$ *holds whenever* $\widetilde{\mathbf{M}}$ *is a Riemannian manifold. The second inclusion* $\mathfrak{p}(\mathbf{T}^{\perp}_{p_0}\mathbf{M}) \subset \mathbf{T}_{p_0}\mathbf{M}$ *is trivially true in co-dimension one. This is because the holonomy group of normal vectors is* $Z_2 = \{-1, 1\}$, *if* $\mathbf{M}$ *is simply connected. Differentiable field of unit normal vectors along paths in* $\mathbf{M} \subset \widetilde{\mathbf{M}}$ *gives a field of vectors that does not depend on the choice of curves, cf.* [7, p. 5].

Theorem 6. *Let* $\mathfrak{p}$ *be as in Proposition 4 and* χ *be a rolling map of a symmetric space* $\mathbf{M} \cong \mathfrak{G}/H$ *embedded in Euclidean space. Then* $(\dot{\chi}\,\chi^{-1})_*$ *is an element of* $\mathfrak{p}$.

Proof: Denote $(\dot{\chi}\,\chi^{-1})_*$ by $u \in \mathfrak{g}$. Let $u = u_{\mathfrak{h}} + u_{\mathfrak{p}}$ be a decomposition of u into components in $\mathfrak{h}$ and $\mathfrak{p}$, respectively. For any vector $V \in \mathbf{T}_{p_0}\mathbf{M}$ there is

$$u \cdot V = (u_{\mathfrak{h}} + u_{\mathfrak{p}}) \cdot V = u_{\mathfrak{h}} \cdot V + u_{\mathfrak{p}} \cdot V,$$

where $u_{\mathfrak{h}} \cdot V \in \mathbf{T}_{p_0}\mathbf{M}$ and $u_{\mathfrak{p}} \cdot V \in \mathbf{T}^{\perp}_{p_0}\mathbf{M}$, by Propositions 2 and 4, respectively. From the tangential part of the *"no-twist"* conditions $u \cdot V \in \mathbf{T}^{\perp}_{p_0}\mathbf{M}$ then it follows that $u_{\mathfrak{h}} \cdot V$ is zero, for all $V \in \mathbf{T}_{p_0}\mathbf{M}$. By a similar reasoning with the normal part of the *"no-twist"* conditions one shows that also $u_{\mathfrak{h}} \cdot V = 0$, for all $V \in \mathbf{T}^{\perp}_{p_0}\mathbf{M}$. Therefore $u_{\mathfrak{h}} \equiv 0$ and $u = u_{\mathfrak{p}} \in \mathfrak{p}$. This completes the proof. ■

If $\chi = (g, s)$ is a rolling map of $\mathbf{M}$ upon its affine tangent space at p_0, then $\sigma(t) = g^{-1}(t) \cdot p_0$ is the rolling curve. To see this it is enough to check that χ with σ conform to the conditions of Definition 1. We start with the rolling conditions. The development curve is $\sigma_0 = p_0 + s$ hence the first rolling condition $\sigma_0 \in \mathbf{T}^{\mathrm{aff}}_{p_0}\mathbf{M}$ requires that $s \in \mathbf{T}_{p_0}\mathbf{M}$. The second rolling condition requires that $\mathbf{T}_{\chi\sigma}(\chi(\mathbf{M})) = \mathbf{T}_{\chi\sigma}\mathbf{T}^{\mathrm{aff}}_{p_0}$. Since $\sigma = g^{-1} \cdot p_0$ then

$\mathbf{T}_\sigma\mathbf{M} = g_*^{-1}(\mathbf{T}_{p_0}\mathbf{M})$ and the left hand side of the above equality is equal to

$$\mathbf{T}_{\chi\sigma}(\chi(\mathbf{M})) = \chi_*(\mathbf{T}_\sigma\mathbf{M}) = g_*\big(g_*^{-1}(\mathbf{T}_{p_0}\mathbf{M})\big) = \mathbf{T}_{p_0}\mathbf{M}.$$

