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SOLVING INTERPOLATION PROBLEMS ON STIEFEL MANIFOLDS USING QUASI-GEODESICS

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ABSTRACT: The main objective of this paper is to propose a new method to generate smooth interpolating curves on Stiefel manifolds. This method is obtained from a modification of the geometric Casteljau algorithm on manifolds and is based on successive quasi-geodesic interpolation. The quasi-geodesics introduced here for Stiefel manifolds have constant speed, constant covariant acceleration and constant geodesic curvature, and in some particular circumstances they are true geodesics.

CONTENTS

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1. Introduction

Stiefel and Graßmann manifolds arise naturally in several vision applications, such as machine learning and pattern recognition, since features and patterns that describe visual objects may be represented as elements in those manifolds. These geometric representations facilitate the analysis of the underlying geometry of the data. The Graßmann manifold is the space of k-dimensional subspaces in $\mathbb{R}^n$ and the Stiefel manifold is the space of k orthonormal vectors in $\mathbb{R}^n$. While a point in the Graßmann manifold represents a subspace, a point in the Stiefel manifold identifies exactly what frame (basis of vectors) is used to specify that subspace.

Although these two manifolds are related, the geometry of the Graßmann manifold is much simpler than that of the Stiefel manifold. This reflects on solutions of simple formulated problems, such as the case of geodesics that join two given points. A formula for the geodesic that joins two points on Graßmann manifolds and depends explicitly only on those points was recently presented in Batzies et al. [3]. Knowing such explicit formulas is also a crucial step to solve other important problems such as, averaging, fitting and interpolation of data. Results about geodesics on Stiefel manifolds are not so easy to obtain. Even the simpler problem of finding a geodesic that starts at a given point with a prescribed velocity is not so straightforward, as can be seen for instance in the work of Edelman et al. in [7].

In the present paper we solve a slightly different but related problem, which consists of joining two points on the Stiefel manifold by quasi-geodesics. These curves have constant speed, constant covariant acceleration, and therefore constant geodesic curvature. Moreover, they are defined explicitly in terms of the points they join. In some cases, depending on those points, the quasi-geodesics are true geodesics. Interestingly enough, these special curves can be used successfully to generate smooth interpolating curves on the Stiefel manifold, as will be explained later. These results may have a great impact in computer vision, since a curve that interpolates a set of time-labeled points on the Stiefel manifold may correspond to the temporal evolution of an event or dynamic scene from which only a limited number of observations was captured, as nicely explained in Su et al. [18].

The organisation of this paper is the following. After this introduction that motivates the reader to the importance of the problems studied here in the context of applications, we introduce in Section 2 the manifolds that play a major role throughout the paper: Graßmann and Stiefel manifolds. This section also includes known results about geodesics and, in particular, a closed formula for the geodesic in the Graßmann manifold that joins two given points. In Section 3 we present quasi-geodesics in the Stiefel manifold. These curves have some interesting properties, such as constant speed, constant intrinsic acceleration and constant geodesic curvature. We provide an explicit formula for quasi-geodesics that join two arbitrary points in the Stiefel manifold and show that in two particular circumstances they are true geodesics. Interpolations problems are formulated in Section 4. We first review the Casteljau algorithm on manifolds, then implement this algorithm to generate a $\mathcal{C}^1$-smooth interpolating curve on the Graßmann manifold satisfying some prescribed boundary conditions and, finally, combine the Casteljau algorithm on the special orthogonal group and on the Graßmannian in order to generate a curve in the Stiefel manifold. Due to the fact that the projection of the Stiefel manifold onto the Graßmann manifold is many to one, the resulting curve is not necessarily continuous. The last section contains the main results in the paper concerning the generation of interpolating problems on the Stiefel manifold. Contrary to the algorithm contained in Section 4, we now solve the interpolation problem on the Stiefel manifold intrinsically, that is, without resorting to other manifolds. To do that we introduce a convenient modification to the Casteljau algorithm, by replacing geodesic interpolation by quasi-geodesic interpolation. This overcomes the difficulties that arise from not knowing explicit formulas for geodesics that join two arbitrary points on the Stiefel manifold and justifies the introduction of quasi-geodesics. The paper ends with some concluding remarks.

2. Preliminaries

In this section we recall the main definitions associated to Graßmann and Stiefel manifolds and several properties that will be used throughout this paper. Due to the important role that these manifolds play in applied areas, they have been studied in the context of numerical algorithms for instance in Edelman et al. [7], Absil et al. [1] and Helmke et al. [8], and in a more abstract form in Kobayashi and Nomizu [11]. Recently, Batzies et al. [3] found a closed form expression for a geodesic in the Graßmann manifold that joins two given points. This formula turns out to be very important for the developments throughout the whole paper. Our main references for this introductory definitions and concepts are Edelman et al. [7] and Batzies et al. [3].

2.1. The Graßmann manifold. Let $\mathfrak{s}(n)$ and $\mathfrak{so}(n)$ denote the set of all $n \times n$ symmetric matrices and the set of all $n \times n$ skew-symmetric matrices respectively.

The (real) Graßmann manifold $\boldsymbol{\mathscr{G}}_{n,k}$ is the set of all k-dimensional linear subspaces in $\mathbb{R}^n$, where $n \geq k \geq 1$. This manifold has a matrix representation

$$\boldsymbol{\mathscr{G}}_{n,k} := \left\{ P \in \mathfrak{s}(n) \ : \ P^2 = P \text{ and } \operatorname{rank}(P) = k \right\}$$

so that it is considered a submanifold of $\mathbb{R}^{n \times n}$ with dimension $k(n-k)$. Graßmann manifold $\boldsymbol{\mathscr{G}}_{n,k}$ can also be viewed as a homogeneous space

$$\boldsymbol{\mathscr{G}}_{n,k} \cong \mathbb{O}(n)/(\mathbb{O}(k) \times \mathbb{O}(n-k)),$$

where $\mathbb{O}(n)$ is the orthogonal Lie group.

Given a point $P \in \boldsymbol{\mathscr{G}}_{n,k}$ define the following sets

$$\begin{aligned} \mathfrak{gl}_P(n) &:= \left\{ X \in \mathfrak{gl}(n) \ : \ X = PX + XP \right\}, \\ \mathfrak{s}_P(n) &:= \mathfrak{s}(n) \cap \mathfrak{gl}_P(n) \quad \text{and} \\ \mathfrak{so}_P(n) &:= \mathfrak{so}(n) \cap \mathfrak{gl}_P(n). \end{aligned}$$

We will need the following properties.

Proposition 1 (Batzies et al. [3])**.** *Let $P \in \boldsymbol{\mathscr{G}}_{n,k}$ and $X \in \mathfrak{gl}_P(n)$ then*
(1) $PX^{2i-1}P = 0$, for any $i \geq 1$,
(2) $PX^{2i} = PX^{2i}P = X^{2i}P$, for any $i \geq 0$,
(3) $[P,[P,X]] = X$.

The tangent space to a point $P \in \boldsymbol{\mathscr{G}}_{n,k}$ is given by

$$\mathbf{T}_P\boldsymbol{\mathscr{G}}_{n,k} = \left\{ [X,P] \ : \ X \in \mathfrak{so}_P(n) \right\}.$$

The Graßmann manifold will be equipped with the metric inherited from the Euclidean space $\mathbb{R}^{n \times n}$, which incidentally coincides with the Frobenius metric, *cf.* [8]

$$\langle [X_1,P],[X_2,P] \rangle = \operatorname{tr}(X_1^{\mathrm{T}} X_2).$$

If $P \in \boldsymbol{\mathscr{G}}_{n,k}$ then $\Theta P \Theta^{\mathrm{T}} \in \boldsymbol{\mathscr{G}}_{n,k}$, for any $\Theta \in \mathbb{O}(n)$. Thus $\gamma\colon (-\varepsilon,\varepsilon) \to \boldsymbol{\mathscr{G}}_{n,k}$ given by $\gamma(t) = \Theta(t) P \Theta^{\mathrm{T}}(t)$, where Θ is a curve in $\mathbb{O}(n)$ satisfying $\Theta(0) = \mathbf{I}$, is a curve in the Graßmann manifold passing through P at $t = 0$.

