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GENERATING SUBLOCALES BY SUBSETS AND RELATIONS: A TANGLE OF ADJUNCTIONS

M. ANDREW MOSHIER, JORGE PICADO AND ALEŠ PULTR

ABSTRACT: Generalizing the obvious representation of a subspace $Y \subseteq X$ as a sublocale in $\Omega(X)$ by the congruence $\{(U,V) \mid U \cap Y = V \cap Y\}$ one obtains the congruence $\{(a,b) \mid \mathfrak{o}(a) \cap S = \mathfrak{o}(b) \cap S\}$, first with sublocales S of a frame L, which (as it is well known) produces back the sublocale S itself, and then with general subsets $S \subseteq L$. The relation of such S with the sublocale produced is studied (the result is not always the sublocale generated by S). Further, one discusses in general the associated adjunctions, in particular that of relations on L and subsets of L and views the aforementioned phenomena in this perspective.

KEYWORDS: frame, locale, sup-lattice, sublocale, sublocale lattice, frame congruence, nucleus, saturation, frame quotient theorem, cozero part.

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Introduction

Consider a subset (subspace) Y of a topological space X. The sublocale (generalized subspace) corresponding to Y in the frame of open sets $\Omega(X)$ is associated with the congruence

$$\{(U,V) \mid U \cap Y = V \cap Y,\ U,V \in \Omega(X)\}$$

(which, of course, comes from the quotient map $\Omega(j)\colon \Omega(X) \to \Omega(Y)$ where $j\colon Y \to X$ is the embedding: $\Omega(j)(U) = j^{-1}[U] = U \cap Y$). More generally, any sublocale S of a frame L is obtained as the quotient L/R_S where

$$R_S = \{(a,b) \mid \mathfrak{o}(a) \cap S = \mathfrak{o}(b) \cap S\}$$

($\mathfrak{o}(x)$ are the open sublocales associated with $x \in L$).

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Now take an arbitrary subset $A \subseteq L$. Unlike a general subset of a topological space, which always carries a subspace, such an A does not, in general, make immediate (point-free) topological sense. But the congruence

$$R_A = \{(a,b) \mid \mathfrak{o}(a) \cap A = \mathfrak{o}(b) \cap A\}$$

produces a sublocale L/R_A — we call it $\mathsf{sat}(A)$ — even if A is not one. One naturally asks what is its relation to the set A (for instance whether it is the sublocale generated by A). The study of this relationship is one of the main motivations of this article.

We start with a much more transparent situation of sup-lattices (generalizing frames) from [8] and their meet-subsets (generalizing sublocales). There, the situation is simple, and we show, in Section 2, that the procedure creates for any subset the smallest meet-set containing it. In connection with that we encounter an adjunction between relations on L and subsets of L; this is described in Section 3.

Turning to frames one soon learns that $\mathsf{sat}(A)$ is not always the smallest sublocale containing A, as one might at the first sight assume. In fact it does not have to contain A at all. This is analyzed in Sections 4 and 5. Among other we show that $A \subseteq \mathsf{sat}(A)$ if and only if the meet-subset generated by A is already a sublocale, and on the other hand we present an example of the set of cozero elements with this inclusion that does not satisfy a condition that otherwise naturally relates meet-subsets and sublocales.

In Section 6 we discuss the adjunction of relations vs. subobjects in this special setting and show how the mentioned phenomena appear in its perspective, and in Section 7 we finish the article presenting a localic version of the frame quotient theorem.

