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STABILITY AND CONVERGENCE OF A LEAP-FROG DISCONTINUOUS GALERKIN METHOD FOR TIME-DOMAIN MAXWELL'S EQUATIONS IN ANISOTROPIC MATERIALS

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ABSTRACT: In this work we discuss the numerical discretization of the time-dependent Maxwell's equations using a fully explicit leap-frog type discontinuous Galerkin (DG) method. We present a sufficient condition for the stability and we derive the convergence properties of the fully discrete method, for cases of typical boundary conditions (either perfect electric, perfect magnetic or first order Silver-Müller). In the model we consider heterogeneous anisotropic permittivity tensors which arise naturally in our application of interest [2]. Numerical results supporting the analysis are provided.

KEYWORDS: Maxwell's equations, leap-frog DG method, stability and convergence.

1. Introduction

The electromagnetic field consists of coupled electric and magnetic fields, known as electric field intensity, E, and magnetic induction, B. The effects of these two fundamental fields on matter, can be characterized by the electric displacement and the magnetic field intensity vectors, frequently denoted by D and H, respectively. The knowledge of the material properties can be used to derive a useful relation between D and E and between B and H. Here we will consider the constitutive relations of the form $D = \epsilon E$ and $B = \mu H$, where ϵ is the medium's electric permittivity and μ is the medium's magnetic permeability.

Maxwell's equations are a fundamental set of partial differential equations which describe electromagnetic wave interactions with materials. In three-dimensional spaces for heterogeneous anisotropic linear media with no source, these equations can be written in the form [9]

$$\epsilon \frac{\partial E}{\partial t} = \text{curl } H, \qquad \mu \frac{\partial H}{\partial t} = -\text{curl } E.$$

The computational modeling of linear light scattering is done via discrete solutions of Maxwell's equations. The finite-difference time-domain (FDTD)

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method, first introduced by Yee in 1966 [22], computes a discrete solution of Maxwell's curl equations in the time domain by applying centered finite difference operators on staggered grids in space and time for each electric and magnetic vector field component in the equations. This method has been applied to a wide range of electromagnetic problems. An application to cellular-level biophotonics was reported in [16], wherein visible light interactions with a retinal photoreceptors were modelled for the two-dimensional transverse magnetic (TM) and transverse electric (TE) polarisation cases. In [6], Dunn and Richards-Kortum pioneered the application of FDTD method to light scattering from cells. And many more examples can be found in literature [19].

Many recent papers on the simulation of electromagnetic waves propagation have shown interest in the use of discontinuous Galerkin time domain methods (DGTD) to solve Maxwell's equations and the advantages of using DGTD methods when compared with classical FDTD methods, finite volume time domain methods or finite element time domain methods, have been reported by several authors (see *e.g.* [9]). DGTD methods gather many desirable features such as being able to achieve high-order accuracy and easily handle complex geometries. Moreover, they are suitable for parallel implementation on modern multi-graphics processing units. Local refinement strategies can be incorporated due to the possibility of considering irregular meshes with hanging nodes and local spaces of different orders.

Despite the relevance of the anisotropic case, most of the formulation of the DGTD methods present in the literature are restricted to isotropic materials [11, 12, 15]. Motivated by our application of interest described in [2], in the present paper we consider a model with an heterogeneous anisotropic permittivity tensor. The treatment of anisotropic materials within a DGTD framework was discussed for instance in [9] (with central fluxes) and in [13] (with upwind fluxes). Here we combine the nodal DG method [12] (considering both central and upwind fluxes) for the integration in space with an explicit leap-frog type method for the time integration. We present a rigorous proof of stability and derive the error estimates. The bounds of the stability region are derived reveling the influence of not only the boundary conditions and the mesh size but also the dependence of the numerical flux and the polynomial degree used in the construction of the finite element space, making possible to balance accuracy and computational efficiency. Moreover a modified method is proposed to improve the order of convergence in time for certain type of boundary conditions. We illustrate the stability condition as well as the convergence order of the fully-discrete scheme, both in space and time, with numerical tests.

2. Electromagnetic waves in anisotropic materials

At the macroscopic scale, a dielectric material is optically isotropic if, at any given spatial location in it, its optical properties are the same for any direction [4]. Then at a given spatial location in that medium, there is only one dielectric permittivity (for a given frequency of light) and, hence, only one refractive index of light. Gas, liquid and amorphous solids constitute relevant optically isotropic dielectric materials. Various general and specific aspects of the propagation and scattering of the electromagnetic field in optically isotropic materials are well understood and well documented [20].

An optically anisotropic dielectric material is one in which the optical properties depend on the chosen direction (also, for a given frequency of light) [4]. The electric permittivity ϵ and magnetic permeability μ are in this case described by tensor functions of position. The electrical properties of atoms and molecules are typically anisotropic.

In a similar fashion to [22], we decompose the electromagnetic wave in a transverse electric (TE) mode and a transverse magnetic (TM) mode, this way reducing significantly the number of equations implemented in our model. Here we shall analyse the time domain Maxwell's equations in the transverse electric (TE) mode, as in [13], where the only non-vanishing components of the electromagnetic fields are E_x, E_y and H_z. For this case, and assuming no conductivity effects, the equations in the non-dimensional form are

$$\epsilon\frac{\partial E}{\partial t} = \nabla \times H \quad \text{in } \Omega \times (0, T_f] \tag{1}$$

$$\mu\frac{\partial H}{\partial t} = -\text{curl } E \quad \text{in } \Omega \times (0, T_f], \tag{2}$$

where $E = (E_x, E_y)$ and $H = (H_z)$. This equations are set and solved on the bounded polygonal domain $\Omega \subset \mathbb{R}^2$. Note that we use the following notation for the vector and scalar curl operators

$$\nabla \times H = \left(\frac{\partial H_z}{\partial y}, -\frac{\partial H_z}{\partial x}\right)^T, \quad \text{curl } E = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}.$$

The electric permittivity of the medium, ϵ and the magnetic permeability of the medium μ are varying in space, being ϵ an anisotropic tensor

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{pmatrix}, \tag{3}$$

while we consider isotropic permeability μ. We assume that electric permittivity tensor ϵ is symmetric and uniformly positive definite for almost every $(x, y) \in \Omega$, and it is uniformly bounded with a strictly positive lower bound, *i.e.*, there are constants $\underline{\epsilon} > 0$ and $\overline{\epsilon} > 0$ such that, for almost every $(x, y) \in \Omega$,

$$\underline{\epsilon}|\xi|^2 \leq \xi^T \epsilon(x, y)\xi \leq \overline{\epsilon}|\xi|^2, \qquad \forall \xi \in \mathbb{R}^2.$$

We also assume that there are constants $\underline{\mu} > 0$ and $\overline{\mu} > 0$ such that, for almost every $(x, y) \in \Omega$,

$$\underline{\mu} \leq \mu(x, y) \leq \overline{\mu}.$$

Let the unit outward normal vector to the boundary be denoted by n. We can define an effective permittivity ([13]) by

$$\epsilon_{eff} = \frac{\det(\epsilon)}{n^T \epsilon n},$$

that is used to characterize the speed with which a wave travels along the direction of the unit normal

$$c = \sqrt{\frac{n^T \epsilon n}{\mu \det(\epsilon)}}.$$

The model equations (1)–(2) must be complemented by proper boundary conditions. Here we consider the most common, either the perfect electric conductor boundary condition (PEC)

$$n \times E = 0 \qquad \text{on } \partial\Omega, \tag{4}$$

the perfect magnetic conductor boundary condition (PMC),

$$n \times H = 0 \qquad \text{on } \partial\Omega, \tag{5}$$

or the first order Silver-Müller absorbing boundary condition

$$n \times E = c\mu n \times (H \times n) \qquad \text{on } \partial\Omega. \tag{6}$$

Initial conditions

$$E(x, y, 0) = E_0(x, y) \quad \text{and} \quad H(x, y, 0) = H_0(x, y) \quad \text{in } \Omega,$$

must also be provided.

We can write Maxwell's equations (1)–(2) in a conservation form

$$Q\frac{\partial q}{\partial t} + \nabla \cdot F(q) = 0 \quad \text{in } \Omega \times (0, T_f], \tag{7}$$

with

$$Q = \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}, \quad q = \begin{pmatrix} E_x \\ E_y \\ H_z \end{pmatrix} \quad \text{and} \quad F(q) = \begin{pmatrix} 0 & H_z \\ -H_z & 0 \\ -E_y & E_x \end{pmatrix}^T,$$

where $\nabla\cdot$ denotes the divergence operator.

3. A leap-frog discontinuous Galerkin method

The aim of this section is to derive our computational method. We will consider a nodal discontinuous Galerkin method for the space discretization and a leap-frog method for the time integration.

3.1. The discontinuous Galerkin method. Assume that the computational domain Ω is partitioned into K triangular elements T_k such that $\overline{\Omega} = \cup_k T_k$. For simplicity, we consider that the resulting mesh $\mathcal{T}_h$ is conforming, that is, the intersection of two elements is either empty or an edge.

Let h_k be the diameter of the triangle $T_k \in \mathcal{T}_h$, and h be the maximum element diameter,

$$h_k = \sup_{P_1, P_2 \in T_K} \|P_1 - P_2\|, \quad h = \max_{T_k \in \mathcal{T}_h}\{h_k\}.$$

We assume that the mesh is regular in the sense that there is a constant $\tau > 0$ such that

$$\forall\, T_k \in \mathcal{T}_h, \quad \frac{h_k}{\tau_k} \le \tau, \tag{8}$$

where τ_k denotes the maximum diameter of a ball inscribed in T_k.

On each element T_k, the solution fields are approximated by polynomials of degree less than or equal to N. The global solution $q(x, y, t)$ is then assumed to be approximated by the piecewise N order polynomials

$$q(x, y, t) \simeq \tilde{q}(x, y, t) = \bigoplus_{k=1}^{K} \tilde{q}_k(x, y, t),$$

defined as the direct sum of the K local polynomial solutions $\tilde{q}_k(x,t) = (\tilde{E}_{x_k}, \tilde{E}_{y_k}, \tilde{H}_{z_k})$. We use the notation

$$\tilde{E}_x(x,y,t) = \bigoplus_{k=1}^{K} \tilde{E}_{x_k}(x,y,t), \quad \tilde{E}_y(x,y,t) = \bigoplus_{k=1}^{K} \tilde{E}_{y_k}(x,y,t),$$

$$\tilde{H}_z(x,y,t) = \bigoplus_{k=1}^{K} \tilde{H}_{z_k}(x,y,t).$$

The finite element space is then taken to be

$$V_N = \{v \in L^2(\Omega)^3 : v|_{T_k} \in P_N(T_k)^3\},$$

where $P_N(T_k)$ denotes the space of polynomials of degree less than or equal to N on T_k. The fields are expanded in terms of interpolating Lagrange polynomials $L_i(x,y)$,

$$\tilde{q}_k(x,y,t) = \sum_{i=1}^{N_p} \tilde{q}_k(x_i,y_i,t) L_i(x,y) = \sum_{i=1}^{N_p} \tilde{q}_{ki}(t) L_i(x,y).$$

Here N_p denotes the number of coefficients that are utilized, which is related with the polynomial order N via $N_p = (N+1)(N+2)/2$.

In order to deduce the method, we start by multiplying equation (7) by test functions $v \in V_N$, usually the Lagrange polynomials, and integrate over each element T_k. The next step is to employ one integration by parts and to substitute in the resulting contour integral the flux F by a numerical flux F^*. Reversing the integration by parts yields

$$\int_{T_k} \left(Q\frac{\partial \tilde{q}}{\partial t} + \nabla \cdot F(\tilde{q}) \right) v(x,y)\, dxdy = \int_{\partial T_k} n \cdot (F(\tilde{q}) - F^*(\tilde{q}))\, v(x,y)\, ds,$$

where n is the outward pointing unit normal vector of the contour.

The approximate fields are allowed to be discontinuous across element boundaries. In this way, we introduce the notation for the jumps of the field values across the interfaces of the elements, $[\tilde{E}] = \tilde{E}^- - \tilde{E}^+$ and $[\tilde{H}] = \tilde{H}^- - \tilde{H}^+$, where the superscript " $+$ " denotes the neighboring element and the superscript " $-$ " refers to the local cell. Furthermore we introduce, respectively, the cell-impedances and cell-conductances $Z^{\pm} = \mu^{\pm} c^{\pm}$

and $Y^{\pm} = (Z^{\pm})^{-1}$ where

$$c^{\pm} = \sqrt{\frac{n^T \epsilon^{\pm} n}{\mu^{\pm} \det(\epsilon^{\pm})}}.$$

At the outer cell boundaries we set $Z^+ = Z^-$.

The coupling between elements is introduced via numerical flux, defined by

$$n \cdot (F(\tilde{q}) - F^*(\tilde{q})) \;=\; \begin{pmatrix} \dfrac{-n_y}{Z^+ + Z^-}\left(Z^+[\tilde{H}_z] - \alpha\left(n_x[\tilde{E}_y] - n_y[\tilde{E}_x]\right)\right) \\ \dfrac{n_x}{Z^+ + Z^-}\left(Z^+[\tilde{H}_z] - \alpha\left(n_x[\tilde{E}_y] - n_y[\tilde{E}_x]\right)\right) \\ \dfrac{1}{Y^+ + Y^-}\left(Y^+\left(n_x[\tilde{E}_y] - n_y[\tilde{E}_x]\right) - \alpha[\tilde{H}_z]\right) \end{pmatrix}.$$

The parameter $\alpha \in [0,1]$ in the numerical flux can be used to control dissipation. Taking $\alpha = 0$ yields a non dissipative central flux while $\alpha = 1$ corresponds to the classic upwind flux.

In order to discretize the boundary conditions we set $[\tilde{E}_x] = 2\tilde{E}_x^-$, $[\tilde{E}_y] = 2\tilde{E}_y^-$, $[\tilde{H}_z] = 0$ and $[\tilde{E}_x] = 0$, $[\tilde{E}_y] = 0$, $[\tilde{H}_z] = 2\tilde{H}_z^-$, for PEC and PMC boundary conditions, respectively. For Silver-Müller absorbing boundary conditions, using the same kind of approach as in [1], we consider, for upwind fluxes $Z^-\tilde{H}_z^+ = n_x\tilde{E}_y^+ - n_y\tilde{E}_x^+$ or equivalently $\tilde{H}_z^+ = Y^-(n_x\tilde{E}_y^+ - n_y\tilde{E}_x^+)$ and, for central fluxes

$$Z^-\tilde{H}_z^+ = (n_x\tilde{E}_y^- - n_y\tilde{E}_x^-)$$

and

$$Y^-(n_x\tilde{E}_y^+ - n_y\tilde{E}_x^+) = \tilde{H}_z^-.$$

This is equivalent to consider, for both upwind and central fluxes, $\alpha = 1$ for numerical flux at the outer boundary and $[\tilde{E}_x] = \tilde{E}_x^-$, $[\tilde{E}_y] = \tilde{E}_y^-$ and $[\tilde{H}_z] = \tilde{H}_z^-$.

3.2. Time discretization. To define a fully discrete scheme, we divide the time interval $[0,T]$ into M subintervals by points $0 = t^0 < t^1 < \cdots < t^M = T$, where $t^m = m\Delta t$, Δt is the time step size and $T + \Delta t/2 \leq T_f$. The unknowns related to the electric field are approximated at integer time-stations t^m and are denoted by $\tilde{E}_k^m = \tilde{E}_k(.,t^m)$. The unknowns related to the magnetic field are approximated at half-integer time-stations $t^{m+1/2} = (m+\frac{1}{2})\Delta t$ and are denoted by $\tilde{H}_k^{m+1/2} = \tilde{H}_k(.,t^{m+1/2})$. With the above setting, we can now formulate the leap-frog DG method: given an initial approximation $(\tilde{E}^0_{x_k}, \tilde{E}^0_{y_k}, \tilde{H}^{1/2}_{z_k}) \in V_N$, for each $m = 0, 1, \ldots, M-1$, find $(\tilde{E}^{m+1}_{x_k}, \tilde{E}^{m+1}_{y_k}, \tilde{H}^{m+1/2}_{z_k}) \in V_N$ such that, $\forall (u_k, v_k, w_k) \in V_N$,

$$\left(\epsilon_{xx}\frac{\tilde{E}^{m+1}_{x_k} - \tilde{E}^m_{x_k}}{\Delta t} + \epsilon_{xy}\frac{\tilde{E}^{m+1}_{y_k} - \tilde{E}^m_{y_k}}{\Delta t}, u_k\right)_{T_k} = \left(\partial_y \tilde{H}^{m+1/2}_{z_k}, u_k\right)_{T_k} + \left(\frac{-n_y}{Z^+ + Z^-}\left(Z^+[\tilde{H}^{m+1/2}_z] - \alpha\left(n_x[\tilde{E}^m_y] - n_y[\tilde{E}^m_x]\right)\right), u_k\right)_{\partial T_k}, \quad (9)$$

$$\left(\epsilon_{yx}\frac{\tilde{E}^{m+1}_{x_k} - \tilde{E}^m_{x_k}}{\Delta t} + \epsilon_{yy}\frac{\tilde{E}^{m+1}_{y_k} - \tilde{E}^m_{y_k}}{\Delta t}, v_k\right)_{T_k} = -\left(\partial_x \tilde{H}^{m+1/2}_{z_k}, v_k\right)_{T_k} + \left(\frac{n_x}{Z^+ + Z^-}\left(Z^+[\tilde{H}^{m+1/2}_z] - \alpha\left(n_x[\tilde{E}^m_y] - n_y[\tilde{E}^m_x]\right)\right), v_k\right)_{\partial T_k}, \quad (10)$$

$$\left(\mu\frac{\tilde{H}^{m+3/2}_{z_k} - \tilde{H}^{m+1/2}_{z_k}}{\Delta t}, w_k\right)_{T_k} = \left(\partial_y \tilde{E}^{m+1}_{x_k} - \partial_x \tilde{E}^{m+1}_{y_k}, w_k\right)_{T_k} + \left(\frac{1}{Y^+ + Y^-}\left(Y^+(n_x[\tilde{E}^{m+1}_y] - n_y[\tilde{E}^{m+1}_x]) - \alpha[\tilde{H}^{m+1/2}_z]\right), w_k\right)_{\partial T_k}, \quad (11)$$

where $(\cdot,\cdot)_{T_k}$ and $(\cdot,\cdot)_{\partial T_k}$ denote the classical $L^2(T_k)$ and $L^2(\partial T_k)$ inner-products. The boundary conditions are considered as described in the previous section.