A geodesic γ in $\mathscr{G}_{n,k}$ starting from P with initial velocity $\dot\gamma(0) = [X, P]$ is given by

$$\gamma(t) = e^{tX} P e^{-tX}.$$

An explicit formula for a geodesic $\gamma\colon [0, 1] \to \mathscr{G}_{n,k}$ joining a point P to a point Q was derived in [3] and gives the initial velocity vector in terms of the initial and final points only. This geodesic is given by

$$\gamma(t) = e^{tX} P e^{-tX}, \quad \text{where} \quad X = \frac{1}{2}\log((\mathbf{I} - 2Q)(\mathbf{I} - 2P)). \tag{1}$$

Here 'log' stands for the *principal logarithm* of a matrix. If the orthogonal matrix $(\mathbf{I}-2Q)(\mathbf{I}-2P)$ has no negative real eigenvalues then this *geodesic is unique*. The following property of the velocity vector field along a geodesic will be used later.

Proposition 2. *If $\gamma(t) = e^{tX} P e^{-tX}$ is a geodesic in $\mathscr{G}_{n,k}$, then $X \in \mathfrak{so}_{\gamma(t)}(n)$.*

Proof: One needs to show that

$$X = \gamma(t)X + X\gamma(t). \tag{2}$$

By the hypothesis, equality (2) holds for $t = 0$. From the commuting properties of the matrix exponential with its argument,

$$\begin{aligned} \gamma(t)X + X\gamma(t) &= e^{tX} P e^{-tX} X + X e^{tX} P e^{-tX} \\ &= e^{tX}(PX + XP)e^{-tX} = e^{tX} X e^{-tX} = X. \end{aligned}$$

■

2.2. The Stiefel manifold. The Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$ has the following matrix representation:

$$\mathscr{S}_{n,k} := \left\{ S \in \mathbb{R}^{n\times k} \ : \ S^{\mathrm{T}}S = \mathbf{I}_k \right\}.$$

This is a submanifold of $\mathbb{R}^{n\times k}$, having dimension $nk - (k + 1)k/2$. Stiefel manifolds $\mathscr{S}_{n,k}$ are homogeneous spaces

$$\mathscr{S}_{n,k} \cong \mathbb{O}(n)/\mathbb{O}(n-k).$$

The tangent space to $\mathscr{S}_{n,k}$ at a point $S \in \mathscr{S}_{n,k}$ can be parametrized as

$$\mathbf{T}_S\mathscr{S}_{n,k} = \left\{ V \in \mathbb{R}^{n\times k} \ : \ V^{\mathrm{T}}S + S^{\mathrm{T}}V = 0 \right\}.$$

The Stiefel manifold is equipped with the *canonical metric*, given by

$$\langle V_1, V_2\rangle = \mathrm{tr}\big(V_1^{\mathrm{T}}(\mathbf{I} - \tfrac{1}{2}SS^{\mathrm{T}})V_2\big), \quad \text{where} \quad V_1, V_2 \in \mathbf{T}_S\mathscr{S}_{n,k}.$$

The projection onto the tangent space $\pi_\top \colon \mathbb{R}^{n\times k} \to \mathbf{T}_S\mathscr{S}_{n,k}$ is given by

$$\pi_\top(X) := S\,\mathbf{skew}(S^\mathrm{T}X) + (\mathbf{I} - SS^\mathrm{T})X,$$

where, for a square matrix A, $\mathbf{skew}(A)$ denotes $(A - A^\mathrm{T})/2$.

Geodesics in the Stiefel manifold satisfy the following second order differential equation, *cf.* [7]

$$\ddot{\gamma} + \dot{\gamma}\dot{\gamma}^\mathrm{T}\gamma + \gamma\Big((\gamma^\mathrm{T}\dot{\gamma})^2 + \dot{\gamma}^\mathrm{T}\dot{\gamma}\Big) = 0.$$

A geodesic γ in $\mathscr{S}_{n,k}$ starting from a point $S = \Theta\Delta$, where $\Theta \in \mathbb{O}(n)$ and $\Delta = \left[\begin{array}{c}\mathbf{I}_k\\ \hline 0\end{array}\right]_{n\times k}$, is given by

$$\gamma(t) = \Theta e^{tX}\Delta,$$

where $X \in \mathfrak{so}(n)$ has the following structure:

$$X = \left[\begin{array}{c|c} A & -B^\mathrm{T} \\ \hline B & 0 \end{array}\right].$$

Compared with what happens for the Graßmann manifolds, solving the geodesic equation for Stiefel manifolds is quite hard. Nevertheless, Edelman et al. in [7] have included formulae for geodesics on Stiefel manifolds that start at a given point with a prescribed velocity vector. But, as far as we know, there are no explicit formulas for the geodesic joining two arbitrary points that depends on these points only.

2.3. Relationships. There are some intimate relationships between $\mathbb{SO}(n)$, $\mathscr{G}_{n,k}$ and $\mathscr{S}_{n,k}$ that can be expressed in terms of the following surjective mappings, where Δ is the matrix defined in the previous section and $\Lambda = \Delta\Delta^\mathrm{T}$.

- The projection $\pi \colon \mathbb{SO}(n) \to \mathscr{S}_{n,k}$ defined by $\pi(\Theta) := \Theta\Delta$;
- The mapping $\varphi \colon \mathbb{SO}(n) \to \mathscr{G}_{n,k}$ defined by $\varphi(\Theta) := \Theta\Lambda\Theta^\mathrm{T}$;
- The mapping $\psi \colon \mathscr{S}_{n,k} \to \mathscr{G}_{n,k}$ defined by $\psi(S) := SS^\mathrm{T}$.

Clearly, $(\Theta\Delta)^{\mathrm{T}}(\Theta\Delta) = \Delta^{\mathrm{T}}\Delta = \mathbf{I}_k \quad \text{and} \quad (\Theta\Delta)(\Theta\Delta)^{\mathrm{T}} = \Theta\Lambda\Theta^{\mathrm{T}}$.
The following commutative diagram summarizes these relationships.

$$\begin{tikzcd} \mathbb{SO}(n) \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\ \boldsymbol{\mathscr{S}}_{n,k} \arrow[r, "\psi"] & \boldsymbol{\mathscr{G}}_{n,k} \end{tikzcd} \tag{3}$$

This also creates relationships between geodesics on these manifolds. In particular, geodesics on Graßmann manifolds and geodesics on Stiefel manifold are projections of special geodesics on $\mathbb{SO}(n)$, as explained next.

Let $\gamma\colon [0,1] \to \mathbb{SO}(n)$ be a geodesic given by $\gamma(t) = e^{tX}\Theta$. Define a curve $\sigma\colon [0,1] \to \boldsymbol{\mathscr{G}}_{n,k}$ by

$$\sigma(t) := (\varphi \circ \gamma)(t) = e^{tX}\Theta\Lambda\Theta^{\mathrm{T}}e^{-tX}.$$

Denote $P = \Theta\Lambda\Theta^{\mathrm{T}} \in \boldsymbol{\mathscr{G}}_{n,k}$ and suppose that $X \in \mathfrak{so}_P(n)$. Then, σ is a geodesic in $\boldsymbol{\mathscr{G}}_{n,k}$ starting from P, with initial velocity equal to $[X, P]$ *cf.* [3]. A simple calculation shows that the condition $X \in \mathfrak{so}_P(n)$ is equivalent to $\Theta^{\mathrm{T}}X\Theta \in \mathfrak{so}_\Lambda(n)$, and the latter implies a particular matrix structure for $\Theta^{\mathrm{T}}X\Theta$, namely

$$\Theta^{\mathrm{T}}X\Theta = \left[\begin{array}{c|c} 0 & -B^{\mathrm{T}} \\ \hline B & 0 \end{array}\right].$$

It has already been mentioned earlier that the minimising geodesic $\sigma\colon [0,1] \to \boldsymbol{\mathscr{G}}_{n,k}$ joining two close enough points P, Q is given by

$$\sigma(t) = e^{tX}Pe^{-tX},$$

where $X = \frac{1}{2}\log((\mathbf{I} - 2Q)(\mathbf{I} - 2P))$.
For any $\Theta \in \varphi^{-1}(P)$, it also happens that the image by φ of the geodesic $\gamma(t) = e^{tX}\Theta$ in $\mathbb{SO}(n)$ is the corresponding geodesic in the Graßman manifold $\boldsymbol{\mathscr{G}}_{n,k}$, i.e.,

$$P = \Theta\Lambda\Theta^{\mathrm{T}}, \qquad Q = e^{X}\Theta\Lambda\Theta^{\mathrm{T}}e^{-X}$$

and

$$\varphi\big(e^{tX}\Theta\big) = e^{tX}\Theta\Lambda\Theta^{\mathrm{T}}e^{-tX} = e^{tX}Pe^{-tX} = \sigma(t).$$

In a similar way, one may analyse which geodesics in $\mathbb{SO}(n)$ project to geodesics in the Stiefel manifold. Edelman et al. in [7] proved that the curve defined by

$$\alpha(t) = e^{tZ}S = e^{tZ}\Theta\Delta = \pi(e^{tZ}\Theta)$$

is a geodesic in $\mathscr{S}_{n,k}$ starting at S when the skew-symmetric matrix Z satisfies the following block structure

$$\Theta^{\mathrm{T}} Z \Theta = \left[\begin{array}{c|c} A & -B^{\mathrm{T}} \\ \hline B & 0 \end{array}\right].$$

But a closed form expression for Z, given the initial and final points, is not yet known.