1. Preliminaries

1.1. Recall from [8] the category **SupLat** of sup-lattices, with complete lattices for objects and $\bigvee$-homomorphisms (mappings preserving arbitrary suprema) for morphisms. The right Galois adjoint of a $\bigvee$-homomorphism $f\colon K \to L$ will be denoted by $f_*\colon L \to K$. The correspondence $f \mapsto f_*$ gives rise to a natural duality

$$\mathbf{SupLat} \cong \mathbf{SupLat}^{\mathrm{op}}.$$

1.2. For a relation $R \subseteq L \times L$ on a sup-lattice L call an $s \in L$ *weakly saturated* (more precisely, *weakly R-saturated*) if

$$aRb \quad \Rightarrow \quad (a \le s \text{ iff } b \le s).$$

Obviously

- a meet of weakly saturated elements is weakly saturated

and hence we have the least weakly saturated upper bound

$$\kappa(x) = \kappa_R(x) = \bigwedge\{s \mid x \le s,\ s \text{ weakly saturated}\}$$

of x giving rise to a monotone mapping $\kappa\colon L \to L$ such that

$$x \le \kappa(x) \quad \text{and} \quad \kappa\kappa(x) = \kappa(x).$$

If we set

$$L/_w R = \kappa[L] = \{x \mid x = \kappa(x)\}$$

we obtain a sup-lattice (with, in general, the suprema differing from those in L) and a $\bigvee$-homomorphism

$$\kappa' = (x \mapsto \kappa(x))\colon L \to L/_w R.$$

It is a standard fact that

- $xRy \;\Rightarrow\; \kappa'(x) = \kappa'(y)$, and
- if for a $\bigvee$-homomorphism $h\colon L \to K$ one has $xRy \Rightarrow h(x) = h(y)$ then there is precisely one $\bigvee$-homomorphism $\overline{h}\colon L/_w R \to K$ such that $\overline{h} \cdot \kappa' = h$; moreover, for $x \in L/_w R$, $\overline{h}(x) = h(x)$.

1.3. A *frame* L is a complete lattice satisfying

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{1.3.1}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* $h\colon L \to K$ is a $\bigvee$-homomorphism that preserves, moreover, all finite meets. Thus, the resulting category of frames **Frm** is a subcategory of **SupLat**, not a full one.

A typical frame is the lattice $\Omega(X)$ of open sets of a topological space X, and a typical frame homomorphism is $\Omega(f) = (U \mapsto f^{-1}[U])$ where $f\colon X \to Y$ is a continuous map. Thus we have a contravariant functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$. Frames can be viewed as generalized spaces (for sober spaces, Ω is a full embedding); hence it is of advantage to modify Ω to a covariant functor $\mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$. The category $\mathbf{Frm}^{\mathrm{op}}$ is called the category of *locales* and denoted by **Loc** (see, e.g., [6]). It is expedient to represent the morphisms of **Loc** as the right adjoints h_* of frame homomorphisms. We then speak of such maps as of *localic maps* and think of them as of *generalized continuous* maps between generalized spaces.

The distributivity law (1.3.1) states that the maps $(x \mapsto x \wedge b)\colon L \to L$ preserve all suprema. Consequently, they have right adjoints $(y \mapsto (b \to y))\colon L \to L$ which results in a Heyting operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c.$$

A frame homomorphism does not necessarily preserve the Heyting operation. Nevertheless, the operation $\to$ plays an important role.

For more about frames see e.g. [7, 9, 10, 11].

1.4. Analogously as in 1.2 we have quotients constructed as follows. We call an $s \in L$ *saturated* (more precisely, *R-saturated*) if

$$aRb \quad \Rightarrow \quad \forall c,\ a \wedge c \leq s \ \text{ iff } \ b \wedge c \leq s$$

(in other words,

$$aRb \quad \Rightarrow \quad a \to s = b \to s).$$

Again, a meet of saturated elements is saturated, we have a monotone mapping $\kappa = (x \mapsto \kappa(x)) = \bigwedge\{s \mid x \leq s,\ s \text{ saturated}\}$ satisfying

$$x \leq \kappa(x), \quad \kappa\kappa(x) = \kappa(x) \quad \text{and, moreover,} \quad \kappa(x \wedge y) = \kappa(x) \wedge \kappa(y)$$

(the *nucleus* of R), and if we set

$$L/R = \{x \mid x = \kappa(x)\}$$

we obtain a frame homomorphism $\kappa' = (x \mapsto \kappa(x))\colon L \to L/R$ such that

- $xRy \ \Rightarrow \ \kappa'(x) = \kappa'(y)$, and
- if for a frame homomorphism $h\colon L \to K$ one has $xRy \Rightarrow h(x) = h(y)$ then there is precisely one frame homomorphism $\overline{h}\colon L/R \to K$ such that $\overline{h} \cdot \kappa' = h$; moreover, for $x \in L/R$, $\overline{h}(x) = h(x)$

(see, e.g. [10]).