On the other hand, for the affine tangent space $\mathbf{T}_{\chi\sigma}\mathbf{T}_{p_0}^{\mathrm{aff}}\mathbf{M} = \mathbf{T}_{p_0}\mathbf{M}$ hence the equality of the tangent spaces at the contact point holds. The *"no-slip"* condition implies that

$$(\dot\chi\,\chi^{-1})\sigma_0 = \dot\chi\,\chi^{-1}\chi\sigma = \dot\chi\sigma = \dot\chi g^{-1}\cdot p_0 = \dot g g^{-1}\cdot p_0 + \dot s = 0,$$

giving a differential equation for s. A solution exists for s in $\mathbf{T}_{p_0}\mathbf{M}$ because $\dot g g^{-1}$ maps $\mathbf{M}$ to $\mathbf{TM}$. The two *"no-twist"* conditions now read as

$$(\dot g g^{-1})_*(\mathbf{T}_{p_0}\mathbf{M}) \subset \mathbf{T}_{p_0}^{\perp}\mathbf{M} \quad\text{and}\quad (\dot g g^{-1})_*(\mathbf{T}_{p_0}^{\perp}\mathbf{M}) \subset \mathbf{T}_{p_0}\mathbf{M}.$$

They are both satisfied because $\chi = (g, s)$ is a rolling map.

Theorem 7. *Let $\chi = (g, s)$ be a rolling map of a symmetric space $\mathbf{M} \cong \mathfrak{G}/H$ and $\sigma(t) = g^{-1}(t)\cdot p_0$ be the corresponding rolling curve. For any $V_0 \in \mathbf{T}_{p_0}\mathbf{M}$ define a vector field along σ by*

$$V(t) := g^{-1}(t)_*\cdot V_0.$$

Then V is a left-invariant parallel vector field along σ.

Proof: Clearly $V(t) \in \mathbf{T}_{\sigma(t)}\mathbf{M}$. We show first that V is left-invariant. Let L_a denote the left translation by $a \in \mathfrak{G}$ then $V = (L_{g^{-1}})_*\cdot V_0$ and

$$V(f\circ L_g) = \big((L_{g^{-1}})_*\cdot V_0\big)(f\circ L_g) = V_0(f\circ L_g\circ L_{g^{-1}}) = V_0(f),$$

for any differentiable f on $\mathbf{M}$. Hence $(L_g)_*V = V_0 = V(0)$ and V is left invariant.

The rolling map χ generates vector field $\widetilde V$ along development curve $\sigma_0(t) = \chi(t)\cdot\sigma(t)$ and since rolling maps preserve covariant differentiation, *cf.* [5], then $D_tV = \widetilde D_t\widetilde V$, where $\widetilde D_t$ is the covariant derivative on the affine tangent space. Because $\widetilde V(t) = \chi(t)_*\cdot V(t) = (g(t)_*\ g^{-1}(t)_*)\cdot V_0 = V_0$ is constant therefore $D_tV = 0$, what was to show. ∎

3.1. Examples. Here we give a few examples of rolling symmetric spaces on their respective affine tangent spaces. These examples illustrate the main ideas behind the structure of the rolling maps and decomposition of a Lie algebra. In all the cases considered here the manifolds are *normal homogeneous* with respect to their metric, i.e., $\mathfrak{p} = \mathfrak{h}^\perp$.