To overcome this difficulty and being able to propose a solution for an interpolating problem on Stiefel manifolds, we are going to introduce, in the next section, other interesting curves in $\mathscr{S}_{n,k}$ that will play an important role.

We end this section with two properties that are immediate consequences of those in Proposition 1 and will be useful later on.

Proposition 3. *Let $S \in \mathscr{S}_{n,k}$ and $X \in \mathfrak{gl}_{SS^{\mathrm{T}}}(n)$. Then*

(1) $S^{\mathrm{T}} X S = 0$;
(2) $SS^{\mathrm{T}} X^2 S = X^2 SS^{\mathrm{T}} S = X^2 S$.

3. Quasi-Geodesics in Stiefel Manifolds

In this section we define certain smooth curves in the Stiefel manifold $\mathscr{S}_{n,k}$ that join two arbitrary points S_1 and S_2 and have many interesting properties. In some cases these curves are geodesics, but in general their velocity vector field may fail to have zero covariant derivative. The generic term for these curves will be quasi-geodesics because they have constant geodesic curvature and are associated to certain retractions on $\mathscr{S}_{n,k}$. We use here the notion of a retraction introduced in Absil et al. [2] for general manifolds.

Definition 4. *A retraction R on the Stiefel manifold $\mathscr{S}_{n,k}$ is a smooth mapping from the tangent bundle $\mathbf{T}\mathscr{S}_{n,k}$ to $\mathscr{S}_{n,k}$ that, when restricted to each tangent space at a point $S \in \mathscr{S}_{n,k}$ (restriction denoted by R_S), satisfies the following properties:*

(i) $R_S(0) = S$.
(ii) $dR_S(0) = id$.

If $V \in \mathbf{T}_S\mathscr{S}_{n,k}$, one can define a smooth curve $\beta_V\colon t \mapsto R_S(tV)$ associated to the retraction R. The curve β_V which satisfies $\beta_V(0) = S$ and $\dot\beta_V(0) = V$ is called a *quasi-geodesic*. In the sequel we will present a quasi-geodesic on the Stiefel manifold, which is different from the example included in [2] or in [14], but has other very interesting properties. Before that, we present a representation of the tangent space to $\mathscr{S}_{n,k}$ at a point S, which differs from that considered in Edelman et al. [7] but will prove to be very important for further derivations.

Proposition 5. *Let $S \in \mathscr{S}_{n,k}$, so that $P = SS^{\mathrm{T}} \in \mathscr{G}_{n,k}$. Then,*

$$\mathbf{T}_S\mathscr{S}_{n,k} = \{XS + S\Omega, \text{ where } X \in \mathfrak{so}_P(n) \text{ and } \Omega \in \mathfrak{so}(k)\}\,. \tag{4}$$

Moreover, if $V = XS + S\Omega \in \mathbf{T}_S\mathscr{S}_{n,k}$, then

$$X = VS^{\mathrm{T}} - SV^{\mathrm{T}} + 2SV^{\mathrm{T}}SS^{\mathrm{T}} \text{ and } \Omega = S^{\mathrm{T}}V.$$

Proof: Let $M := \{XS + S\Omega, \text{ where } X \in \mathfrak{so}_P(n) \text{ and } \Omega \in \mathfrak{so}(k)\}$. Notice that the dimensions of M and of $\mathbf{T}_S\mathscr{S}_{n,k}$ match. Indeed,

$$\dim(\mathfrak{s}_P(n)) = \dim(\mathbf{T}_P\mathscr{G}_{n,k}) = k(n-k),\ \ \dim(\mathfrak{so}(k)) = k(k-1)/2,$$

and so,

$$\dim(M) = k(n-k) + k(k-1)/2 = nk - k(k+1)/2 = \dim(\mathbf{T}_S\mathscr{S}_{n,k}).$$

To show that (4) is a good parametrization of $\mathbf{T}_S\mathscr{S}_{n,k}$, we must prove that $M \subset \mathbf{T}_S\mathscr{S}_{n,k}$ and $\mathbf{T}_S\mathscr{S}_{n,k} \subset M$. For the first part, a trivial calculation shows that if $V = XS + S\Omega \in M$ then, since X and Ω are skew symmetric, V satisfies the equation $V^{\mathrm{T}}S + S^{\mathrm{T}}V = 0$, that is, $V \in \mathbf{T}_S\mathscr{S}_{n,k}$. For the second part, we show that if $V \in \mathbf{T}_S\mathscr{S}_{n,k}$, there exists $\Omega \in \mathfrak{so}(k)$ and $X \in \mathfrak{so}_P(n)$ such that $V = XS + S\Omega$. This is done by construction:

$$\Omega := S^{\mathrm{T}}V \text{ and } X := VS^{\mathrm{T}} - SV^{\mathrm{T}} + 2SV^{\mathrm{T}}SS^{\mathrm{T}}.$$

It is just a matter of simple calculations, using the fact that $V \in \mathbf{T}_S\mathscr{S}_{n,k}$, to check that indeed $V = XS + S\Omega$, $\Omega \in \mathfrak{so}(k)$, $X \in \mathfrak{so}(n)$, and moreover $X = XSS^{\mathrm{T}} + SS^{\mathrm{T}}X$, that is $X \in \mathfrak{so}_P(n)$.

The last statement in the proposition follows from the previous considerations. ■

Proposition 6. *Let S, X, and Ω be as in the Proposition 5. Then, the mapping $R\colon \mathbf{T}\mathscr{S}_{n,k} \to \mathscr{S}_{n,k}$ whose restriction to $\mathbf{T}_S\mathscr{S}_{n,k}$ is defined by $R_S(V) = e^X S e^\Omega$ is a retraction on the Stiefel manifold, and $\beta\colon t \mapsto e^{tX}Se^{t\Omega}$ is a quasi-geodesic in $\mathscr{S}_{n,k}$ that satisfies*

(1) $\beta(0) = S$;
(2) $\dot\beta(t) = e^{tX}(XS + S\Omega)e^{t\Omega}$;
(3) $\ddot\beta(t) = e^{tX}(X^2S + 2XS\Omega + S\Omega^2)e^{t\Omega}$.

Proof: This statement is true because the mapping R satisfies both conditions of the Definition 4 and β is the quasi-geodesic associated to the retraction. The formulas for the derivatives of β are also straightforward. ■

3.1. Joining points in the Stiefel manifold by quasi-geodesics. Given two distinct points S_1 and S_2 in the Stiefel manifold, our objective now is to choose $X \in \mathfrak{so}(n)$ and $\Omega \in \mathfrak{so}(k)$ so that the quasi geodesic defined by $\beta(t) = e^{tX} S_1 e^{t\Omega}$ joins the point S_1 (at $t = 0$) to the point S_2 (at $t = 1$). Some restrictions on S_1 and S_2 are expected and they will be determined by the existence of logarithms of some matrices that appear in the next theorem. We recall that, a nonsingular matrix Y without negative eigenvalues always has a unique logarithm whose spectrum lies in the horizontal strip $\{z \in \mathbb{C} : -\pi < \operatorname{Im}(z) < \pi\}$. This unique matrix is called the principal logarithm of Y and is denoted by $\log(Y)$ (Horn and Johnson [9]).

