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A CALCULUS OF LAX FRACTIONS

LURDES SOUSA

ABSTRACT: We present a notion of category of lax fractions, where lax fraction stands for a formal composition $s_* f$ with $s_* s = \mathrm{id}$ and $ss_* \leq \mathrm{id}$, and a corresponding calculus of lax fractions which generalizes the Gabriel-Zisman calculus of fractions.

1. Introduction

Given a class Σ of morphisms of a category $\mathcal{X}$, we can construct a category of fractions $\mathcal{X}[\Sigma^{-1}]$ where all morphisms of Σ are invertible. More precisely, we can define a functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma^{-1}]$ which takes the morphisms of Σ to isomorphisms, and, moreover, P_Σ is universal with respect to this property. As shown in [13], if Σ admits a calculus of fractions, then the morphisms of $\mathcal{X}[\Sigma^{-1}]$ can be expressed by equivalence classes of cospans (f,g) of morphisms of $\mathcal{X}$ with $g \in \Sigma$, which correspond to the formal compositions $g^{-1} f$.

We recall that categories of fractions are closely related to reflective subcategories and orthogonality. In particular, if $\mathcal{A}$ is a full reflective subcategory of $\mathcal{X}$, the class Σ of all morphisms inverted by the corresponding reflector functor – equivalently, the class of all morphisms with respect to which $\mathcal{A}$ is orthogonal – admits a left calculus of fractions; and $\mathcal{A}$ is, up to equivalence of categories, a category of fractions of $\mathcal{X}$ for Σ. In [3] we presented a Finitary Orthogonality Deduction System inspired by the left calculus of fractions, which can be looked as a generalization of the Implicational Logic of [20], see [4].

Assume now that $\mathcal{X}$ is an order-enriched category, that is, its hom-sets $\mathcal{X}(X,Y)$ are endowed with a partial order satisfying the condition $f \leq g \Rightarrow hfj \leq hgj$ for every morphisms $f,g : X \to Y$, $j : Z \to X$ and $h : Y \to W$. We call a morphism $f : X \to Y$ of $\mathcal{X}$ a *left adjoint section* if it is a left adjoint and

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has a left inverse; equivalently, there is a morphism $f_*: Y \to X$ such that $f_*f = \mathrm{id}_X$ and $ff_* \leq \mathrm{id}_Y$. We are interested in a category of lax fractions in the sense that, given a class Σ of morphisms of $\mathcal{X}$, we want a category $\mathcal{X}[\Sigma_*]$ and an order-enriched functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ which takes morphisms of Σ to left adjoint sections of $\mathcal{X}[\Sigma_*]$ and, moreover, P_Σ is universal with respect to that property. This problem is connected with the study of KZ-monads and Kan-injectivity as explained next.

In recent papers ([1, 8]) we have studied a lax version of orthogonality in order-enriched categories: Kan-injectivity. An object A is said to be (left) Kan-injective with respect to a morphism $h : X \to Y$ provided that for every morphism $f : X \to A$ there is a left Kan extension of f along h, denoted f/h, and, moreover, $f = (f/h)h$. And a morphism $k : A \to B$ is said to be Kan-injective with respect to h if A and B are so and k preserves left Kan extensions along h, i.e., $(kf)/h = k(f/h)$. Let $\mathcal{A}$ be a subcategory of an order-enriched category $\mathcal{X}$. We say that $\mathcal{A}$ is KZ-reflective if it is reflective and the monad induced in $\mathcal{X}$ by the reflector functor $F : \mathcal{X} \to \mathcal{A}$ is a KZ-monad, i.e., the unit η satisfies the inequalities $F\eta_X \leq \eta_{FX}$ for all objects X of $\mathcal{X}$ ([18, 12]). If, moreover, $\mathcal{A}$ is an Eilenberg-Moore category of a KZ-monad over $\mathcal{X}$, we say that $\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$. Let $\mathcal{A}^{\mathrm{KInj}}$ denote the class of all morphisms with respect to which all objects and morphisms of $\mathcal{A}$ are Kan-injective. As shown in [8], if $\mathcal{A}$ is KZ-reflective in $\mathcal{X}$, $\mathcal{A}^{\mathrm{KInj}}$ consists precisely of all morphisms of $\mathcal{X}$ whose images through the reflector functor are left adjoint sections.

In this paper we present the notion of category of lax fractions $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ and a calculus of lax fractions which generalize the usual non-lax versions. But now Σ is not just a class of morphisms, as in the ordinary case; instead, it is a subcategory of the arrow category $\mathcal{X}^{\to}$. And the calculus of lax fractions is expressed as a calculus of squares (called Σ-squares) which represent formal equalities of the form $fr_* = s_*g$ (see Section 4). This way, we obtain a description of the category of lax fractions of $\mathcal{X}$, for Σ a subcategory of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions, in terms of formal fractions s_*f represented by cospans $\bullet \xrightarrow{f} \bullet \xleftarrow{s} \bullet$ with s an object of Σ (Theorem 4.11). The idea of "calculating" with squares of the base category $\mathcal{X}$ instead of just with morphisms of $\mathcal{X}$ is also used in the paper in preparation [2] in order to obtain a Kan-Injectivity Logic generalizing the Orthogonality Logic of [3].

Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, let $\mathcal{A}^{\underline{\mathsf{KInj}}}$ denote the subcategory of $\mathcal{X}^{\to}$ whose objects are the morphisms of $\mathcal{A}^{\mathsf{KInj}}$, and whose morphisms between them are those of the form $(u,v) : (s : X \to Y) \longrightarrow (s' : Z \to W)$ such that $(fu)/s = (f/s')v$ for all f with domain Z and codomain in $\mathcal{A}$. We show that, for $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$, if $\mathcal{A}$ is a KZ-reflective subcategory of $\mathcal{X}$, the category $\mathcal{X}[\Sigma_*]$ is the Kleisli category for the monad induced by the reflector functor $F : \mathcal{X} \to \mathcal{A}$, and F differs from the functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ at most by closedness under left adjoint retractions (Theorem 3.7); moreover, Σ admits a left calculus of lax fractions (Proposition 4.5).

We finish up with some properties on cocompleteness. We show that whenever $\mathcal{X}$ has weighted colimits, any subcategory of $\mathcal{X}^{\to}$ of the form $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ also has weighted colimits (Theorem 5.1) and admits a left calculus of lax fractions, and the corresponding category of lax fractions $\mathcal{X}[\Sigma_*]$ has (small) conical coproducts. Moreover, we present conditions on any subcategory Σ under which $\mathcal{X}[\Sigma_*]$ has finite conical coproducts, provided $\mathcal{X}$ has them.

Several examples of subcategories Σ of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions are provided in Example 4.4 for $\mathcal{X}$ the category Pos of posets and monotone maps, the category Loc of locales and localic maps, and the category Top_0 of T_0 topological spaces and continuous maps.

The study of constructions of categories by freely adding adjoints to the arrows of a category has been addressed before. Although the present approach is completely different, it is worth mentioning here the works [10] and [11] of Dawson, Paré and Pronk.

2. Preliminaries

Along this paper we work in the order-enriched context. More precisely, we consider categories and functors enriched in the category Pos of posets and monotone maps. For a category $\mathcal{X}$ this means that each one of its hom-sets $\mathcal{X}(X,Y)$ is equipped with a partial order $\leq$ which is preserved by composition on the left and on the right. And a functor between order-enriched categories is order-enriched if it preserves the partial order of the morphisms. A subcategory of an order-enriched category $\mathcal{X}$ will be considered order-enriched via the restriction of the order on the morphisms of $\mathcal{X}$ to the morphisms of $\mathcal{A}$.

In this section, we recall the notions of Kan-injectivity and KZ-reflective subcategory, and some of their properties, which are presented in [8] and [**?**].

2.1. Kan-injectivity. In an order-enriched category $\mathcal{X}$, an object A is said to be *left Kan-injective* (or just *Kan-injective*) with respect to a morphism $h : X \to Y$, if, for every morphism $f : X \to A$, there is a morphism $f/h : Y \to A$ such that

(i) $(f/h)h = f$, and
(ii) $f \leq gh \Rightarrow f/h \leq g$, for every morphism $g : Y \to A$.

A morphism $k : A \to B$ is said to be (left) *Kan-injective* with respect to h provided that A and B are so, and the equality $(kf)/h = k(f/h)$ holds for all $f : X \to A$.

(Left) Kan-injectivity may be equivalently defined as follows: An object A is left Kan-injective with respect to a morphism $h : X \to Y$, if and only if the hom-map $\mathcal{X}(h,A) : \mathcal{X}(Y,A) \to \mathcal{X}(X,A)$ is a right adjoint retraction (short for a morphism which is simultaneously a right adjoint and a retraction) in the category Pos. In this case, if $(\mathcal{X}(h,A))^* : \mathcal{X}(X,A) \to \mathcal{X}(Y,A)$ is the left adjoint of $\mathcal{X}(h,A)$, then we have that $(\mathcal{X}(h,A))^*(f) = f/h$.

Given a class $\mathcal{H}$ of morphisms of $\mathcal{X}$, the objects and morphisms of $\mathcal{X}$ which are Kan-injective with respect to all morphisms of $\mathcal{H}$ constitute a subcategory, denoted by

$$\mathsf{KInj}(\mathcal{H})$$

and said to be a *Kan-injective subcategory*. And, given a subcategory $\mathcal{A}$ of $\mathcal{X}$, we denote by

$$\mathcal{A}^{\mathsf{KInj}}$$

the class of all morphisms with respect to which all objects and morphisms of $\mathcal{A}$ are Kan-injective.

2.2. KZ-reflective subcategories. We recall that a *KZ-monad* (or *lax idempotent monad*) on $\mathcal{X}$ is a monad $T : \mathcal{X} \to \mathcal{X}$ whose unit η satisfies the inequalities $T\eta_X \leq \eta_{TX}$, $X \in \mathcal{X}$ ([18], [12]). Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. $\mathcal{A}$ is said to be a *KZ-reflective subcategory* of $\mathcal{X}$ if it is reflective in $\mathcal{X}$ and the monad over $\mathcal{X}$ induced by the corresponding adjunction is of KZ type; that is, the left adjoint $F : \mathcal{X} \to \mathcal{A}$ and the unit η satisfy the inequalities

$$F\eta_X \leq \eta_{FX},\ X \in \mathcal{X}. \tag{1}$$

The Eilenberg-Moore categories of KZ-monads over $\mathcal{X}$ are, up to isomorphism of categories, KZ-reflective subcategories, called then *KZ-monadic subcategories*. Thus the concept of KZ-monadic subcategory is a lax version of the one of replete full reflective subcategory. In [8] we showed that KZ-monadic subcategories are precisely the KZ-reflective categories closed under left adjoint retractions (i.e., the equality $gx = yf$ between morphisms of $\mathcal{X}$ with f in $\mathcal{A}$ and x and y both left adjoint retractions implies that g also belongs to $\mathcal{A}$). In [**?**] we proved that in well-behaved categories, namely in locally ranked ones, every Kan-injective subcategory $\mathsf{KInj}(\mathcal{H})$ with $\mathcal{H}$ a set is indeed a KZ-monadic subcategory.

When $\mathcal{A}$ is KZ-reflective in $\mathcal{X}$, with $F : \mathcal{X} \to \mathcal{A}$ the corresponding reflector functor, $\mathcal{A}^{\mathsf{KInj}}$ is precisely the class of all morphisms f of $\mathcal{X}$ such that Ff is a left adjoint section in $\mathcal{A}$, that is, there is a morphism $(Fh)_*$ in $\mathcal{A}$ with $(Fh)_*Fh = \mathrm{id}$ and $Fh(Fh)_* \leq \mathrm{id}$. We call this kind of morphisms *F-embeddings*, following the terminology of M. Escardó [12].

3. Categories of lax fractions

It is well known that if $\mathcal{A}$ is a full reflective subcategory of an ordinary category $\mathcal{X}$ with reflector functor $F : \mathcal{X} \to \mathcal{A}$, then $\mathcal{A}$ is, up to equivalence of categories, the category of fractions of $\mathcal{X}$ for the class of morphisms inverted by F. Indeed, this category of fractions is the Kleisli category of the idempotent monad induced by the corresponding adjunction. Formally we can think of a "fraction" as a composition of the form $h^{-1}f$ where h^{-1} is a formal inverse of h. Here we use the term "lax fraction" evoking a composition of the form h_*f where h_* is a *formal* left inverse and right adjoint of h (that is, h_* is thought as satisfying $h_*h = \mathrm{id}$ and $\mathrm{id} \leq h_*h$). We show that, in the order-enriched context, a KZ-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, with reflector $F : \mathcal{X} \to \mathcal{A}$, is also closely related to the category of lax fractions of $\mathcal{X}$ for the F-embeddings of $\mathcal{X}$. And this category of lax fractions coincides with the Kleisli category of the corresponding KZ-monad too.