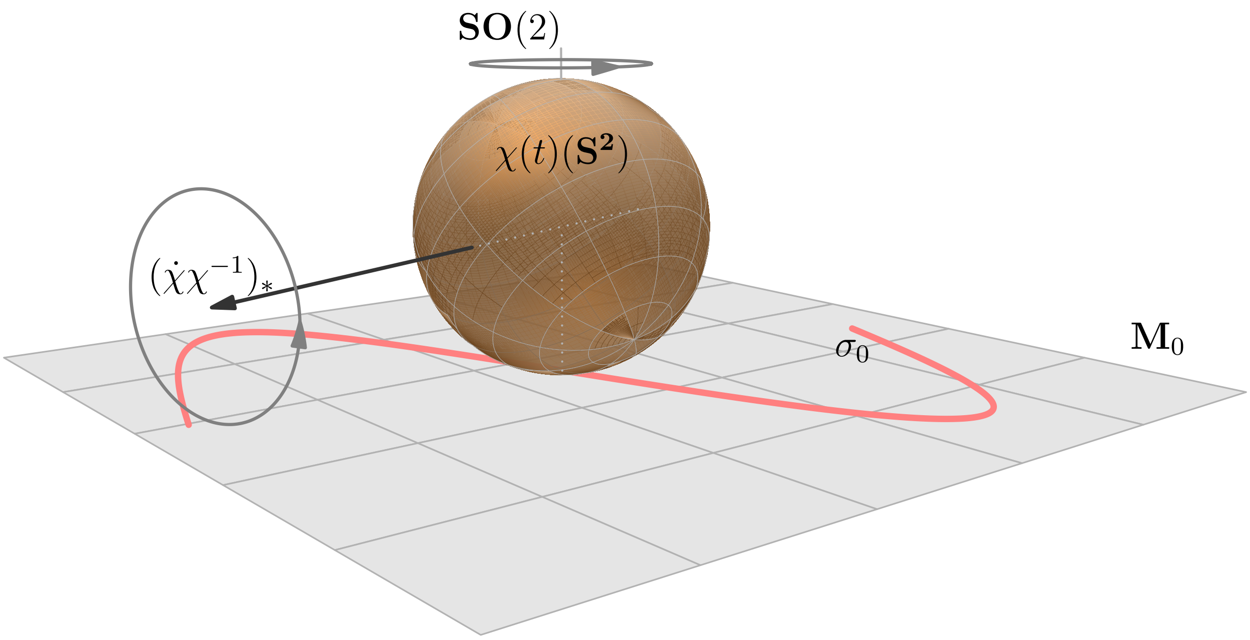

FIGURE 2. A sphere $\mathbf{S}^2$ is rolling upon $\mathbf{M}_0$ along the development curve σ_0 without slipping or twisting; the infinitesimal action $(\dot{\chi}\,\chi^{-1})_*$ is orthogonal to the Lie algebra of the isotropy group

Example 1 (the sphere)**.** *Consider the well studied problem of rolling the sphere $\mathbf{S}^n$ on its affine tangent space. Since $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ is homogeneous space, take any $p_0 \in \mathbf{S}^n$, then $H = \mathbf{SO}(n)$ is an isotropy group leaving p_0 fixed.*

To be more precise, choose $p_0 = \big(0, \dots, 0, -1\big)$ to be the "south pole" of $\mathbf{S}^n$. The Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ splits into the direct sum $\mathfrak{p} \oplus \mathfrak{h}$, where

$$\mathfrak{h} = \left\{ x \in \mathfrak{so}(n+1) \ : \ x = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A \in \mathfrak{so}(n) \right\}$$

and $\mathfrak{p} = \mathfrak{h}^\perp$ is given by

$$\mathfrak{p} = \left\{ x \in \mathfrak{so}(n+1) \ : \ x = \begin{bmatrix} 0 & m \\ -m^{\mathrm{T}} & 0 \end{bmatrix} \quad \text{and} \quad m \in \mathbb{R}^{n\times 1} \right\} \cong \mathbf{T}_{p_0}\mathbf{S}^n.$$

It is easy to see that $\mathfrak{p} \cdot p_0 = \mathbf{T}_{p_0}\mathbf{S}^n$ and $\mathfrak{h} \cdot p_0 = 0$. Note that $\operatorname{span}(p_0) = \mathbf{T}^\perp_{p_0}\mathbf{S}^n$. *Let χ be the rolling map and let $u = (\dot{\chi}\ \chi^{-1})_*$ then $u \in \mathfrak{g}$ and $\big\langle u \cdot (\mathfrak{p} \cdot p_0), p_0 \big\rangle = -\big\langle \mathfrak{p} \cdot p_0, u \cdot p_0 \big\rangle$. From the tangential part of the* "no-twist" *condition it follows that $u \in \mathfrak{p}$.*

Figure 2 illustrates this situation for the two dimensional sphere.