Theorem 7. *Let S_1 and S_2 be two distinct points in $\mathscr{S}_{n,k}$ so that, for $i = 1, 2$, $P_i = S_i S_i^{\mathrm{T}} \in \mathscr{G}_{n,k}$. Then, if*

$$X = \frac{1}{2}\log\big((\mathbf{I} - 2S_2S_2^{\mathrm{T}})(\mathbf{I} - 2S_1S_1^{\mathrm{T}})\big) \ \textit{and} \ \Omega = \log\big(S_1^{\mathrm{T}} e^{-X} S_2\big), \tag{5}$$

the quasi-geodesic defined by

$$\beta(t) := e^{tX} S_1 e^{t\Omega}, \tag{6}$$

has the following properties:

(1) $\beta(0) = S_1$;
(2) $\beta(1) = S_2$;
(3) $\|\dot\beta(t)\|^2 = -\operatorname{tr}\big(S_1^{\mathrm{T}} X^2 S_1 + \frac{1}{2}\Omega^2\big)$ *(constant speed)*;
(4) $D_t\dot\beta(t) = X\beta(t)\Omega$;
(5) $\|D_t\dot\beta(t)\|^2 = \operatorname{tr}\big(\Omega^2 S_1^{\mathrm{T}} X^2 S_1\big)$ *(constant covariant acceleration)*.

Proof: Before starting the proof, we show that X and Ω agree with the parametrisation of the tangent space given in Proposition 5. According to (1), $[X, P_1]$ is the initial velocity vector of the geodesic in the Graßmann manifold that joins the point P_1 (at $t = 0$) to P_2 (at $t = 1$), so $X \in \mathfrak{so}_{P_1}(n)$. Moreover, $S_1^{\mathrm{T}} e^{-X} S_2$ is orthogonal as can be easily checked. Indeed, from the expression for X we immediately get

$$\big(S_1^{\mathrm{T}} e^{-X} S_2\big)\big(S_1^{\mathrm{T}} e^{-X} S_2\big)^{\mathrm{T}} = S_1^{\mathrm{T}} e^{-X} S_2 S_2^{\mathrm{T}} e^{X} S_1 = S_1^{\mathrm{T}} S_1 S_1^{\mathrm{T}} S_1 = \mathbf{I}_k;$$
$$\big(S_1^{\mathrm{T}} e^{-X} S_2\big)^{\mathrm{T}}\big(S_1^{\mathrm{T}} e^{-X} S_2\big) = S_2^{\mathrm{T}} e^{X} S_1 S_1^{\mathrm{T}} e^{-X} S_2 = S_2^{\mathrm{T}} S_2 S_2^{\mathrm{T}} S_2 = \mathbf{I}_k.$$

Now, the first two properties follow from the definition of the curve β. To simplify notations we omit the dependency on t. With the canonical metric on the Stiefel manifold, we can write

$$\|\dot\beta\|^2 = \langle\dot\beta,\dot\beta\rangle = \operatorname{tr}\Big(\dot\beta^{\mathrm{T}}(\mathbf{I}-\tfrac{1}{2}\beta\beta^{\mathrm{T}})\dot\beta\Big) = \operatorname{tr}\Big(\dot\beta^{\mathrm{T}}\dot\beta-\tfrac{1}{2}\dot\beta^{\mathrm{T}}\beta\beta^{\mathrm{T}}\dot\beta\Big)$$

But

$$\begin{aligned}
&\dot\beta^{\mathrm{T}}\dot\beta = -e^{-t\Omega}S_1^{\mathrm{T}}X^2S_1e^{t\Omega}-\Omega^2,\\
&\beta^{\mathrm{T}}\dot\beta = \Omega, \quad \text{hence} \quad \dot\beta^{\mathrm{T}}\beta = -\Omega.
\end{aligned}$$

So

$$\begin{aligned}
\|\dot\beta\|^2 &= -\operatorname{tr}\big(S_1^{\mathrm{T}}X^2S_1\big)-\operatorname{tr}\big(\Omega^2\big)+\tfrac{1}{2}\operatorname{tr}\big(\Omega^2\big)\\
&= -\operatorname{tr}\big(S_1^{\mathrm{T}}X^2S_1+\tfrac{1}{2}\Omega^2\big).
\end{aligned}$$

This concludes the proof of (3).

To prove property (4) we take into consideration the formulas for $\dot\beta$ and $\ddot\beta$ in Proposition 6 and the following formula for the covariant derivative of $\dot\beta$ along β, given in [7]:

$$D_t\dot\beta = \ddot\beta+\dot\beta\dot\beta^{\mathrm{T}}\beta+\beta\Big((\beta^{\mathrm{T}}\dot\beta)^2+\dot\beta^{\mathrm{T}}\dot\beta\Big).$$

Using the properties in Proposition 3, this can be simplified to obtain $D_t\dot\beta(t) = X\beta(t)\Omega$. Finally, we show that the acceleration vector field along β is constant. This requires some lengthy calculations that are partially omitted because simplifications only require properties that have already been used before.

$$\begin{aligned}
\|D_t\dot\beta\|^2 &= \operatorname{tr}\Big((D_t\dot\beta)^{\mathrm{T}}(\mathbf{I}-\tfrac{1}{2}\beta\beta^{\mathrm{T}})(D_t\dot\beta)\Big)\\
&= \operatorname{tr}\big(\Omega\beta^{\mathrm{T}}X(X\beta\Omega-\tfrac{1}{2}\beta\beta^{\mathrm{T}}X\beta\Omega)\big)\\
&= \operatorname{tr}\big(\Omega S_1^{\mathrm{T}}X^2S_1\Omega\big) = \operatorname{tr}\big(\Omega^2S_1^{\mathrm{T}}X^2S_1\big).
\end{aligned}$$

This completes the proof. ■

There are two situations when the quasi-geodesic defined in (6) is a true geodesic, as will be detailed in the next Corollary. Figures 1 and 2 illustrate these situations. For the situation in Figure 3, no explicit form for the geodesic joining the frames S_1 and S_2 is known. In this case, what can be easily exhibited is a two-piece broken geodesic joining those frames, each piece being a geodesic of one of the two types illustrated in Figures 1 and 2.

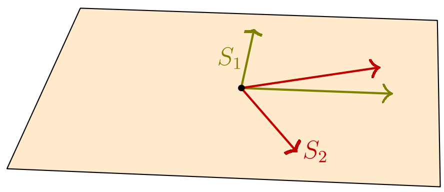

FIGURE 1. The frames S_1 and S_2 span the same subspace

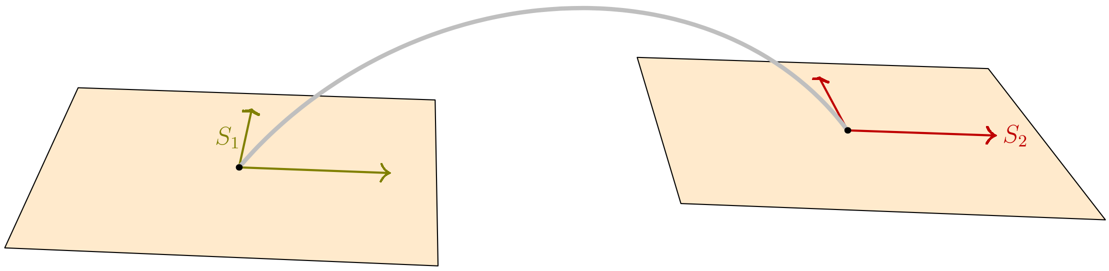

FIGURE 2. The frames S_1 and S_2 span two different subspaces, but $e^X S_1 = S_2$

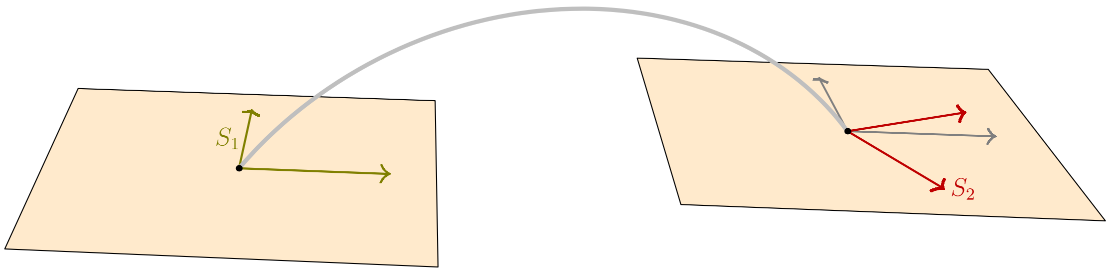

FIGURE 3. The frames S_1 and S_2 span two different subspaces, but $e^X S_1 \neq S_2$

Corollary 8.