Given a full subcategory $\mathcal{A}$ of any category $\mathcal{X}$, some of the nice properties of the class $\mathcal{A}^{\mathsf{Orth}}$ of all morphisms with respect to which $\mathcal{A}$ is orthogonal are obtained by looking at $\mathcal{A}^{\mathsf{Orth}}$ as a full subcategory of the arrow category $\mathcal{X}^{\to}$. This is the case, for instance, of the closedness under colimits of $\mathcal{A}^{\mathsf{Orth}}$ in $\mathcal{X}^{\to}$, when $\mathcal{X}$ is cocomplete (cf. [21]). Let $\mathcal{X}$ be an order-enriched category, and let $\mathcal{X}^{\to}$ be order-enriched with the coordinatewise order. KZ-reflective subcategories are not full, in general. Thus it is not surprising that, in order to generalize orthogonality properties to Kan-injectivity ones, we need to consider $\mathcal{A}^{\mathsf{KInj}}$ as a subcategory of $\mathcal{X}^{\to}$ which is not necessarily full. In the same vein, we define categories of lax fractions for subcategories Σ of $\mathcal{X}^{\to}$.

Definition 3.1. Let $\mathcal{X}$ be a category and Σ a subcategory of the arrow category $\mathcal{X}^{\to}$. A *category of lax fractions* of $\mathcal{X}$ for Σ consists of a (quasi)category $\mathcal{X}[\Sigma_*]$ and a functor $P_{\Sigma} : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ such that:

(i) $P_{\Sigma} h$ is a left adjoint section, for every object h of Σ.

(ii) For every morphism $(u,v) : h \to h'$ in Σ, $P_{\Sigma} u \cdot (P_{\Sigma} h)_* = (P_{\Sigma} h')_* \cdot P_{\Sigma} v$.

(iii) If $G : \mathcal{X} \to \mathcal{C}$ is another functor enjoying the properties (i) and (ii), then there is a unique functor $H : \mathcal{X}[\Sigma_*] \to \mathcal{C}$ such that $HP_{\Sigma} = G$.

Remark 3.2. If we think of an ordinary category $\mathcal{X}$ as an order-enriched one via the discrete order, i.e., the order $=$, then (ii) trivially holds, and Definition 3.1 becomes the usual definition of category of fractions.

Definition 3.3. Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, we will denote by

$$\mathcal{A}^{\underline{\mathsf{KInj}}}$$

the subcategory of the arrow category $\mathcal{X}^{\to}$ consisting of:

(i) Objects: all morphisms h of $\mathcal{X}$ such that all objects and morphisms of $\mathcal{A}$ are left-Kan injective with respect to h. That is, the class of objects of $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is $\mathcal{A}^{\mathsf{KInj}}$.

(ii) Morphisms: those morphisms $(u,v) : (X \xrightarrow{h} Y) \to (X' \xrightarrow{h'} Y')$, with h and h' in $\mathcal{A}^{\mathsf{KInj}}$, such that, for every $g : X' \to A$, with $A \in \mathcal{A}$, we have that $(gu)/h = (g/h')v$:

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\
X' \arrow[r, "h'"] \arrow[d, "g"'] & Y' \arrow[dl, "g/h'"] \\
A &
\end{tikzcd}$$

In other words, a morphism $(u,v) : (X \xrightarrow{h} Y) \to (X' \xrightarrow{h'} Y')$ of $\mathcal{X}^{\to}$ is a morphism of $\mathcal{A}^{\underline{\mathsf{KInj}}}$ iff it satisfies the equality $\mathcal{X}(h,A)^* \cdot \mathcal{X}(u,A) = \mathcal{X}(v,A) \cdot \mathcal{X}(h',A)^*$ for all objects $A \in \mathcal{A}$.

The next lemmas are going to be used in the proof of the main result of this section, Theorem 3.7.

Lemma 3.4. *Let $\mathcal{A}$ be a KZ-reflective subcategory of $\mathcal{X}$ with reflector functor $F:\mathcal{X}\to\mathcal{A}$. Then, for every morphism $h:X\to Y$ in $\mathcal{X}$ and every morphism $(u,v):h\to h'$ in $\mathcal{X}^{\to}$, we have that:*

(i) $h\in\mathcal{A}^{\mathsf{KInj}}$ iff Fh is a left adjoint section in $\mathcal{A}$; and

(ii) for h and h' in $\mathcal{A}^{\mathsf{KInj}}$, a morphism $(u,v):h\to h'$ lies in $\mathcal{A}^{\underline{\mathsf{KInj}}}$ iff $Fu\,(Fh)_*=(Fh')_*Fv$

Proof: (i) was proved in [8] (see the last paragraph of 2.2).

(ii) It is easy to verify, and it was observed in [12], that, under the present conditions, given $a:X\to A$ with $A\in\mathcal{A}$, we have that

$$a/h=\varepsilon_A\cdot Fa\cdot(Fh)_*\cdot\eta_Y, \tag{2}$$

where η and ε are the corresponding unit and counit. Let $(u,v):h\to h'$ be a morphism of $\mathcal{A}^{\underline{\mathsf{KInj}}}$:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "h'"'] & Y' \end{tikzcd}$$

Then, for $\eta_{X'}:X'\to FX'$, we have $(\eta_{X'}/h')v=(\eta_{X'}u)/h$, that is, by (2), $\varepsilon_{FX'}F\eta_{X'}(Fh')_*\eta_{Y'}v=\varepsilon_{FX'}F(\eta_{X'}u)(Fh)_*\eta_Y$. Consequently, $(Fh')_*\eta_{Y'}v=Fu(Fh)_*\eta_Y$, i.e., $(Fh')_*Fv\eta_Y=Fu(Fh)_*\eta_Y$; thus, $(Fh')_*Fv=Fu(Fh)_*$, since from (i) we know that $(Fh')_*Fv$ and $Fu(Fh)_*$ are both morphisms of $\mathcal{A}$.

Conversely, if the equality $(Fh')_*Fv=Fu(Fh)_*$ holds, for $d:X'\to D$, with $D\in\mathcal{A}$, we have that $(d/h')v=\varepsilon_DFd(Fh')_*\eta_{Y'}v=\varepsilon_DFd(Fh')_*Fv\eta_Y=\varepsilon_DFdFu(Fh)_*\eta_Y=\varepsilon_DF(du)(Fh)_*\eta_Y=(du)/h$. ■

Remark 3.5. ([8]) Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{X}$, with reflector functor F, unit η and counit ε. Then $\mathcal{A}$ is KZ-reflective if and only if $F\varepsilon_A\geq\varepsilon_{FA}$, $A\in\mathcal{A}$, if and only if $\eta_A\varepsilon_A\geq\mathrm{id}_{FA}$, $A\in\mathcal{A}$. Then, when $\mathcal{A}$ is KZ-reflective, ε_A is a left adjoint retraction, with $(\varepsilon_A)_*=\eta_A$. Moreover, every $F\eta_X$ is a left adjoint section, with $(F\eta_X)_*=\varepsilon_{FX}$. Thus, ε_{FX} is simultaneously a right adjoint and a left adjoint satisfying the inequalities $F\eta_X\varepsilon_{FX}\leq\mathrm{id}_{F^2X}\leq\eta_{FX}\varepsilon_{FX}$.

Lemma 3.6. *Let $\mathcal{A}$ be a KZ-reflective subcategory of $\mathcal{X}$, with reflector F and unit η. Then, for every $f:X\to Y$, $(f,Ff):\eta_X\to\eta_Y$ is a morphism of the category $\mathcal{A}^{\underline{\mathsf{KInj}}}$.*

Proof: Indeed, with respect to the commutative square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & FX \arrow[d, "Ff"] \\ Y \arrow[r, "\eta_Y"'] & FY \end{tikzcd}$$

using Remark 3.5, we have that $Ff(F\eta_X)_* = Ff\varepsilon_{FX} = \varepsilon_{FY}F^2f = (F\eta_Y)_*F^2f$; hence, by Lemma 3.4, the morphism (f, Ff) lies in $\mathcal{A}^{\underline{\mathsf{KInj}}}$. ∎

Theorem 3.7. *Let $\mathcal{A}$ be a KZ-reflective subcategory of $\mathcal{X}$ with reflector functor $F : \mathcal{X} \to \mathcal{A}$. Then there exists a category $\mathcal{X}[\Sigma_*]$ and a functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ forming a category of lax fractions of $\mathcal{X}$ for $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$. Moreover, if $H : \mathcal{X}[\Sigma_*] \to \mathcal{A}$ is the unique functor with $HP_\Sigma = F$, then for every $f : A \to B$ in $\mathcal{A}$ there is some $g : X \to Y$ in $\mathcal{X}[\Sigma_*]$ and a commutative diagram* $\begin{tikzcd} HX \arrow[r, "Hg"] \arrow[d, "r"'] & HY \arrow[d, "r'"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$ *in $\mathcal{X}$*

with r and r' left adjoint retractions.

Proof: Let η and ε be the corresponding unit and counit of the KZ-reflection of $\mathcal{X}$ into $\mathcal{A}$. Define a category $\mathcal{X}[\Sigma_*]$ and a functor $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ as follows:

- $|\mathcal{X}[\Sigma_*]| = |\mathcal{X}|$, where $|\mathcal{X}|$ denotes the class of objects of $\mathcal{X}$.
- for every $X, X' \in |\mathcal{X}|$, the poset $\mathcal{X}[\Sigma_*](X, X')$ is $\mathcal{A}(FX, FX')$;
- for every object X of $\mathcal{X}[\Sigma_*](X, X')$ the identity id_X is just id_{FX}, and the composition is defined as in $\mathcal{A}$;
- $P_\Sigma X = X$ and $P_\Sigma f = Ff$, for every object X and every morphism f of $\mathcal{X}$.

$\mathcal{X}[\Sigma_*]$ is, up to isomorphism of categories, the Kleisli category of the monad induced in $\mathcal{X}$ by F, and $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ is the corresponding reflection of $\mathcal{X}$ in it (cf. [19]). We show that $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ is a category of lax fractions for $\mathcal{A}^{\underline{\mathsf{KInj}}}$.

The satisfaction by P_Σ of conditions (i) and (ii) of Definition 3.1 follows immediately from the definition of P_Σ and Lemma 3.4.

Concerning (iii), let $G : \mathcal{X} \to \mathcal{C}$ be a functor satisfying conditions (i) and (ii) of Definition 3.1. We want to define a functor $H : \mathcal{X}[\Sigma_*] \to \mathcal{C}$ such that $HP_\Sigma = G$ and show that there is a unique such functor H.

First observe that if this functor H exists, then we have

$$HX = HP_\Sigma X = GX, \tag{3}$$

for every $X \in |\mathcal{X}[\Sigma_*]|$; and, for every morphism f of $\mathcal{X}$ for which $(Ff)_*$ exists,

$$H((Ff)_*) = (HFf)_* = (Gf)_*, \tag{4}$$

since we are dealing with order-enriched functors, which preserve adjunctions and retractions. In particular (see Remark 3.5),

$$H(\varepsilon_{FX}) = H((F\eta_X)_*) = (G\eta_X)_*. \tag{5}$$

Moreover, given $f \in \mathcal{X}[\Sigma_*](X, X')$, i.e., $f : FX \to FX'$ in $\mathcal{A}$, we have that $Hf = H(f\varepsilon_{FX}F\eta_X) = H(\varepsilon_{FX'} \cdot Ff \cdot F\eta_X)$; then, by (5),

$$Hf = (G\eta_{X'})_* \cdot Gf \cdot G(\eta_X). \tag{6}$$

The satisfaction of (3) and (6) defines H uniquely, and the equality $HP_\Sigma = G$ is easily verified.

It remains to show that H is indeed a functor. The preservation of identities is clear. To prove that H preserves composition, let $f : FX \to FY$ and $g : FY \to FZ$ be two morphisms of $\mathcal{X}[\Sigma_*](X, Y)$ and $\mathcal{X}[\Sigma_*](Y, Z)$, respectively. We want to show that $H(gf) = Hg \cdot Hf$.