Example 2 (the Graßmann manifold)**.** *We now look at the Graßmann manifold rolling on its affine tangent space, cf.* [6]*. The Graßmann manifold* $\mathscr{G}_{\mathbf{k},\mathbf{n}}$ *is defined by* $\mathscr{G}_{\mathbf{k},\mathbf{n}} := \{ P \in \mathfrak{s}(n) \ : \ P^2 = P \text{ and } \operatorname{rank}(P) = k \}$ *and considered embedded in* $\mathfrak{s}(n)$*, where* $\mathfrak{s}(n)$ *is the set of* $n \times n$ *symmetric matrices. Group* $\mathfrak{G} = \mathbf{SO}(n)$ *acts transitively on* $\mathscr{G}_{\mathbf{k},\mathbf{n}}$ *by* $(X, P) \mapsto X \cdot P \cdot X^{\mathrm{T}}$*. This action induces Lie algebra action* $(a, V) \mapsto a \cdot V + V \cdot a^{\mathrm{T}}$*. Take*

$$P_0 = \begin{bmatrix} \mathbf{1}_k & 0 \\ 0 & 0 \end{bmatrix}$$

and let $H \subset \mathfrak{G}$ *be the isotropy group leaving* P_0 *fixed. Then*

$$H = \left\{ \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \ : \ H_1 \in \mathbf{SO}(k) \quad \text{and} \quad H_2 \in \mathbf{SO}(n-k) \right\}.$$

Then Lie algebra $\mathfrak{h}$ *of the group* H *is*

$$\mathfrak{h} = \left\{ \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \ : \ h_1 \in \mathfrak{so}(k) \quad \text{and} \quad h_2 \in \mathfrak{so}(n-k) \right\}.$$

The orthogonal complement $\mathfrak{p} = \mathfrak{h}^{\perp}$ *is therefore*

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & m \\ -m^{\mathrm{T}} & 0 \end{bmatrix} \ : \ m \in \mathbb{R}^{k \times (n-k)} \right\}.$$

The tangent and normal spaces at P_0 *are given by*

$$\mathbf{T}_{P_0}\mathscr{G}_{\mathbf{k},\mathbf{n}} = \left\{ \begin{bmatrix} 0 & Z \\ Z^{\mathrm{T}} & 0 \end{bmatrix} \ : \ Z \in \mathbb{R}^{k \times (n-k)} \right\} \quad \text{and}$$
$$\mathbf{T}^{\perp}_{P_0}\mathscr{G}_{\mathbf{k},\mathbf{n}} = \left\{ \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \ : \ S_1 \in \mathfrak{s}(k), \quad S_2 \in \mathfrak{s}(n-k) \right\}.$$

According to the developments in [6]*, if* $\chi = (R, s)$ *is a rolling map of* $\mathscr{G}_{\mathbf{k},\mathbf{n}}$ *upon its affine tangent space at* P_0 *and* $A \in \mathfrak{s}(n)$*, we have*

$$(\dot{\chi}\,\chi^{-1})_* A = \dot{R}^{\mathrm{T}} R A - A R^{\mathrm{T}} \dot{R} = [\Omega, A],$$

where $\Omega = \dot{R}^{\mathrm{T}} R \in \mathfrak{so}(n)$*. Partitioning* Ω *as*

$$\Omega = \begin{bmatrix} m_1 & m_2 \\ -m_2^{\mathrm{T}} & m_3 \end{bmatrix}, \quad \text{where} \quad m_1 = -m_1^{\mathrm{T}} \quad \text{and} \quad m_3 = -m_3^{\mathrm{T}},$$

and taking $A = \left[\begin{smallmatrix} S_1 & 0 \\ 0 & S_2 \end{smallmatrix}\right] \in \mathbf{T}^{\perp}_{p_0}\mathscr{G}_{\mathbf{k},\mathbf{n}}$, *the normal part of the* "no-twist" *conditions implies that* $\big[\Omega, A\big] \in \mathbf{T}_{p_0}\mathscr{G}_{\mathbf{k},\mathbf{n}}$. *That is*

$$\begin{aligned} &\begin{bmatrix} m_1 & m_2 \\ -m_2^{\mathrm{T}} & m_3 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} + \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} m_1^{\mathrm{T}} & -m_2 \\ m_2^{\mathrm{T}} & m_3^{\mathrm{T}} \end{bmatrix} \\ &\qquad\qquad = \begin{bmatrix} \big[m_1, S_1\big] & m_2 \cdot S_2 - S_1 \cdot m_2 \\ -m_2^{\mathrm{T}} \cdot S_1 + S_2 \cdot m_2^{\mathrm{T}} & \big[m_3, S_2\big] \end{bmatrix} \end{aligned}$$

yields $\big[m_1, S_1\big] = 0$ *and* $\big[m_3, S_2\big] = 0$, *for any symmetric* S_1 *and* S_2. *This is only possible when* $m_1 = 0$ *and* $m_3 = 0$, *hence* $(\dot\chi\, \chi^{-1})_* \in \mathfrak{p}$, *as expected.*