(1) *If the frames S_1 and S_2 generate the same subspace, then the curve β defined in* (6) *is a geodesic in $\mathscr{S}_{n,k}$ joining S_1 to S_2.*

(2) *If the frames S_1 and S_2 do not generate the same subspace but the frame $e^X S_1$, where $X = \frac{1}{2}\log\big((\mathbf{I} - 2S_2S_2^{\mathrm{T}})(\mathbf{I} - 2S_1S_1^{\mathrm{T}})\big)$, coincides with S_2, then the curve β defined in* (6) *is also a geodesic in $\mathscr{S}_{n,k}$ joining S_1 to S_2.*

Proof: In the first case $X = 0$ and $\beta(t) = S_1 e^{t\Omega}$. In the second case $\Omega = 0$ and $\beta(t) = e^{tX} S_1$. It follows imediatly from Theorem 7 - *(4)* that in both cases $D_t\dot{\beta}(t) \equiv 0$.

■

Remark 9. *For the two extreme cases when $k = 1$ and $k = n$, the Stiefel manifold becomes respectively a sphere and an orthogonal group. The quasi-geodesics are then true geodesics since both cases fit into the two exceptions above. Indeed, $\Omega = 0$ if $k = 1$ and $X = 0$ if $k = n$. In this case $\Omega = \log(S_1^{\mathrm{T}} S_2) \in \mathfrak{so}(n)$.*

Proposition 10. *The geodesic curvature κ of the quasi-geodesic defined in (6) is constant and given by*

$$\kappa = -\frac{\sqrt{\operatorname{tr}\left(\Omega^2 S_1^{\mathrm{T}} X^2 S_1\right)}}{\operatorname{tr}\left(S_1^{\mathrm{T}} X^2 S_1 + \frac{1}{2}\Omega^2\right)}.$$

Moreover, $0 \leq \kappa < 1$.

Proof: We use the following formula, obtained from [12, p. 137], for computing the geodesic curvature

$$\kappa = \frac{\|D_t \dot{\beta}\|}{\|\dot{\beta}\|^2} - \frac{\langle D_t \dot{\beta}, \dot{\beta}\rangle}{\|\dot{\beta}\|^3}. \tag{7}$$

Since β has constant speed, the second term in (7) vanishes and the expression for the geodesic curvature reduces to the first term that is immediately obtained from the formulas in Theorem 7.

To show that $0 \leq \kappa < 1$, we use a trace inequality due to von Neumann [19], which states that for any $k \times k$ complex matrices A and B with singular values $a_1 \geq a_2 \geq \cdots \geq a_k$ and $b_1 \geq b_2 \geq \cdots \geq b_k$ respectively, $|\operatorname{tr}(AB)| \leq \sum_i a_i b_i$. If we consider $A = -S_1^{\mathrm{T}} X^2 S_1$ and $B = -\Omega^2$ which are real symmetric and nonnegative definite, their singular values coincide with their eigenvalues and so $\operatorname{tr}(AB) \leq \sum_i a_i b_i$. Consequently,

$$\begin{aligned} \kappa^2 &= \frac{\operatorname{tr}(AB)}{\operatorname{tr}^2(A + \frac{1}{2}B)} = \frac{\operatorname{tr}(AB)}{\operatorname{tr}^2(A) + \frac{1}{4}\operatorname{tr}^2(B) + \operatorname{tr}(A)\operatorname{tr}(B)} \\ &\leq \frac{\sum_i (a_i b_i)}{(\sum_i a_i)^2 + \frac{1}{4}(\sum_i b_i)^2 + \sum_{i \neq j}(a_i b_j) + \sum_i (a_i b_i)}. \end{aligned}$$

Since the eigenvalues of A and B are nonnegative and not simultaneously equal to zero and κ is nonnegative, the geodesic curvature has the required bounds. ■

Note that the geodesic curvature is zero whenever A or B is zero, that is, when $X = 0$ or $\Omega = 0$. As expected, this result is consistent with the statements in Corollary 8.

4. The Casteljau Algorithm on manifolds

The classical Casteljau algorithm, introduced independently by Casteljau [6] and Bézier [4], is a geometric construction to generate polynomial curves in $\mathbb{R}^n$ based on successive linear interpolation techniques. After the basic idea of Park and Ravani [15] of replacing linear interpolation by geodesic interpolation, the Casteljau algorithm was generalized to accommodate geometric polynomial curves and also interpolating splines on Riemannian manifolds (see, for instance, the work of Crouch, Kun and Silva Leite [5], Popiel and Noakes [16], and Nava-Yazdani and Polthier [13]). We next give a succinct description of this algorithm for generating polynomials of degree m on a complete Riemannian manifold $\mathbf{M}$, where for the sake of simplicity we parametrize the curves on the $[0, 1]$ interval.

If $x_0, \ldots, x_m$ are distinct points in $\mathbf{M}$ and $\sigma_1(t, x_i, x_{i+1})$ is the geodesic arc joining x_i (at $t = 0$) to x_{i+1} (at $t = 1$), a smooth curve $t \mapsto \sigma_m(t)$, joining x_0 (at $t = 0$) to x_m (at $t = 1$), may be constructed by recursive geodesic interpolation and depends on the given points. The curves produced by this recursive process which involves m steps are defined by

$$\begin{aligned} \sigma_k(t, x_i, \ldots, x_{i+k}) &= \sigma_1\left(t, \sigma_{k-1}(t, x_i, \ldots, x_{i+k-1}), \sigma_{k-1}(t, x_{i+1}, \ldots, x_{i+k})\right), \\ &\quad k = 2, \ldots, m; \quad i = 0, \ldots, m-k. \end{aligned}$$

The curve σ_m obtained in the last step, that is $\sigma_m(t) := \sigma_m(t, x_0, \ldots, x_m)$, generalises the Euclidean polynomials of degree m and is called *geometric polynomial* in $\mathbf{M}$. This curve doesn't interpolate the points $x_1, \ldots, x_{m-1}$. They are only used to generate the curve that joins x_0 to x_m but, of course, influence the shape of the curve. For that reason, they are called *control points*. Alternatively, one can prescribe other boundary conditions, such as $m-1$ initial conditions $D_t^k \dot{\sigma}_m(0)$, $k = 0, \ldots, m-2$, and compute from them the control points needed for the algorithm. This is theoretically possible, but the complexity of the computations increases significantly with m. This algorithm can also be used to generate *geometric polynomial splines*, which are interpolating curves obtained by piecing together several geometric polynomials in a smooth manner.

Figure 4 illustrates the idea behind the Casteljau algorithm in the 2-sphere. It shows how to generate several points of a quadratic curve.

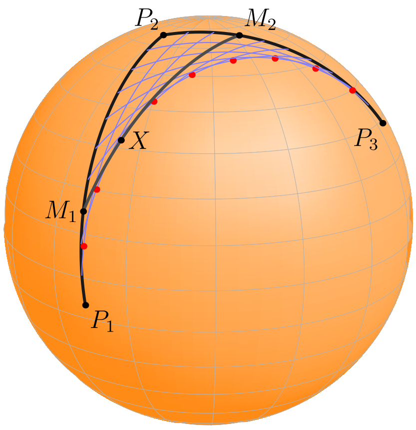

FIGURE 4. Illustration of the Casteljau algorithm to generate a quadratic polynomial in S^2

4.1. A modification of the Casteljau algorithm. Although the Casteljau algorithm appeared as a geometric tool to construct polynomials of any order by successive linear interpolation, it can be modified to accommodate curves with other properties. This has been done, for instance, in [10] and [17]. On manifolds where explicit formulas for geodesics are not available, this is particularly useful and will also be used to generate a $\mathcal{C}^1$ interpolating curve on the Stiefel manifold in the next section. For now we proceed with some generic results for curves obtained with only one control point and two steps and first prove a result involving the initial and final velocity of a curve generated with two steps, but not necessarily a quadratic polynomial.

Given a set of three points $\{ x_i \}_{i=0}^2$ in a manifold $\mathbf{M}$, let $t \mapsto \sigma_1(t, x_i, x_{i+1})$ be curves joining x_i to x_{i+1}, for $i = 0, 1$. Define a family of curves $\gamma \colon [0,1] \times [0,1] \to \mathbf{M}$ as follows. For a fixed $t_0 \in [0,1]$, the map $t \mapsto \gamma(t, t_0)$ is a curve joining $\sigma_1(t_0, x_0, x_1)$ to $\sigma_1(t_0, x_1, x_2)$, as illustrated in Figure 5. Then $\sigma_2 \colon [0,1] \to \mathbf{M}$ given by $\sigma_2(t) = \gamma(t,t)$ is a curve joining x_0 to x_2. If the curves considered above are geodesics then σ_2 is a second order polynomial.