1.5. Onto frame homomorphisms h are precisely the extremal monomorphisms in **Frm**; consequently, the associated one-to-one localic morphisms $f = h_*$ (the extremal epimorphisms in **Loc**) naturally model embeddings of subspaces. This leads to the concept of a sublocale $S \subseteq L$ as a subset satisfying

(S1) $A \subseteq S \ \Rightarrow \ \bigwedge A \in S$, and
(S2) if $x \in L$ and $s \in S$ then $x \to s \in S$.

Sublocales are precisely the images $j[K]$ of one-to-one localic maps j, which is the same as the L/R obtained from arbitrary relations $R \subseteq L \times L$ (see, e.g. [10]).

In the context of sup-lattices we will consider the *meet-sets* $M \subseteq L$ satisfying

(M) $A \subseteq M \;\Rightarrow\; \bigwedge A \in M$.

1.5.1. The set of all sublocales of a frame L will be denoted by $\mathcal{S}(L)$. Ordered by inclusion it is a complete lattice. The meets in $\mathcal{S}(L)$ coincide with the intersections, and the joins are defined by

$$\bigvee S_i = \{\bigwedge A \mid A \subseteq \bigcup S_i\}. \tag{$*$}$$

$\mathcal{S}(L)$ is a co-frame, that is, the opposite $\mathcal{S}(L)^{\mathrm{op}}$ is a frame.

Similarly we have, for any sup-lattice L the complete lattice $\mathscr{M}(L)$ of all the meet-sets. Again, the meets coincide with the intersections and the joins are given by the formula $(*)$. Thus, if L is a frame then $\mathcal{S}(L)$ is a complete sublattice of $\mathscr{M}(L)$.

The intersection of any system of sublocales of a frame (resp. meet-subsets of a sup-lattice) is a sublocale (resp. meet-set). Thus, for any subset A of a frame (resp. sup-lattice) we have the smallest sublocale $\mathsf{sl}(A)$ containing A resp. the smallest meet-set $\mathsf{m}(A)$ containing A. Thus we have monotone maps

$$\mathsf{sl}\colon \mathscr{P}(L) \to \mathcal{S}(L) \quad \text{resp.} \quad \mathsf{m}\colon \mathscr{P}(L) \to \mathscr{M}(L)$$

($\mathscr{P}(L)$ is the power-set of L), obviously right adjoints to the inclusion maps $j\colon \mathcal{S}(L) \subseteq \mathscr{P}(L)$ resp. $j\colon \mathscr{M}(L) \subseteq \mathscr{P}(L)$. By abuse of notation, we will also use the symbol sl for the restriction of sl to $\mathscr{M}(L) \to \mathcal{S}(L)$. Note that, trivially,

$$\mathsf{m}(A) = \{\bigwedge B \mid B \subseteq A\}.$$

1.6. In analogy with closed (open) subspaces of spaces we have closed sublocales

$$\mathfrak{c}(a) = {\uparrow}a = \{x \mid x \geq a\}$$

and open sublocales

$$\mathfrak{o}(a) = \{x \mid a{\to}x = x\} = \{a{\to}x \mid x \in L\}.$$

$\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other and we have $\mathfrak{o}(a{\wedge}b) = \mathfrak{o}(a){\cap}\mathfrak{o}(b)$, $\mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$, $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b)$ and $\mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$.