Due to the equality $(F\eta_X)_* = \varepsilon_{FX}$, given in Remark 3.5, we have that, for every morphism $f : FX \to FY$ of $\mathcal{A}$, $f = (F\eta_Y)_* \cdot Ff \cdot F(\eta_X)$. Taking this into account and the fact that G preserves adjunctions, we have:

$$GgGf = (GF\eta_Z)_* \cdot GFg \cdot GF(\eta_Y) \cdot (GF\eta_Y)_* \cdot GFf \cdot GF(\eta_X).$$

Multiplying by $(G\eta_Z)_*$ on the left-hand side and by $G\eta_X$ on the right-hand side, and using (6), we obtain:

$$H(gf) = (G\eta_Z)_* \cdot (GF\eta_Z)_* \cdot GFg \cdot GF(\eta_Y) \cdot (GF\eta_Y)_* \cdot GFf \cdot GF(\eta_X) \cdot G\eta_X. \tag{7}$$

But the diagram

$$\begin{tikzcd}
GFY \arrow[d, "Gg"'] & GF^2Y \arrow[l, "(G\eta_{FY})_*"'] \arrow[d, "GFg"] \\
GFZ \arrow[d, "(G\eta_Z)_*"'] & GF^2Z \arrow[l, "(G\eta_{FZ})_*"'] \arrow[d, "(GF\eta_Z)_*"] \\
GZ & GFZ \arrow[l, "(G\eta_Z)_*"']
\end{tikzcd}$$

is commutative: the top square commutes, because $(g, Fg) : \eta_{FY} \to \eta_{FZ}$ is a morphism of $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$, from Lemma 3.6, and G satisfies condition (ii) of Definition 3.1; the bottom square commutes because all morphisms η_Z, $F\eta_Z$ and η_{FZ} belong to Σ, thus $(G\eta_Z)_*$, $(GF\eta_Z)_*$ and $(G\eta_{FZ})_*$ are defined and, from the equality $F\eta_Z \cdot \eta_Z = \eta_{FZ} \cdot \eta_Z$, it follows the required equality. Consequently, we have:

$$(G\eta_Z)_* \cdot (GF\eta_Z)_* \cdot GFg = (G\eta_Z)_* \cdot Gg \cdot (G\eta_{FY})_*. \tag{8}$$

Moreover,

$$GFf \cdot GF(\eta_X) \cdot G\eta_X = GFf \cdot G\eta_{FX} \cdot G\eta_X = G(\eta_{FY}) \cdot Gf \cdot G\eta_X. \tag{9}$$

Therefore, by applying (8) and (9) to the right-hand side of (7), we get

$$H(gf) = (G\eta_Z)_* \cdot Gg \cdot (G\eta_{FY})_* \cdot GF(\eta_Y) \cdot (GF\eta_Y)_* \cdot G(\eta_{FY}) \cdot Gf \cdot G\eta_X.$$

In order to conclude that the right-hand side of the last equality is precisely

$$Hg \cdot Hf = (G(\eta_Z))_* G(g)G(\eta_Y)(G(\eta_Y))_* G(f)G(\eta_X),$$

it suffices to show that $(G\eta_{FY})_* \cdot GF(\eta_Y) \cdot (GF\eta_Y)_* \cdot G(\eta_{FY}) = G(\eta_Y)(G(\eta_Y))_*$. This is easy:

$$\begin{aligned}(G\eta_{FY})_* \cdot GF(\eta_Y) \cdot (GF\eta_Y)_* \cdot G(\eta_{FY}) &= G\eta_Y \cdot (G\eta_Y)_* \cdot (GF\eta_Y)_* \cdot G(\eta_{FY}), \quad \text{by 3.6}\\ &= G\eta_Y \cdot (G\eta_Y)_* \cdot \mathrm{id}_{FY}\\ &= G\eta_Y \cdot (G\eta_Y)_*.\end{aligned}$$

The order-enrichment of H is immediate from the definition of H, since G is so.

Finally, from Lemma 3.4, we know that the reflector functor $F : \mathcal{X} \to \mathcal{A}$ satisfies conditions (i) and (ii). Thus, as we have just seen, the unique functor $H : \mathcal{X}[\Sigma_*] \to \mathcal{A}$ such that $HP_\Sigma = F$ is defined by $HX = FX$ and $Hf = (F\eta_Y)_* \cdot Ff \cdot F\eta_X = \varepsilon_{FY} \cdot Ff \cdot F\eta_X = f \cdot \varepsilon_{FX} \cdot F\eta_X = f$. For every morphism $g : A \to A'$ of $\mathcal{A}$, we have $Fg \in \mathcal{X}[\Sigma_*](A, A')$, with $H(Fg) = (Fg : FA \to FA')$, and thus we have a commutative diagram of the form

$$\begin{tikzcd} HA \arrow[r, "H(Fg)"] \arrow[d, "\varepsilon_A"'] & HA' \arrow[d, "\varepsilon_{A'}"] \\ A \arrow[r, "g"'] & A' \end{tikzcd}$$

with ε_A and $\varepsilon_{A'}$ left adjoint retractions in $\mathcal{X}$ (see Remark 3.5). ■

Remark 3.8. Under the conditions of the above theorem, let $E : \mathcal{A} \to \mathcal{X}$ be the corresponding inclusion functor and put $K = P_\Sigma E : \mathcal{A} \to \mathcal{X}[\Sigma_*]$. Then K is faithful, because, for every morphism $f : A \to A'$ of $\mathcal{A}$, we have that $f = \varepsilon_{A'} F f \eta_A$. And it has the property that, for every morphism $g : X \to X'$ in $\mathcal{X}[\Sigma_*]$, there are a morphism $f : A \to A'$ in $\mathcal{A}$ and a commutative diagram

$$\begin{tikzcd} KA \arrow[r, "Kf"] \arrow[d, "r"'] & KA' \arrow[d, "r'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

in $\mathcal{X}[\Sigma_*]$ with r and r' retractions which are simultaneously left and right adjoints. Indeed, it suffices to take $r = \varepsilon_{FX}$ and $r' = \varepsilon_{FX'}$ (see Remark 3.5).

Remark 3.9. As observed before, the category $\mathcal{X}[\Sigma_*]$ described in the proof of the above theorem is the Kleisli category for the monad over $\mathcal{X}$ induced by its KZ-reflection into $\mathcal{A}$. We point out that in [14] the authors show that, for every monad, the Kleisli category can always be seen as a category of (generalized) fractions.

4. A left calculus of lax fractions

In this section we introduce the notion of a left calculus of lax fractions relatively to a subcategory Σ of the arrow category $\mathcal{X}^{\to}$, which generalizes the usual left calculus of fractions and allows us to describe the category of lax fractions of $\mathcal{X}$ for Σ in terms of formal fractions $s_* f$ represented by cospans $\bullet \xrightarrow{f} \bullet \xleftarrow{s} \bullet$ with s an object of Σ.

Σ-squares, as described next, are going to be used to define and manipulate the left calculus of lax fractions.

Terminology 4.1. Given a subcategory Σ of $\mathcal{X}^{\to}$, we use a square of the form

$$\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

to indicate that f, g, r and s are morphisms of $\mathcal{X}$ such that $(f, g) : r \to s$ is a morphism of Σ, and a square of this type is called a Σ-*square*.

Moreover, by a a Σ-*span* we mean a span $\bullet \xleftarrow{r} \bullet \xrightarrow{f} \bullet$ with r an object of Σ. And a Σ-*cospan* from A to B is a cospan $A \xrightarrow{g} J \xleftarrow{s} B$ with s an object of Σ.

When we have (r,f) and (g,s) forming a Σ-square as above, we say that the Σ-span (r,f) *covers* the Σ-cospan (g,s).

Thinking of a Σ-span $\bullet \xleftarrow{r} \bullet \xrightarrow{f} \bullet$ as a formal representation of the (lax) fraction fr_*, and of the Σ-cospan $\bullet \xrightarrow{g} \bullet \xleftarrow{s} \bullet$ as a formal representation of the (lax) fraction s_*g, the above Σ-square represents the formal equality $fr_* = s_*g$.

Definition 4.2. A subcategory Σ of $\mathcal{X}^{\to}$ is said *to admit a left calculus of lax fractions of* $\mathcal{X}$ if it satisfies the following conditions:

1. *Identity.* The identities of $\mathcal{X}$ are objects of Σ and $\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "s"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$ for all objects s of Σ.
2. *Composition.* If we have $\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$ and $\begin{tikzcd} \bullet \arrow[r, "r'"] \arrow[d, "g"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "s'"'] & \bullet \end{tikzcd}$ then also $\begin{tikzcd} \bullet \arrow[r, "r'r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "s's"'] & \bullet \end{tikzcd}$.
3. *Square.* For every Σ-span $\bullet \xleftarrow{r} \bullet \xrightarrow{f} \bullet$, there are morphisms r' and f' such that

$$\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "r'"'] & \bullet \end{tikzcd}.$$

4. *Coinsertion.* Given a diagram
$$\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "f"'] & \bullet \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$
where the inner square is a Σ-square, and such that $gr \leq hr$, then there is a morphism t, whose domain is the codomain of s, satisfying the following conditions:
$$tg \leq th \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r, "ts"'] & \bullet \end{tikzcd}.$$

Remark 4.3. Combining the composition of morphisms in the category Σ with the one given by *Composition*, we have that any square obtained by finite horizontal and vertical compositions of Σ-squares is a Σ-square. This is going to be very useful in the proofs of this section.

Examples 4.4. 1. Recall that a class of morphisms Σ of an ordinary category $\mathcal{X}$ *admits a left calculus of fractions* if it satisfies the following conditions:

1'. Σ contains all identities of $\mathcal{X}$.

2'. Σ is closed under composition.

3'. For every span $\bullet \xleftarrow{r} \bullet \xrightarrow{f} \bullet$ with $r \in \Sigma$, there is a cospan $\bullet \xrightarrow{f'} \bullet \xleftarrow{r'} \bullet$ with $r' \in \Sigma$ and $f'r = r'f$.

4'. If we have a diagram $\bullet \xrightarrow{r} \bullet \overset{h}{\underset{g}{\rightrightarrows}} \bullet$ with $r \in \Sigma$ and $gr = hr$ then there is some $t \in \Sigma$ with $tg = th$.

Let $\mathcal{X}$ be an ordinary category, equivalently, a category enriched with the discrete order $=$. Let Σ be a class of morphisms of $\mathcal{X}$, regarded as a full subcategory of $\mathcal{X}^{\to}$. Then Σ admits a left calculus of lax fractions if and only if it admits a left calculus of fractions in the usual sense. Indeed, the equivalence of the three first conditions is immediately seen. To show that, in the presence of 1-3, 4 implies 4', let g and h be a pair of morphisms equalized by a morphism r of Σ. For $f = gr = hr$ and $s = \mathrm{id}$ we obtain a diagram as the first one in Definition 4.2.4, which is a Σ-square because of the fullness of Σ. Consequently, there is some morphism t under the conditions of the second diagram of Definition 4.2.4; since s is the identity, we conclude that $t \in \Sigma$. Conversely, given a diagram as the first one in Definition 4.2.4, with $gr = hr$, let t be a morphism of Σ such that $tg = th$. Then, the second diagram of Definition 4.2.4 is indeed a Σ-square, since $ts \in \Sigma$.

In this case $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ is just the category of fractions $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma^{-1}]$. Moreover, (i) every map of $\mathcal{X}[\Sigma^{-1}]$ can be represented as $(P_\Sigma s)^{-1} P_\Sigma f$ with $s \in \Sigma$, and (ii) $(P_\Sigma s)^{-1} P_\Sigma f = (P_\Sigma t)^{-1} P_\Sigma g$ iff there is a commutative diagram in $\mathcal{X}$ of the form

$$\begin{tikzcd} & \bullet \arrow[d, "x"] & \\ \bullet \arrow[ur, "f"] \arrow[dr, "g"'] & \bullet & \bullet \arrow[ul, "s"'] \arrow[dl, "t"] \\ & \bullet \arrow[u, "y"'] & \end{tikzcd}$$

with $xs = yt$ in Σ. In [7], J. Bénabou presents a calculus of fractions which provides necessary and sufficient conditions on Σ for (i) and (ii).

2. Let Σ be the subcategory of $\mathcal{X}^{\to}$ whose objects are all left adjoint sections of $\mathcal{X}$, and the morphisms between them are all $(f,g) : r \to s$ with $fr_* = s_*g$. Then Σ is clearly a subcategory of $\mathcal{X}$, and it admits a left calculus of lax fractions. To show *Coinsertion*, given a morphism $(f,g) : r \to s$, let h be a morphism of $\mathcal{X}$ with $gr \le hr$; then s_* plays the role of t in Definition 4.2, the inequality being obtained as follows: $s_*g = s_*sfr_* = s_*grr_* \le s_*hrr_* \le s_*h$.

3. Let $\mathcal{X}$ be an order-enriched category with conical pushouts (see Section 5). A morphism $e : X \to Y$ of $\mathcal{X}$ is said to be order-epic if, for every pair of morphisms $f,g : Y \to Z$ with $fe \le ge$, we have that $f \le g$. It is easily seen that every (conical) pushout of an order-epic morphism along an arbitrary morphism is also order-epic. Let Σ be the subcategory of $\mathcal{X}^{\to}$ defined as follows. The objects are all order-epic morphisms, and the morphisms are all morphisms of $\mathcal{X}^{\to}$ of the form $(\mathrm{id}, e) : \mathrm{id} \to e$ with e order-epic, represented by the square $\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[d, "e"] \\ \bullet \arrow[r, "e"'] & \bullet \end{tikzcd}$, all morphisms $(f,g) : e \to e'$ of $\mathcal{X}$

such that the square $\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "e'"'] & \bullet \end{tikzcd}$ is a pushout, and all morphisms of $\mathcal{X}^{\to}$

obtained by finite horizontal and vertical composition of these two types of squares. It is easy to see that Σ is indeed a subcategory of $\mathcal{X}^{\to}$ which admits a left calculus of lax fractions.