We are interested in finding out how the velocities of σ_2 at the end points are related with the velocities of the curves used along the algorithm steps. The answer is given in the following proposition.

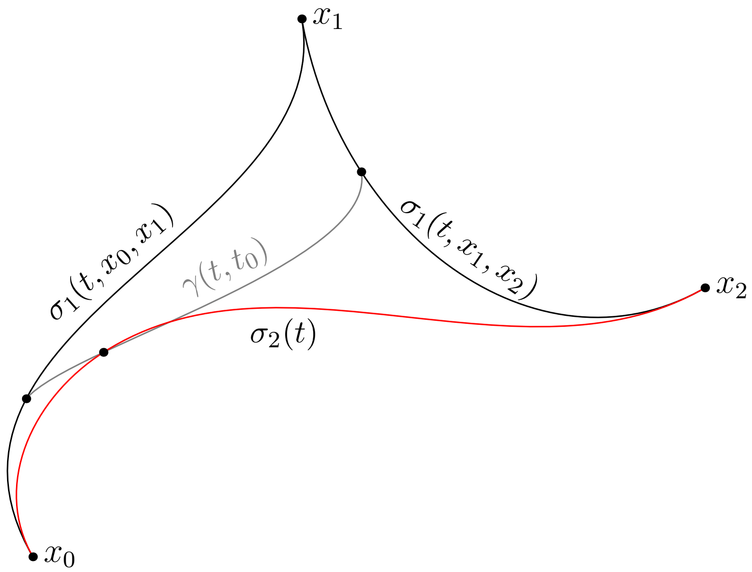

FIGURE 5. A modification of the two-step Casteljau algorithm

Proposition 11. *Suppose that the curves σ_1 are differentiable and that*

$$\gamma(t, 0) = \sigma_1(t, x_0, x_1) \quad \textit{and} \quad \gamma(t, 1) = \sigma_1(t, x_1, x_2). \tag{8}$$

Then

$$\dot{\sigma}_2(0) = 2\dot{\sigma}_1(0, x_0, x_1) \quad \textit{and} \quad \dot{\sigma}_2(1) = 2\dot{\sigma}_1(1, x_1, x_2).$$

Proof: The following identities follow from the definition of the curve γ:

$$\gamma(0, t) = \sigma_1(t, x_0, x_1) \quad \text{and} \quad \gamma(1, t) = \sigma_1(t, x_1, x_2). \tag{9}$$

Note that

$$\dot{\sigma}_2(t) = \left.\frac{\partial}{\partial s}\right|_{s=t} \gamma(s, t) + \left.\frac{\partial}{\partial s}\right|_{s=t} \gamma(t, s).$$

Therefore, from the hypothesis (8) and by identities (9) it follows that

$$\dot{\sigma}_2(0) = \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(s, 0) + \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(0, s) = 2\dot{\sigma}_1(0, x_0, x_1).$$

Similarly

$$\dot{\sigma}_2(1) = \left.\frac{\partial}{\partial s}\right|_{s=1} \gamma(s, 1) + \left.\frac{\partial}{\partial s}\right|_{s=1} \gamma(1, s) = 2\dot{\sigma}_1(1, x_1, x_2).$$

∎

This result easily generalises to higher order curves.

Our main purpose is to generate simple and smooth spline curves interpolating a set of data points on the manifold. We are particularly interested in spline curves that are differentiable and obtained by piecing together quadratic polynomials. This can be achieved by using, for instance, the Casteljau algorithm in each time interval $[i, i+1]$, starting with an arbitrary control point in the first interval (or, equivalently, prescribing the velocity at $t = 0$), and computing the control points for the remaining intervals so that the whole curve is $\mathcal{C}^1$-smooth.

Since this algorithm is based on linear interpolation, its implementation on a specific Riemannian manifold requires that an explicit formula for the geodesic that joins two points is available. Although this is not the case in general, it happens that for the Graßmann manifold such a formula has been derived recently in Batzies *et al.* [3].

4.2. The Casteljau algorithm on Graßmann manifolds. The generalisation of the classical Casteljau algorithm will be used to solve the following problem on the Graßmann manifolds.

Problem 1. *Given a set of points $P_0, \ldots, P_m$ in the Graßmann manifold $\boldsymbol{\mathscr{G}}_{n,k}$ and $\Omega_0 \in \mathfrak{so}_{P_0}(n)$, find a $\mathcal{C}^1$-smooth curve σ that interpolates the points P_i at time i and has initial velocity equal to $[\Omega_0, P_0]$, that is*

$$\sigma(i) = P_i, \quad \textit{for} \quad i = 0, 1, \ldots, m \quad \textit{and} \quad \dot{\sigma}(0) = [\Omega_0, P_0].$$

4.2.1. *Solving this problem using the Casteljau algorithm.* The curve σ may be generated by piecing together quadratic polynomials defined on each subinterval $[i, i+1]$, and joining P_i to P_{i+1} with control point C_i, that is

$$\sigma(t)\big|_{[i,i+1]} = \sigma_2(t - i, P_i, C_i, P_{i+1}).$$

The first control point C_0 is computed from P_0, P_1 and Ω_0, Figure 6. In order to ensure that σ is $\mathcal{C}^1$-smooth, the initial velocity of each subsequent spline segment must equal the final velocity of the previous segment.

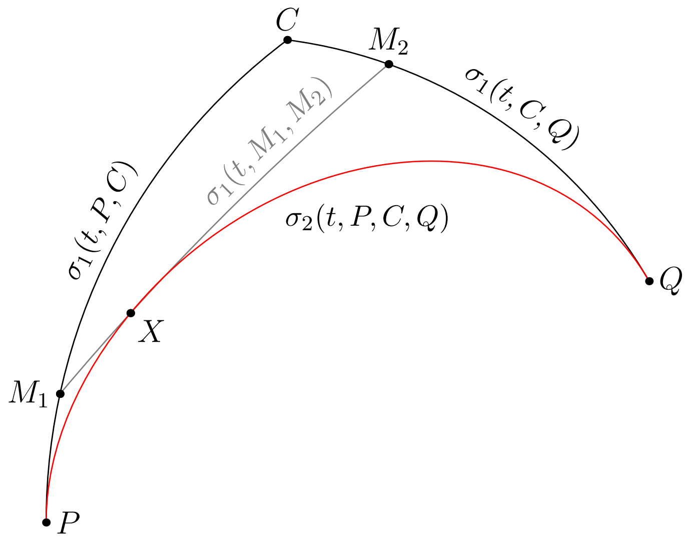

FIGURE 6. The generalised Casteljau algorithm; segments $\sigma_1(t, P, C)$ and $\sigma_1(t, C, Q)$ form the first step of the algorithm. Then, given $t_0 \in [0, 1]$, $X = \sigma_1(t_0, M_1, M_2)$ is a point in the spline, where $M_1 = \sigma_1(t_0, P, C)$ and $M_2 = \sigma_1(t_0, C, Q)$

4.2.2. *Generating a second order spline.* Given a sequence of data points $P_0, P_1, \ldots, P_m$ and an initial Ω_0, the algorithm produces a second order spline $\sigma\colon [0, m] \to \boldsymbol{\mathscr{G}}_{n,k}$, passing through the data points, such that

$$\sigma(i) = P_i, \quad \text{for} \quad 0 \le i \le m \quad \text{and} \quad \dot{\sigma}(0) = [\Omega_0, P_0].$$

Each segment $\sigma([i, i{+}1])$ joins P_i to P_{i+1}. The algorithm is based on a general version of the Casteljau algorithm described at the beginning of the section.

To find a point $\sigma(t)$ of the spline, first iterate Algorithm 2 to find the component of initial velocity vector Ω_i, for the segment $\sigma([i, i+1])$, where $t \in [i, i+1]$. Then with the triple Ω_i, P_i and P_{i+1}, apply Algorithm 1 to get the desired point.

4.3. A modified Casteljau algorithm on Graßmann manifolds. The objective here is to use the relationships between the rotation group $\mathbb{SO}(n)$, the Graßmann manifold and the Stiefel manifold, presented in Section 2, to produce a curve in the latter. This uses a modification of the previous Casteljau algorithm.