Example 3 (the Essential manifold)**.** *The essential manifold is defined as* $\mathscr{E} = \mathscr{G}_{\mathbf{2},\mathbf{3}} \times \mathbf{SO}(3)$. *We consider this a manifold embedded in* $\mathfrak{s}(3) \times \mathbb{R}^{3\times 3}$ *equipped with the Euclidean (Frobenius) norm. Points in* $\mathscr{E}$ *are represented by pairs* $\big(UE_0U^{\mathrm{T}}, R\big)$, *where* $U, R \in \mathbf{SO}(3)$ *and* $E_0 = \left[\begin{smallmatrix} \mathbf{1}_2 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. *At the point* $P_0 = \big(E_0, \mathbf{1}\big)$ *the tangent and normal space to* $\mathscr{E}$ *at* P_0 *are given by*

$$\begin{aligned} \mathbf{T}_{P_0}\mathscr{E} &= \left\{ \left(\begin{bmatrix} 0 & \Lambda \\ \Lambda^{\mathrm{T}} & 0 \end{bmatrix}, C \right) \;:\; \Lambda \in \mathbb{R}^{2\times 1} \quad \textit{and} \quad C \in \mathfrak{so}(3) \right\}; \\ \mathbf{T}^{\perp}_{P_0}\mathscr{E} &= \left\{ \left(\begin{bmatrix} B & 0 \\ 0 & b \end{bmatrix}, S \right) \;:\; B \in \mathfrak{s}(2), \quad b \in \mathbb{R} \quad \textit{and} \quad S \in \mathfrak{s}(3) \right\}. \end{aligned}$$

Rolling maps for the essential manifold have been studied in [11]. *We refer to this paper for details.*

The action of the Lie group $\mathfrak{G} = \mathbf{SO}(3) \times \mathbf{SO}(3) \times \mathbf{SO}(3)$ *on* $\mathscr{E}$, *defined by* $(U, V, W) \cdot (P, R) := (UPU^{\mathrm{T}}, VRW^{\mathrm{T}})$, *is transitive. The isotropy subgroup of* $\mathfrak{G}$ *that leaves* $P_0 = (E_0, \mathbf{1})$ *invariant is the set*

$$H = \left\{ \big(U, V, V\big) \;:\; V \in \mathbf{SO}(3) \quad \textit{and} \quad U = \begin{bmatrix} \mathbf{SO}(2) & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

The Lie algebra of $\mathfrak{G}$, $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, *decomposes as* $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$, *where*

$$\mathfrak{h} := \left\{ \big(\vartheta, \zeta, \zeta\big) \;:\; \vartheta = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, \quad \beta \in \mathfrak{so}(2), \quad \zeta \in \mathfrak{so}(3) \right\}$$

and

$$\mathfrak{p} := \left\{ \big(\vartheta, \zeta, -\zeta\big) \;:\; \vartheta = \begin{bmatrix} 0 & 0 & m_1 \\ 0 & 0 & m_2 \\ -m_1 & -m_2 & 0 \end{bmatrix}, \quad \zeta \in \mathfrak{so}(3) \right\} = \mathfrak{h}^{\perp}.$$

It is easy to check that $\big[\mathfrak{p},\mathfrak{p}\big]\subset\mathfrak{h}$, *and* $\big[\mathfrak{h},\mathfrak{p}\big]\subset\mathfrak{p}$. *Therefore* $\mathscr{E}$ *is a* symmetric Riemannian homogeneous space, *cf.* [3].