4. In the category Pos, we say that a morphism $m : X \to Y$ is an *(order) embedding* if it satisfies the condition $m(x) \leq m(x') \Rightarrow x \leq x'$, for all $x, x' \in X$. We know that, in Pos, every complete lattice is Kan-injective with respect to embeddings, and given $f : X \to C$ with C a complete lattice f/m is defined by (see [6] and [1])

$$(f/m)(b) = \bigvee_{m(x) \leq b} f(x). \tag{10}$$

Moreover, embeddings are precisely those morphisms $m : X \to Y$ with respect to which the two-element chain $D = (0 < 1)$ is Kan-injective; indeed, given $a, a' \in X$ with $m(a) \leq m(a')$, define $f : X \to D$ by $f(x) = 1$ if $a \leq x$, otherwise $f(x) = 0$. Then, if D is Kan-injective with respect to m, we have $1 = f(a) = (f/m)m(a) \leq (f/m)m(a') = f(a')$, and this implies the equality $f(a') = 1$, i.e. $a \leq a'$.

Let Σ be the subcategory of $\mathrm{Pos}^{\to}$ consisting of:

- Objects: all embeddings;
- Morphisms: all morphism $(u, v) : m \to n$, with $m : X \to Y$ and $n : Z \to W$ embeddings, satisfying the following condition, for all $y \in Y$ and $z \in Z$:

$$n(z) \leq v(y) \implies \text{there is some } x \in X \text{ with } z \leq u(x) \text{ and } m(x) \leq y. \tag{11}$$

We show that $\Sigma = D^{\underline{\mathsf{KInj}}}$. As a consequence, Σ admits a left calculus of lax fractions. Indeed, in Proposition 5.3 we will see that if $\mathcal{X}$ has finite weighted colimits then, for every subcategory $\mathcal{A}$ of $\mathcal{X}$, $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ always admits a left calculus of fractions.

Since we already have seen that embeddings are precisely the morphisms of $\mathcal{X}$ with respect to which D is Kan-injective, it remains to show that (11) characterizes the morphisms of $D^{\underline{\mathsf{KInj}}}$. Let then the morphism $(u, v) : m \to n$ of $\mathrm{Pos}^{\to}$ satisfy (11), and consider a morphism $f : Z \to D$. We want to show that $(fu)/m = (f/n)v$. Since $(fu)/m \leq (f/n)v$ always holds, it suffices to show that, for each $y \in Y$, $((f/n)v)(y) = 1$ implies $((fu)/m)(y) = 1$; in other words, taking into account (10), if $y \in Y$ and $z \in Z$ are such that $f(z) = 1$ and $n(z) \leq v(y)$, then there is some $x \in X$ with $fu(x) = 1$ and $m(x) \leq y$. But the satisfaction of this last condition is clearly ensured by (11). Conversely, let $(u, v) : m \to n$ be a morphism of $D^{\underline{\mathsf{KInj}}}$, and consider $y \in Y$ and

$z \in Z$ with $n(z) \le v(y)$. Let $f : Z \to D$ be defined by $f(z') = 1$ if $z \le z'$, otherwise, $f(z') = 0$. Since $f(z) = 1$ and $n(z) \le v(y)$, we have that $((f/n)v)(y) = 1$. Thus also $((fu)/m)(y) = 1$. But this means that there is some $x \in X$ with $m(x) \le y$ and $(fu)(x) = 1$, the last equality meaning that $z \le u(x)$.

Let Ω_0 be the contravariant endofunctor of Pos sending every poset X to the poset $\Omega_0 X$ of its lower sets, and every monotone map $f : X \to Y$ to the preimage map $\Omega_0 f : \Omega_0 Y \to \Omega_0 X$. In [2], we show that condition (11) above is equivalent to the Beck-Chevalley condition $(\Omega_0 u)^* \cdot \Omega_0 m = \Omega_0 n \cdot (\Omega_0 b)^*$, where $-^*$ stands for the left adjoint.

5. (cf. [2]) Let Loc be the category of locales (i.e., frames) and localic maps, i.e., maps f preserving all infima and whose left adjoint f^* preserves finite meets. Recall that embeddings in Loc are precisely the localic maps h made split monomorphisms by its left adjoint: $h^*h = \mathrm{id}$ ([15]).

Let Σ_0 be the subcategory of $\mathrm{Loc}^{\to}$ having all embeddings as objects and whose morphisms are those $(u, v) : m \to n$ of $\mathrm{Loc}^{\to}$ satisfying the Beck-Chevalley condition $v^*n = mu^*$. We are going to show that Σ_0 admits a left calculus of lax fractions.

In [9] we showed that for every finitely generated frame F, given an embedding $m : X \to Y$ and $f : X \to F$, the map mf^* is a frame homomorphism, thus $(mf^*)_*$ is localic, and moreover

$$f/m = (mf^*)_*. \tag{12}$$

We also proved that embeddings are precisely the localic maps with respect to which the free frame F_1 generated by $1 = \{0\}$ is Kan-injective. In order to conclude that Σ_0 admits a left calculus of lax fractions we show that $\Sigma_0 = F_1^{\mathsf{KInj}}$. Then, since Loc has finite weighted colimits, the result follows from Proposition 5.3.

Indeed, assume that in the commutative square

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "n"'] & W \end{tikzcd}$$

m and n are embeddings and $mu^* = v^*n$. Then, for every $f : Z \to F_1$, we have:

$$(f/n)v = (nf^*)_*v = (v^*(nf^*))_* = (mu^*f^*)_* = (m(fu)^*)_* = (fu)/m.$$

Conversely, assume that $(u,v) : m \to n$ lies in $F_1^{\underline{\mathsf{KInj}}}$. We show $mu^* = v^*n$. Given $z \in Z$, let $g : F_1 \to Z$ be the frame homomorphism sending the element 0 to z. The localic map $g_* : Z \to F_1$ satisfies the equality $(g_*/n)v = (g_*u)/m$, i.e., by (12), $(ng)_*v = (mu^*g)_*$; then, by applying the operator $-^*$ to the last equality, we obtain $v^*ng = mu^*g$, thus $v^*n(z) = v^*ng(0) = mu^*g(0) = mu^*(z)$.

6. Recall that in Loc dense embeddings are those preserving the bottom $\perp$, and flat embeddings are those preserving finite joins. Let now F_0, F_1 and F_2 be the free frames generated by the empty set, $1 = \{0\}$ and $2 = \{0,1\}$, respectively, and let $f_i : F_i \to F_1$, $i = 0,2$, be the localic maps determined by $f_0(\perp) = 0$, $f_2(0 \vee 1) = 0$ and $f_2(x) = \perp$ for $x \neq \top, 0 \vee 1$. In [9] dense embeddings were characterized as precisely the localic maps with respect to which the morphism f_0 is Kan-injective. And flat embeddings were characterized there as precisely those morphisms with respect to which both f_0 and f_2 are Kan-injective. Let Σ_1 and Σ_2 be the full subcategories of the category $\Sigma_0 = F_1^{\underline{\mathsf{KInj}}}$, described in 5, consisting of all dense embeddings, and all flat embeddings, respectively. Both Σ_1 and Σ_2 admit a left calculus of lax fractions. Indeed, by using the same arguments as in the previous example, we see that $\Sigma_1 = \{f_0\}^{\underline{\mathsf{KInj}}}$ and $\Sigma_2 = \{f_0, f_2\}^{\underline{\mathsf{KInj}}}$.

7. Let Top_0 be the category of T_0-topological spaces and continuous maps, considered as an order-enriched category via the dual of the specialization order. Let $\mathrm{Lc} : \mathsf{Top}_0 \to \mathsf{Loc}$ be the functor taking every space X to the frame of its open sets ΩX, and every continuous map $f : X \to Y$ to the right adjoint of the preimage map $f^{-1} : \Omega Y \to \Omega X$. Then the subcategory Σ of $\mathsf{Top}_0^{\to}$ consisting of all (topological) embeddings and all morphisms $(u,v) : m \to n$ between embeddings such that $(\mathrm{Lc}(u), \mathrm{Lc}(v)) : \mathrm{Lc}(m) \to \mathrm{Lc}(n)$ belongs to the category Σ_0 described above (in 5) admits a left calculus of lax fractions. Indeed as shown in [2], Σ is precisely $\mathbf{S}^{\underline{\mathsf{KInj}}}$ in Top_0 where $\mathbf{S}$ is the Sierpiński space.

A collection of examples of subcategories $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions (which indeed includes Examples 3, 5 and 6 of 4.4 (see [9]), is obtained from the next proposition.

Proposition 4.5. *If $\mathcal{A}$ is a KZ-reflective subcategory of $\mathcal{X}$, then $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ admits a left calculus of fractions.*

Proof: Using Lemma 3.4, the satisfaction of *Identity* and *Composition* is clear. To obtain *Square*, in 4.2.3 let X be the domain of r and let Y and Z be the codomains of r and f, respectively; put $r' = \eta_Z$ and $f' = Ff(Fr)_*\eta_Y$. From Remark 3.5, we know that $(F\eta_Z)_* = \varepsilon_{FZ}$, and then, since $F(Fr)_* \cdot F\eta_Y \cdot \eta_Y = F(Fr)_* \cdot \eta_{FY} \cdot \eta_Y = \eta_{FX} \cdot (Fr)_* \cdot \eta_Y$, we have that

$$(F\eta_Z)_* \cdot F^2 f \cdot F(Fr)_* \cdot F\eta_Y \cdot \eta_Y = Ff \cdot \varepsilon_{FX} \cdot \eta_{FX} \cdot (Fr)_* \cdot \eta_Y = Ff \cdot (Fr)_* \cdot \eta_Y.$$

Since $(F\eta_Z)_* \cdot F(Ff \cdot (Fr)_* \cdot \eta_Y)$ and $Ff \cdot (Fr)_*$ are both morphisms of $\mathcal{A}$ (see 2.2), we conclude that they are equal; that is, by Lemma 3.4 again, our square is of Σ type.

To show *Coinsertion*, let us have a diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "f"'] & Y \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\ Z \arrow[r, "s"'] & W \end{tikzcd}$$

where the inner square is a Σ-square and with $gr \leq hr$. Put $t = (Fs)_*\eta_W$. Then, $tg = (Fs)_*\eta_W g = (Fs)_* Fg\eta_Y = Ff(Fr)_*\eta_Y = (Fs)_* FsFf(Fr)_*\eta_Y$. But $FsFf(Fr)_* = FgFr(Fr)_* \leq FhFr(Fr)_* \leq Fh$. Thus

$$tg \leq (Fs)_* Fh\eta_Y = (Fs)_*\eta_W h = th.$$

Moreover, we have $ts = (Fs)_*\eta_W s = (Fs)_* Fs\eta_Y = \eta_Y$; hence, by Lemma 3.4 and Remark 3.5, $ts \in \Sigma$. To show that $(\mathrm{id}, t) : s \to ts$ is a morphism of Σ we also use property (ii) of Lemma 3.4: $(F(ts))_* Ft = (F\eta_Y)_* F(Fs)_* F\eta_W = \varepsilon_{FY} F(Fs)_* F\eta_W = (Fs)_*\varepsilon_{FW} F\eta_W = (Fs)_*$. ∎

In Proposition 5.3 we will see that if $\mathcal{X}$ has finite weighted colimits then, for every subcategory $\mathcal{A}$ of $\mathcal{X}$, $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ always admits a left calculus of fractions.

Let Σ be a subcategory of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions. We are going to see that then we obtain a category of lax fractions as follows: the objects of $\mathcal{X}[\Sigma_*]$ are those of $\mathcal{X}$, and the morphisms are going to be equivalence classes of Σ-cospans. In general, $\mathcal{X}[\Sigma_*]$ is not locally small (even if $\mathcal{X}$ is so), analogously to what happens in the ordinary case to $\mathcal{X}[\Sigma^{-1}]$ for Σ admitting a left calculus of fractions.

The following definitions and lemmas are a preparation for Theorem 4.11 below.

4.6. The relation $\leq$ between Σ-cospans. A Σ-cospan from A to B of the form

$$A \xrightarrow{f} I \xleftarrow{s} B$$

will be denoted by (f, I, s) or just by (f, s).

Given objects A and B of $\mathcal{X}$, we consider a relation $\leq$ between Σ-cospans from A to B given by

$$(f, I, s) \leq (g, J, t)$$

if there is a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dd, equal] \arrow[dr, phantom, "\leq"] & I \arrow[d, "x"'] & B \arrow[l, "s"'] \arrow[d, equal] \arrow[dl, phantom, "\Sigma"] \\ & X & B \arrow[l] \arrow[d, equal] \arrow[dl, phantom, "\Sigma"] \\ A \arrow[r, "g"'] & J \arrow[u, "y"] & B \arrow[l, "t"] \end{tikzcd}$$

where, as indicated, $xf \leq yg$, and the two squares on the right-hand side are Σ-squares, i.e., $(\mathrm{id}, x) : s \to sx$ and $(\mathrm{id}, y) : t \to yt$ are morphisms of Σ with $xs = yt$.