We want to emphasize that the scheme (9)–(11) is fully explicit in time, in opposition to [14], where the scheme is defined with the upwind fluxes involving the unknowns E^{m+1}_k and $H^{m+3/2}_k$ and to [9], where the scheme that is defined with the central fluxes leads to a locally implicit time method in the case of Silver-Müller absorbing boundary conditions.

4. Stability Analysis

The aim of this section is to provide a sufficient condition for the L^2-stability of the leap-frog DG method (9)–(11).

Choosing

$$u_k = \Delta t \tilde{E}^{[m+1/2]}_{x_k}, v_k = \Delta t \tilde{E}^{[m+1/2]}_{y_k} \text{ and } w_k = \Delta t \tilde{H}^{[m+1]}_{z_k},$$

where

$$\tilde{E}^{[m+1/2]} = \left(\tilde{E}^m + \tilde{E}^{m+1}\right)/2 \text{ and } \tilde{H}^{[m+1]} = \left(\tilde{H}^{m+1/2} + \tilde{H}^{m+3/2}\right)/2,$$

we have

$$\begin{aligned}\left(\epsilon \tilde{E}_k^{m+1}, \tilde{E}_k^{m+1}\right)_{T_k} - \left(\epsilon \tilde{E}_k^{m}, \tilde{E}_k^{m}\right)_{T_k} &= 2\Delta t \left(\nabla \times \tilde{H}_{z_k}^{m+1/2}, \tilde{E}_k^{[m+1/2]}\right)_{T_k} \\ &+2\Delta t \left(\frac{-n_y}{Z^+ + Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}] - \alpha\left(n_x[\tilde{E}_y^m] - n_y[\tilde{E}_x^m]\right)\right), \tilde{E}_{x_k}^{[m+1/2]}\right)_{\partial T_k} \\ &+2\Delta t \left(\frac{n_x}{Z^+ + Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}] - \alpha\left(n_x[\tilde{E}_y^m] - n_y[\tilde{E}_x^m]\right)\right), \tilde{E}_{y_k}^{[m+1/2]}\right)_{\partial T_k},\end{aligned} \tag{12}$$

$$\begin{aligned}\left(\mu \tilde{H}_{z_k}^{m+3/2}, \tilde{H}_{z_k}^{m+3/2}\right)_{T_k} - \left(\mu \tilde{H}_{z_k}^{m+1/2}, \tilde{H}_{z_k}^{m+1/2}\right)_{T_k} &= -2\Delta t \left(\text{curl } \tilde{E}_k^{m+1}, \tilde{H}_{z_k}^{[m+1]}\right)_{T_k} \\ &+2\Delta t \left(\frac{1}{Y^+ + Y^-}\left(Y^+\left(n_x[\tilde{E}_y^{m+1}] - n_y[\tilde{E}_x^{m+1}]\right) - \alpha[\tilde{H}_z^{m+1/2}]\right), \tilde{H}_{z_k}^{[m+1]}\right)_{\partial T_k}.\end{aligned} \tag{13}$$

Using the identity,

$$\begin{aligned}\left(\text{curl } \tilde{E}_k^{m+1}, \tilde{H}_{z_k}^{[m+1]}\right)_{T_k} &= \left(\nabla \times \tilde{H}_{z_k}^{[m+1]}, \tilde{E}_k^{m+1}\right)_{T_k} \\ &+ \left(n_x \tilde{E}_{y_k}^{m+1} - n_y \tilde{E}_{x_k}^{m+1}, \tilde{H}_{z_k}^{[m+1]}\right)_{\partial T_k},\end{aligned}$$

summing (12) and (13) from $m = 0$ to $m = M - 1$, and integrating by parts, we get

$$\begin{aligned}\left(\epsilon \tilde{E}_k^{M}, \tilde{E}_k^{M}\right)_{T_k} + \left(\mu \tilde{H}_{z_k}^{M+1/2}, \tilde{H}_{z_k}^{M+1/2}\right)_{T_k} &= \left(\epsilon \tilde{E}_k^{0}, \tilde{E}_k^{0}\right)_{T_k} + \left(\mu \tilde{H}_{z_k}^{1/2}, \tilde{H}_{z_k}^{1/2}\right)_{T_k} \\ +\Delta t \left(\nabla \times \tilde{H}_{z_k}^{1/2}, \tilde{E}_k^0\right)_{T_k} - \Delta t \left(\nabla \times \tilde{H}_{z_k}^{M+1/2}, \tilde{E}_k^M\right)_{T_k} &+ 2\Delta t \sum_{m=0}^{M-1} A_k^m,\end{aligned} \tag{14}$$

where

$$
\begin{aligned}
A_k^m = & \left(\frac{-n_y}{Z^+ + Z^-} \left(Z^+[\tilde{H}_z^{m+1/2}] - \alpha \left(n_x[\tilde{E}_y^m] - n_y[\tilde{E}_x^m] \right) \right), \tilde{E}_{x_k}^{[m+1/2]} \right)_{\partial T_k} \\
& + \left(\frac{n_x}{Z^+ + Z^-} \left(Z^+[\tilde{H}_z^{m+1/2}] - \alpha \left(n_x[\tilde{E}_y^m] - n_y[\tilde{E}_x^m] \right) \right), \tilde{E}_{y_k}^{[m+1/2]} \right)_{\partial T_k} \\
& + \left(\frac{1}{Y^+ + Y^-} \left(Y^+ \left(n_x[\tilde{E}_y^{m+1}] - n_y[\tilde{E}_x^{m+1}] \right) - \alpha[\tilde{H}_z^{m+1/2}] \right), \tilde{H}_{z_k}^{[m+1]} \right)_{\partial T_k} \\
& - \left(n_x \tilde{E}_{y_k}^{m+1} - n_y \tilde{E}_{x_k}^{m+1}, \tilde{H}_{z_k}^{[m+1]} \right)_{\partial T_k}.
\end{aligned}
$$

Let us denote by F^{int} the set of internal edges and F^{ext} the set of edges that belong to the boundary $\partial\Omega$. Let ν_k be the set of indices of the neighboring elements of T_k. For each $i \in \nu_k$, we consider the internal edge $f_{ik} = T_i \cap T_k$, and we denote by n_{ik} the unit normal oriented from T_i towards T_k. For each boundary edge $f_k = T_k \cap \partial\Omega$, n_k is taken to be the unitary outer normal vector to f_k. Summing over all elements $T_k \in \mathcal{T}_h$ we obtain

$$
\sum_{T_k \in \mathcal{T}_h} A_k^m = B_1^m + B_2^m + B_3^m + B_4^m,
$$

where

$$
\begin{aligned}
B_1^m = & \sum_{f_{ik} \in F^{int}} \int_{f_{ik}} \left(\frac{-(n_y)_{ki}}{Z_i + Z_k} \left(Z_i[\tilde{H}_{z_k}^{m+1/2}] - \alpha \left((n_x)_{ki}[\tilde{E}_{y_k}^m] - (n_y)_{ki}[\tilde{E}_{x_k}^m] \right) \right) \tilde{E}_{x_k}^{[m+1/2]} \right. \\
& + \frac{-(n_y)_{ik}}{Z_i + Z_k} \left(Z_k[\tilde{H}_{zi}^{m+1/2}] - \alpha \left((n_x)_{ik}[\tilde{E}_{yi}^m] - (n_y)_{ik}[\tilde{E}_{xi}^m] \right) \right) \tilde{E}_{xi}^{[m+1/2]} \\
& - \frac{Y_i (n_y)_{ki}}{Y_i + Y_k} [\tilde{E}_{x_k}^{m+1}] \tilde{H}_{z_k}^{[m+1]} - \frac{Y_k (n_y)_{ik}}{Y_i + Y_k} [\tilde{E}_{xi}^{m+1}] \tilde{H}_{zi}^{[m+1]} \\
& \left. + (n_y)_{ki} \tilde{E}_{x_k}^{m+1} \tilde{H}_{z_k}^{[m+1]} + (n_y)_{ik} \tilde{E}_{xi}^{m+1} \tilde{H}_{zi}^{[m+1]} \right) ds, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
B_2^m \;=\; \sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \Bigg(& \frac{(n_x)_{ki}}{Z_i+Z_k}\left(Z_i[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^m] - (n_y)_{ki}[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{y_k}^{[m+1/2]} \\
& + \frac{(n_x)_{ik}}{Z_i+Z_k}\left(Z_k[\tilde{H}_{zi}^{m+1/2}] - \alpha\left((n_x)_{ik}[\tilde{E}_{yi}^m] - (n_y)_{ik}[\tilde{E}_{xi}^m]\right)\right)\tilde{E}_{yi}^{[m+1/2]} \\
& + \frac{Y_i(n_x)_{ki}}{Y_i+Y_k}[\tilde{E}_{y_k}^{m+1}]\tilde{H}_{z_k}^{[m+1]} + \frac{Y_k(n_x)_{ik}}{Y_i+Y_k}[\tilde{E}_{yi}^{m+1}]\tilde{H}_{zi}^{[m+1]} \\
& - (n_x)_{ki}\tilde{E}_{y_k}^{m+1}\tilde{H}_{z_k}^{[m+1]} - (n_x)_{ik}\tilde{E}_{yi}^{m+1}\tilde{H}_{zi}^{[m+1]}\Bigg)\, ds, \qquad (16)
\end{aligned}
$$

$$
B_3^m = -\sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \left(\frac{\alpha}{Y_i+Y_k}[\tilde{H}_{z_k}^{m+1/2}]\tilde{H}_{z_k}^{[m+1]} + \frac{\alpha}{Y_i+Y_k}[\tilde{H}_{zi}^{m+1/2}]\tilde{H}_{zi}^{[m+1]} \right) ds \qquad (17)
$$

and B_4^m has the terms related with the outer boundary

$$
\begin{aligned}
B_4^m \;=\; \sum_{f_k\in F^{ext}} \int_{f_k} \Bigg(& \frac{-(n_y)_k}{2Z_k}\left(Z_k[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_k[\tilde{E}_{y_k}^m] - (n_y)_k[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{x_k}^{[m+1/2]} \\
& + \frac{(n_x)_k}{2Z_k}\left(Z_k[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_k[\tilde{E}_{y_k}^m] - (n_y)_k[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{y_k}^{[m+1/2]} \\
& + \frac{1}{2Y_k}\left(Y_k\left((n_x)_k[\tilde{E}_{y_k}^{m+1}] - (n_y)_k[\tilde{E}_{x_k}^{m+1}]\right) - \alpha[\tilde{H}_{z_k}^{m+1/2}]\right)\tilde{H}_{z_k}^{[m+1]} \\
& - \left((n_x)_k\tilde{E}_{y_k}^{m+1} - (n_y)_k\tilde{E}_{x_k}^{m+1}\right)\tilde{H}_{z_k}^{[m+1]}\Bigg)\, ds. \qquad (18)
\end{aligned}
$$

Lemma 1. *Let B_1^m, B_2^m and B_3^m be defined by (15), (16) and (17), respectively. Then*

$$\begin{aligned}\sum_{m=0}^{M-1}(B_1^m+B_2^m) \;\le\; & \sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{1}{4(Z_i+Z_k)}\Bigg(-\alpha\left((n_y)_{ki}[\tilde{E}^0_{x_k}]-(n_x)_{ki}[\tilde{E}^0_{y_k}]\right)^2\\
&-2(n_y)_{ki}\left(Z_i\tilde{E}^0_{x_k}+Z_k\tilde{E}^0_{xi}\right)[\tilde{H}^{1/2}_{z_k}]\\
&+2(n_x)_{ki}\left(Z_i\tilde{E}^0_{y_k}+Z_k\tilde{E}^0_{yi}\right)[\tilde{H}^{1/2}_{z_k}]\\
&+\alpha\left((n_y)_{ki}[\tilde{E}^M_{x_k}]-(n_x)_{ki}[\tilde{E}^M_{y_k}]\right)^2\\
&+2(n_y)_{ki}\left(Z_i\tilde{E}^M_{x_k}+Z_k\tilde{E}^M_{xi}\right)[\tilde{H}^{M+1/2}_{z_k}]\\
&-2(n_x)_{ki}\left(Z_i\tilde{E}^M_{y_k}+Z_k\tilde{E}^M_{yi}\right)[\tilde{H}^{M+1/2}_{z_k}]\Bigg)\,ds\end{aligned}$$

and

$$\sum_{m=0}^{M-1}B_3^m \;\le\; -\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{\alpha}{4(Y_i+Y_k)}\left([\tilde{H}^{1/2}_{z_k}]^2-[\tilde{H}^{M+1/2}_{z_k}]^2\right)\,ds.$$

Proof: Since

$$\frac{Z_i}{Z_i+Z_k}+\frac{Y_i}{Y_i+Y_k}=\frac{Z_k}{Z_i+Z_k}+\frac{Y_k}{Y_i+Y_k}=1$$

and

$$\frac{Z_i}{Z_i+Z_k}=\frac{Y_k}{Y_i+Y_k},\qquad \frac{Z_k}{Z_i+Z_k}=\frac{Y_i}{Y_i+Y_k},$$

we have, for B_1^m,

$$\begin{aligned}
B_1^m \;=\; & \frac{1}{2}\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\left(\frac{-(n_y)_{ki}}{Z_i+Z_k}\left(Z_i[\tilde{H}_{z_k}^{m+1/2}]-\alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^m]-(n_y)_{ki}[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{x_k}^m\right.\\
&+\frac{-(n_y)_{ki}}{Z_i+Z_k}\left(-\alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^m]-(n_y)_{ki}[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{x_k}^{m+1}\\
&+\frac{-(n_y)_{ik}}{Z_i+Z_k}\left(Z_k[\tilde{H}_{zi}^{m+1/2}]-\alpha\left((n_x)_{ik}[\tilde{E}_{yi}^m]-(n_y)_{ik}[\tilde{E}_{xi}^m]\right)\right)\tilde{E}_{xi}^m\\
&+\frac{-(n_y)_{ik}}{Z_i+Z_k}\left(-\alpha\left((n_x)_{ik}[\tilde{E}_{yi}^m]-(n_y)_{ik}[\tilde{E}_{xi}^m]\right)\right)\tilde{E}_{xi}^{m+1}\\
&-\frac{Y_i(n_y)_{ki}}{Y_i+Y_k}[\tilde{E}_{x_k}^{m+1}]\tilde{H}_{z_k}^{m+3/2}-\frac{Y_k(n_y)_{ik}}{Y_i+Y_k}[\tilde{E}_{xi}^{m+1}]\tilde{H}_{zi}^{m+3/2}\\
&\left.+(n_y)_{ki}\tilde{E}_{x_k}^{m+1}\tilde{H}_{z_k}^{m+3/2}+(n_y)_{ik}\tilde{E}_{xi}^{m+1}\tilde{H}_{zi}^{m+3/2}\right)ds.
\end{aligned}$$

Summing from $m=0$ to $m=M-1$ we conclude that

$$\begin{aligned}
\sum_{m=0}^{M-1}B_1^m \;=\; & \sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{(n_y)_{ki}}{2(Z_i+Z_k)}\left(-\left(Z_i\tilde{E}_{x_k}^0+Z_k\tilde{E}_{xi}^0\right)[\tilde{H}_{z_k}^{1/2}]\right.\\
&+\alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^0]-(n_y)_{ki}[\tilde{E}_{x_k}^0]\right)[\tilde{E}_{x_k}^0]\\
&+\alpha\sum_{m=0}^{M-1}\left((n_x)_{ki}[\tilde{E}_{y_k}^{m+1}]-(n_y)_{ki}[\tilde{E}_{x_k}^{m+1}]\right.\\
&\qquad\left.+(n_x)_{ki}[\tilde{E}_{y_k}^m]-(n_y)_{ki}[\tilde{E}_{x_k}^m]\right)[\tilde{E}_{x_k}^{m+1}]\\
&+\left(Z_i\tilde{E}_{x_k}^M+Z_k\tilde{E}_{xi}^M\right)[\tilde{H}_{z_k}^{M+1/2}]\\
&\left.-\alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^M]-(n_y)_{ki}[\tilde{E}_{x_k}^M]\right)[\tilde{E}_{x_k}^M]\right)ds.
\end{aligned}$$