2. The relation associated with subspaces and sublocales

2.1. Let X be a topological space and let $A \subseteq X$ be a subset. The resulting subspace can be represented as the sublocale

$$\Omega(X)/\rho(A)$$

where

$$\rho(A) = \{(U, V) \mid U \cap A = V \cap A\}.$$

2.2. In the point-free context we have a more general theorem about sublocales (see [10, VI.1.4.1]).

Proposition. *Let S be a sublocale of a frame L. Set*

$$\rho(S) = \{(a, b) \in L \times L \mid \mathfrak{o}(a) \cap S = \mathfrak{o}(b) \cap S\}.$$

Then

$$S = L/\rho(S).$$

2.3. Obviously, $\rho(S)$ can be equivalently defined as

$$\rho(S) = \{(a, b) \in L \times L \mid \mathfrak{c}(a) \cap S = \mathfrak{c}(b) \cap S\} = \{(a, b) \mid {\uparrow}a \cap S = {\uparrow}b \cap S\}.$$

2.3.1. The last formula can be adopted for sup-lattices. In analogy with 2.2 we have

Proposition. *If $M \subseteq L$ is a meet-set and $\rho(M) = \{(a, b) \mid {\uparrow}a \cap M = {\uparrow}b \cap M\}$ then $L/_w\, \rho(M) = M$.*

Proof: Indeed, set $\overline{x} = \bigwedge\{m \in M \mid x \leq m\}$. Since M is a meet-set, $\overline{x} \in M$. Obviously, $x \leq y \Rightarrow \overline{x} \leq \overline{y}$, $\overline{\overline{x}} = \overline{x}$, and $(a, b) \in \rho(M)$ iff $\overline{a} = \overline{b}$. If $m \in M$, $(a, b) \in \rho(M)$, and $a \leq m$ then $b \leq \overline{b} = \overline{a} \leq \overline{m} = m$, and we see m is weakly saturated. If m is weakly saturated then, as $\overline{m} = \overline{\overline{m}}$, we have $(m, \overline{m}) \in \rho(M)$ and hence $\overline{m} \leq m$ and $m = \overline{m} \in M$. ∎

3. The contravariant adjunction ρ vs. ε

3.1. The formula from 2.3.1 defines a

$$\rho\colon \mathscr{P}(L) \to \mathrm{Rel}(L) = \mathscr{P}(L \times L).$$

On the other hand, for a relation $R \subseteq L \times L$ consider the set

$$\varepsilon(R) = L/_w R.$$

This set can be expediently described as follows. For $(a, b) \in L \times L$ set

$$Q(a, b) = \{x \mid a \le x \text{ iff } b \le x\}.$$

Then obviously

$$\varepsilon(R) = \bigcap\{Q(a, b) \mid (a, b) \in R\}. \tag{3.1.1}$$

3.2. Proposition. *ε is an antitone map* $\mathrm{Rel}(L) \to \mathscr{P}(L)$*, and ε and ρ are contravariantly adjoint on the right, that is,*

$$R \subseteq \rho(A) \quad \textit{iff} \quad A \subseteq \varepsilon(R).$$

Proof: Let $R \subseteq \rho(A)$ and let $x \in A$. If $(a, b) \in R$ then $(a, b) \in \rho(A)$ and hence $x \in Q(a, b)$. Since this holds for all $(a, b) \in R$, $x \in \bigcap\{Q(a, b) \mid (a, b) \in R\} = \varepsilon(R)$.

On the other hand, let $A \subseteq \varepsilon(R)$ and let $(a, b) \in R$. Then in particular $A \subseteq Q(a, b)$ and for all $x \in A$, $x \ge a$ iff $x \ge b$. Thus, ${\uparrow}a \cap A = {\uparrow}b \cap A$, and $(a, b) \in \rho(A)$. ■

3.3. Observation. *For every relation $R \subseteq L \times L$, $\varepsilon(R)$ is a meet-set, and for every $A \subseteq L$, $\rho(A)$ is a $\bigvee$-congruence.*

(The first statement follows from the definition; further, if ${\uparrow}a_i \cap A = {\uparrow}b_i \cap A$ for all i and if $\bigvee a_i \le x \in A$ then $a_i \le x$ for all i, hence $b_i \le x$ for all i, and $\bigvee b_i \le x$.)