Lemma 4.7. *For Σ admitting a left calculus of lax fractions, let* $A \xleftarrow{r} D \xrightarrow{d} B$ *be a Σ-span covering the two Σ-cospans* $A \xrightarrow{f_i} I_i \xleftarrow{s_i} B$ *, $i = 1, 2$ (see Terminology 4.1). Then $(f_1, I_1, s_1) \leq (f_2, I_2, s_2)$, and, analogously, $(f_2, I_2, s_2) \leq (f_1, I_1, s_1)$.*

Proof: We show that $(f_1, I_1, s_1) \leq (f_2, I_2, s_2)$. Using *Square*, form the Σ-square

$$\begin{tikzcd} B \arrow[r, "s_1"] \arrow[d, "s_2"'] \arrow[dr, phantom, "\Sigma"] & I_1 \arrow[d, "r_1"] \\ I_2 \arrow[r, "r_2"'] & J \end{tikzcd} . \tag{13}$$

Since, by hypothesis, $(d, f_i) : r \to s_i$ is a morphism of Σ for $i = 1, 2$, by vertical composition of Σ-squares, we obtain the Σ-square $\begin{tikzcd} D \arrow[r, "r"] \arrow[d, "s_2 d"'] \arrow[dr, phantom, "\Sigma"] & A \arrow[d, "r_1 f_1"] \\ I_2 \arrow[r, "r_2"'] & J \end{tikzcd}$.

Moreover, $(r_1 f_1)r = r_1 s_1 d = r_2 s_2 d = (r_2 f_2)r$. Consequently, by *Coinsertion*, there is some morphism $p : J \to I_0$ such that $p(r_1 f_1) \leq p(r_2 f_2)$, and

$$\begin{tikzcd} B \arrow[r, "r_2"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & J \arrow[d, "p"] \\ B \arrow[r, "pr_2"'] & I_0 \end{tikzcd} . \tag{14}$$

To conclude that $(f_1, I_1, s_1) \preceq (f_2, I_2, s_2)$, it remains to verify that the two squares on the right-hand side of the following diagram are of Σ type:
$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[dd, equal] & I_1 \arrow[d, "pr_1"] & B \arrow[l, "s_1"'] \arrow[d, equal] \\ \arrow[r, phantom, "\rotatebox{90}{$\leq$}"] & I_0 & B \arrow[l] \\ A \arrow[r, "f_2"'] & I_2 \arrow[u, "pr_2"'] & B \arrow[l, "s_2"] \arrow[u, equal] \end{tikzcd}$$
. Concerning the bottom one, it follows from the composition of the following Σ-squares, where we use (14), the fact that Σ is a subcategory of $\mathcal{X}^{\to}$, and *Identity*:

$$\begin{tikzcd} B \arrow[r, "s_2"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & I_1 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & I_1 \arrow[d, "r_2"] \\ B \arrow[r, "s_2"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & I_1 \arrow[r, "r_2"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & Z \arrow[d, "p"] \\ B \arrow[r, "s_2"'] & X \arrow[r, "pr_2"'] & I_0 \end{tikzcd} \tag{15}$$

Concerning the top one, observe that, from (13), *Identity* and *Composition*, we have that the outside square of the diagram
$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & B \arrow[r, "s_1"] \arrow[d, "s_2"'] \arrow[dr, phantom, "\Sigma"] & I_1 \arrow[d, "r_1"] \\ B \arrow[r, "s_2"'] & I_2 \arrow[r, "r_2"'] & J \end{tikzcd}$$
is a Σ one. Now, composing vertically with the Σ-square given by the composition of the two Σ-squares in the bottom of (15), and taking into account that $r_2 s_2 = r_1 s_1$, we obtain the desired Σ-square.

Analogously, we can show that $(f_2, I_2, s_2) \preceq (f_1, I_1, s_1)$. ∎

Lemma 4.8. *The relation $\preceq$ on the class of all Σ-cospans is reflexive and transitive.*

Proof: Reflexivity holds since the identity morphism $(\mathrm{id},\mathrm{id}) : s \to s$ is in Σ.

Concerning transitivity, let (f,I,s), (g,J,t) and (h,K,u) be Σ-cospans from A to B such that $(f,I,s) \leq (g,J,t)$ and $(g,J,t) \leq (h,K,u)$ through the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[d, "x"'] \arrow[r, phantom, shift right=4, "\Sigma"] & B \arrow[l, "s"'] \arrow[d, equal] \\
\arrow[r, phantom, "\leq"] & X & B \arrow[l] \arrow[d, equal] \\
A \arrow[r, "g"] \arrow[dd, equal] & J \arrow[u, "y"] \arrow[d, "z"'] & B \arrow[l, "t"'] \arrow[d, equal] \\
\arrow[r, phantom, "\leq"] & Z & B \arrow[l] \arrow[d, equal] \\
A \arrow[r, "h"'] & K \arrow[u, "w"] & B \arrow[l, "u"]
\end{tikzcd}$$

Then we have that the Σ-span $B \xleftarrow{\mathrm{id}_B} B \xrightarrow{t} J$ covers both the Σ-cospans $J \xrightarrow{y} X \xleftarrow{yt} B$ and $J \xrightarrow{z} Z \xleftarrow{zt} B$. Consequently, by Lemma 4.7, $(y,yt) \leq (z,zt)$. Therefore, there are morphisms $a : X \to Y$ and $b : Z \to Y$ with which we obtain the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, equal] & & I \arrow[d, "x"'] & B \arrow[l, "s"'] \arrow[d, equal] \\
A \arrow[r, "g"] \arrow[ddd, equal] & J \arrow[r, "y"] \arrow[ddd, equal] & X \arrow[d, "a"'] & B \arrow[l, "yt"'] \arrow[d, equal] \\
 & \arrow[r, phantom, "\leq"] & Y & B \arrow[l] \arrow[d, equal] \\
 & & & B \arrow[d, equal] \\
A \arrow[r, "g"'] \arrow[d, equal] & J \arrow[r, "z"'] & Z \arrow[uu, "b"] & B \arrow[l, "zt"] \arrow[d, equal] \\
A \arrow[rr, "h"'] & & K \arrow[u, "w"] & B \arrow[l, "u"]
\end{tikzcd}$$

with $(ax)f \leq ayg \leq bzg \leq (bw)h$. Thus $(f,s) \leq (h,u)$. ■

4.9. The equivalence classes of Σ-cospans and their composition. We say that two Σ-cospans (f,s) and (g,t) with the same domain and codomain are *equivalent*, and write

$$(f,s) \equiv (g,t)$$

whenever $(f,s) \preceq (g,t)$ and $(g,t) \preceq (f,s)$.

Since $\preceq$ is reflexive and transitive, $\equiv$ is an equivalence relation.

We denote the equivalence class of a Σ-cospan (f,s) by $[(f,s)]$. When there is no reason for confusion, we also represent the equivalence class by one of its elements.

Since $\preceq$ is reflexive and transitive, we obtain a partial order $\leq$ between equivalence classes of Σ-cospans with the same domain and codomain as follows:

$$[(f,s)] \leq [(g,t)] \quad \text{whenever} \quad (f,s) \preceq (g,t).$$

In particular, we conclude that, for two Σ-cospans as in Lemma 4.7, $(f_1,I_1,s_1) \equiv (f_2,I_2,s_2)$.

Next we define a composition between equivalence classes of Σ-cospans, for Σ admitting a left calculus of lax fractions. We give the definition and we show that it is well-defined and that it is preserved by the order $\leq$ defined between equivalence classes of Σ-cospans.

Given two Σ-cospans $(f,I,s) : A \to B$ and $(g,J,t) : B \to C$, we define

$$[(g,J,t)] \cdot [(f,I,s)]$$

as being the equivalence class of any Σ-cospan $(g'f,K,s't) : A \to C$ obtained by forming a Σ-square as follows:

$$\begin{tikzcd} A \arrow[r, "f"] & I \arrow[d, "g'"'] \arrow[dr, phantom, "\Sigma"] & B \arrow[l, "s"'] \arrow[d, "g"] & \\ & K & J \arrow[l, "s'"] & C \arrow[l, "t"] \end{tikzcd}$$

From now on *a composition of two Σ-cospans* $(f,I,s) : A \to B$ and $(g,J,t) : B \to C$ will be denoted by

$$(g,J,t) \circ (f,I,s)$$

and it refers to any Σ-cospan $(g'f,K,s't) : A \to C$ obtained as described above.

The above composition is well-defined, that is, if $I \xrightarrow{g'} K \xleftarrow{s'} J$ and $I \xrightarrow{\hat{g}} M \xleftarrow{\hat{s}} J$ are two Σ-cospans covered by the Σ-span $I \xleftarrow{s} B \xrightarrow{g} J$, then $(g'f, K, s't) \equiv (\hat{g}f, M, \hat{s}t)$.

Indeed, in that case, by Lemma 4.7, $(g', K, s') \leq (\hat{g}, M, \hat{s})$, thus we have a diagram of the form

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[r, "g'"] \arrow[dd, equal] & K \arrow[d, "a"'] \arrow[r, phantom, bend left=0, "\Sigma" description, shift right=12] & J \arrow[l, "s'"'] \arrow[d, equal] \arrow[r, phantom, "\Sigma" description, shift right=12] & C \arrow[l, "t"'] \arrow[d, equal] \\
 & \arrow[r, phantom, "\leq" description] & N & J \arrow[l] \arrow[r, phantom, "\Sigma" description, shift right=12] & C \arrow[l, "t"'] \\
A \arrow[r, "f"'] & I \arrow[r, "\hat{g}"'] & M \arrow[u, "b"] \arrow[r, phantom, "\Sigma" description, shift left=12] & J \arrow[l, "\hat{s}"] \arrow[u, equal] & C \arrow[l, "t"] \arrow[u, equal]
\end{tikzcd}
$$

showing that $(g'f, K, s't) \leq (\hat{g}f, M, \hat{s}t)$; and analogously, we have $(\hat{g}f, M, \hat{s}t) \leq (g'f, K, s't)$.

Lemma 4.10. *The relation $\leq$ is compactible with composition, i.e., if we have a diagram of Σ-cospans*

$$
\begin{tikzcd}
A \arrow[r, shift left, "(f_2, s_2)"] \arrow[r, shift right, "(f_1, s_1)"'] & B \arrow[r, shift left, "(g_2, t_2)"] \arrow[r, shift right, "(g_1, t_1)"'] & C
\end{tikzcd}
$$

with $(f_1, s_1) \leq (f_2, s_2)$ and $(g_1, t_1) \leq (g_2, t_2)$, then any composition of the two lower Σ-cospans is $\leq$-related to any composition of the two upper Σ-cospans.

Proof: It suffices to prove that the property holds for

(A) $(f, s) = (f_1, s_1) = (f_2, s_2)$, and
(B) $(g, t) = (g_1, t_1) = (g_2, t_2)$.

(A) Let us have the inequality $(g_1, t_1) \leq (g_2, t_2)$ through the diagram

$$
\begin{tikzcd}
B \arrow[r, "g_1"] \arrow[dd, equal] & J_1 \arrow[d, "y_1"'] \arrow[r, phantom, "\Sigma" description, shift right=12] & C \arrow[l, "t_1"'] \arrow[d, equal] \\
\arrow[r, phantom, "\leq" description] & J_0 & C \arrow[l] \\
B \arrow[r, "g_2"'] & J_2 \arrow[u, "y_2"] \arrow[r, phantom, "\Sigma" description, shift left=12] & C \arrow[l, "t_2"] \arrow[u, equal]
\end{tikzcd}
$$

and, using *Square*, consider the compositions $(g_i, J_i, t_i) \circ (f, I, s)$, $i = 1, 2$, given by

$$\begin{tikzcd}
A \arrow[r, "f"] & I \arrow[d, "g_i'"'] \arrow[dr, phantom, "\leq"] & B \arrow[l, "s"'] \arrow[d, "g_i"] \\
& K_i & J_i \arrow[l, "s_i"] & C \arrow[l, "t_i"]
\end{tikzcd} \quad . \tag{16}$$

Square also ensures the existence of the following first two Σ-squares, which in turn, combined with (16), give rise to the third diagram:

$$\begin{tikzcd}
J_i \arrow[r, "s_i"] \arrow[d, "y_i"'] \arrow[dr, phantom, "\Sigma"] & K_i \arrow[d, "y_i'"] \\
J_0 \arrow[r, "s_i'"'] & L_i
\end{tikzcd} \;, \; i = 1, 2, \qquad
\begin{tikzcd}
J_0 \arrow[r, "s_1'"] \arrow[d, "s_2'"'] \arrow[dr, phantom, "\Sigma"] & L_1 \arrow[d, "r_1"] \\
L_2 \arrow[r, "r_2"'] & M
\end{tikzcd} \qquad
\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "s_2' y_1 g_1"'] & I \arrow[d, shift right, "r_1 y_1' g_1'"'] \arrow[d, shift left, "r_2 y_2' g_2'"] \\
L_2 \arrow[r, "r_2"'] & N
\end{tikzcd} \quad . \tag{17}$$

In the last diagram the inner square is of Σ type, because of *Composition*, and, furthermore, we have that $(r_1 y_1' g_1')s = r_1 y_1' s_1 g_1 = r_1 s_1' y_1 g_1 = r_2 s_2' y_1 g_1 \leq r_2 s_2' y_2 g_2 = r_2 y_2' s_2 g_2 = (r_2 y_2' g_2')s$. Consequently, by *Coinsertion*, there is $p : M \to P$ such that

$$pr_1 y_1' g_1' \leq pr_2 y_2' g_2' \qquad \text{and} \qquad
\begin{tikzcd}
L_2 \arrow[r, "r_2"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & M \arrow[d, "p"] \\
L_2 \arrow[r, "pr_2"'] & P
\end{tikzcd} \quad . \tag{18}$$