Algorithm 1: calculate a point $\sigma(t)$, for $t \in [0,1]$, such that: $\sigma(0) = P$, $\sigma(1) = Q$, and $\dot\sigma(0) = 2[\Omega, P]$ *(note that C and Ω do not depend on t and can be pre-computed to improve the efficiency)*

Input: $t \in [0,1]$, $P, Q \in \boldsymbol{\mathcal{G}}_{n,k}$, $\Omega \in \mathfrak{so}_P(n)$

Output: $X = \sigma(t)$

1 Calculate control point C:

$$C = \exp(\Omega) \cdot P \cdot \exp(-\Omega)$$

Calculate first step end points M_1 and M_2:

$$\begin{aligned} M_1 &= \exp(t\Omega) \cdot P \cdot \exp(-t\Omega) \\ \Theta_0 &= \frac{1}{2}\log\big((\mathbf{I} - 2\,Q) \cdot (\mathbf{I} - 2\,C)\big) \\ M_2 &= \exp(t\Theta_0) \cdot C \cdot \exp(-t\Theta_0) \end{aligned}$$

2 Compute the point on the geodesic from M_1 to M_2 at t:

$$\begin{aligned} \Theta_1 &= \frac{1}{2}\log\big((\mathbf{I} - 2\,M_2) \cdot (\mathbf{I} - 2\,M_1)\big) \\ X &= \exp(t\Theta_1) \cdot M_1 \cdot \exp(-t\Theta_1) \end{aligned}$$

return X

Algorithm 2: calculate $\widetilde{\Omega}$ so that the final velocity $\dot\sigma_1(1, C, Q) = [\widetilde{\Omega}, Q]$, where σ satisfies: $\sigma(0) = P$, $\sigma(1) = Q$, and $\dot\sigma_1(0, P, C) = [\Omega, P]$ *(note that to improve the efficiency, all quantities can be computed in advance, once the whole data is known)*

Input: $P, Q \in \boldsymbol{\mathcal{G}}_{n,k}$, $\Omega \in \mathfrak{so}_P(n)$

Output: $\widetilde{\Omega}$ such that $\dot\sigma(1) = [\widetilde{\Omega}, Q]$

1 Calculate control point C:

$$C = \exp(\Omega) \cdot P \cdot \exp(-\Omega)$$

2 Calculate a component of the initial velocity vector for a geodesic from C to Q:

$$\widetilde{\Omega} = \frac{1}{2}\log\big((\mathbf{I} - 2\,Q) \cdot (\mathbf{I} - 2\,C)\big)$$

3 **return** $\widetilde{\Omega}$

Problem 2. *Given a set of points $P_0, \ldots, P_m$ in the Graßmann manifold $\boldsymbol{\mathcal{G}}_{n,k}$ and $\Omega_0 \in \mathfrak{so}_{P_0}(n)$, find a $\mathcal{C}^1$-smooth curve σ that interpolates the points P_i at time i and has initial velocity equal to $[\Omega_0, P_0]$, that is*

$$\sigma(i) = P_i, \quad \textit{for} \quad i = 0, 1, \ldots, m, \quad \textit{and} \quad \dot\sigma(0) = [\Omega_0, P_0].$$

Additionally, we require an accompanying piecewise $\mathcal{C}^1$-smooth curve β in the Stiefel manifold that satisfies

$$\beta(t) \cdot \beta(t)^{\mathrm{T}} = \sigma(t), \quad \textit{for} \quad t \in [0, m]. \tag{10}$$

4.4. Solving Problem 2 using a modified Casteljau algorithm. To accommodate the additional condition (10) we *"lift"* the problem to $\mathbb{SO}(n)$, perform the Casteljau algorithm there and then project the resulting curve onto the Graßmann and the Stiefel manifolds, as explained in Section 2.3.

Curve σ may be generated by piecing together $\mathcal{C}^1$-smooth curves defined on each subinterval $[i, i+1]$, and joining P_i to P_{i+1} with control point C_i, that is

$$\sigma(t)|_{[i,i+1]} = \widetilde{\sigma}_2(t - i, P_i, C_i, P_{i+1}),$$

where $\widetilde{\sigma}_2$ is a curve obtained from the modified Casteljau algorithm. The first control point C_0 is computed from P_0, P_1 and Ω_0. In order to ensure that σ is $\mathcal{C}^1$-smooth, the initial velocity of each subsequent spline segment must equal the final velocity of the previous segment.

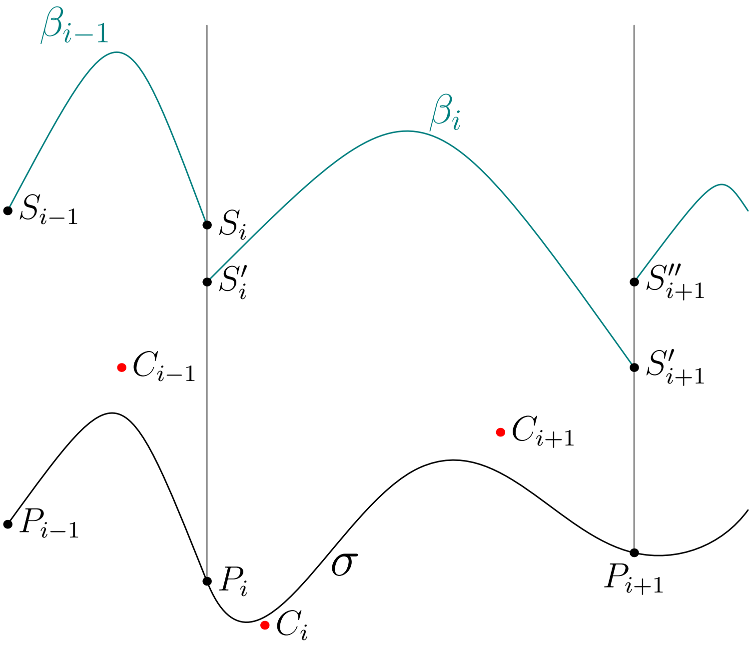

FIGURE 7. A scheme of the generalised Casteljau algorithm; segments β_i in the Stiefel manifold project into $\mathcal{C}^1$-smooth curve σ in the Graßmann, i.e., $\psi \circ \beta_i = \sigma$ by (3), however β_i segments depend on the initial points S_i, so continuity is not guaranteed

Figure 7 illustrates the following modification of the Casteljau algorithm. Let

$$\begin{aligned}
\sigma_0(t) &= e^{t\Omega_0} P_0 e^{-t\Omega_0}, && \text{where} \quad \Omega_0 = \frac{1}{2}\log\big((\mathbf{I}-2P_1)(\mathbf{I}-2P_0)\big),\\
\sigma_1(t) &= e^{t\Omega_1} P_1 e^{-t\Omega_1}, && \text{where} \quad \Omega_1 = \frac{1}{2}\log\big((\mathbf{I}-2P_2)(\mathbf{I}-2P_1)\big),\\
\sigma(t) &= e^{tX(t)}\sigma_0(t)e^{-tX(t)}, && \text{where} \quad X(t) = \log\big(e^{t\Omega_1}e^{(1-t)\Omega_0}\big),\\
\beta(t) &= e^{tX(t)}e^{t\Omega_0}S_0, && \text{such that} \quad S_0\, S_0^{\mathrm{T}} = P_0.
\end{aligned}$$

4.4.1. *Generating a second order spline.* Given a sequence of data points $P_0, P_1, \ldots, P_m$ and an initial Ω_0, the algorithm produces a second order spline $\sigma\colon [0,m] \to \mathscr{G}_{n,k}$, passing through the data points, such that

$$\sigma(i) = P_i, \quad \text{for} \quad 0 \le i \le m \quad \text{and} \quad \dot\sigma(0) = [\Omega_0, P_0].$$

As a consequence, segments $\sigma([i, i+1])$ join P_i to P_{i+1}.

To find a point $\sigma(t)$ of the spline, first iterate Algorithm 2 to find the component of initial velocity vector Ω_i, for the segment $\sigma([i, i+1])$, where $t \in [i, i+1]$. Then with the triple Ω_i, P_i and P_{i+1}, apply Algorithm 3 to get the desired points.