Let χ *be the rolling map of* $\mathscr{E}$ *on its affine tangent space at* P_0 *and* $(U,V,W)\in\mathfrak{G}$ *be the first component of* χ. *Then, according to the developments in* [11], *if* (A,C) *is a vector in the embedding space, we have*

$$(\dot\chi\,\chi^{-1})_*\cdot(A,C)=\Big(\dot U^{\mathrm T}UA+AU^{\mathrm T}\dot U,\ \dot V^{\mathrm T}VC+CW^{\mathrm T}\dot W\Big),$$

or, defining skew-symmetric matrices $\Omega_U:=\dot U^{\mathrm T}U$, $\Omega_V:=\dot V^{\mathrm T}V$ *and* $\Omega_W:=\dot W^{\mathrm T}W$,

$$(\dot\chi\,\chi^{-1})_*\cdot(A,C)=\big(\big[\Omega_U,A\big],\,\Omega_VC-C\Omega_W\big).$$

Since the tangential "no-twist" *condition requires that*

$$(\dot\chi\,\chi^{-1})_*\cdot(A,C)\in\mathbf{T}^{\perp}_{P_0}\mathscr{E},\quad \textit{for}\quad (A,C)\in\mathbf{T}_{P_0}\mathscr{E},$$

we can do some computations to conclude that $u=(\dot\chi\,\chi^{-1})_*=(\Omega_U,\Omega_V,-\Omega_W)$, *where* Ω_U *is of the form* $\left[\begin{smallmatrix}0&0&m_1\\0&0&m_2\\-m_1&-m_2&0\end{smallmatrix}\right]$ *and* $\Omega_V=\Omega_W$. *So,*

$$u=(\dot\chi\,\chi^{-1})_*=(\Omega_U,\Omega_V,-\Omega_V)\in\mathfrak{p}.$$

Example 4 (the Lorentzian sphere)**.** *We now look at the pseudo-Riemannian case, cf.* [8]. *The embedding space is* $\mathbb{R}^{n+1}$ *endowed with the Minkowski metric with the signature* $(n,1)$, *denoted by* J. *Let* $\mathbf{S}^{n,1}$ *be the surface defined by*

$$\mathbf{S}^{n,1}:=\left\{\,x\in\mathbb{R}^{n+1}\ :\ \big\langle x,x\big\rangle_J=1\,\right\}.$$

Surface $\mathbf{S}^{n,1}$ *is called the* Lorentzian sphere *also known as* de Sitter space. *The symmetry group acting transitively on* $\mathbf{S}^{n,1}$ *is* $\mathbf{SO}(n,1)$ *defined as*

$$\mathbf{SO}(n,1):=\Big\{\,X\in\mathbb{R}^{(n+1)\times(n+1)}\ :\ X^{\mathrm T}JX=J\quad\textit{and}\quad\det X=1\,\Big\},$$

with its Lie algebra

$$\mathfrak{so}(n,1):=\Big\{\,\Omega\in\mathbb{R}^{(n+1)\times(n+1)}\ :\ \Omega^{\mathrm T}J=-J\Omega\,\Big\}.$$

It is known that $\mathbf{S}^{n,1}=\mathbf{SO}(n,1)/\mathbf{SO}(n-1,1)$ *is a symmetric space. Choose* $p_0=\big(1,0,\ldots,0\big)$ *and* $n>1$ *then the isotropy group becomes*

$$H=\left\{\,X\in\mathbf{SO}(n,1)\ :\ X=\begin{bmatrix}1&0\\0&\mathbf{SO}(n-1,1)\end{bmatrix}\,\right\}.$$

Its Lie algebra is therefore

$$\mathfrak{h} = \left\{ x \in \mathfrak{so}(n,1) \ : \ x = \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{so}(n-1,1) \end{bmatrix} \right\}$$

and its orthogonal complement is

$$\mathfrak{p} = \left\{ x \in \mathfrak{so}(n,1) \ : \ x = J \cdot \begin{bmatrix} 0 & -m^{\mathrm{T}} \\ m & 0 \end{bmatrix} \quad \text{and} \quad m \in \mathbb{R}^{n\times 1} \right\}.$$

This is consistent with the results in [8].