In the same way, for B_2^m we have

$$\begin{aligned}
\sum_{m=0}^{M-1} B_2^m &= \sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \frac{(n_x)_{ki}}{2(Z_i+Z_k)} \Bigg(\left(Z_i \tilde{E}^0_{y_k} + Z_k \tilde{E}^0_{yi}\right) [\tilde{H}^{1/2}_{z_k}] \\
&\quad - \left((n_x)_{ki}[\tilde{E}^0_{y_k}] - (n_y)_{ki}[\tilde{E}^0_{x_k}]\right)[\tilde{E}^0_{y_k}] \\
&\quad -\alpha \sum_{m=0}^{M-1} \Big((n_x)_{ki}[\tilde{E}^{m+1}_{y_k}] - (n_y)_{ki}[\tilde{E}^{m+1}_{x_k}] \\
&\qquad + (n_x)_{ki}[\tilde{E}^{m}_{y_k}] - (n_y)_{ki}[\tilde{E}^{m}_{x_k}]\Big)[\tilde{E}^{m+1}_{y_k}] \\
&\quad - \left(Z_i \tilde{E}^M_{y_k} + Z_k \tilde{E}^M_{yi}\right)[\tilde{H}^{M+1/2}_{z_k}] \\
&\quad +\alpha \left((n_x)_{ki}[\tilde{E}^M_{y_k}] - (n_y)_{ki}[\tilde{E}^M_{x_k}]\right)[\tilde{E}^M_{y_k}]\Bigg)\, ds,
\end{aligned}$$

and for B_3^m

$$\sum_{m=0}^{M-1} B_3^m = -\sum_{m=0}^{M-1} \sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \frac{\alpha}{2(Y_i+Y_k)} [\tilde{H}^{m+1/2}_{z_k}] \left([\tilde{H}^{m+1/2}_{z_k}] + [\tilde{H}^{m+3/2}_{z_k}]\right)\, ds.$$

Observing that, for general sequences $\{a^m\}$ and $\{b^m\}$, hold

$$\sum_{m=0}^{M-1} \left(a^{m+1}+a^m\right) a^{m+1} = \frac{1}{2}\left(-(a^0)^2 + (a^M)^2 + \sum_{m=0}^{M-1}\left(a^m + a^{m+1}\right)^2\right),$$

$$\begin{aligned}
\sum_{m=0}^{M-1} \left(a^{m+1}+a^m\right) b^{m+1} &= \frac{1}{2}\Bigg(-a^0b^0 + a^Mb^M \\
&\qquad + \sum_{m=0}^{M-1}\left(a^mb^m + 2a^mb^{m+1} + a^{m+1}b^{m+1}\right)\Bigg),
\end{aligned}$$

we get

$$
\begin{aligned}
\sum_{m=0}^{M-1} (B_1^m + B_2^m) \;\leq\; & \sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \frac{1}{4(Z_i+Z_k)} \Bigg(-\alpha(n_y)_{ki}^2 \left(-[\tilde{E}_{x_k}^0]^2 + [\tilde{E}_{x_k}^M]^2\right) \\
& +\alpha(n_x)_{ki}(n_y)_{ki} \left(-[\tilde{E}_{x_k}^0][\tilde{E}_{y_k}^0] + [\tilde{E}_{x_k}^M][\tilde{E}_{y_k}^M]\right) \\
& -2(n_y)_{ki} \left(Z_i\tilde{E}_{x_k}^0 + Z_k\tilde{E}_{xi}^0\right)[\tilde{H}_{z_k}^{1/2}] \\
& +2\alpha(n_y)_{ki} \left((n_x)_{ki}[\tilde{E}_{y_k}^0] - (n_y)_{ki}[\tilde{E}_{x_k}^0]\right)[\tilde{E}_{x_k}^0] \\
& +2(n_y)_{ki} \left(Z_i\tilde{E}_{x_k}^M + Z_k\tilde{E}_{xi}^M\right)[\tilde{H}_{z_k}^{M+1/2}] \\
& -2\alpha(n_y)_{ki} \left((n_x)_{ki}[\tilde{E}_{y_k}^M] - (n_y)_{ki}[\tilde{E}_{x_k}^M]\right)[\tilde{E}_{x_k}^M] \\
& -\alpha(n_x)_{ki}^2 \left(-[\tilde{E}_{y_k}^0]^2 + [\tilde{E}_{y_k}^M]^2\right) \\
& +\alpha(n_x)_{ki}(n_y)_{ki} \left(-[\tilde{E}_{x_k}^0][\tilde{E}_{y_k}^0] + [\tilde{E}_{x_k}^M][\tilde{E}_{y_k}^M]\right) \\
& +2(n_x)_{ki} \left(Z_i\tilde{E}_{y_k}^0 + Z_k\tilde{E}_{yi}^0\right)[\tilde{H}_{z_k}^{1/2}] \\
& -2\alpha(n_x)_{ki} \left((n_x)_{ki}[\tilde{E}_{y_k}^0] - (n_y)_{ki}[\tilde{E}_{x_k}^0]\right)[\tilde{E}_{y_k}^0] \\
& -2(n_x)_{ki} \left(Z_i\tilde{E}_{y_k}^M + Z_k\tilde{E}_{yi}^M\right)[\tilde{H}_{z_k}^{M+1/2}] \\
& +2\alpha(n_x)_{ki} \left((n_x)_{ki}[\tilde{E}_{y_k}^M] - (n_y)_{ki}[\tilde{E}_{x_k}^M]\right)[\tilde{E}_{y_k}^M] \Bigg)\, ds.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\sum_{m=0}^{M-1} B_3^m \;=\; & -\sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \frac{\alpha}{4(Y_i+Y_k)} \Bigg([\tilde{H}_{z_k}^{1/2}]^2 - [\tilde{H}_{z_k}^{M+1/2}]^2 \\
& + \sum_{m=0}^{M-1} \left([\tilde{H}_{z_k}^{m+1/2}] + [\tilde{H}_{z_k}^{m+3/2}]\right)^2 \Bigg)\, ds \\
\leq\; & -\sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \frac{\alpha}{4(Y_i+Y_k)} \left([\tilde{H}_{z_k}^{1/2}]^2 - [\tilde{H}_{z_k}^{M+1/2}]^2\right) ds,
\end{aligned}
$$

which concludes the proof. ■

Let us now analyze the term B_4^m for different kinds of boundary conditions.

Lemma 2. *Let B_4^m be defined by (2). Then*

$$\begin{aligned}\sum_{m=0}^{M-1} B_4^m \;\leq\; & \sum_{f_k\in F^{ext}} \int_{f_k} \frac{\beta_1}{4Z_k}\Bigg(-\Big((n_y)_k\tilde{E}^0_{x_k}-(n_x)_k\tilde{E}^0_{y_k}\Big)^2 \\ &\qquad + \Big((n_y)_k\tilde{E}^M_{x_k}-(n_x)_k\tilde{E}^M_{y_k}\Big)^2\Bigg) \\ &\quad +\frac{\beta_2}{2}\Bigg(\tilde{H}^{1/2}_{z_k}\left((n_x)_k\tilde{E}^0_{y_k}-(n_y)_k\tilde{E}^0_{x_k}-\frac{\beta_3}{2Y_k}\tilde{H}^{1/2}_{z_k}\right) \\ &\qquad -\tilde{H}^{M+1/2}_{z_k}\left((n_x)_k\tilde{E}^M_{y_k}-(n_y)_k\tilde{E}^M_{x_k}-\frac{\beta_3}{2Y_k}\tilde{H}^{M+1/2}_{z_k}\right)\Bigg)\,ds,\end{aligned}$$

where $\beta_1=\alpha, \beta_2=0$ for PEC, $\beta_1=0, \beta_2=1$, $\beta_3=\alpha$ for PMC, and $\beta_1=\beta_2=\frac{1}{2}$, $\beta_3=1$ for Silver-Müller boundary conditions.

Proof: First we consider PEC boundary conditions. We have

$$\begin{aligned}B_4^m \;=\; & \sum_{f_k\in F^{ext}} \int_{f_k} \frac{\alpha}{Z_k}\Bigg((n_y)_k\left((n_x)_k\tilde{E}^m_{y_k}-(n_y)_k\tilde{E}^m_{x_k}\right)\tilde{E}^{[m+1/2]}_{x_k} \\ &\qquad -(n_x)_k\left((n_x)_k\tilde{E}^m_{y_k}-(n_y)_k\tilde{E}^m_{x_k}\right)\tilde{E}^{[m+1/2]}_{y_k}\Bigg)\,ds.\end{aligned}$$

Summing from $m = 0$ to $m = M - 1$ we obtain

$$
\begin{aligned}
\sum_{m=0}^{M-1} B_4^m &= \sum_{f_k \in F^{ext}} \int_{f_k} \frac{\alpha}{4Z_k} \Bigg(-\left((n_x)_k \tilde{E}_{y_k}^0 - (n_y)_k \tilde{E}_{x_k}^0 \right)^2 \\
&\qquad + \left((n_x)_k \tilde{E}_{y_k}^M - (n_y)_k \tilde{E}_{x_k}^M \right)^2 \\
&\qquad - 4 \sum_{m=0}^{M-1} \left((n_x)_k \tilde{E}_{y_k}^{[m+1/2]} - (n_y)_k \tilde{E}_{x_k}^{[m+1/2]} \right)^2 \Bigg) ds \\
&\leq \sum_{f_k \in F^{ext}} \int_{f_k} \frac{\alpha}{4Z_k} \Bigg(-\left((n_x)_k \tilde{E}_{y_k}^0 - (n_y)_k \tilde{E}_{x_k}^0 \right)^2 \\
&\qquad + \left((n_x)_k \tilde{E}_{y_k}^M - (n_y)_k \tilde{E}_{x_k}^M \right)^2 \Bigg) ds.
\end{aligned}
$$

For PMC boundary conditions we have

$$
\begin{aligned}
B_4^m &= \sum_{f_k \in F^{ext}} \int_{f_k} \Bigg(\tilde{H}_{z_k}^{m+1/2} \left((n_x)_k \tilde{E}_{y_k}^{[m+1/2]} - (n_y)_k \tilde{E}_{x_k}^{[m+1/2]} \right) \\
&\qquad - \left(\frac{\alpha}{Y_k} \tilde{H}_{z_k}^{m+1/2} + (n_x)_k \tilde{E}_{y_k}^{m+1} - (n_y)_k \tilde{E}_{x_k}^{m+1} \right) \tilde{H}_{z_k}^{[m+1]} \Bigg) ds.
\end{aligned}
$$

Summing from $m = 0$ to $m = M - 1$ results

$$
\begin{aligned}
\sum_{m=0}^{M-1} B_4^m \;=\;& \sum_{f_k\in F^{ext}} \int_{f_k} \left(\frac{\tilde{H}_{z_k}^{1/2}}{2} \left((n_x)_k \tilde{E}_{y_k}^0 - (n_y)_k \tilde{E}_{x_k}^0 \right) - \frac{\alpha}{4Y_k} \left(\tilde{H}_{z_k}^{1/2} \right)^2 \right. \\
& - \sum_{m=0}^{M-1} \frac{\alpha}{Y_k} \left(\tilde{H}_{z_k}^{[m+1]} \right)^2 \\
& \left. - \frac{\tilde{H}_{z_k}^{M+1/2}}{2} \left((n_x)_k \tilde{E}_{y_k}^M - (n_y)_k \tilde{E}_{x_k}^M \right) - \frac{\alpha}{4Y_k} \left(\tilde{H}_{z_k}^{M+1/2} \right)^2 \right) ds \\
\leq\;& \frac{1}{2} \sum_{f_k\in F^{ext}} \int_{f_k} \left(\tilde{H}_{z_k}^{1/2} \left((n_x)_k \tilde{E}_{y_k}^0 - (n_y)_k \tilde{E}_{x_k}^0 - \frac{\alpha}{2Y_k} \tilde{H}_{z_k}^{1/2} \right) \right. \\
& \left. - \tilde{H}_{z_k}^{M+1/2} \left((n_x)_k \tilde{E}_{y_k}^M - (n_y)_k \tilde{E}_{x_k}^M - \frac{\alpha}{2Y_k} \tilde{H}_{z_k}^{M+1/2} \right) \right) ds.
\end{aligned}
$$

For Silver-Müller absorbing boundary conditions we have

$$
\begin{aligned}
B_4^m \;=\;& \frac{1}{2} \sum_{f_k\in F^{ext}} \int_{f_k} \left(\left(-(n_y)_k \tilde{H}_{z_k}^{m+1/2} + \frac{(n_y)_k}{Z_k} \left((n_x)_k \tilde{E}_{y_k}^m - (n_y)_k \tilde{E}_{x_k}^m \right) \right) \tilde{E}_{x_k}^{[m+1/2]} \right. \\
& + \left((n_x)_k \tilde{H}_{z_k}^{m+1/2} - \frac{(n_x)_k}{Z_k} \left((n_x)_k \tilde{E}_{y_k}^m - (n_y)_k \tilde{E}_{x_k}^m \right) \right) \tilde{E}_{y_k}^{[m+1/2]} \\
& \left. - \left(\frac{1}{Y_k} \tilde{H}_{z_k}^{m+1/2} + (n_x)_k \tilde{E}_{y_k}^{m+1} - (n_y)_k \tilde{E}_{x_k}^{m+1} \right) \tilde{H}_{z_k}^{[m+1]} \right) ds.
\end{aligned}
$$

Summing from $m = 0$ to $m = M - 1$, and taking into account the previous cases, we deduce that

$$\sum_{m=0}^{M-1} B_4^m \;\leq\; \sum_{f_k \in F^{ext}} \int_{f_k} \frac{1}{8Z_k}\Bigg(- \Big((n_y)_k \tilde{E}^0_{x_k} - (n_x)_k \tilde{E}^0_{y_k}\Big)^2 + \Big((n_y)_k \tilde{E}^M_{x_k} - (n_x)_k \tilde{E}^M_{y_k}\Big)^2 \Bigg)$$
$$+\frac{1}{4}\Bigg(\tilde{H}^{1/2}_{z_k}\Big((n_x)_k \tilde{E}^0_{y_k} - (n_y)_k \tilde{E}^0_{x_k} - \frac{1}{2Y_k}\tilde{H}^{1/2}_{z_k}\Big) - \tilde{H}^{M+1/2}_{z_k}\Big((n_x)_k \tilde{E}^M_{y_k} - (n_y)_k \tilde{E}^M_{x_k} - \frac{1}{2Y_k}\tilde{H}^{M+1/2}_{z_k}\Big)\Bigg)\, ds,$$

which concludes the proof. ■

Theorem 1. *Let us consider the leap-frog DG method (9)–(11) complemented with the discrete boundary conditions defined in Section 3.1. If the time step* Δt *is such that*

$$\Delta t < \frac{\min\{\underline{\epsilon}, \underline{\mu}\}}{\max\{C_E, C_H\}} \min\{h_k\}, \tag{19}$$

where

$$C_E = \frac{1}{2} C^2_{inv} N^2 + C^2_\tau (N+1)(N+2)\left(2 + \beta_2 + \frac{\alpha + \beta_1}{2\min\{Z_k\}}\right),$$

$$C_H = \frac{1}{2} C^2_{inv} N^2 + C^2_\tau (N+1)(N+2)\left(2 + \beta_2 + \frac{\alpha + 2\beta_2\beta_3}{2\min\{Y_k\}}\right),$$

with C_τ *defined by (45) of Lemma 7 and* C_{inv} *defined by (46) of Lemma 8, and* $\beta_1 = \alpha, \beta_2 = 0$ *for PEC,* $\beta_1 = 0, \beta_2 = 1$, $\beta_3 = \alpha$ *for PMC, and* $\beta_1 = \beta_2 = \frac{1}{2}$, $\beta_3 = 1$ *for Silver-Müller boundary conditions, then the method is stable.*

Proof: From (14) and the previous lemmata, considering the Cauchy-Schwarz's inequality and taking into account that $Z_i/(Z_i + Z_k) < 1$, we obtain

$$
\begin{aligned}
&\sum_{T_k\in\mathcal{T}_h}\left(\left(\epsilon\tilde{E}_k^M,\tilde{E}_k^M\right)_{T_k}+\left(\mu\tilde{H}_{z_k}^{M+1/2},\tilde{H}_{z_k}^{M+1/2}\right)_{T_k}\right)\leq\\
&\sum_{T_k\in\mathcal{T}_h}\left(\left(\epsilon\tilde{E}_k^0,\tilde{E}_k^0\right)_{T_k}+\left(\mu\tilde{H}_{z_k}^{1/2},\tilde{H}_{z_k}^{1/2}\right)_{T_k}\right)\\
&+\Delta t\sum_{T_k\in\mathcal{T}_h}\left(\|\nabla\times\tilde{H}_{z_k}^{1/2}\|_{L^2(T_k)}\|\tilde{E}_k^0\|_{L^2(T_k)}+\|\nabla\times\tilde{H}_{z_k}^{M+1/2}\|_{L^2(T_k)}\|\tilde{E}_k^M\|_{L^2(T_k)}\right)\\
&+\frac{\alpha\Delta t}{4\min\{Z_k\}}\sum_{f_{ik}\in F^{int}}\|[\tilde{E}_k^M]\|_{L^2(f_{ik})}^2\\
&+2\Delta t\sum_{f_{ik}\in F^{int}}\left(\|\tilde{E}_k^M\|_{L^2(f_{ik})}\|[\tilde{H}_{z_k}^{M+1/2}]\|_{L^2(f_{ik})}+\|\tilde{E}_k^0\|_{L^2(f_{ik})}\|[\tilde{H}_{z_k}^{1/2}]\|_{L^2(f_{ik})}\right)\\
&+\frac{\alpha\Delta t}{4\min\{Y_k\}}\sum_{f_{ik}\in F^{int}}\|[\tilde{H}_{z_k}^{M+1/2}]\|_{L^2(f_{ik})}^2\\
&+\frac{\beta_1\Delta t}{2\min\{Z_k\}}\sum_{f_k\in F^{ext}}\|\tilde{E}_k^M\|_{L^2(f_k)}^2+\frac{\beta_2\beta_3\Delta t}{\min\{Y_k\}}\sum_{f_k\in F^{ext}}\|\tilde{H}_{z_k}^{M+1/2}\|_{L^2(f_k)}^2\\
&+2\beta_2\Delta t\sum_{f_k\in F^{ext}}\left(\|\tilde{H}_{z_k}^{1/2}\|_{L^2(f_k)}\|\tilde{E}_k^0\|_{L^2(f_k)}+\|\tilde{H}_{z_k}^{M+1/2}\|_{L^2(f_k)}\|\tilde{E}_k^M\|_{L^2(f_k)}\right).
\end{aligned}
$$