Algorithm 3: calculate a point $\sigma(t)$ in $\mathscr{G}_{n,k}$ and $\beta(t)$ in $\mathscr{S}_{n,k}$, for $t \in [0,1]$, such that: $\sigma(0) = P$, $\sigma(1) = Q$, and $\dot\sigma_1(0, P, C) = [\Omega, P]$ and $\beta(t)\cdot\beta(t)^{\mathrm{T}} = \sigma(t)$ *(note that C_i and Θ_i do not depend on t and can be pre-computed to improve the efficiency)*

Input: $t \in [0,1]$, $\Omega \in \mathfrak{so}_P(n)$, $P, Q \in \mathscr{G}_{n,k}$, $S \in \mathscr{S}_{n,k}$ such that $SS^{\mathrm{T}} = P$

Output: $Z = \sigma(t)$ and $B = \beta(t)$

1 Calculate control point C:

$$C = \exp(\Omega)\cdot P\cdot \exp(-\Omega)$$

2 Calculate first segment end points M_1 and M_2:

$$\begin{aligned}
M_1 &= \exp(t\Omega)\cdot P\cdot\exp(-t\Omega)\\
\Theta &= \frac{1}{2}\log\big((\mathbf{I}-2Q)\cdot(\mathbf{I}-2C)\big)\\
M_2 &= \exp(t\Theta)\cdot C\cdot\exp(-t\Theta)
\end{aligned}$$

3 Compute the point on the spline segment from M_1 to M_2:

$$\begin{aligned}
X &= \log\big(\exp(t\Theta)\cdot\exp((1-t)\Omega)\big)\\
Z &= \exp(tX)\cdot M_1\cdot\exp(-tX)\\
B &= \exp(tX)\cdot\exp(t\Omega)\cdot S
\end{aligned}$$

return Z, B

---

5. Solving Interpolation problems on Stiefel manifolds

In the previous section we presented a procedure to find an interpolating curve on the Stiefel manifold based on solving an interpolating problem on the Graßmann manifold. That procedure had a major drawback due to the fact that to each point P on the Graßmann there are multiple frames S_i on the Stiefel that satisfy $S_i S_i^{\mathrm{T}} = P$.

The objective of this section is to solve a smooth interpolation problem on Stiefel which is intrinsic to this manifold and results from replacing geodesics in the Casteljau algorithm by the quasi-geodesics introduced in Section 3.

Problem 3. *Given a set of points $\{ S_i \}_{i=0}^{m}$ belonging to the Stiefel manifold $\mathscr{S}_{n,k}$, and a vector $V_0 \in T_{S_0}\mathscr{S}_{n,k}$, find a $\mathcal{C}^1$ interpolating curve passing through these points and having initial velocity equal to V_0.*

5.1. Solving the interpolation problem using quasi-geodesics. Since every two points on the Stiefel manifold can be joined by a quasi geodesic, we use these curves to perform a modified Casteljau algorithm where successive linear interpolation is replaced by successive quasi-linear interpolation.

The crucial procedure is the generation of the first curve segment, joining S_0 to S_1 and having prescribed initial velocity equal to V_0. Without loss of generality, we assume that all segments are parameterized in the $[0, 1]$ time interval.

5.2. Generating the first curve segment. First we need to find a control point C_0, which is the end point of the quasi-geodesic that starts at the point S_0 with initial velocity equal to $\frac{1}{2}V_0$. This quasi-geodesic is given by

$$\beta_0(t) = e^{tX_0} S_0 e^{t\Omega_0},$$

where, according to (11) in Proposition 5,

$$X_0 = \frac{1}{2}V_0 S_0^{\mathrm{T}} - \frac{1}{2}S_0 V_0^{\mathrm{T}} + S_0 V_0^{\mathrm{T}} S_0 S_0^{\mathrm{T}} \quad \text{and} \quad \Omega_0 = \frac{1}{2}S_0^{\mathrm{T}} V_0. \tag{11}$$

So, $C_0 = e^{X_0} S_0 e^{\Omega_0}$ defines the control point.

We now proceed to the construction of the second quasi-geodesic β_1 that joins C_0 to S_1, using Theorem 7 with the obvious adaptations. The first curve segment, joining S_0 to S_1 with prescribed initial velocity equal to V_0 can now be obtained from quasi-linear interpolation of β_0 and β_1. The procedure to generate this curve is summarized in the algorithms 4 and 5.

Algorithm 4: calculate a point $\sigma(t)$ in $\mathscr{S}_{n,k}$, for $t \in [0,1]$, such that: $\sigma(0) = S$, $\sigma(1) = Q$

Input: $t \in [0,1]$, $S, C, Q \in \mathscr{S}_{n,k}$, $V_0 \in \mathbf{T}_S\mathscr{S}_{n,k}$, where S is the initial point, Q is the final point and C is the control point

Output: $Z = \sigma(t)$, $X_1 \in \mathfrak{so}_{CC^{\mathrm{T}}}(n)$, $\Omega_1 \in \mathfrak{so}(k)$

1 Calculate velocity components X_0 and Ω_0:

$$X_0 = \frac{1}{2}\log\big((\mathbf{I} - 2CC^{\mathrm{T}}) \cdot (\mathbf{I} - 2SS^{\mathrm{T}})\big)$$
$$\Omega_0 = \log(S^{\mathrm{T}} \cdot \exp(-X_0) \cdot C)$$

2 Calculate velocity components X_1 and Ω_1:

$$X_1 = \frac{1}{2}\log\big((\mathbf{I} - 2QQ^{\mathrm{T}}) \cdot (\mathbf{I} - 2CC^{\mathrm{T}})\big)$$
$$\Omega_1 = \log(C^{\mathrm{T}} \cdot \exp(-X_1) \cdot Q)$$

3 Calculate quasi-geodesics $\beta_0(t)$ and $\beta_1(t)$:

$$\beta_0(t) = \exp(tX_0) \cdot S \cdot \exp(t\Omega_0)$$
$$\beta_1(t) = \exp(tX_1) \cdot C \cdot \exp(t\Omega_1)$$

4 Calculate velocity components $X(t)$ and $\Omega(t)$ for the joining segment:

$$X(t) = \frac{1}{2}\log\Big(\big(\mathbf{I} - 2\,\beta_1(t)\,\beta_1^{\mathrm{T}}(t)\big)\big(\mathbf{I} - 2\,\beta_0(t)\,\beta_0^{\mathrm{T}}(t)\big)\Big)$$
$$\Omega(t) = \log\big(\beta_0^{\mathrm{T}}(t) \cdot \exp(-X(t)) \cdot \beta_1(t)\big)$$

5 Compute the point on the spline segment:

$$Z = \exp(tX(t)) \cdot \beta_0(t) \cdot \exp(t\Omega(t))$$

return $[Z, X_1, \Omega_1]$

Algorithm 5: calculate the control point C given the initial point for the segment and velocity components of the previous segment

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Input: $Q \in \mathscr{S}_{n,k}$, $X_1 \in \mathfrak{so}_{QQ^{\mathrm{T}}}(n)$ and $\Omega_1 \in \mathfrak{so}(k)$

Output: $C \in \mathscr{S}_{n,k}$

1 Calculate the control point C:

$$C = \exp(X_1) \cdot Q \cdot \exp(\Omega_1)$$

2 **return** C

5.3. Generating consecutive segments: We now explain how to continue in order to find a smooth curve that joins S_1 to S_2 and is $\mathcal{C}^1$ at S_1. One needs to find the control point C_1 for this segment. Since the curve must be $\mathcal{C}^1$ at S_1, the initial velocity for the second curve segment must equal the end velocity of the previous curve segment, which is known. So, we are reduced to the generation of a curve segment that joins S_1 to S_2 and whose initial velocity at S_1 is equal to $\dot{\sigma}(1)$.

The other consecutive segments are generated similarly.

Remark 12. *Note that the mapping $\psi\colon \mathscr{S}_{n,k} \to \mathscr{G}_{n,k}$ defined by $\psi(S) := SS^{\mathrm{T}}$ transforms quasi-geodesics on the Stiefel manifold into geodesics on the Graßmann manifold. As a consequence, the algorithm presented in this section projects on the Graßmann manifold $\mathscr{G}_{n,k}$ as the true Casteljau algorithm presented in Subsection 4.2.*

6. Conclusion

The generalisation of the classical Casteljau algorithm can be used to generate interpolating polynomial splines on manifolds. It is based on successive linear interpolation and can be successfully used whenever explicit formulas for the geodesic joining two points are available. We have implemented this algorithm on the Graßmann manifold, but without explicit formulas for geodesics on the Stiefel manifold the algorithm couldn't be applied. However, we have been able to present a very successful alternative based on a convenient modification of the Casteljau algorithm, where instead of geodesics we have used quasi-geodesics. These curves happen to have constant speed, constant covariant acceleration and constant geodesic curvature not greater than one. Optimal properties of these curves are under investigation.

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