Example 5 (the ellipsoid)**.** *Consider the rolling ellipsoid $\mathcal{E}^n$ isometrically embedded in the Riemannian structure $\widetilde{\mathbf{M}} = \left(\mathbb{R}^{n+1}, D^{-2}\right)$ induced by a positive definite matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_{n+1}) \succ 0$, cf.* [10]*. Then*

$$\mathcal{E}^n := \left\{ p \in \widetilde{\mathbf{M}} \ : \ \|p\|_{D^{-2}} = 1 \right\}.$$

Here, the group acting on $\mathcal{E}^n$ is $\mathfrak{G} = D \cdot \mathbf{SO}(n+1) \cdot D^{-1}$. Since $R \mapsto D\,R\,D^{-1}$ is the group isomorphism $\mathbf{SO}(n+1) \cong \mathfrak{G}$, this example is similar to the rolling sphere covered in Example 1. However, the metric considered here is left-invariant*.*

Let $p_0 = \left(0, \ldots, 0, -d_{n+1}\right) = -D\,e_{n+1}$ be the "south pole" of $\mathcal{E}^n$. The subgroup $H = D \cdot \mathbf{SO}(n) \cdot D^{-1}$ is an isotropy group leaving p_0 fixed. The Lie algebra $\mathfrak{g} = D \cdot \mathfrak{so}(n+1) \cdot D^{-1}$ splits into the direct sum $\mathfrak{p} \oplus \mathfrak{h}$, where

$$\mathfrak{h} = \left\{ x \in \mathfrak{g} \ : \ x = D \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} D^{-1} \quad \text{and} \quad A \in \mathfrak{so}(n) \right\}$$

and $\mathfrak{p} = \mathfrak{h}^{\perp}$ is given by

$$\mathfrak{p} = \left\{ x \in \mathfrak{g} \ : \ x = D \begin{bmatrix} 0 & m \\ -m^{\mathrm{T}} & 0 \end{bmatrix} D^{-1} \quad \text{and} \quad m \in \mathbb{R}^{n\times 1} \right\}.$$

Clearly $\mathbf{T}_{p_0}\mathcal{E}^n \cong \mathfrak{p}$ because

$$\mathfrak{p} \cdot p_0 = -D \cdot \begin{bmatrix} m \\ 0 \end{bmatrix} = \mathbf{T}_{p_0}\mathcal{E}^n \quad \text{and} \quad \mathfrak{h} \cdot p_0 = 0.$$

Let χ be the rolling map and let $u = (\dot{\chi}\ \chi^{-1})_*$. The "no-twist" *conditions become*

$$u \cdot \left(\mathbf{T}_{p_0}\mathcal{E}^n\right) \subset \mathbf{T}^{\perp}_{p_0}\mathcal{E}^n \quad \text{and} \quad u \cdot \left(\mathbf{T}^{\perp}_{p_0}\mathcal{E}^n\right) \subset \mathbf{T}_{p_0}\mathcal{E}^n.$$

By the same reasoning as in the spherical case we reach the conclusion that $u \in \mathfrak{p}$ which in an agreement with results in [10].

Example 6 (pseudo-orthogonal groups)**.** *Let* $J = \operatorname{diag}(1, \dots, 1, -1, \dots, -1)$ *be the diagonal matrix with* k *ones and* $(n-k)$ *minus ones. For any matrix* $A \in \mathbb{R}^{n\times n}$ *define*

$$A^J := J^{\mathrm{T}} A^{\mathrm{T}} J.$$

Then $\mathbb{R}^{n\times n}$ *may be endowed with the indefinite inner product*

$$\big\langle U, V \big\rangle_J := \operatorname{trace}(U^J V). \tag{4}$$

To each J *one may associate a matrix Lie group which is the connected component containing the identity of*

$$\mathbf{SO}(k, n-k) := \left\{ X \in \mathbb{R}^{n\times n} \ : \ X^{\mathrm{T}} J X = J \quad \textit{and} \quad \det X = 1 \right\}$$

with its Lie algebra

$$\mathfrak{so}(k, n-k) := \left\{ \Omega \in \mathbb{R}^{n\times n} \ : \ \Omega^{\mathrm{T}} J = -J\Omega \right\}.$$