Using the inequality (45) of Lemma 7 and the inequality (46) of Lemma 8 (both in Appendix), we get

$$
\begin{aligned}
&\min\{\underline{\epsilon},\underline{\mu}\}\left(\|\tilde{E}^M\|_\Omega^2+\|\tilde{H}_z^{M+1/2}\|_\Omega^2\right)\leq\\
&\max\{\bar{\epsilon},\bar{\mu}\}\left(\|\tilde{E}^0\|_\Omega^2+\|\tilde{H}_z^{1/2}\|_\Omega^2\right)\\
&+\frac{\Delta t}{2}C_{inv}^2N^2\max\left\{h_k^{-1}\right\}\left(\|\tilde{H}_z^{1/2}\|_\Omega^2+\|\tilde{E}^0\|_\Omega^2+\|\tilde{H}_z^{M+1/2}\|_\Omega^2+\|\tilde{E}^M\|_\Omega^2\right)\\
&+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+\beta_1}{2\min\{Z_k\}}\right)\|\tilde{E}^M\|_\Omega^2\\
&+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+2\beta_2\beta_3}{2\min\{Y_k\}}\right)\|\tilde{H}_z^{M+1/2}\|_\Omega^2\\
&+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}(2+\beta_2)\left(\|\tilde{E}^0\|_\Omega^2+\|\tilde{H}_z^{1/2}\|_\Omega^2\right).
\end{aligned}
$$

and so, taking $C_0 = \frac{1}{2}C_{inv}^2 N^2 + C_\tau^2 (N+1)(N+2)\left(2+\beta_2\right)$,

$$\left(\min\{\underline{\epsilon},\underline{\mu}\} - \Delta t \max\left\{h_k^{-1}\right\}\max\{C_E,C_H\}\right)\left(\|\tilde{E}^M\|^2_{L^2(\Omega)} + \|\tilde{H}_z^{M+1/2}\|^2_{L^2(\Omega)}\right) \leq$$
$$\left(\max\{\bar{\epsilon},\bar{\mu}\} + \Delta t \max\left\{h_k^{-1}\right\} C_0\right)\left(\|\tilde{E}^0\|^2_{L^2(\Omega)} + \|\tilde{H}_z^{1/2}\|^2_{L^2(\Omega)}\right),$$

which concludes the proof. ∎

The stability condition (19) shows that the method is conditionally stable, which is natural since we considered an explicit time discretization. Further, it discloses the influence of the values of α, h_{min} and N on the bounds of the stable region. This is of utmost importance to balance accuracy *versus* stability.

5. Error Estimate

The main result of this section is Theorem 2, where the error estimates are presented. The key idea for the proof is to find a variational system for the difference between the the numerical solution and a projection of (E_x, E_y, H_z) onto the space V_N. Lemma 9 in the Appendix furnishes an optimal error estimation which plays a central role in our derivation.

To provide a proper functional setting, we need to define spaces involving time-dependent functions ([8]). Let X denote a Banach space with norm $\|.\|_X$. The spaces $L^2(0,T;X)$ and $L^\infty(0,T;X)$ consist, respectively, of all measurable functions $v:[0,T]\to X$ with

$$\|v\|_{L^2(0,T;X)} = \left(\int_0^T \|v(t)\|_X^2\, dt\right)^{1/2} < \infty$$

and

$$\|v\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0\leq t\leq T} \|v(t)\|_X < \infty.$$

In what follows, X is shorthand for any of the usual Sobolev spaces $H^p(\Omega)$ or the Banach space $L^\infty(\Omega)$.

We start by integrating (1) from t^m to t^{m+1} and (2) from $t^{m+1/2}$ to $t^{m+3/2}$. Then, multiplying the resultant by $(u_k, v_k, w_k) \in V_N$ with respect to the L^2-inner product over T_k, we obtain

$$\left(\epsilon_{xx}\frac{E_{x_k}^{m+1}-E_{x_k}^m}{\Delta t} + \epsilon_{xy}\frac{E_{y_k}^{m+1}-E_{y_k}^m}{\Delta t}, u_k\right)_{T_k} = \frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}} \frac{\partial H_{z_k}}{\partial y} dt, u_k\right)_{T_k}, \quad (20)$$

$$\left(\epsilon_{yx}\frac{E_{x_k}^{m+1}-E_{x_k}^{m}}{\Delta t}+\epsilon_{yy}\frac{E_{y_k}^{m+1}-E_{y_k}^{m}}{\Delta t},v_k\right)_{T_k}=-\frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial x}dt,v_k\right)_{T_k},\quad(21)$$

$$\begin{aligned}\left(\mu\frac{H_{z_k}^{m+3/2}-H_{z_k}^{m+1/2}}{\Delta t},w_k\right)_{T_k} &= \frac{1}{\Delta t}\left(\int_{t^{m+1/2}}^{t^{m+3/2}}\frac{\partial E_{x_k}}{\partial y}dt,w_k\right)_{T_k}\\ &\quad -\frac{1}{\Delta t}\left(\int_{t^{m+1/2}}^{t^{m+3/2}}\frac{\partial E_{y_k}}{\partial x}dt,w_k\right)_{T_k}. \qquad (22)\end{aligned}$$

Let $(\mathcal{P}_N E_x,\mathcal{P}_N E_y,\mathcal{P}_N H_z)\in V_N$ be an interpolant of (E_x,E_y,H_z) having the optimal approximation errors (47)–(48). On the external boundary we define the jumps $[\mathcal{P}_N E_x]=2\mathcal{P}_N E_x$, $[\mathcal{P}_N E_y]=2\mathcal{P}_N E_y$, $[\mathcal{P}_N H_z]=0$ and $[\mathcal{P}_N E_x]=0$, $[\mathcal{P}_N E_y]=0$, $[\mathcal{P}_N H_z]=2\mathcal{P}_N H_z$, for PEC and PMC boundary conditions, respectively. For Silver-Müller absorbing boundary conditions we consider $[\mathcal{P}_N E_x]=\mathcal{P}_N E_x$, $[\mathcal{P}_N E_y]=\mathcal{P}_N E_y$, $[\mathcal{P}_N H_z]=\mathcal{P}_N H_z$. Subtracting (9)–(11) from (20)–(22), and using the notation $\xi_{x_k}^m=\mathcal{P}_N E_{x_k}^m-\tilde{E}_{x_k}^m$, $\rho_{x_k}^m=\mathcal{P}_N E_{x_k}^m-E_{x_k}^m$, $\xi_{y_k}^m=\mathcal{P}_N E_{y_k}^m-\tilde{E}_{y_k}^m$, $\rho_{y_k}^m=\mathcal{P}_N E_{y_k}^m-E_{y_k}^m$, $\eta_{z_k}^{m+1/2}=\mathcal{P}_N H_{z_k}^{m+1/2}-\tilde{H}_{z_k}^{m+1/2}$, $\phi_{z_k}^{m+1/2}=\mathcal{P}_N H_{z_k}^{m+1/2}-H_{z_k}^{m+1/2}$, we obtain

$$\begin{aligned}&\left(\epsilon_{xx}\frac{\xi_{x_k}^{m+1}-\xi_{x_k}^{m}}{\Delta t},u_k\right)_{T_k}-\left(\epsilon_{xx}\frac{\rho_{x_k}^{m+1}-\rho_{x_k}^{m}}{\Delta t},u_k\right)_{T_k}\\
+&\left(\epsilon_{xy}\frac{\xi_{y_k}^{m+1}-\xi_{y_k}^{m}}{\Delta t},u_k\right)_{T_k}-\left(\epsilon_{xy}\frac{\rho_{y_k}^{m+1}-\rho_{y_k}^{m}}{\Delta t},u_k\right)_{T_k}\\
=&\frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial y}dt,u_k\right)_{T_k}-\left(\frac{\partial}{\partial y}\left(\mathcal{P}_N H_{z_k}^{m+1/2}\right),u_k\right)_{T_k}+\left(\frac{\partial \eta_{z_k}^{m+1/2}}{\partial y},u_k\right)_{T_k}\\
+&\left(\frac{n_y}{Z^++Z^-}\left(Z^+[\mathcal{P}_N H_{z_k}^{m+1/2}]-\alpha\left(n_x[\mathcal{P}_N E_{y_k}^m]-n_y[\mathcal{P}_N E_{x_k}^m]\right)\right),u_k\right)_{\partial T_k}\\
+&\left(\frac{-n_y}{Z^++Z^-}\left(Z^+[\eta_{z_k}^{m+1/2}]-\alpha\left(n_x[\xi_{y_k}^m]-n_y[\xi_{x_k}^m]\right)\right),u_k\right)_{\partial T_k}, \qquad (23)\end{aligned}$$

$$
\begin{aligned}
&\left(\epsilon_{yx}\frac{\xi_{x_k}^{m+1}-\xi_{x_k}^{m}}{\Delta t}, v_k\right)_{T_k} - \left(\epsilon_{yx}\frac{\rho_{x_k}^{m+1}-\rho_{x_k}^{m}}{\Delta t}, v_k\right)_{T_k} \\
+ &\left(\epsilon_{yy}\frac{\xi_{y_k}^{m+1}-\xi_{y_k}^{m}}{\Delta t}, v_k\right)_{T_k} - \left(\epsilon_{yy}\frac{\rho_{y_k}^{m+1}-\rho_{y_k}^{m}}{\Delta t}, v_k\right)_{T_k} \\
= &-\frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial x}dt, v_k\right)_{T_k} + \left(\frac{\partial}{\partial x}\left(\mathcal{P}_N H_{z_k}^{m+1/2}\right), v_k\right)_{T_k} - \left(\frac{\partial \eta_{z_k}^{m+1/2}}{\partial x}, v_k\right)_{T_k} \\
- &\left(\frac{n_x}{Z^+ + Z^-}\left(Z^+[\mathcal{P}_N H_{z_k}^{m+1/2}] - \alpha\left(n_x[\mathcal{P}_N E_{y_k}^{m}] - n_y[\mathcal{P}_N E_{x_k}^{m}]\right)\right), v_k\right)_{\partial T_k} \\
+ &\left(\frac{n_x}{Z^+ + Z^-}\left(Z^+[\eta_{z_k}^{m+1/2}] - \alpha\left(n_x[\xi_{y_k}^{m}] - n_y[\xi_{x_k}^{m}]\right)\right), v_k\right)_{\partial T_k} \qquad (24)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\mu\frac{\eta_{z_k}^{m+3/2}-\eta_{z_k}^{m+1/2}}{\Delta t}, w_k\right)_{T_k} - \left(\mu\frac{\phi_{z_k}^{m+3/2}-\phi_{x_k}^{m+1/2}}{\Delta t}, w_k\right)_{T_k} \\
= &\ \frac{1}{\Delta t}\left(\int_{t^{m+1/2}}^{t^{m+3/2}}\frac{\partial E_{x_k}}{\partial y}dt, w_k\right) - \frac{1}{\Delta t}\left(\int_{t^{m+1/2}}^{t^{m+3/2}}\frac{\partial E_{y_k}}{\partial x}dt, w_k\right)_{T_k} \\
- &\left(\frac{\partial}{\partial y}\left(\mathcal{P}_N E_{x_k}^{m+1}\right), w_k\right)_{T_k} + \left(\frac{\partial \xi_{x_k}^{m+1}}{\partial y}, w_k\right)_{T_k} \\
+ &\left(\frac{\partial}{\partial x}\left(\mathcal{P}_N E_{y_k}^{m+1}\right), w_k\right)_{T_k} - \left(\frac{\partial \xi_{y_k}^{m+1}}{\partial x}, w_k\right)_{T_k} \\
- &\left(\frac{1}{Y^+ + Y^-}\left(Y^+(n_x[\mathcal{P}_N E_{y_k}^{m+1}] - n_y[\mathcal{P}_N E_{x_k}^{m+1}]\right) - \alpha[\mathcal{P}_N H_{z_k}^{m+1/2}], w_k\right)_{\partial T_k} \\
+ &\left(\frac{1}{Y^+ + Y^-}\left(Y^+(n_x[\xi_{y_k}^{m+1}] - n_y[\xi_{x_k}^{m+1}]\right) - \alpha[\eta_{z_k}^{m+1/2}], w_k\right)_{\partial T_k}. \qquad (25)
\end{aligned}
$$

In (23)–(25), let $u_k = \xi_{x_k}^{m} + \xi_{x_k}^{m+1}$, $v_k = \xi_{y_k}^{m} + \xi_{y_k}^{m+1}$ and $w_k = \eta_{z_k}^{m+1/2} + \eta_{z_k}^{m+3/2}$. Summing from $m = 0$ to $m = M-1$ and using the symmetry property of the permittivity tensor ϵ, we get

$$\left(\epsilon\xi_k^M,\xi_k^M\right)_{T_k}+\left(\mu\eta_{z_k}^{M+1/2},\eta_{z_k}^{M+1/2}\right)_{T_k}=\left(\epsilon\xi_k^0,\xi_k^0\right)_{T_k}+\left(\mu\eta_{z_k}^{1/2},\eta_{z_k}^{1/2}\right)_{T_k}$$
$$+\Delta t\left(\nabla\times\eta_{z_k}^{1/2},\xi_k^0\right)_{T_k}-\Delta t\left(\nabla\times\eta_{z_k}^{M+1/2},\xi_k^M\right)_{T_k}+2\Delta t\sum_{m=0}^{M-1}R_k^m, \tag{26}$$

with

$$R_k^m=S_{1,k}^m+S_{2,k}^m+S_{3,k}^m+S_{4,k}^m,$$

being $S_{1,k}^m$, $S_{2,k}^m$, $S_{3,k}^m$ and $S_{4,k}^m$ defined below using the average notation $\xi_{x_k}^{[m+1/2]}=(\xi_{x_k}^m+\xi_{x_k}^{m+1})/2$, $\xi_{y_k}^{[m+1/2]}=(\xi_{y_k}^m+\xi_{y_k}^{m+1})/2$ and $\eta_k^{[m+1]}=(\eta_{z_k}^{m+1/2}+\eta_{z_k}^{m+3/2})/2$,

$$\begin{aligned}S_{1,k}^m &= \left(\epsilon_{xx}\frac{\rho_{x_k}^{m+1}-\rho_{x_k}^m}{\Delta t},\xi_{x_k}^{[m+1/2]}\right)_{T_k}+\left(\epsilon_{xy}\frac{\rho_{y_k}^{m+1}-\rho_{y_k}^m}{\Delta t},\xi_{x_k}^{[m+1/2]}\right)_{T_k}\\ &+ \left(\epsilon_{yx}\frac{\rho_{x_k}^{m+1}-\rho_{x_k}^m}{\Delta t},\xi_{y_k}^{[m+1/2]}\right)_{T_k}+\left(\epsilon_{yy}\frac{\rho_{y_k}^{m+1}-\rho_{y_k}^m}{\Delta t},\xi_{y_k}^{[m+1/2]}\right)_{T_k}\\ &+ \left(\mu\frac{\phi_{z_k}^{m+3/2}-\phi_{z_k}^{m+1/2}}{\Delta t},\eta_{z_k}^{[m+1]}\right)_{T_k},\end{aligned} \tag{27}$$

$$\begin{aligned}S_{2,k}^m &= -\left(\frac{\partial}{\partial y}\left(\mathcal{P}_N H_{z_k}^{m+1/2}\right),\xi_{x_k}^{[m+1/2]}\right)_{T_k}+\frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial y}dt,\xi_{x_k}^{[m+1/2]}\right)_{T_k}\\ &+ \left(\frac{\partial}{\partial x}\left(\mathcal{P}_N H_{z_k}^{m+1/2}\right),\xi_{y_k}^{[m+1/2]}\right)_{T_k}-\frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial x}dt,\xi_{y_k}^{[m+1/2]}\right)_{T_k}\\ &- \left(\frac{\partial}{\partial y}\left(\mathcal{P}_N E_{x_k}^{m+1}\right),\eta_{z_k}^{[m+1]}\right)_{T_k}+\frac{1}{\Delta t}\left(\int_{t^{m+1/2}}^{t^{m+3/2}}\frac{\partial E_{x_k}}{\partial y}dt,\eta_{z_k}^{[m+1]}\right)_{T_k}\\ &+ \left(\frac{\partial}{\partial x}\left(\mathcal{P}_N E_{y_k}^{m+1}\right),\eta_{z_k}^{[m+1]}\right)_{T_k}-\frac{1}{\Delta t}\left(\int_{t^{m+1/2}}^{t^{m+3/2}}\frac{\partial E_{y_k}}{\partial x}dt,\eta_{z_k}^{[m+1]}\right)_{T_k},\end{aligned} \tag{28}$$