Therefore, we have the following diagram, where $t = y_i t_i$,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[r, "y_1' g_1'"] \arrow[dd, equal] & L_1 \arrow[d, "pr_1"'] \arrow[dr, phantom, "1"] & J_0 \arrow[l, "s_1'"'] \arrow[d, equal] \arrow[dr, phantom, "\leq"] & C \arrow[l, "t"'] \arrow[d, equal] \\
& \leq & P & J_0 \arrow[l] & C \arrow[l, "t"'] \\
A \arrow[r, "f"'] & I \arrow[r, "y_2' g_2'"'] & L_2 \arrow[u, "pr_2"] \arrow[ur, phantom, "2"] & J_0 \arrow[l, "s_2'"] \arrow[u, equal] \arrow[ur, phantom, "\leq"] & C \arrow[l, "t"] \arrow[u, equal]
\end{tikzcd} \tag{19}$$

with both squares ① and ② of Σ type. Indeed ① and ② are the outside squares of the following diagrams obtained by vertical and horizontal composition of Σ-squares:

$$
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[r, "s_1'"] \arrow[d, "s_2'"'] & \bullet \arrow[d, "r_1"] \\
\bullet \arrow[r, "s_2'"] \arrow[d, equal] & \bullet \arrow[r, "r_2"] \arrow[d, equal] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "s_2'"'] & \bullet \arrow[r, "pr_2"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "s_2'"] \arrow[d, equal] & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[d, "r_2"] \\
\bullet \arrow[r, "s_2'"] \arrow[d, equal] & \bullet \arrow[r, "r_2"] \arrow[d, equal] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "s_2'"'] & \bullet \arrow[r, "pr_2"'] & \bullet
\end{tikzcd}
$$

From (16) and the first diagram of (17), with $t = y_i t_i$, we obtain the commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & I \arrow[r, "g_i'"] \arrow[d, equal] & K_i \arrow[d, "y_i'"'] & J_i \arrow[l, "s_i"'] \arrow[d, "y_i"'] & C \arrow[l, "t_i"'] \arrow[d, equal] \\
A \arrow[r, "f"'] & I \arrow[r, "y_i'g_i'"'] & M_i & J_0 \arrow[l, "s_i'"] & C \arrow[l, "t"]
\end{tikzcd} . \tag{20}
$$

Now, the diagram obtained by composing vertically first the diagram (20) with $i = 1$, next the diagram (19), and lastly the diagram (20) with $i = 2$, shows that $(g_1'f, s_1't_1) \leq (g_2'f, s_2't_2)$, as desired.

(B) Let us have the inequality $(f_1, s_1) \leq (f_2, s_2)$ through the diagram

$$
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[dd, equal] & I_1 \arrow[d, "x_1"'] & B \arrow[l, "s_1"'] \arrow[d, equal] \\
 & I_0 & C \arrow[l] \\
A \arrow[r, "f_2"'] & I_2 \arrow[u, "x_2"] & B \arrow[l, "s_2"] \arrow[u, equal]
\end{tikzcd} .
$$

Then the following diagram, where $(\tilde{g}, \tilde{s})$ is a Σ-cospan obtained by *Square* applied to the Σ-span (s, g),

$$\begin{tikzcd}
A \arrow[r, "f_i"] & I_i \arrow[d, "x_i"'] \arrow[dr, phantom, "\leq"] & B \arrow[l, "s_i"'] \arrow[d, equal] & \\
& I_0 \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\leq"] & B \arrow[l, "s"'] \arrow[d, "g"] & \\
& M & J \arrow[l, "\tilde{s}"] & C \arrow[l, "t"]
\end{tikzcd}$$

shows that, for $i = 1, 2$, $(\tilde{g}x_i f_i, \tilde{s}t)$ is a composition of (f_i, s_i) with (g, t). Thus, the diagram

$$\begin{tikzcd}
A \arrow[r, "\tilde{g}x_1 f_1"] \arrow[d, equal] \arrow[dr, phantom, "\leq"] & M \arrow[d, equal] \arrow[dr, phantom, "\leq"] & C \arrow[l, "\tilde{s}t"'] \arrow[d, equal] \\
A \arrow[r, "\tilde{g}x_2 f_2"'] & M & C \arrow[l, "\tilde{s}t"]
\end{tikzcd}$$

tells us that $(g, t) \circ (f_1, s) \leq (g, t) \circ (f_2, s)$. ∎

Now we are ready to prove the announced theorem:

Theorem 4.11. *Let Σ be a subcategory of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions. Then the category of fractions $P_\Sigma : \mathcal{X} \to \mathcal{X}[\Sigma_*]$ can be described as follows:*

- *the objects of $\mathcal{X}[\Sigma_*]$ are those of $\mathcal{X}$;*
- *the morphisms of $\mathcal{X}[\Sigma_*]$ are $\equiv$-equivalence classes of Σ-cospans with the composition and order on morphisms as described in 4.9;*
- *$P_\Sigma A = A$ and $P_\Sigma f = [(f, id)]$ for all objects A and morphisms f of $\mathcal{X}$.*

Proof: (A) $\mathcal{X}[\Sigma_*]$, as described above, is actually a category.

The identity on an object A is the equivalence class of $(\mathrm{id}_A, \mathrm{id}_A)$. Indeed, given $(f, I, s) : A \to B$, using the fact that Σ is a subcategory of $\mathcal{X}^{\to}$, *Square* and *Identity*, we obtain the diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] & I \arrow[d, "\mathrm{id}_I"'] \arrow[r, phantom, "\leq" near end] & B \arrow[l, "s"'] \arrow[d, "\mathrm{id}_B"] & \\
& I & B \arrow[l, "s"] & B \arrow[l, "\mathrm{id}_B"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & I \arrow[r, equal] \arrow[d, "d"'] \arrow[dr, phantom, "\leq"] & I \arrow[d, equal] \arrow[dr, phantom, "\leq"] & B \arrow[l, "s"'] \arrow[d, equal] \\
A \arrow[r, "f'"'] & I & B \arrow[l, "d"] & B \arrow[l, "s"]
\end{tikzcd}$$

which show that $(\mathrm{id}_B,\mathrm{id}_B)\circ(f,s)\equiv(f,s)$ and $(f,s)\circ(\mathrm{id}_A,\mathrm{id}_A)\equiv(f',ds)\equiv(f,s)$.

Moreover, the associativity of the composition is illustrated by the following diagram, which shows that $(h''g'f, s''t'u)$ is simultaneously a composition of the form $((h,u)\circ(g,t))\circ(f,s)$ and a composition of the form $(h,u)\circ((g,t)\circ(f,s))$:

$$\begin{tikzcd}
A \arrow[r, "f"] & I \arrow[d, "g'"'] \arrow[dr, phantom, "\text{Ʒ}"] & B \arrow[l, "s"'] \arrow[d, "g"] & & \\
& M_1 \arrow[d, "h''"'] \arrow[dr, phantom, "\text{Ʒ}"] & J \arrow[l, "s'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\text{Ʒ}"] & C \arrow[l, "t"'] \arrow[d, "h"] & \\
& M_0 & M_2 \arrow[l, "s''"] & K \arrow[l, "t'"] & D \arrow[l, "u"]
\end{tikzcd}$$

(B) P_Σ is clearly a functor, since $P_\Sigma(\mathrm{id}_A)=(\mathrm{id}_A,\mathrm{id}_A)$, and, given $f:A\to B$ and $g:B\to C$ in $\mathcal{X}$, we have that $P_\Sigma(g)\cdot P_\Sigma(f)\equiv(g,\mathrm{id}_C)\circ(f,\mathrm{id}_B)\equiv(gf,\mathrm{id}_C)\equiv P_\Sigma(gf)$; to see that indeed $(g,\mathrm{id}_C)\circ(f,\mathrm{id}_B)\equiv(gf,\mathrm{id}_C)$, let $(g'f,d)$ be a composition of (g,id) with (f,id), i.e., $(g,g'):\mathrm{id}\to d$ is a morphism of Σ, obtained by *Square*; then, using *Identity*, we have the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, equal] \arrow[d, "d"'] \arrow[dr, phantom, "\text{Ʒ}"] & C \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, "g'"'] & \bullet & C \arrow[l, "d"]
\end{tikzcd}$$

which shows that $[(gf,\mathrm{id}_C)]=[(g,\mathrm{id}_C)]\cdot[(f,\mathrm{id}_B)]$.

Furthermore, P_Σ is order-enriched: given $f,g:A\to B$ with $f\le g$, then $P_\Sigma f\le P_\Sigma g$.

(C) To verify that P_Σ satisfies condition (i) of Definition 3.1, let $s:A\to B$ be an object of Σ. We show that $P_\Sigma s=[(s,\mathrm{id}_B)]$ is a left adjoint section, by showing that $[(\mathrm{id}_B,s)]\cdot[(s,\mathrm{id}_B)]=[(\mathrm{id}_A,\mathrm{id}_A)]$ and $(s,\mathrm{id}_B)\circ(\mathrm{id}_B,s)\le(\mathrm{id}_B,\mathrm{id}_B)$; thus, in particular, we have that $([(s,\mathrm{id})])_*=[(\mathrm{id},s)]$. The Σ-cospan (s,s) is clearly a composition of the form $(\mathrm{id}_B,s)\circ(s,\mathrm{id}_B)$, and the fact that $(s,s)\equiv$

$(\mathrm{id}_A, \mathrm{id}_A)$ follows from the diagram

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[dd, equal] & B \arrow[d, equal] \arrow[dr, phantom, "\textcircled{1}"] & A \arrow[l, "s"'] \arrow[d, equal] \\
& B & A \arrow[l, "s"'] \arrow[d, equal] \\
A \arrow[r, "\mathrm{id}_A"'] & A \arrow[u, "s"] \arrow[ur, phantom, "\textcircled{2}"] & A \arrow[l, "\mathrm{id}_A"]
\end{tikzcd}$$

where ① is a Σ-square because it is the identity morphism on the object s of Σ, and ② is a Σ-square because of *Identity*. In order to conclude that $(s, \mathrm{id}_B) \circ (\mathrm{id}_B, s) \le (\mathrm{id}_B, \mathrm{id}_B)$, let (s_1, s_2) be a composition of (s, id_B) with (id_B, s), as illustrated by the following diagram:

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}_B"] & B \arrow[d, "s_1"'] \arrow[dr, phantom, "\mathfrak{Z}"] & A \arrow[l, "s"'] \arrow[d, "s"] & \\
& C & B \arrow[l, "s_2"] & B \arrow[l, "\mathrm{id}_B"]
\end{tikzcd}$$

Since $s_1 s = s_2 s$, by *Coinsertion* we know that there is a morphism $d : C \to D$ such that $ds_1 \le ds_2$ and the Σ-span (s_2, id_B) covers the Σ-cospan (d, ds_2). We obtain then the diagram

$$\begin{tikzcd}
B \arrow[r, "s_1"] \arrow[dd, equal] & C \arrow[d, "d"'] \arrow[dr, phantom, "\mathfrak{Z}"] & B \arrow[l, "s_2"'] \arrow[d, equal] \\
& D & B \arrow[l] \arrow[d, equal] \\
B \arrow[r, "\mathrm{id}_B"'] & B \arrow[u, "ds_2"] \arrow[ur, phantom, "\mathfrak{Z}"] & B \arrow[l, "\mathrm{id}_B"]
\end{tikzcd}$$

with $ds_1 \le ds_2$. That is, $(s_1, s_2) \le (\mathrm{id}_B, \mathrm{id}_B)$, where (s_1, s_2) is a representative of $[(s, \mathrm{id}_B) \circ (\mathrm{id}_B, s)]$.

Now, the satisfaction of (ii) of Definition 3.1 is easily seen since, given a morphism $(u, v) : r \to s$ in Σ, it is clear that $(u, \mathrm{id}) \circ (\mathrm{id}, r) \equiv (v, s) \equiv (\mathrm{id}, s) \circ (v, \mathrm{id})$.

(D) P_Σ is universal. Let $F : \mathcal{X} \to \mathcal{X}$ be a functor such that Fs is a left adjoint section for every $s \in \Sigma$, and, moreover, for every morphism $(f, g) : r \to s$ in Σ, the equality $(Fs)_* g = f(Fr)_*$ holds. We define $H : \mathcal{X}[\Sigma_*] \to \mathcal{X}$ by

$$HX = FX \quad \text{ and } \quad H[(f,I,s)] = (Fs)_*Ff.$$

First we show that, assuming that H is a functor, it is the unique one such that $HP_\Sigma = F$. Indeed we have $H(P_\Sigma X) = HX = FX$; and $H(P_\Sigma f) = H(f,\mathrm{id}) = (F(\mathrm{id}))_*Ff = Ff$. Furthermore, if $\bar{H} : \mathcal{X}[\Sigma_*] \to \mathcal{X}$ is another functor such that $\bar{H}P_\Sigma = F$, taking into account that we are dealing with order-enriched functors, we have that:

$$\begin{array}{rcl}
\bar{H}X & = & \bar{H}(P_\Sigma X) = FX; \text{ and} \\
\bar{H}[(f,I,s)] & = & \bar{H}[(\mathrm{id}_I,I,s)] \cdot \bar{H}[(f,I,\mathrm{id}_I)] \\
 & = & \Big(\bar{H}[(s,I,\mathrm{id}_I)]\Big)_* \cdot \bar{H}[(f,I,\mathrm{id}_I)] \\
 & = & \Big(\bar{H}P_\Sigma s\Big)_* \cdot \Big(\bar{H}P_\Sigma f\Big) \\
 & = & (Fs)_*Ff \\
 & = & H[(f,I,s)].
\end{array}$$

It remains to show that $H : \mathcal{X}[\Sigma_*] \to \mathcal{X}$ is indeed a functor.