For simplicity, from now on we also use the notation $\mathbf{SO}(k, n-k)$ *for the connected component containing the identity. Let*

$$\mathfrak{s}(k, n-k) := \left\{ \Omega \in \mathbb{R}^{n\times n} \ : \ \Omega^{\mathrm{T}} J = J\Omega \right\}$$

then the tangent and normal space to $\mathbf{SO}(k, n-k)$ *at a point* P_0 *are given by*

$$\mathbf{T}_{P_0}\mathbf{SO}(k, n-k) = \left\{ P_0\Omega \ : \ \Omega \in \mathfrak{so}(k, n-k) \right\} \quad \textit{and}$$
$$\mathbf{T}^{\perp}_{P_0}\mathbf{SO}(k, n-k) = \left\{ P_0\Omega \ : \ \Omega \in \mathfrak{s}(k, n-k) \right\}.$$

Rolling maps for the pseudo-orthogonal groups have been studied in [1]*. We refer to this paper for details.*

Let $\mathfrak{G} = \mathbf{SO}(k, n-k) \times \mathbf{SO}(k, n-k)$ *then* $\mathfrak{G}$ *acts transitively on* $\mathbf{SO}(k, n-k)$ *by*

$$\big((X, Y), R\big) \mapsto X \cdot R \cdot Y^{-1}.$$

Take any $P_0 \in \mathbf{SO}(k, n-k)$ *and let* $H \subset \mathfrak{G}$ *be the isotropy group leaving* P_0 *fixed. Then*

$$H = \left\{ (X, Y) \in \mathfrak{G} \ : \ X \cdot P_0 \cdot Y^{-1} = P_0 \right\}.$$

For the point $P_0 = \mathbf{1}$ *the Lie algebra of* H *is*

$$\mathfrak{h} = \{ (x, x) \in \mathfrak{so}(k, n-k) \times \mathfrak{so}(k, n-k) \}$$

and its orthogonal complement $\mathfrak{p} = \mathfrak{h}^{\perp}$ *with respect to the product metric induced by* (4) *is*

$$\mathfrak{p} = \{ (x, -x) \in \mathfrak{so}(k, n-k) \times \mathfrak{so}(k, n-k) \}.$$

By the normal part of the "no-twist" *conditions there must be*

$$u \cdot \Omega \in \mathfrak{so}(k, n-k), \quad \textit{for any} \quad \Omega \in \mathfrak{s}(k, n-k).$$

Let $u = \big(m_1, m_2\big) \in \mathfrak{so}(k, n-k) \times \mathfrak{so}(k, n-k)$ *then*

$$\big(m_1, m_2\big) \cdot \Omega = m_1 \Omega - \Omega m_2 \in \mathfrak{so}(k, n-k).$$

Hence

$$\big(m_1 \Omega - \Omega m_2\big)^{\mathrm{T}} J = -J\big(m_1 \Omega - \Omega m_2\big) \tag{5}$$

where the left hand side of the above equality is equal to

$$\Omega^{\mathrm{T}} m_1^{\mathrm{T}} J - m_2^{\mathrm{T}} \Omega^{\mathrm{T}} J = -\Omega^{\mathrm{T}} J m_1 - m_2^{\mathrm{T}} J \Omega = -J \Omega m_1 + J m_2 \Omega$$

therefore the above condition (5) *now reads*

$$-J\big(\Omega m_1 - J m_2 \Omega\big) = -J\big(m_1 \Omega - \Omega m_2\big)$$

which is equivalent to

$$\big[m_1, \Omega\big] + \big[m_2, \Omega\big] = \big[m_1 + m_2, \Omega\big], \quad \textit{for all} \quad \Omega \in \mathfrak{s}(k, n-k).$$

Then $m_1 + m_2 = 0$ *hence* $u \in \mathfrak{p}$ *as desired.*

When $J = \mathbf{1}$ *we are reduced to the special orthogonal group* $\mathbf{SO}(n)$.

4. Final Remarks

We have proven that the natural decomposition of the Lie algebra associated to a symmetric space embedded in a Euclidean space or of co-dimension one provides the structure for the kinematic equations that describe the rolling motion of that space upon its affine tangent space at a point. Several examples have been provided to illustrate the results. Based on the analysis of several examples, one of which is included at the end of the last section, we strongly believe that the theory developed here is more general.

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