$$
\begin{aligned}
S_{3,k}^m &= \left(\frac{n_y}{Z^+ + Z^-} \left(Z^+[\mathcal{P}_N H_{z_k}^{m+1/2}] - \alpha \left(n_x[\mathcal{P}_N E_{y_k}^m] - n_y[\mathcal{P}_N E_{x_k}^m] \right) \right), \xi_{x_k}^{[m+1/2]} \right)_{\partial T_k} \\
&\quad - \left(\frac{n_x}{Z^+ + Z^-} \left(Z^+[\mathcal{P}_N H_{z_k}^{m+1/2}] - \alpha \left(n_x[\mathcal{P}_N E_{y_k}^m] - n_y[\mathcal{P}_N E_{x_k}^m] \right) \right), \xi_{y_k}^{[m+1/2]} \right)_{\partial T_k} \\
&\quad - \left(\frac{1}{Y^+ + Y^-} \left(Y^+ (n_x[\mathcal{P}_N E_{y_k}^{m+1}] - n_y[\mathcal{P}_N E_{x_k}^{m+1}]) - \alpha[\mathcal{P}_N H_{z_k}^{m+1/2}] \right), \eta_{z_k}^{[m+1]} \right)_{\partial T_k},
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
S_{4,k}^m &= \left(\frac{-n_y}{Z^+ + Z^-} \left(Z^+[\eta_{z_k}^{m+1/2}] - \alpha \left(n_x[\xi_{y_k}^m] - n_y[\xi_{x_k}^m] \right) \right), \xi_{x_k}^{[m+1/2]} \right)_{\partial T_k} \\
&\quad + \left(\frac{n_x}{Z^+ + Z^-} \left(Z^+[\eta_{z_k}^{m+1/2}] - \alpha \left(n_x[\xi_{y_k}^m] - n_y[\xi_{x_k}^m] \right) \right), \xi_{y_k}^{[m+1/2]} \right)_{\partial T_k} \\
&\quad + \left(\frac{1}{Y^+ + Y^-} \left(Y^+ (n_x[\xi_{y_k}^{m+1}] - n_y[\xi_{x_k}^{m+1}]) - \alpha[\eta_{z_k}^{m+1/2}], \eta_{z_k}^{[m+1]} \right) \right)_{\partial T_k} \\
&\quad + \left(n_y \xi_{x_k}^{m+1}, \eta_{z_k}^{[m+1]} \right)_{\partial T_k} - \left(n_x \xi_{y_k}^{m+1}, \eta_{z_k}^{[m+1]} \right)_{\partial T_k}.
\end{aligned} \tag{30}
$$

Next we will derive upper bounds for $S_{1,k}^m$, $S_{2,k}^m$, $S_{3,k}^m$ and $S_{4,k}^m$.

Lemma 3. *Let $S_{1,k}^m$ be defined by (27). Then*

$$
\begin{aligned}
\sum_{m=0}^{M-1} \sum_{T_k \in \mathcal{T}_h} S_{1,k}^m &\le \frac{C h^{2\sigma}}{N^{2p}} \left(\frac{\overline{\epsilon}}{\delta} \int_0^T \left\| \frac{\partial E}{\partial t} \right\|_{H^p(\Omega)}^2 dt + \frac{\overline{\mu}}{\delta} \int_{\Delta t/2}^{T+\Delta t/2} \left\| \frac{\partial H_z}{\partial t} \right\|_{H^p(\Omega)}^2 dt \right) \\
&\quad + \frac{\delta}{2} \left(\|\xi^0\|_{L^2(\Omega)}^2 + 2 \sum_{m=1}^{M-1} \|\xi^m\|_{L^2(\Omega)}^2 + \|\xi^M\|_{L^2(\Omega)}^2 \right) \\
&\quad + \frac{\delta}{2} \left(\|\eta_z^{1/2}\|_{L^2(\Omega)}^2 + 2 \sum_{m=1}^{M-1} \|\eta_z^{m+1/2}\|_{L^2(\Omega)}^2 + \|\eta_z^{M+1/2}\|_{L^2(\Omega)}^2 \right),
\end{aligned} \tag{31}
$$

where $p \ge 0$, $\sigma = \min(p, N+1)$, C is a constant independent of $(E_{x_k}, E_{y_k}, H_{z_k})$, h and N, and δ is an arbitrary positive constant.

Proof. We start by observing that

$$\|\rho_{x_k}^{m+1} - \rho_{x_k}^{m}\|_{L^2(T_k)}^2 = \left\| \int_{t^m}^{t^{m+1}} \frac{\partial \rho_{x_k}}{\partial t} dt \right\|_{L^2(T_k)}^2 .$$

Applying the Cauchy-Schwarz's inequality and the approximation property of Lemma 9, we get

$$\begin{aligned} \left\| \int_{t^m}^{t^{m+1}} \frac{\partial \rho_{x_k}}{\partial t} dt \right\|_{L^2(T_k)}^2 &\leq \Delta t \int_{t^m}^{t^{m+1}} \left\| \frac{\partial \rho_{x_k}}{\partial t} \right\|_{L^2(T_k)}^2 dt \\ &\leq \Delta t C \frac{h_k^{2\sigma}}{N^{2p}} \int_{t^m}^{t^{m+1}} \left\| \frac{\partial E_{x_k}}{\partial t} \right\|_{H^p(T_k)}^2 dt. \end{aligned} \tag{32}$$

Using Cauchy-Schwarz's inequality, the estimate (32) and the Young's inequality in the form

$$ab \leq \frac{\delta}{2} a^2 + \frac{1}{2\delta} b^2,$$

where δ is an arbitrary positive constant, we obtain

$$\begin{aligned} S_{1,k}^m \leq\; & \frac{\overline{\epsilon} C}{\delta} \frac{h_k^{2\sigma}}{N^{2p}} \int_{t^m}^{t^{m+1}} \left\| \frac{\partial E_{x_k}}{\partial t} \right\|_{H^p(T_k)}^2 dt + \frac{\delta}{2} \|\xi_{x_k}^{[m+1/2]}\|_{L^2(T_k)}^2 \\ +\; & \frac{\overline{\epsilon} C}{\delta} \frac{h_k^{2\sigma}}{N^{2p}} \int_{t^m}^{t^{m+1}} \left\| \frac{\partial E_{y_k}}{\partial t} \right\|_{H^p(T_k)}^2 dt + \frac{\delta}{2} \|\xi_{y_k}^{[m+1/2]}\|_{L^2(T_k)}^2 \\ +\; & \frac{\overline{\mu} C}{\delta} \frac{h_k^{2\sigma}}{N^{2p}} \int_{t^{m+1/2}}^{t^{m+3/2}} \left\| \frac{\partial H_{z_k}}{\partial t} \right\|_{H^p(T_k)}^2 dt + \frac{\delta}{2} \|\eta_{z_k}^{[m+1]}\|_{L^2(T_k)}^2 . \end{aligned}$$

Summing from $m = 0$ to $M - 1$, we arrive at (31). ∎

Lemma 4. *Let $S_{2,k}^m$ be defined by (28). Then*

$$
\begin{aligned}
\sum_{m=0}^{M-1}\sum_{T_k\in\mathcal{T}_h} S_{2,k}^m \;\le\;& \frac{CMh^{2\sigma-2}}{N^{2p-2}}\left(\frac{1}{\delta}\|E\|^2_{L^\infty(0,T;H^p(\Omega))}+\frac{1}{\delta}\|H_z\|^2_{L^\infty(0,T;H^p(\Omega))}\right)\\
&+\; C\Delta t^3\left(\frac{1}{\delta}\int_{\Delta t/2}^{T+\Delta t/2}\left|\frac{\partial^2 E}{\partial t^2}\right|^2_{H^1(\Omega)}dt+\frac{1}{\delta}\int_0^T\left|\frac{\partial^2 H_z}{\partial t^2}\right|^2_{H^1(\Omega)}dt\right)\\
&+\; \delta\left(\frac{1}{2}\|\xi^0\|^2_{L^2(\Omega)}+\sum_{m=1}^{M-1}\|\xi^m\|^2_{L^2(\Omega)}+\frac{1}{2}\|\xi^M\|^2_{L^2(\Omega)}\right)\\
&+\; \delta\left(\|\eta_z^{1/2}\|^2_{L^2(\Omega)}+2\sum_{m=1}^{M-1}\|\eta_z^{m+1/2}\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\right), \qquad (33)
\end{aligned}
$$

where $p\ge 0$, $\sigma=\min(p,N+1)$, C is a constant independent of $(E_{x_k},E_{y_k},H_{z_k})$, h and N, and δ is an arbitrary positive constant.

Proof. It is easy to check that

$$
\begin{aligned}
&-\left(\frac{\partial}{\partial y}\left(\mathcal{P}_N H_{z_k}^{m+1/2}\right),\xi_{x_k}^{[m+1/2]}\right)_{T_k}+\frac{1}{\Delta t}\left(\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial y}(s)ds,\xi_{x_k}^{[m+1/2]}\right)_{T_k}\\
=\;& -\left(\frac{\partial \phi_{z_k}^{m+1/2}}{\partial y},\xi_{x_k}^{[m+1/2]}\right)_{T_k}-\left(\frac{\partial H_{z_k}^{m+1/2}}{\partial y}-\frac{1}{\Delta t}\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial y}(s)ds,\xi_{x_k}^{[m+1/2]}\right)_{T_k}.
\end{aligned}
$$

From Lemma 9 we obtain

$$
\left\|\frac{\partial \phi_{z_k}^{m+1/2}}{\partial y}\right\|_{L^2(T_k)}\le C\frac{h_k^{\sigma-1}}{N^{p-1}}\|H_{z_k}^{m+1/2}\|_{H^p(T_k)}, \qquad (34)
$$

where $\sigma=\min(p,N+1)$. Applying Cauchy-Schwarz's inequality and using (34), we arrive at

$$
\left(\frac{\partial \phi_{z_k}^{m+1/2}}{\partial y},\xi_{x_k}^{[m+1/2]}\right)_{T_k}\le\frac{C}{\delta}\frac{h_k^{2\sigma-2}}{N^{2p-2}}\|H_{z_k}^{m+1/2}\|^2_{H^p(T_k)}+\frac{\delta}{2}\|\xi_{x_k}^{[m+1/2]}\|^2_{L^2(T_k)}.
$$

Using the Taylor expansion with an integral remainder yields to

$$
\left\|\frac{\partial H_{z_k}^{m+1/2}}{\partial y}-\frac{1}{\Delta t}\int_{t^m}^{t^{m+1}}\frac{\partial H_{z_k}}{\partial y}dt\right\|^2_{L^2(T_k)}\le\frac{\Delta t^3}{64}\int_{t^m}^{t^{m+1}}\left\|\frac{\partial^2}{\partial t^2}\left(\frac{\partial H_{z_k}}{\partial y}\right)\right\|^2_{L^2(T_k)}dt
$$

and by Cauchy-Schwarz's and Young's inequalities follows

$$
\begin{aligned}
&\left(\frac{\partial H_{z_k}^{m+1/2}}{\partial y} - \frac{1}{\Delta t}\int_{t^m}^{t^{m+1}} \frac{\partial H_{z_k}}{\partial y}\, dt, \xi_{x_k}^{[m+1/2]}\right)_{T_k} \\
&\qquad\qquad \le \frac{\Delta t^3}{128\,\delta}\int_{t^m}^{t^{m+1}} \left\| \frac{\partial^2}{\partial t^2}\left(\frac{\partial H_{z_k}}{\partial y}\right)\right\|_{L^2(T_k)}^2 dt + \frac{\delta}{2}\|\xi_{x_k}^{[m+1/2]}\|_{L^2(T_k)}^2.
\end{aligned}
$$

The other terms in $S_{2,k}^m$ can be bounded in a similar way. Therefore,

$$
\begin{aligned}
S_{2,k}^m \ \le\ & \frac{C}{\delta}\frac{h_k^{2\sigma-2}}{N^{2p-2}}\|E_{x_k}^{m+1}\|_{H^p(T_k)}^2 + \frac{C}{\delta}\frac{h_k^{2\sigma-2}}{N^{2p-2}}\|E_{y_k}^{m+1}\|_{H^p(T_k)}^2 \\
+\ & \frac{2C}{\delta}\frac{h_k^{2\sigma-2}}{N^{2p-2}}\|H_{z_k}^{m+1/2}\|_{H^p(T_k)}^2 \\
+\ & \frac{\Delta t^3}{128\,\delta}\int_{t^{m+1/2}}^{t^{m+3/2}} \left\|\frac{\partial^2}{\partial t^2}\left(\frac{\partial E_{x_k}}{\partial y}\right)\right\|_{L^2(T_k)}^2 + \left\|\frac{\partial^2}{\partial t^2}\left(\frac{\partial E_{y_k}}{\partial x}\right)\right\|_{L^2(T_k)}^2 dt \\
+\ & \frac{\Delta t^3}{128\,\delta}\int_{t^{m}}^{t^{m+1}} \left\|\frac{\partial^2}{\partial t^2}\left(\frac{\partial H_{z_k}}{\partial y}\right)\right\|_{L^2(T_k)}^2 + \left\|\frac{\partial^2}{\partial t^2}\left(\frac{\partial H_{z_k}}{\partial x}\right)\right\|_{L^2(T_k)}^2 dt \\
+\ & \delta\|\xi_{x_k}^{[m+1/2]}\|_{L^2(T_k)}^2 + \delta\|\xi_{y_k}^{[m+1/2]}\|_{L^2(T_k)}^2 + 2\delta\|\eta_{z_k}^{[m+1]}\|_{L^2(T_k)}^2.
\end{aligned}
$$

Summing from $m = 0$ to $M-1$ we arrive at (33). ∎

Lemma 5. *Let $S_{3,k}^m$ be defined by (29). Then*

$$
\begin{aligned}
\sum_{m=0}^{M-1}\sum_{T_k\in\mathcal{T}_h} S_{3,k}^m \;\leq\;& \frac{CMh^{2\sigma-2}}{\delta N^{2p+1}}C_\tau^2(N+1)(N+2)\left(1+\frac{\alpha}{\min\{Z_k^2\}}\right)\|E\|^2_{L^\infty(0,T;H^p(\Omega))}\\
+\;& \frac{CMh^{2\sigma-2}}{\delta N^{2p+1}}C_\tau^2(N+1)(N+2)\left(1+\frac{\alpha}{\min\{Y_k^2\}}\right)\|H_z\|^2_{L^\infty(0,T;H^p(\Omega))}\\
+\;& \frac{\beta_4 C_\tau^2(N+1)(N+2)\Delta t}{16\delta\min\{Z_k^2\}}\int_0^{T-\Delta t/2}\left\|\frac{\partial E}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt\\
+\;& \frac{\beta_4 C_\tau^2(N+1)(N+2)\Delta t}{32\delta\min\{Y_k^2\}}\int_{\Delta t/2}^{T}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt\\
+\;& \frac{\delta}{4}(1+\beta_4)\Big(\|\xi^0\|^2_{L^2(\Omega)}+2\sum_{m=1}^{M-1}\|\xi^m\|^2_{L^2(\Omega)}+\|\xi^M\|^2_{L^2(\Omega)}\Big)\\
+\;& \frac{\delta}{4}(1+\beta_4)\Big(\|\eta_z^{1/2}\|^2_{L^2(\Omega)}+2\sum_{m=1}^{M-1}\|\eta_z^{m+1/2}\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\Big),
\end{aligned}
\tag{35}
$$

where $p\geq 0$, $\sigma=\min(p,N+1)$, C and C_τ are constants independent of $(E_{x_k},E_{y_k},H_{z_k})$, h and N, and δ is an arbitrary positive constant. Moreover, $\beta_4=0$ for PEC and PMC boundary conditions and $\beta_4=1$ for Silver-Müller absorbing boundary conditions.