H is well-defined on equivalence classes and is order-enriched. In order to conclude these both assertions, taking into account that $\equiv$ is defined by means of $\leq$, it suffices to prove that, for a pair of Σ-cospans $(f,I,s), (g,J,t) : A \to B$ with $(f,I,s) \leq (g,J,t)$, we have that $(Fs)_*Ff \leq (Ft)_*Fg$. Indeed, if $(f,I,s) \leq (g,J,t)$, then we have a diagrama as follows:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[d, "x"'] \arrow[dr, phantom, "\Sigma"] & B \arrow[l, "s"'] \arrow[d, equal] \\
{} \arrow[r, phantom, "\leq"] & K \arrow[dr, phantom, "\Sigma"] & B \arrow[l] \arrow[d, equal] \\
A \arrow[r, "g"'] & J \arrow[u, "y"] & B \arrow[l, "t"]
\end{tikzcd}$$

The fact that the two squares on the right-hand side are of Σ type implies that $(F(xs))_*Fx = (Fs)_*$ and $(Ft)_* = (F(yt))_*Fy$, by assumption on F. Hence,

$$(Fs)_*Ff = (F(xs))_*FxFf \leq (F(xs))_*FyFg = (F(yt))_*FyFg = (Ft)_*Fg.$$

H is functorial. Indeed, H preserves identities since

$$H[(\mathrm{id}_A,\mathrm{id}_A)] = (F\mathrm{id}_A)_*(F\mathrm{id}_A) = \mathrm{id}_{FA}.$$

In order to show that H preserves composition, given Σ-cospans $(f,s) : A \to B$ and $(g,t) : B \to C$, let $(\tilde{g}f, \tilde{s}t)$ be a composition of them, that is,

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "g"'] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "\tilde{g}"] \\ \bullet \arrow[r, "\tilde{s}"'] & \bullet \end{tikzcd}$$
. Then we have that

$H([(g,t)]\cdot[(f,s)]) = H([(\tilde{g}f,\tilde{s}t)]) = (F(\tilde{s}t))_*F(\tilde{g}f) = (Ft)_*(F\tilde{s})_*F\tilde{g}Ff$. But, by hypothesis, $(F\tilde{s})_*F\tilde{g} = Fg(Fs)_*$. Consequently, we obtain $H([(g,t)]\cdot[(f,s)]) = (Ft)_*Fg(Fs)_*Ff = H([(g,t)])\cdot H([(f,s)])$. ■

5. The cocompleteness of $\mathcal{A}^{\underline{\mathsf{KInj}}}$

We recall from [17] that an order-enriched category $\mathcal{X}$ has weighted colimits if and only if it has conical coproducts and coinserters. We also recall that $\mathcal{X}$ has conical coproducts whenever it has coproducts and the corresponding injections are collectively order-epic, that is, for every coproduct $\nu_i : X_i \to \coprod_{i\in I} X_i$ and every pair of morphisms $f,g : \coprod_{i\in I} X_i \to Y$ with $f\nu_i \le g\nu_i$, $i\in I$, we have $f \le g$. The coinserter of a pair of morphisms $f,g : X \to Y$ is an order-epic morphism $c : Y \to C$ such that $cf \le cg$ and every morphism $d : Y \to D$ with $df \le dg$ factorizes uniquely through c; briefly, $c = \mathrm{coins}(f,g)$.

If $\mathcal{X}$ has weighted colimits, then the arrow category $\mathcal{X}^{\to}$ also has weighted colimits, and they are constructed coordinatewise. We are going to see that $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is closed under weighted colimits in $\mathcal{X}^{\to}$.

Theorem 5.1. *Let $\mathcal{X}$ have weighted colimits. Then, for every subcategory $\mathcal{A}$ of $\mathcal{X}$, the category $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is closed under weighted colimits in $\mathcal{X}^{\to}$.*

Proof: It suffices to show that $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is closed under conical coproducts and coinserters.

Concerning conical coproducts, let $h_i : X_i \to Y_i$ belong to $\mathcal{A}^{\mathsf{KInj}}$, and form the conical coproduct in $\mathcal{X}^{\to}$:

$$\begin{tikzcd} X_i \arrow[r, "h_i"] \arrow[d, "\nu_i^X"'] & Y_i \arrow[d, "\nu_i^Y"] \\ \coprod_{i\in I} X_i \arrow[r, "h"] & \coprod_{i\in I} Y_i \end{tikzcd} \tag{21}$$

First we show that $h \in \mathcal{A}^{\underline{\mathsf{KInj}}}$ and (ν_i^X, ν_i^Y) are morphisms of $\mathcal{A}^{\underline{\mathsf{KInj}}}$. Given $g : \coprod_{i\in I} X_i \to A$, with $A \in \mathcal{A}$, we have a unique morphism g/h with

$$g/h : \coprod_{i\in I} Y_i \to A \ \text{ is such that } (g/h)\nu_i^Y = (g\nu_i^X)/h_i,\ i\in I. \tag{22}$$

We show that g/h deserves its designation. Indeed,

$$(g/h)h\nu_i^X = (g/h)\nu_i^Y h_i = ((g\nu_i^X)/h_i)h_i = g\nu_i^X,\ i \in I,$$

hence $(g/h)h = g$. And, for $s : \coprod_{i\in I} Y_i \to A$ with $g \leq sh$, we have $g\nu_i^X \leq sh\nu_i^X = s\nu_i^Y h_i$, thus $(g\nu_i^X)/h_i \leq s\nu_i^Y$, that is, $(g/h)\nu_i^Y \leq s\nu_i^Y$. Since this holds for all i, $g/h \leq s$. Moreover, since g/h is defined by (22), it is clear that all (ν_i^X, ν_i^Y) are morphisms of $\mathcal{A}^{\underline{\mathsf{KInj}}}$.

Let now have morphisms $(r_i, s_i) : h_i \to t$ in $\mathcal{A}^{\underline{\mathsf{KInj}}}$, $i \in I$. Then, in $\mathcal{X}^{\to}$, we have a unique morphism $(r,s) : h \to t$ such that $(r,s)\cdot(\nu_i^X, \nu_i^Y) = (r_i, s_i)$, $i \in I$:

$$\begin{tikzcd}
X_i \arrow[rrr, "h_i"] \arrow[dr, "\nu_i^X"] \arrow[dd, "r_i"'] & & & Y_i \arrow[dl, "\nu_i^Y"'] \arrow[dd, "s_i"] \\
& \coprod_{i\in I} X_i \arrow[r, "h"] \arrow[dl, "r"'] & \coprod_{i\in I} Y_i \arrow[dr, "s"] & \\
R \arrow[rrr, "t"] & & & S
\end{tikzcd} \tag{23}$$

We show that (r,s) is a morphism of $\mathcal{A}^{\underline{\mathsf{KInj}}}$. Consider $a : R \to A$ with $A \in \mathcal{A}$. Then, using the fact that (ν_i^X, ν_i^Y) and (r_i, s_i) are both morphisms of $\mathcal{A}^{\underline{\mathsf{KInj}}}$ and formula (22), we have:

$$(a/t)s\nu_i^Y = (a/t)s_i = (ar_i)/h_i = (ar\nu_i^X)/h_i = ((ar)/h)\nu_i^Y.$$

Consequently, $(a/t)s = (ar)/h$.

Concerning coinserters, let $(u_1, v_1), (u_2, v_2) : f \to g$ be two morphisms in $\mathcal{A}^{\underline{\mathsf{KInj}}}$ and let (c,d) be the coinserter of $((u_1, v_1), (u_2, v_2))$ in $\mathcal{X}^{\to}$:

$$\begin{tikzcd}
X \arrow[r, shift left, "u_2"] \arrow[r, shift right, "u_1"'] \arrow[d, "f"'] & Z \arrow[r, "c"] \arrow[d, "g"] & C \arrow[d, "t"] \\
Y \arrow[r, shift left, "v_2"] \arrow[r, shift right, "v_1"'] & W \arrow[r, "d"'] & D
\end{tikzcd} \tag{24}$$

In particular, $c = \mathrm{coins}(u_1, u_2)$, $d = \mathrm{coins}(v_1, v_2)$, and t is the unique morphism for which $tc = dg$. We want to show that the morphism (c,d) is also the coinserter of (u_1, v_1) and (u_2, v_2) in $\mathcal{A}^{\underline{\mathsf{KInj}}}$.

First we show that the object t and the morphism $(c,d) : g \to t$ lie in $\mathcal{A}^{\underline{\mathsf{KInj}}}$. For that, consider $k : C \to A$ with A in $\mathcal{A}$. Taking into account that (u_i, v_i), $i = 1, 2$, are morphisms in $\mathcal{A}^{\underline{\mathsf{KInj}}}$, and that $cu_1 \leq cu_2$, we have that

$$((kc)/g)\,v_1 = (kcu_1)/f \leq (kcu_2)/f = ((kc)/g)\,v_2,$$

and, consequently, since $d = \mathrm{coins}(v_1, v_2)$, there is a unique morphism $w : D \to A$ with

$$wd = (kc)/g. \tag{25}$$

We show that $w = k/t$. Indeed, $wtc = wdg = ((kc)/g)g = kc$, thus $wt = k$, since c is order-epic, in particular, an epimorphism. Moreover, if $w' : D \to A$ is such that $k \le w't$, then $kc \le w'tc = w'dg$, then $(kc)/g \le w'd$, and we have that $wd = (kc)/g \le w'd$. Now, since d is order-epic, it follows that $w \le w'$.

The conclusion that $(c,d) : g \to t$ is a morphism in $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is immediate from the definition of w in (25).

Let us now have $t' : C' \to D'$ and a morphism $(c', d') : g \to t'$ in $\mathcal{A}^{\underline{\mathsf{KInj}}}$ with $(c', d') \cdot (u_1, v_1) \le (c', d') \cdot (u_2, v_2)$.

$$\begin{tikzcd}
 & & & C' \arrow[ddd, "t'"] \\
X \arrow[r, shift left, "u_2"] \arrow[r, shift right, "u_1"'] \arrow[d, "f"'] & Z \arrow[r, "c"] \arrow[d, "g"] \arrow[urr, "c'"] & C \arrow[d, "t"] \arrow[ur, dotted, "a"'] & \\
Y \arrow[r, shift left, "v_2"] \arrow[r, shift right, "v_1"'] & W \arrow[r, "d"] \arrow[drr, "d'"'] & D \arrow[dr, dotted, "b"] & \\
 & & & D'
\end{tikzcd} \tag{26}$$

Since $(c,d) = \mathrm{coins}((u_1, v_1), (u_2, v_2))$ in $\mathcal{X}^{\to}$, there is a unique morphism $(a,b) : t \to t'$ such that $(ac, bd) = (c', d')$. We want to show that (a,b) lies in $\mathcal{A}^{\underline{\mathsf{KInj}}}$. Let then $l : C' \to A$ have codomain in $\mathcal{A}$. From above, we know that $(la)/t$ is the unique morphism such that $((la)/t)d = (lac)/g$. But, by hypothesis, $(l/t')bd = (lac)/g$, thus $(l/t')bd = ((la)/t)d$ and, consequently, $(l/t')b = (la)/t$, as desired. ∎

Remark 5.2. Moreover, under the conditions of the above theorem, $\mathcal{A}^{\underline{\mathsf{KInj}}}$ is a coinserter-ideal. That is, given a parallel pair of morphisms $(u_1, v_1), (u_2, v_2) : f \to g$ in $\mathcal{X}^{\to}$, if (u_1, v_1) belongs to $\mathcal{A}^{\underline{\mathsf{KInj}}}$ then also the coinserter of $((u_1, v_1), (u_2, v_2))$ lies in $\mathcal{A}^{\underline{\mathsf{KInj}}}$. Indeed, in the above proof of the closedness of $\mathcal{A}^{\underline{\mathsf{KInj}}}$ under coinserters we only used the fact that (u_1, v_1) belongs to $\mathcal{A}^{\underline{\mathsf{KInj}}}$.

Next we show that the existence of finite weighted colimits in $\mathcal{X}$ allows $\mathcal{A}^{\underline{\mathsf{KInj}}}$ to admit a left calculus of fractions.

Proposition 5.3. *Let $\mathcal{X}$ have finite weighted colimits and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. Then $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ admits a left calculus of lax fractions.*

Proof: *Identity* is obvious, since we always have that, supposing that g/s is defined, $(g \cdot \mathrm{id})/\mathrm{id} = g = (g/s)s$.