Proof. In order to estimate $\sum_{T_k\in\mathcal{T}_h}S_{3,k}^m$, let us write $\|[\mathcal{P}_N E_{x_k}^m]\|_{f_{ik}}$, $f_{ik}\subset F^{int}$, as

$$
\begin{aligned}
\|[\mathcal{P}_N E_{x_k}^m]\|_{L^2(f_{ik})} &= \|\mathcal{P}_N E_{x_k}^{m-}-E_{x_k}^m+E_{x_k}^m-\mathcal{P}_N E_{x_k}^{m+}\|_{L^2(f_{ik})}\\
&\leq \|\mathcal{P}_N E_{x_k}^{m-}-E_{x_k}^m\|_{L^2(f_{ik})}+\|E_{x_k}^m-\mathcal{P}_N E_{x_k}^{m+}\|_{L^2(f_{ik})}.
\end{aligned}
$$

By Lemma 9 we deduce that

$$\|\mathcal{P}_N E_{x_k}^{m-}-E_{x_k}^m\|^2_{L^2(f_{ik})}\leq C\frac{h_k^{2\sigma-1}}{N^{2p-1}}\|E_{x_k}^m\|^2_{H^p(T_k)},$$

where $\sigma=\min(p,N+1)$ and $p>\frac{1}{2}$. In the same way, we obtain

$$\|\mathcal{P}_N E_{x_k}^{m+}-E_{x_k}^m\|^2_{L^2(f_{ik})}\leq C\frac{h_k^{2\sigma-1}}{N^{2p-1}}\|E_{x_k}^m\|^2_{H^p(T_i)},$$

Similar estimates hold for $\|[\mathcal{P}_N E^m_{y_k}]\|^2_{L^2(f_{ik})}$ and $\|[\mathcal{P}_N H^m_{z_k}]\|^2_{L^2(f_{ik})}$.

Let us now consider the edges that belong to the external boundary $\partial\Omega$. In the case of PEC boundary condition we have $[\mathcal{P}_N H^{m+1/2}_{z_k}] = 0$. Since $n_x E^m_{y_k} - n_y E^m_{x_k} = 0$, then

$$n_x[\mathcal{P}_N E^m_{y_k}] - n_y[\mathcal{P}_N E^m_{x_k}] = 2(n_x(\mathcal{P}_N E^m_{y_k} - E^m_{y_k}) - n_y(\mathcal{P}_N E^m_{x_k} - E^m_{x_k})).$$

For PMC boundary conditions we have $[\mathcal{P}_N E^m_{y_k}] = 0$ and $[\mathcal{P}_N E^m_{x_k}] = 0$. Since $H^{m+1/2}_{z_k} = 0$, then

$$[\mathcal{P}_N H^{m+1/2}_{z_k}] = 2(\mathcal{P}_N H^{m+1/2}_{z_k} - H^{m+1/2}_{z_k}).$$

Applying Cauchy-Schwarz's inequality, (45) of Lemma 7 and Young's inequality, for the cases of PEC or PMC boundary conditions, we obtain

$$\begin{aligned}\sum_{T_k\in\mathcal{T}_h} S^m_{3,k} \;\leq\; \sum_{T_k\in\mathcal{T}_h}\Bigg(& \\ & \frac{C}{\delta}C^2_\tau(N+1)(N+2)\left(1+\frac{\alpha}{\min\{Y^2_k\}}\right)\sum_{i\in\nu_k}\frac{h_i^{2\sigma-2}}{N^{2p+1}}\|H^{m+1/2}_{z_i}\|^2_{H^p(T_i)} \\ & +\frac{C}{\delta}C^2_\tau(N+1)(N+2)\left(1+\frac{\alpha}{\min\{Z^2_k\}}\right)\sum_{i\in\nu_k}\frac{h_i^{2\sigma-2}}{N^{2p+1}}\|E^m_i\|^2_{H^p(T_i)} \\ & +\frac{\delta}{2}\|\xi^{[m+1/2]}_{x_k}\|^2_{L^2(T_k)}+\frac{\delta}{2}\|\xi^{[m+1/2]}_{y_k}\|^2_{L^2(T_k)}+\frac{\delta}{2}\|\eta^{[m+1]}_{z_k}\|^2_{L^2(T_k)}\Bigg). \qquad (36)\end{aligned}$$

In the case of Silver-Müller absorbing boundary condition, on the edges that belong to the external boundary, we observe that

$$Z^+H^{m+1/2}_{z_k} - (n_x E^{m+1/2}_{y_k} - n_y E^{m+1/2}_{x_k}) = 0,\ Y^+(n_x E^{m+1}_{y_k} - n_y E^{m+1}_{x_k}) - H^{m+1}_{z_k} = 0.$$

Thus,

$$\begin{aligned} & Z^+[\mathcal{P}_N H^{m+1/2}_{z_k}] - (n_x[\mathcal{P}_N E^m_{y_k}] - n_y[\mathcal{P}_N E^m_{x_k}]) \\ & \quad = Z^+\mathcal{P}_N H^{m+1/2}_{z_k} - (n_x\mathcal{P}_N E^m_{y_k} - n_y\mathcal{P}_N E^m_{x_k}) \\ & \quad = Z^+(\mathcal{P}_N H^{m+1/2}_{z_k} - H^{m+1/2}_{z_k}) \\ & \qquad -(n_x(\mathcal{P}_N E^m_{y_k} - E^m_{y_k}) - n_y(\mathcal{P}_N E^m_{x_k} - E^m_{x_k})) \\ & \qquad +n_x(E^{m+1/2}_{y_k} - E^m_{y_k}) - n_y(E^{m+1/2}_{x_k} - E^m_{x_k}) \end{aligned}$$

and

$$\begin{aligned}
Y^+(n_x[\mathcal{P}_N E_{y_k}^{m+1}] - n_y[\mathcal{P}_N E_{x_k}^{m+1}]) - [\mathcal{P}_N H_{z_k}^{m+1/2}] \\
= Y^+(n_x \mathcal{P}_N E_{y_k}^{m+1} - n_y \mathcal{P}_N E_{x_k}^{m+1}) - \mathcal{P}_N H_{z_k}^{m+1/2} \\
= Y^+(n_x(\mathcal{P}_N E_{y_k}^{m+1} - E_{y_k}^{m+1}) - n_y(\mathcal{P}_N E_{x_k}^{m+1} - E_{x_k}^{m+1}) \\
-(\mathcal{P}_N H_{z_k}^{m+1/2} - H_{z_k}^{m+1/2}) + H_{z_k}^{m+1} - H_{z_k}^{m+1/2}.
\end{aligned}$$

For $f_k \in F^{ext}$, we obtain

$$\left\| \frac{n_y}{Z^+ + Z^-}(n_x(E_{y_k}^{m+1/2} - E_{y_k}^m) - n_y(E_{x_k}^{m+1/2} - E_{x_k}^m)) \right\|_{L^\infty(f_k)} \leq \frac{1}{2\min\{Z_k\}} \int_{t^m}^{t^{m+1/2}} \left\| \frac{\partial E}{\partial t} \right\|_{L^\infty(f_k)} dt.$$

In the same way we get the estimate

$$\left\| \frac{1}{Y^+ + Y^-}(H_{z_k}^{m+1} - H_{z_k}^{m+1/2}) \right\|_{L^\infty(f_k)} \leq \frac{1}{2\min\{Y_k\}} \int_{t^{m+1/2}}^{t^{m+1}} \left\| \frac{\partial H_z}{\partial t} \right\|_{L^\infty(f_k)} dt.$$

Therefore, to estimate $S_{3,k}^m$ we use (45) of Lemma 7, and we observe that we need to add the terms

$$\frac{1}{2\delta} \frac{C_\tau^2 (N+1)(N+2)\Delta t}{8\min\{Z_k^2\}} \int_{t^m}^{t^{m+1/2}} \left\| \frac{\partial E}{\partial t} \right\|_{L^\infty(f_k)}^2 dt, \quad \frac{\delta}{2} \|\xi_k^{[m+1/2]}\|_{L^2(T_k)}^2,$$

$$\frac{1}{2\delta} \frac{C_\tau^2 (N+1)(N+2)\Delta t}{16\min\{Y_k^2\}} \int_{t^{m+1/2}}^{t^{m+1}} \left\| \frac{\partial H_z}{\partial t} \right\|_{L^\infty(f_k)}^2 dt, \quad \frac{\delta}{2} \|\eta_{z_k}^{[m+1]}\|_{L^2(T_k)}^2,$$

to the right hand side of (36).

Summing from $m = 0$ to $M - 1$, leads to the estimation (35). ■

Lemma 6. *Let $S^m_{4,k}$ be defined by (30). Then*

$$\sum_{m=0}^{M-1}\sum_{T_k\in\mathcal{T}_h} S^m_{4,k} \leq$$

$$\frac{1}{2}C_\tau^2(N+1)(N+2)\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+\beta_1}{2\min\{Z_k\}}\right)\|\xi^M\|^2_{L^2(\Omega)}$$
$$+\frac{1}{2}C_\tau^2(N+1)(N+2)\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+2\beta_2\beta_3}{2\min\{Y_k\}}\right)\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}$$
$$+\frac{1}{2}C_\tau^2(N+1)(N+2)\max\left\{h_k^{-1}\right\}(2+\beta_2)\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{1/2}\|^2_{L^2(\Omega)}\right). \quad (37)$$

Proof. We can find an upper bound for $\sum_{T_k\in\mathcal{T}_h} S^m_{4,k}$ following the arguments we used to estimate $\sum_{T_k\in\mathcal{T}_h} A^m_k$ in the previous section. We obtain

$$\begin{aligned}\sum_{m=0}^{M-1}\sum_{T_k\in\mathcal{T}_h} S^m_{4,k} \leq\ & \frac{\alpha}{8\min\{Z_k\}}\sum_{f_{ik}\in F^{int}}\|[\xi_k^M]\|^2_{L^2(f_{ik})}\\
&+\sum_{f_{ik}\in F^{int}}\left(\|\xi_k^M\|_{L^2(f_{ik})}\|[\eta_{z_k}^{M+1/2}]\|_{L^2(f_{ik})}+\|\xi_k^0\|_{L^2(f_{ik})}\|[\eta_{z_k}^{1/2}]\|_{L^2(f_{ik})}\right)\\
&+\frac{\alpha}{8\min\{Y_k\}}\sum_{f_{ik}\in F^{int}}\|[\eta_{z_k}^{M+1/2}]\|^2_{L^2(f_{ik})}\\
&+\frac{\beta_1}{4\min\{Z_k\}}\sum_{f_k\in F^{ext}}\|\xi_k^M\|^2_{L^2(f_k)}+\frac{\beta_2\beta_3}{2\min\{Y_k\}}\sum_{f_k\in F^{ext}}\|\eta_{z_k}^{M+1/2}\|^2_{L^2(f_k)}\\
&+\beta_2\sum_{f_k\in F^{ext}}\left(\|\eta_{z_k}^{1/2}\|_{L^2(f_k)}\|\xi_k^0\|_{L^2(f_k)}+\|\eta_{z_k}^{M+1/2}\|_{L^2(f_k)}\|\xi_k^M\|_{L^2(f_k)}\right),\end{aligned}$$

where $\beta_1=\alpha,\beta_2=0$ for PEC, $\beta_1=0,\beta_2=1$, $\beta_3=\alpha$ for PMC, and $\beta_1=\beta_2=\frac{1}{2}$, $\beta_3=1$ for Silver-Müller boundary conditions. As in the proof of Theorem 1, using the inequality (45) of Lemma 7 and the inequality (46) of Lemma 8 (both in Appendix) we obtain the estimate (37). ∎

Theorem 2. *Let us consider the leap-frog DG method (9)–(11) complemented with the discrete boundary conditions defined in Section 3.1 and suppose that the solution of the Maxwell's equations (1)–(2) complemented by (4), (5) or (6) has the following regularity:*

$$E_x, E_y, H_z \in L^\infty(0,T_f;H^{s+1}(\Omega)),$$

$$\frac{\partial E_x}{\partial t}, \frac{\partial E_y}{\partial t}, \frac{\partial H_z}{\partial t} \in L^2(0,T_f; H^{s+1}(\Omega) \cap L^\infty(\partial\Omega))$$

and

$$\frac{\partial^2 E_x}{\partial t^2}, \frac{\partial^2 E_y}{\partial t^2}, \frac{\partial^2 H_z}{\partial t^2} \in L^2(0,T_f; H^1(\Omega)), s \geq 0.$$

If the time step Δt *satisfies*

$$\Delta t \leq \frac{\min\{\underline{\epsilon}, \underline{\mu}\}}{\max\{C_E, C_H\}} \min\{h_k\}(1-\delta), \quad 0 < \delta < 1, \tag{38}$$

where C_E *and* C_H *are the constants defined in Theorem 1, then, for the case of PEC and PMC boundary conditions, holds*

$$\max_{1 \leq m \leq M} \left(\|E^m - \tilde{E}^m\|_{L^2(\Omega)} + \|H_z^{m+1/2} - \tilde{H}_z^{m+1/2}\|_{L^2(\Omega)} \right) \leq C(\Delta t^2 + h^{\min\{s,N\}}) + C\left(\|E^0 - \tilde{E}^0\|_{L^2(\Omega)} + \|H_z^{1/2} - \tilde{H}_z^{1/2}\|_{L^2(\Omega)} \right)$$

and, for the case of Silver-Müller absorbing boundary conditions, holds

$$\max_{1 \leq m \leq M} \left(\|E^m - \tilde{E}^m\|_{L^2(\Omega)} + \|H_z^{m+1/2} - \tilde{H}_z^{m+1/2}\|_{L^2(\Omega)} \right) \leq C(\Delta t + h^{\min\{s,N\}}) + C\left(\|E^0 - \tilde{E}^0\|_{L^2(\Omega)} + \|H_z^{1/2} - \tilde{H}_z^{1/2}\|_{L^2(\Omega)} \right),$$

where C *is a generic constant independent of* Δt *and the mesh size* h*.*

Proof. From (26) and taking into account Lemma 8 and the estimates from previous lemmata, we obtain

$$
\begin{aligned}
&\min\{\underline{\epsilon},\underline{\mu}\}\left(\|\xi^M\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\right)\leq \max\{\bar{\epsilon},\bar{\mu}\}\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{1/2}\|^2_{L^2(\Omega)}\right)\\
&+\frac{\Delta t}{2}C^2_{inv}N^2\max\left\{h_k^{-1}\right\}\left(\|\eta_z^{1/2}\|^2_{L^2(\Omega)}+\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}+\|\xi^M\|^2_{L^2(\Omega)}\right)\\
&+\Delta t\delta(7+\beta_4)\left(\sum_{m=1}^{M-1}\|\xi^m\|^2_{L^2(\Omega)}+\sum_{m=1}^{M-1}\|\eta_z^{m+1/2}\|^2_{L^2(\Omega)}\right)\\
&+\frac{\Delta t\delta}{2}(7+\beta_4)\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\xi^M\|^2_{L^2(\Omega)}+\|\eta_{z_k}^{1/2}\|^2_{L^2(\Omega)}+\|\eta_{z_k}^{M+1/2}\|^2_{L^2(\Omega)}\right)\\
&+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+\beta_1}{2\min\{Z_k\}}\right)\|\xi^M\|^2_{L^2(\Omega)}\\
&+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+2\beta_2\beta_3}{2\min\{Y_k\}}\right)\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\\
&+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}(2+\beta_2)\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{1/2}\|^2_{L^2(\Omega)}\right)\\
&+\frac{Ch^{2\sigma}\Delta t}{\delta N^{2p}}\left(\overline{\epsilon}\int_0^T\left\|\frac{\partial E}{\partial t}\right\|^2_{H^p(\Omega)}dt+\overline{\mu}\int_{\Delta t/2}^{T+\Delta t/2}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{H^p(\Omega)}dt\right)\\
&+\frac{2C\Delta t^4}{\delta}\left(\int_{\Delta t/2}^{T+\Delta t/2}\left|\frac{\partial^2 E}{\partial t^2}\right|^2_{H^1(\Omega)}dt+\int_0^T\left|\frac{\partial^2 H_z}{\partial t^2}\right|^2_{H^1(\Omega)}dt\right)\\
&+\frac{2CTh^{2\sigma-2}}{\delta N^{2p+1}}\left(N^3+\left(1+\frac{\alpha}{\min\{Z_k^2\}}\right)C_\tau^2(N+1)(N+2)\right)\|E\|^2_{L^\infty(0,T;H^p(\Omega))}\\
&+\frac{2CTh^{2\sigma-2}}{\delta N^{2p+1}}\left(N^3+\left(1+\frac{\alpha}{\min\{Y_k^2\}}\right)C_\tau^2(N+1)(N+2)\right)\|H_z\|^2_{L^\infty(0,T;H^p(\Omega))}\\
&+\frac{\beta_4C_\tau^2(N+1)(N+2)\Delta t^2}{8\delta\min\{Z_k^2\}}\int_0^{T-\Delta t/2}\left\|\frac{\partial E}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt\\
&+\frac{\beta_4C_\tau^2(N+1)(N+2)\Delta t^2}{16\delta\min\{Y_k^2\}}\int_{\Delta t/2}^{T}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt,
\end{aligned}
$$

where $p\geq 0$, $\sigma=\min(p,N+1)$, C and C_τ are constants independent of $(E_{x_k},E_{y_k},H_{z_k})$, h_k and N, δ is an arbitrary positive constant, $\beta_4=0$ for

PEC and PMC boundary conditions and $\beta_4 = 1$ for Silver-Müller absorbing boundary conditions.

If (38) holds, using the discrete Gronwall's Lemma (see *e.g.* [7, 21]), we obtain

$$\|\xi^M\|^2_{L^2(\Omega)} + \|\eta_z^{M+1/2}\|^2_{L^2(\Omega)} \leq C(\epsilon,\mu,N)\Big(\|\xi^0\|^2_{L^2(\Omega)} + \|\eta_z^{1/2}\|^2_{L^2(\Omega)}$$
$$+\Delta t h^{2\sigma}\int_0^T \left\|\frac{\partial E}{\partial t}\right\|^2_{H^p(\Omega)} dt + \Delta t h^{2\sigma}\int_{\Delta t/2}^{T+\Delta t/2} \left\|\frac{\partial H_z}{\partial t}\right\|^2_{H^p(\Omega)} dt$$
$$+\Delta t^4\int_0^T \left|\frac{\partial^2 E}{\partial t^2}\right|^2_{H^1(\Omega)} dt + \Delta t^4\int_{\Delta t/2}^{T+\Delta t/2} \left|\frac{\partial^2 H_z}{\partial t^2}\right|^2_{H^1(\Omega)} dt$$
$$+h^{2\sigma-2}\|E^m\|^2_{L^\infty(0,T;H^p(\Omega))} + h^{2\sigma-2}\|H_z^{m+1/2}\|^2_{L^\infty(0,T;H^p(\Omega))}$$
$$+\beta_4\Delta t^2\int_0^{T-\Delta t/2} \left\|\frac{\partial E}{\partial t}\right\|^2_{L^\infty(\partial\Omega)} dt + \beta_4\Delta t^2\int_{\Delta t/2}^{T} \left\|\frac{\partial H_z}{\partial t}\right\|^2_{L^\infty(\partial\Omega)} dt\Big).$$

We complete the proof by using the triangle inequality and Lemma 9. ∎

Remark 1. *We want to remark that in the case of Silver-Müller absorbing boundary conditions we only get first order convergence in time. A possible way to recover second order convergence is to consider a locally implicit time scheme (see e.g.* [9]*). In order to keep efficiency, we propose an alternative which is explicit and second order convergent in time: for each time step solve (9)–(11) and save the solution in the variables* $(\tilde{\tilde{E}}^{m+1}_{x_k}, \tilde{\tilde{E}}^{m+1}_{y_k}, \tilde{\tilde{H}}^{m+3/2}_{z_k})$*. Then the numerical solution* $(\tilde{E}^{m+1}_{x_k}, \tilde{E}^{m+1}_{y_k}, \tilde{H}^{m+3/2}_{z_k})$ *is computed replacing in (9)–(11) the numerical flux by the following expression*

$$\begin{pmatrix} \dfrac{-n_y}{Z^+ + Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}] - \alpha\left(n_x\dfrac{[\tilde{E}_y^m] + [\tilde{\tilde{E}}_y^{m+1}]}{2} - n_y\dfrac{[\tilde{E}_x^m] + [\tilde{\tilde{E}}_x^{m+1}]}{2}\right)\right) \\ \dfrac{n_x}{Z^+ + Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}] - \alpha\left(n_x\dfrac{[\tilde{E}_y^m] + [\tilde{\tilde{E}}_y^{m+1}]}{2} - n_y\dfrac{[\tilde{E}_x^m] + [\tilde{\tilde{E}}_x^{m+1}]}{2}\right)\right) \\ \dfrac{1}{Y^+ + Y^-}\left(Y^+\left(n_x[\tilde{E}_y^{m+1}] - n_y[\tilde{E}_x^{m+1}]\right) - \alpha\dfrac{[\tilde{H}_z^{m+1/2}] + [\tilde{\tilde{H}}_z^{m+3/2}]}{2}\right) \end{pmatrix}.$$

6. Numerical results

In this section we present numerical results that support the theoretical results derived in the previous sections, namely the stability condition and the error estimates.

6.1. Stability condition. We can check numerically that (19) defines a sharp stability condition, in terms of the influence of N and $h_{min} = \min\{h_k\}$. In our experiments, we computed C that satisfies

$$\Delta t_{max} = \frac{C}{(N+1)(N+2)} h_{min}, \tag{39}$$

where Δt_{max} is the maximum observed value of Δt such that the method is stable. For these tests, the domain is the square $\Omega = (-1,1)^2$, the simulation final time is fixed at $T = 1$, the permittivity tensor is the identity matrix and $\mu = 1$. We consider equations (1)–(2) with initial conditions $E_x(x,y,0) = 0, E_y(x,y,0) = 0, H_z(x,y,\Delta t/2) = \cos(\pi x)\cos(\pi y)\cos(\pi\Delta t/2)$ in the case of PEC boundary conditions and $E_x(x,y,0) = 0, E_y(x,y,0) = 0, H_z(x,y,\Delta t/2) = \sin(\pi\Delta t/2)\sin(\pi xy)$ in the case of Silver-Müller absorbing boundary conditions.

In Table 1 and Table 2 the constant C is computed for different mesh sizes for PEC boundary conditions, considering respectively central and upwind fluxes in the DG method. In Table 3, C is computed for Silver-Müller boundary conditions, considering upwind fluxes in the DG method, with leap-frog time integration and modified leap-frog time integration.

h_{min}	$N=1$		$N=2$		$N=3$		$N=4$		$N=5$	
	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C
0.5657	0.1	1.06	0.058	1.23	0.037	1.30	0.026	1.37	0.019	1.41
0.2828	0.052	1.10	0.029	1.23	0.018	1.27	0.013	1.37	0.0095	1.41
0.1414	0.025	1.06	0.014	1.18	0.0093	1.31	0.0064	1.35	0.0047	1.39
0.0707	0.012	1.01	0.0072	1.22	0.0046	1.30	0.0032	1.35	0.0023	1.36
0.0354	0.0064	1.08	0.0036	1.22	0.0023	1.29	0.0016	1.35	0.001	1.18
0.0177	0.0032	1.08	0.0018	1.22	0.001	1.29	0.0008	1.28	0.00056	1.32

TABLE 1. Δt_{max} such that the method is stable and C computed by (39) for PEC boundary conditions and central flux.

h_{min}	$N=1$		$N=2$		$N=3$		$N=4$		$N=5$	
	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C
0.5657	0.055	0.58	0.031	0.66	0.019	0.67	0.013	0.68	0.0096	0.71
0.2828	0.026	0.55	0.014	0.59	0.0094	0.66	0.0064	0.67	0.0047	0.69
0.1414	0.013	0.55	0.0072	0.61	0.0046	0.65	0.0031	0.65	0.0023	0.68
0.0707	0.0065	0.55	0.0035	0.59	0.0023	0.65	0.0015	0.63	0.0011	0.65
0.0354	0.0032	0.54	0.0017	0.58	0.0010	0.56	0.00078	0.66	0.00057	0.67
0.0177	0.0016	0.54	0.00088	0.60	0.00057	0.64	0.00037	0.62	0.00026	0.61

TABLE 2. Δt_{max} such that the method is stable and C computed by (39) for PEC boundary conditions and upwind flux.

h_{min}	Leap-frog						Modified leap-frog					
	$N=1$		$N=2$		$N=3$		$N=1$		$N=2$		$N=3$	
	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C	Δt_{max}	C
0.5657	0.060	0.64	0.032	0.67	0.019	0.67	0.076	0.80	0.038	0.80	0.023	0.81
0.2828	0.028	0.59	0.015	0.64	0.0094	0.66	0.034	0.72	0.017	0.72	0.011	0.70
0.1414	0.013	0.55	0.0072	0.61	0.0046	0.65	0.016	0.67	0.0084	0.71	0.0055	0.77
0.0707	0.0065	0.55	0.0035	0.59	0.0023	0.65	0.0077	0.65	0.0041	0.69	0.0027	0.76
0.0354	0.0032	0.54	0.0017	0.58	0.0011	0.62	0.0037	0.62	0.0020	0.67	0.0013	0.73
0.0177	0.0016	0.54	0.00088	0.60	0.00057	0.64	0.0018	0.61	0.0010	0.67	0.00066	0.74

TABLE 3. Δt_{max} such that the method is stable and C computed by (39) for Silver-Müller boundary conditions and upwind flux.

As expected from the condition (19), the numerical examples in tables 1 and 2 show that the stability regions corresponding to central fluxes are slightly bigger when compared to the regions obtained using upwind fluxes. The same is observed when we compare the regions of stability for PEC boundary conditions and Silver-Müller boundary conditions, which is also according to (19). From Table 3, we perceive a gain in terms of stability on the modified leap-frog method. From all the examples presented, we may deduce that the right hand side of (19) is a sharp bound for Δt_{max}. Moreover, we can also conclude that Δt_{max} is directly proportional h_{min} and inversely proportional to $(N+1)(N+2)$.

6.2. Order of convergence. In this section, we will illustrate the theoretical results of convergence. We consider the model problem

$$\begin{aligned} \epsilon_{xx}\frac{\partial E_x}{\partial t}+\epsilon_{xy}\frac{\partial E_y}{\partial t} &= \frac{\partial H_z}{\partial y}+P(x,y,t), && (40)\\ \epsilon_{yx}\frac{\partial E_x}{\partial t}+\epsilon_{yy}\frac{\partial E_y}{\partial t} &= -\frac{\partial H_z}{\partial x}+Q(x,y,t), && (41)\\ \mu\frac{\partial H_z}{\partial t}=-\frac{\partial E_y}{\partial x} &+ \frac{\partial E_x}{\partial y}+R(x,y,t), && (42) \end{aligned}$$

defined in the square $\Omega=(-1,1)^2$. The source terms $P(x,y,t)$, $Q(x,y,t)$ and $R(x,y,t)$ were introduced in order to make it easier to find examples with known exact solution and consequently with the possibility to compute the error of the numerical solution. The simulation time is fixed at $T=1$ and in all tests we assume that $\mu=1$.

Two distinct situations are considered: in the first, the permittivity tensor ϵ is constant in the whole computational domain; in the second, the permittivity tensor varies in space.

For the first test we consider a symmetric and positive definite anisotropic constant permittivity tensor (3), with $\epsilon_{xx}=5, \epsilon_{xy}=\epsilon_{yx}=1$ and $\epsilon_{yy}=3$.

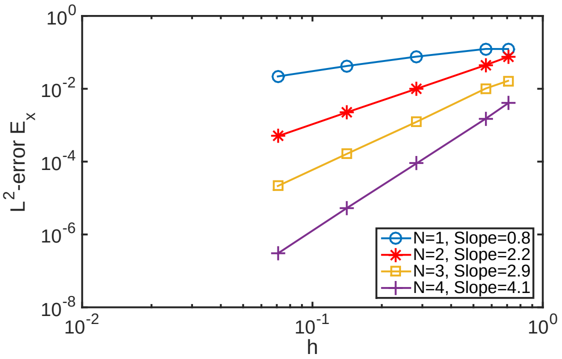
The problem (40)–(42) with PEC boundary conditions is completed with initial conditions and source terms P, Q and R such that is has the solution

$$\begin{aligned} E_x(x,y,t) &= \frac{-\pi}{\omega\epsilon_{xx}}\cos(\pi x)\sin(\pi y)\sin(\omega t),\\ E_y(x,y,t) &= \frac{\pi}{\omega\epsilon_{yy}}\sin(\pi x)\cos(\pi y)\sin(\omega t),\\ H_z(x,y,t) &= \cos(\pi x)\cos(\pi y)\cos(\omega t), \end{aligned}$$

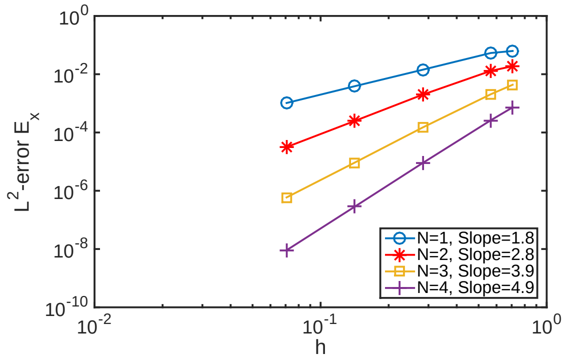
whith $\omega=\pi\sqrt{\frac{1}{\epsilon_{xx}}+\frac{1}{\epsilon_{yy}}}$.

For Silver-Müller absorbing boundary conditions, the initial conditions and source terms P, Q and R are defined such that the problem has the solution

$$\begin{aligned} E_x(x,y,t) &= -\sqrt{\frac{\epsilon_{yy}}{\det(\epsilon)}}\sin(\pi t)\sin(\pi x),\\ E_y(x,y,t) &= \sqrt{\frac{\epsilon_{xx}}{\det(\epsilon)}}\sin(\pi t)\sin(\pi y),\\ H_z(x,y,t) &= \sin(\pi t)\sin(\pi xy). \end{aligned}$$

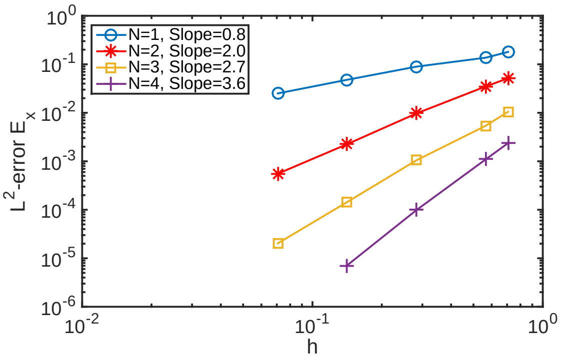


(A) Central flux.

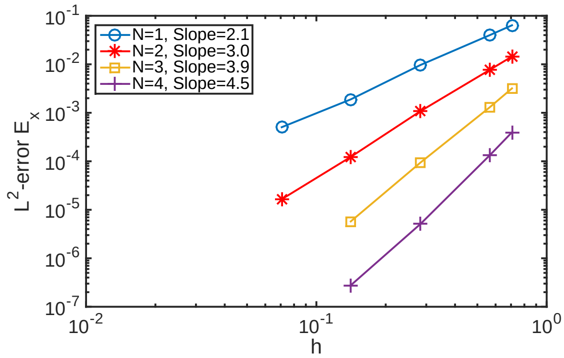


(B) Upwind flux.

FIGURE 1. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$ *versus* h, for constant permittivity tensor and PEC boundary conditions.



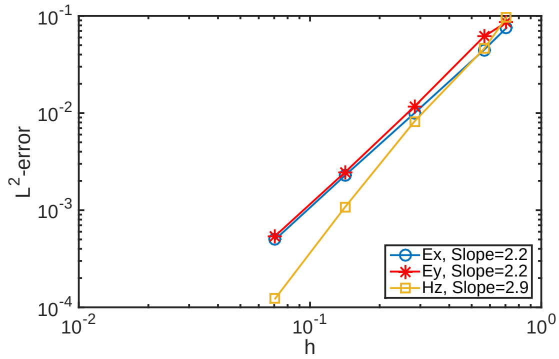
(A) Central flux.



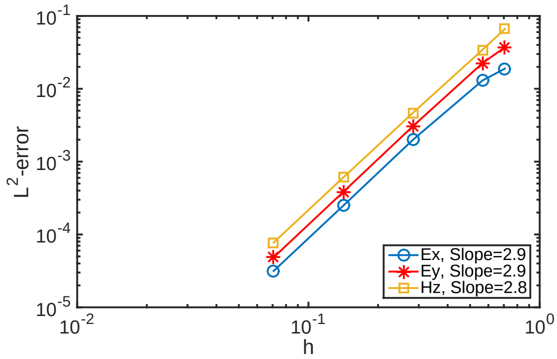
(B) Upwind flux.

FIGURE 2. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$ *versus* h, for constant permittivity tensor and Silver-Müller boundary conditions.

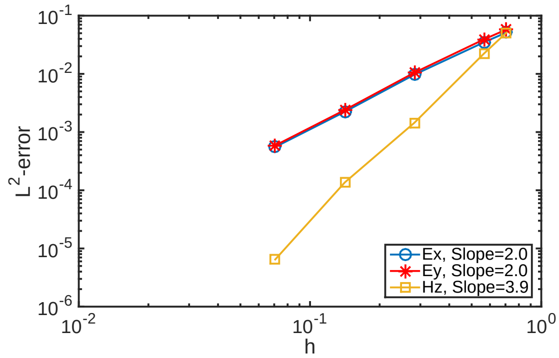
To illustrate the order of convergence in space, we fix $\Delta t = 10^{-5}$, except for figures 2 and 7, where we consider $\Delta t = 10^{-6}$ for $N = 4$. We plot of the error depending on the maximum element diameter for each mesh, where both the vertical and horizontal axis are scaled logarithmically. The numerical convergence rate is approximated by the slope of the linear regression line. In Figure 1, for PEC boundary conditions, and in Figure 2, for Silver-Müller absorbing boundary conditions, we plot the discrete L^2-error of the $\tilde{E}_x$ component of electric field, considering different degrees for the polynomial approximation while the spatial mesh is refined for both central and upwind fluxes. For central flux, the numerical convergence rate is close to the value estimated in Theorem 2, $\mathcal{O}(h^N)$, and for upwind flux we observe higher order



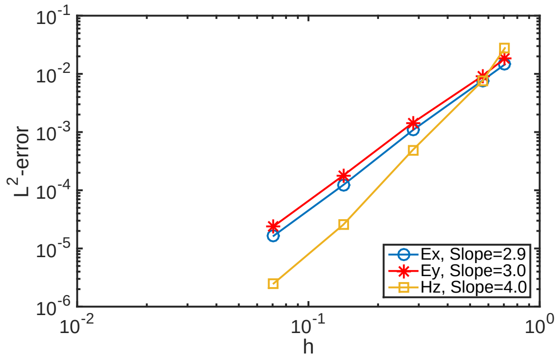
(A) PEC boundary conditions, central flux.



(B) PEC boundary conditions, upwind flux.



(C) Silver-Müller boundary conditions, central flux.