Concerning *Composition*, given two Σ-squares as the two first ones in Definition 4.2.2, let $a : \bullet \to A$, with A in $\mathcal{A}$, be composable with f. It is easy to see that, given a composition $\bullet \xrightarrow{s} \bullet \xrightarrow{s'} \bullet$ with s and s' in $\mathcal{A}^{\mathsf{KInj}}$, then $a/(s's) = (a/s)/s'$ (see [8]). Thus, we have: $(af)/(r'r) = ((af)/r)/r' = ((a/s)g)/r' = ((a/s)/s')h = (a/(s's))h$.

To obtain *Square*, we show that every pushout
$$\begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "f"'] & \bullet \arrow[d, "f'"] \\ \bullet \arrow[r, "r'"'] & \bullet \end{tikzcd}$$
in $\mathcal{X}$ with $r \in \Sigma$ is a Σ-square. This follows from the closedness of $\mathcal{A}^{\mathsf{KInj}}$ under pushouts proven in [8], and can be derived from Theorem 5.1: the diagram

$$\begin{tikzcd} \mathrm{id} \arrow[r, "(\mathrm{id},r)"] \arrow[d, "(f,f)"'] & r \arrow[d, "(f,f')"] \\ \mathrm{id} \arrow[r, "(\mathrm{id},r')"'] & r' \end{tikzcd}$$

is a pushout in $\mathcal{X}^{\to}$, and (id, r) and (f, f) are easily seen to be morphisms in $\mathcal{A}^{\underline{\mathsf{KInj}}}$; thus, by the above theorem the same holds to $(f, f') : r \to r'$.

To show *Coinsertion*, given a diagram
$$\begin{tikzcd} U \arrow[r, "r"] \arrow[d, "f"'] & V \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\ W \arrow[r, "s"'] & X \end{tikzcd}$$
with the inner square of Σ type and $gr \leq hr$, let $t : X \to T$ be the coinserter of (g,h). Thus $tg \leq th$. We show that the morphism ts lies in Σ and $(\mathrm{id}, t) : s \to ts$ is a morphism of Σ. Indeed, given $a : W \to A$ with $A \in \mathcal{A}$, $af = (a/s)sf = (a/s)gr \leq (a/s)hr$, thus $(af)/r \leq (a/s)h$. But, by hypothesis, $(af)/r = (a/s)g$. Thus, $(a/s)g \leq (a/s)h$ and, consequently, there is a unique morphism $u : T \to A$ such that $ut = a/s$. It is easy to see that $u = a/(ts)$. For, if, for $v : T \to A$, we have $a \leq v(ts)$, then $a/s \leq vt$, that is, $ut \leq vt$, and, since t is an order-epimorphism, $u \leq v$. Moreover, we have $(a \cdot \mathrm{id})/s = a/s = ut = (a/(ts))t$, that is, $(\mathrm{id}, t) : s \to ts$ is a morphism of Σ. ∎

In the ordinary case, we know that if Σ is a class of morphisms of a finitely cocomplete category $\mathcal{X}$ admitting a left calculus of fractions then the category of fractions $\mathcal{X}[\Sigma^{-1}]$ has finite colimits ([13]).

In the following we see that if $\mathcal{X}$ has finite conical coproducts then, for Σ a subcategory of $\mathcal{X}^{\to}$ admitting a left calculus of lax fractions and satisfying an extra condition, $\mathcal{X}[\Sigma_*]$ has finite conical coproducts too. Moreover, if $\mathcal{X}$ has weighted colimits then any (quasi)category $\mathcal{X}[\Sigma_*]$ with $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ has (small) conical coproducts.

Definition 5.4. For $\mathcal{X}$ an order-enriched category, a subcategory Σ of $\mathcal{X}^{\to}$ is said to satisfy the *Coequalization* condition if given Σ-squares

$$\begin{tikzcd} U \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & V \arrow[d, "g_i"] \\ W \arrow[r, "s"'] & X \end{tikzcd},$$

$i = 1, 2$, there exists some morphism $t : X \to Y$ with $tg_1 = tg_2$ and

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r, "ts"'] & \bullet \end{tikzcd}.$$

Remark 5.5. 1. Let $\mathcal{X}$ have weighted colimits. An argument similar to the one used for *Coinsertion* in the proof of Proposition 5.3 shows that $\mathcal{A}^{\underline{\mathsf{KInj}}}$ also satisfies *Coequalization*, for every subcategory $\mathcal{A}$ of $\mathcal{X}$.

2. Let Σ be a subcategory of $\mathcal{X}^{\to}$ satisfying the four conditions of a left calculus of lax fractions together with *Coequalization*. Then, by using arguments analogous to the ones of the proof of Lemma 4.7, we conclude that, given two Σ-cospans (f, s) and (g, t) from A to B, we have that $(f, s) \equiv (g, t)$ if and only if there is a commutative diagram of the following form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[d, "x"'] \arrow[dr, phantom, "\Sigma"] & B \arrow[l, "s"'] \arrow[d, equal] \\ & X \arrow[dr, phantom, "\Sigma"] & B \arrow[l] \arrow[d, equal] \\ A \arrow[r, "g"'] & J \arrow[u, "y"] & B \arrow[l, "t"] \end{tikzcd}$$

Proposition 5.6. *1. If $\mathcal{X}$ has weighted colimits and $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$ for some subcategory $\mathcal{A}$ of $\mathcal{X}$, then the (quasi)category $\mathcal{X}[\Sigma_*]$ has, and P_Σ preserves, (small) conical coproducts.*

2. If $\mathcal{X}$ has finite conical coproducts and Σ is a subcategory of $\mathcal{X}^{\to}$ satisfying the four conditions of a left calculus of lax fractions together with Coequalization, *then $\mathcal{X}[\Sigma_*]$ has, and P_Σ preserves, finite conical coproducts.*

Proof: 1. Given $X_i \in \mathcal{X}[\Sigma_*]$, $i \in I$, let $\nu_i : X_i \to \coprod_{i\in I} X_i$ be a conical coproduct in $\mathcal{X}$. We show that $[(\nu_i, \mathrm{id})] : X_i \to \coprod_{i\in I} X_i$ constitutes a conical coproduct in $\mathcal{X}[\Sigma_*]$. First, we see that the morphisms $[(\nu_i, \mathrm{id})]$ are collectively order-epic. For that, let us have two Σ-cospans

$$\begin{tikzcd} \coprod\limits_{i\in I} X_i \arrow[r, shift left, "{(g,J,t)}"] \arrow[r, shift right, "{(f,I,s)}"'] & Y \end{tikzcd}$$

with $(f,s) \circ (\nu_i, \mathrm{id}) \leq (g,t) \circ (\nu_i, \mathrm{id})$. It is easy to see that $(f\nu_i, s) \equiv (f,s) \circ (\nu_i, \mathrm{id})$, since, for $(\mathrm{id}, d) : f \to f'$ a morphism of Σ given by *Square*, we have $(f\nu_i, s) \equiv (f'\nu_i, ds)$. Analogously for $(g\nu_i, s)$. Thus $(f\nu_i, s) \leq (g\nu_i, t)$. We show that then $(f,s) \leq (g,t)$. By hypothesis, there are diagrams of the form

$$\begin{tikzcd} X_i \arrow[r, "\nu_i"] \arrow[dd, equal] & \coprod X_i \arrow[r, "f"] \arrow[dd, phantom, "\leq"] & I \arrow[d, "x_i"'] \arrow[dr, phantom, "\leq"] & Y \arrow[l, "s"'] \arrow[d, equal] \\ & & K_i & Y \arrow[l] \\ X_i \arrow[r, "\nu_i"'] & \coprod X_i \arrow[r, "g"'] & J \arrow[u, "y_i"] \arrow[ur, phantom, "\leq"] & Y \arrow[l, "t"] \arrow[u, equal] \end{tikzcd}$$

where all morphisms $x_i s (= y_i t)$ are objects of Σ. Since, by *Identity*, the morphisms $(\mathrm{id}_Y, x_i s) : \mathrm{id}_Y \to (x_i s)$ of $\mathcal{X}^{\to}$ lie in $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$, it follows from Theorem 5.1 that their multipushout

$$\begin{tikzcd} \mathrm{id} \arrow[r, "{(\mathrm{id}, x_i s)}"] \arrow[dr, "{(\mathrm{id}, x)}"'] & x_i s \arrow[d, "{(\mathrm{id}, u_i)}"] \\ & x \end{tikzcd} \tag{27}$$

also lies in Σ. In particular, we have Σ-squares
$$\begin{tikzcd} \bullet \arrow[r, "x_is"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \bullet \arrow[d, "u_i"] \\ \bullet \arrow[r, "x"'] & K \end{tikzcd}$$
; and then, by vertical composition of Σ-squares, we also have
$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & I \arrow[d, "u_ix_i"] \\ Y \arrow[r, "x"'] & K \end{tikzcd}$$
with $u_ix_is = u_jx_js$ for all $i, j \in I$. Let $c : X \to C$ be the coequalizer of all morphisms u_ix_i. Then $(\mathrm{id}, c) : x \to cx$ is the coequalizer of all $(\mathrm{id}, u_ix_i) : s \to x$ in Σ, and, in particular, we obtain the Σ-square
$$\begin{tikzcd} Y \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & K \arrow[d, "c"] \\ Y \arrow[r, "cx"'] & C \end{tikzcd}$$
. Now we have that $cu_ix_ifv_i \leq cu_iy_igv_i$, with $cu_ix_i = cu_jx_j$, $i, j \in I$. Since $(\mathrm{id}, u_i) : y_it = x_is \to x$ is a morphism of Σ (see (27)), using vertical composition, we also obtain the Σ-square
$$\begin{tikzcd} Y \arrow[r, "t"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & J \arrow[d, "cu_iy_i"] \\ Y \arrow[r, "cx"'] & C \end{tikzcd}$$
with $cu_iy_it = cu_jy_jt$, $i, j \in I$. Consequently, for the coequalizer $d : C \to D$ of all morphisms cu_iy_i we have that all morphisms dcu_iy_i are equal and
$$\begin{tikzcd} Y \arrow[r, "cx"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & C \arrow[d, "d"] \\ Y \arrow[r, "dcx"'] & D \end{tikzcd}$$
. Putting $a = dcu_ix_i$ and $b = dcu_iy_i$, it follows that $afv_i \leq bgv_i$ for all i; then $af \leq bg$. Now we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dd, equal] & I \arrow[d, "a"'] \arrow[dr, phantom, "\Sigma"] & Y \arrow[l, "s"'] \arrow[d, equal] \\ \arrow[r, phantom, "\leq"] & D & Y \arrow[l] \\ X \arrow[r, "g"'] & J \arrow[u, "b"] \arrow[ur, phantom, "\Sigma"] & Y \arrow[l, "t"] \arrow[u, equal] \end{tikzcd}$$

which shows that $(f, I, s) \leq (g, J, t)$, as desired.

Let now $(f_i, I_i, s_i) : X_i \to Y$ be a family of Σ-cospans indexed by I. Let

$$\begin{tikzcd} Y \arrow[r, "s_i"] \arrow[dr, "s"'] & I_i \arrow[d, "t_i"] \\ & I \end{tikzcd}$$

be the multipushout of the morphisms $s_i : Y \to I_i$ in $\mathcal{X}$. Then, by Theorem 5.1, arguing as for (27), we obtain the Σ-square $\begin{tikzcd} Y \arrow[r, "s_i"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & I_i \arrow[d, "t_i"] \\ Y \arrow[r, "s"'] & I \end{tikzcd}$. By the universality of the coproduct in $\mathcal{X}$, there is a unique morphism $w : \coprod X_i \to I$ in $\mathcal{X}$ with $wv_i = t_i f_i$, for all i. Then, composing Σ-cospans, we have: $(w, s) \circ (v_i, \mathrm{id}) \equiv (wv_i, s) = (t_i f_i, s) \equiv (f_i, s_i)$. Hence $[(w, s)]$ is a morphism of $\mathcal{X}[\Sigma_*]$ with $[(w, s)] \cdot [(v_i, \mathrm{id})] = [(f_i, s_i)]$. The uniqueness of $[(w, s)]$ follows from the fact already proved that the morphisms $[(v_i, \mathrm{id})]$ are collectively order-epic.

By the above description of the coproducts in $\mathcal{X}[\Sigma_*]$ it is clear that P_Σ preserves coproducts.

2. The fact that $\mathcal{X}[\Sigma_*]$ has binary coproducts is proved in a completely analogous way to 1. Just in the situations where we needed to construct a multipushout, we use now *Square*, and in the places where we needed coequalizers, we use *Coequalization*. It is easy to see that the initial object of $\mathcal{X}$ is also the initial object of $\mathcal{X}[\Sigma_*]$. ■

Remark 5.7. We leave as an open question the existence of coinserters in $\mathcal{X}[\Sigma_*]$ for $\Sigma = \mathcal{A}^{\underline{\mathsf{KInj}}}$, when $\mathcal{X}$ has weighted colimits.

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Lurdes Sousa
Centre for Mathematics of the University of Coimbra & Polytechnic Institute of Viseu, Portugal
E-mail address: sousa@estv.ipv.pt