(D) Silver-Müller boundary conditions, upwind flux.

FIGURE 3. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$, $\|E_y^M - \tilde{E}_y^M\|_{L^2(\Omega)}$ and $\|H_z^{M+1/2} - \tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}$ *versus* h, for constant permittivity tensor.

of convergence, up to $\mathcal{O}(h^{N+1})$ in some cases. Similar results were obtained for $\tilde{E}_y$ and $\tilde{H}_z$ components as illustrated in Figure 3 for $N = 2$.

To visualize the convergence in time, the polynomials degree and the number of elements have been set to $N = 8$ and $K = 800$, respectively. For the case of PEC boundary conditions, the results in Figure 4 illustrate the second order of convergency established by Theorem 2. These results correspond to upwind flux and similar results are observed for central flux.

For Silver-Müller absorbing boundary conditions, the data plotted in Figure 5a illustrates the first order of convergency established by Theorem 2. The second order of convergence in time can be obtained considering the modified leap-frog time integration method proposed in Remark 1 as illustrated in Figure 5b.

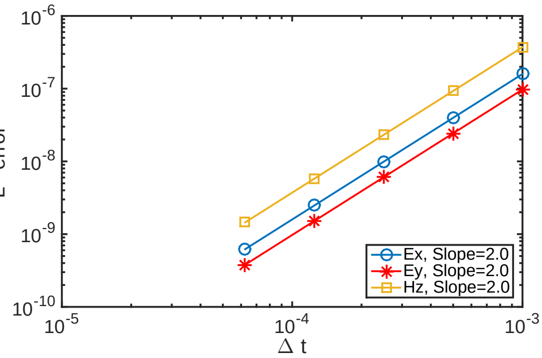
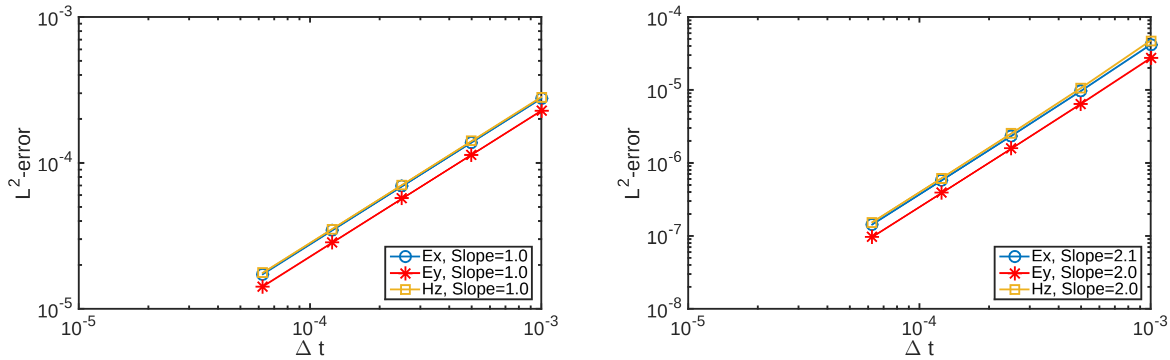


FIGURE 4. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$, $\|E_y^M - \tilde{E}_y^M\|_{L^2(\Omega)}$ and $\|H_z^{M+1/2} - \tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}$ *versus* Δt, for constant permittivity tensor and PEC boundary conditions.



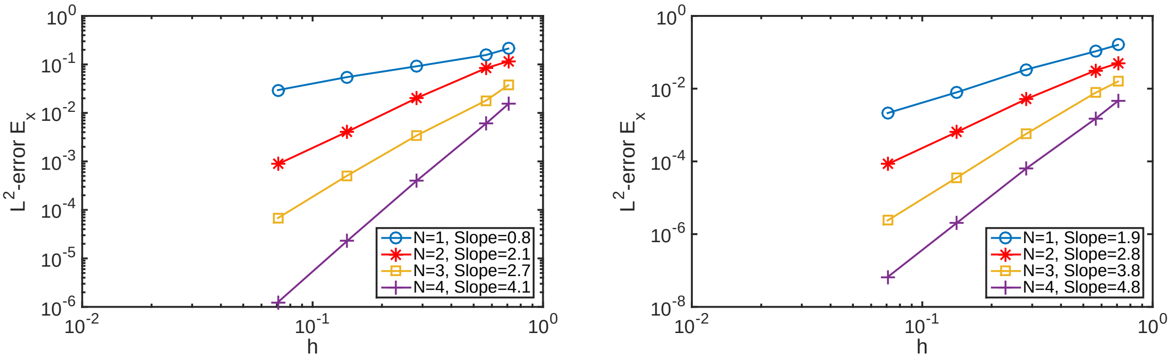
(A) Leap-frog time integration. (B) Modified leap-frog time integration.

FIGURE 5. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$, $\|E_y^M - \tilde{E}_y^M\|_{L^2(\Omega)}$ and $\|H_z^{M+1/2} - \tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}$ *versus* Δt, for constant permittivity tensor and Silver-Müller boundary conditions.

Now we consider the case were the permittivity tensor (3) is space dependent. For the numerical tests we consider

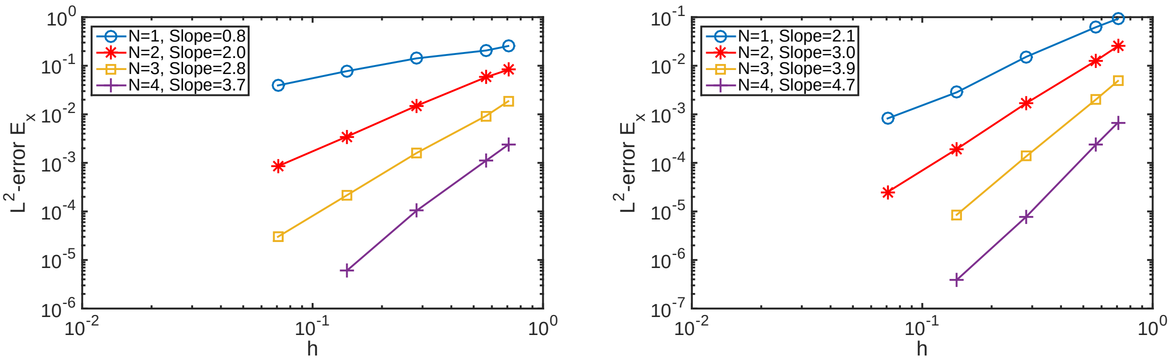
$$\epsilon(x,y) = \begin{pmatrix} 4x^2 + y^2 + 1 & \sqrt{x^2+y^2} \\ \sqrt{x^2+y^2} & x^2+1 \end{pmatrix}.$$

The exact solution for both PEC and Silver-Müller boundary conditions is the same as previous test. The source terms P, Q and R in (40)–(42) are changed due to dependency of the tensor to space. In this test we use the same parameters and repeat the experiments. The results for the spatial convergency for both PEC and Silver-Müller boundary conditions are plotted



(A) Central flux. (B) Upwind flux.

FIGURE 6. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$ *versus* h, for space dependent permittivity tensor and PEC boundary conditions.



(A) Central flux. (B) Upwind flux.

FIGURE 7. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$ *versus* h, for space dependent permittivity tensor and Silver-Müller boundary conditions.

in figures 6 and 7. As in the first example, for central flux, the order of the error is near $\mathcal{O}(h^N)$, and for upwind flux we observe higher order. The results plotted in Figure 8 show the second order convergency in time for PEC. For the case of Silver-Müller boundary conditions, in Figure 9a we observe first order convergency in time, while in Figure 9b we perceive that the second order is recovered with the modified method presented in Remark 1.

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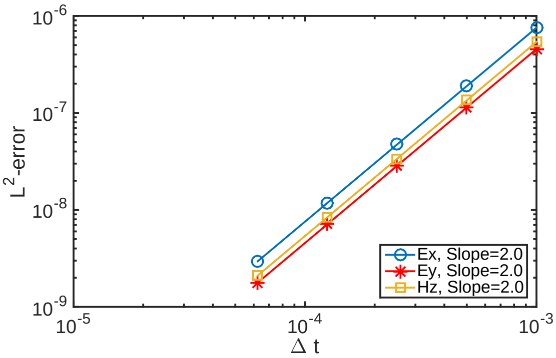
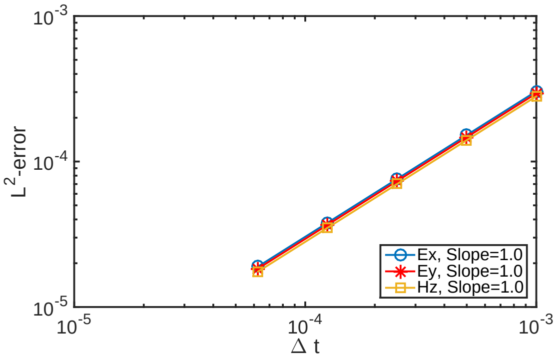
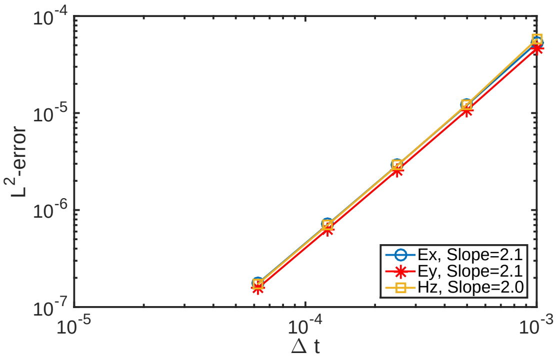


FIGURE 8. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$, $\|E_y^M - \tilde{E}_y^M\|_{L^2(\Omega)}$ and $\|H_z^{M+1/2} - \tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}$ *versus* Δt, for space dependent permittivity tensor and PEC boundary conditions.



(A) Leap-frog time integration.



(B) Modified leap-frog time integration.

FIGURE 9. $\|E_x^M - \tilde{E}_x^M\|_{L^2(\Omega)}$, $\|E_y^M - \tilde{E}_y^M\|_{L^2(\Omega)}$ and $\|H_z^M - \tilde{H}_z^M\|_{L^2(\Omega)}$ *versus* Δt, for space dependent permittivity tensor and Silver-Müller boundary conditions.

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References

[1] J. Álvarez González. *A Discontinuous Galerkin Finite Element Method for the Time-Domain Solution of Maxwell Equations*. PhD thesis, Universidad de Granada, 2014.

[2] A. Araújo, S. Barbeiro, L. Pinto, F. Caramelo, A. L. Correia, M. Morgado, P. Serranho, A. S. C. Silva, and R. Bernardes. Maxwell's equations to model electromagnetic wave's propagation through eye's structures. *Proceedings of the 13th International Conference on Computational and Mathematical Methods in Science and Engineering,CMMSE 2013, Almeria, Ian Hamilton and Jesús Vigo-Aguiar Eds*, 1:121–129, June 2013.

[3] L. Babuška and M. Suri. The optimal convergence rate of the p-version of the finite element method. *SIAM Journal on Numerical Analysis*, 24(4):750–776, 1987.

[4] M. Bon and E. Wolf. *Principles of Optics*. Cambridge: Cambridge University Press, 7 edition, 1999.

[5] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. Studies in mathematics and its applications. North-Holland, Amsterdam, New-York, 1980.

[6] A. Dunn and R. Richards-Kortum. Three-dimensional computation of light scattering from cells. *IEEE Journal of Selected Topics in Quantum Electronics*, 2(4):898–905, Dec 1996.

[7] E. Emmrich. Stability and error of the variable two-step BDF for semilinear parabolic problems. *Journal of Applied Mathematics and Computing*, 19(1-2):33–55, 2005.

[8] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[9] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno. Convergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell's equations on unstructured meshes. *ESAIM: Mathematical Modeling and Numerical Analysis*, 39(6):1149–1176, 2005.

[10] E. H. Georgoulis. Inverse-type estimates on *hp*-finite element spaces and applications. *Mathematics of Computation*, 77(261):201–219, 2008.

[11] J. Hesthaven and T. Warburton. *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Springer Publishing Company, Incorporated, 1st edition, 2008.

[12] J. S. Hesthaven and T. Warburton. Discontinuous Galerkin methods for the time-domain Maxwell's equations: An introduction. *ACES Newsletter*, 19:10–29, 2004.

[13] M. König, K. Busch, and J. Niegemann. The discontinuous Galerkin time-domain method for Maxwell's equations with anisotropic materials. *Photonics and Nanostructures-Fundamentals and Applications*, 8(4):303–309, 2010.

[14] J. Li, J. W. Waters, and E. A. Machorro. An implicit leap-frog discontinuous Galerkin method for the time-domain Maxwell's equations in metamaterials. *Computer Methods in Applied Mechanics and Engineering*, 223:43–54, 2012.

[15] T. Lu, P. Zhang, and W. Cai. Discontinuous Galerkin methods for dispersive and lossy Maxwell's equations and PML boundary conditions. *Journal of Computational Physics*, 200(2):549–580, 2004.

[16] M. J. Piket-May, J. B. Troy, and A. Taflove. Electrodynamics of visible-light interactions with the vertebrate retinal rod. *Optics Letters*, 18(8):568–570, 1993.

[17] B. Riviére. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008.

[18] B. Riviére, M. F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM Journal on Numerical Analysis*, 39(3):902–931, 2001.

[19] A. Taflove and S. Hagness. *Computational Electrodynamics: The Finite-difference Time-domain Method*. Artech House antennas and propagation library. Artech House, 2005.

[20] H. C. van de Hulst. *Light Scattering by Small Particles*. Dover Publications, New York, 1981.

[21] J. Wloka. *Partial Differential Equations*. Cambridge University Press, Cambridge, 1987.

[22] K. S. Yee. Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Transaction on Antennas and Propagation*, 14(3):302–307, 1966.

Appendix A. Technical lemmata

The lemmata included this section are technical tools needed to derive the stability condition and the convergence estimates.

We consider the following trace inequalities (see *e.g.* [17]).

Lemma 7. *Let T_k be an element of $\mathcal{T}_h$ with diameter h_k and let f_k be an edge of T_k. There exists a positive constant C independent of h_k such that, for any $u \in H^1(T_k)$,*

$$\|u\|_{L^2(f_k)} \leq C\sqrt{\frac{|f_k|}{|T_k|}}\left(\|u\|_{L^2(T_k)} + h_k\|\nabla u\|_{L^2(T_k)}\right). \tag{43}$$

Moreover, if u is a polynomials of degree less than or equal to N, there exists a positive constant C_{trace} independent of h_k and u but dependent on the polynomials degree N, such that

$$\|u\|_{L^2(f_k)} \leq C_{trace}\sqrt{\frac{|f_k|}{|T_k|}}\|u\|_{L^2(T_k)}.$$

An exact expression for the constant C_{trace} can be given as a function of the polynomials degree, and the following inequality holds for any $u \in P_N(T_k)$

$$\|u\|_{L^2(f_k)} \leq \sqrt{\frac{(N+1)(N+2)}{2}\frac{|f_k|}{|T_k|}}\|u\|_{L^2(T_k)}. \tag{44}$$

Consequently, there exists a positive constant C_τ independent of h_k and N but depent on the shape-regularity h_k/τ_k, where τ_k is the diameter of the largest inscribed ball contained in T_k (see (8)), such that, for any $u \in P_N(T_k)$,

$$\|u\|_{L^2(f_k)} \leq C_\tau\sqrt{(N+1)(N+2)}h_k^{-1/2}\|u\|_{L^2(T_k)}. \tag{45}$$

The next result is an inverse-type estimate ([5, 10]), where we present explicitly the dependence of the constant on the polynomials degree.

Lemma 8. *Let us consider $T_k \in \mathcal{T}_h$ with diameter h_k. There exists a positive constant C_{inv} independent of h_k and N such that, for any $u \in P_N(T_k)$,*

$$\|u\|_{H^q(T_k)} \leq C_{inv}N^{2q}h_k^{-q}\|u\|_{L^2(T_k)}, \tag{46}$$

where $q \geq 0$.

Note that C_{inv} depends on the shape-regularity h_k/τ_k, where τ_k is the diameter of the largest inscribed ball contained in T_k. A sharper estimate reads

$$\forall u \in P_N(T_k), \quad \|u\|_{H^q(T_k)} \leq \tilde{C}_{inv} N^{2q} \tau_k^{-q} \|u\|_{L^2(T_k)}.$$

The reader can refer to [3] or [18] for the following approximation properties.

Lemma 9. *Let $T_k \in \mathcal{T}_h$ and $u \in H^p(T_k)$. Then there exits a constant C depending on p and on the shape-regularity of T_k but independent of u, h_k and N and a sequence $\mathcal{P}_N u \in P_N(T_k)$, $N = 1, 2, \ldots$, such that, for any $0 \leq q \leq p$*

$$\|u - \mathcal{P}_N u\|_{H^q(T_k)} \leq C \frac{h_k^{\sigma - q}}{N^{p-q}} \|u\|_{H^p(T_k)}, \quad p \geq 0, \tag{47}$$

$$\|u - \mathcal{P}_N u\|_{L^2(f_k)} \leq C \frac{h_k^{\sigma - 1/2}}{N^{p-1/2}} \|u\|_{H^p(T_k)}, \quad p > \frac{1}{2}, \tag{48}$$

where $\sigma = \min(p, N+1)$ and f_k is an edge of T_k.

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