3.4. Proposition. $\varepsilon\rho(A) = L/_w\rho(A) = \mathsf{m}(A)$*, the smallest meet-set containing A.*

Proof: By 3.2 $A \subseteq \varepsilon(\rho(A))$ and $\varepsilon(\rho(A))$ is a meet-set. Now let M be a meet-set and let $A \subseteq M$. Then $\varepsilon\rho(A) \subseteq \varepsilon\rho(M) = L/_w\rho(M) = M$ by 2.3.1. ■

3.5. Comparing Propositions 2.3.1 and 3.4 with 2.2 one may conjecture that in the frame context we should have $L/\rho(A) = \mathsf{sl}(A)$, the sublocale generated by $A \subseteq L$. But this is not generally true: in fact, A is not necessarily a subset of $L/\rho(A)$. We will discuss the relation of $A \subseteq L$ and $\mathsf{sl}(A)$ in the next section.

4. Saturation and generating sublocales

4.1. If A is a subset of a frame L we will say that an $s \in L$ is A-saturated if it is $\rho(A)$-saturated (recall 1.4), that is, if

$$\uparrow a \cap A = \uparrow b \cap A \;\Rightarrow\; a \to s = b \to s. \tag{4.1.1}$$

Obviously this condition is equivalent to

$$\uparrow a \cap A \subseteq \uparrow b \;\Rightarrow\; a \to s \leq b \to s. \tag{4.1.2}$$

The set of all A-saturated elements, that is, the sublocale $L/\rho(A)$ will be denoted by

$$\mathsf{sat}(A).$$

4.2. Lemma. $\mathsf{sat}(A) \subseteq \mathsf{m}(A)$.

Proof: Let $s \in \mathsf{sat}(A)$ and let x be a lower bound of $A \cup \uparrow s$. Then $A \cup \uparrow s \subseteq \uparrow x$ and hence $1 = s \to s \leq x \to s$, hence $x \leq s$, and we see that $s = \bigwedge(A \cap \uparrow s)$, and $A \cap \uparrow s \subseteq A$. ■

4.3. Proposition. *The following statements on a subset A in a frame are equivalent.*

(1) $\mathsf{sl}(A) = \mathsf{sat}(A) = \mathsf{m}(A)$.
(2) $\mathsf{sl}(A) = \mathsf{sat}(A)$.
(3) $A \subseteq \mathsf{sat}(A)$.

Proof: Trivially (1)⇒(2) and (2)⇒(3).
(3)⇒(1): Let $A \subseteq \mathsf{sat}(A)$ Then by 2.2, $\mathsf{sl}(A) \subseteq \mathsf{sat}(A)$ and by 4.2, $\mathsf{sl}(A) \subseteq \mathsf{sat}(A) \subseteq \mathsf{m}(A) \subseteq \mathsf{sl}(A)$. ■

4.4. Here is a simple criterion for $A \subseteq \mathsf{sat}(A)$ (and hence $\mathsf{sat}(A) = L/\rho(A) = \mathsf{sl}(A)$).

Proposition. *Let A be such that*

$$\forall a \in A,\ \forall x \in L, \quad x \to a \in A. \tag{H}$$

Then $A \subseteq \mathsf{sat}(A)$.

Proof: Let $A \cap {\uparrow}u \subseteq {\uparrow}v$. Let $a \in A$ and set $x = u \to a$. Then $u \le x \to a \in A$ and hence $v \le x \to a$ and finally $u \to a = x \le v \to a$. ■

4.5. We will see that the property (H) is a precise counterpart of the meet-structure extending a sup-lattice to a frame, and hence the fact comes as no surprise (see 5.4 below). Rather, it may be a slight surprise that the condition is not necessary, and that there is an important case of an A with $A \subseteq \mathsf{sat}(A)$ which does not satisfy (H).

Recall the relation $\prec$ ($a \prec b$ iff $a^* \vee b = 1$ where a^* is the pseudocomplement – see the standard literature on frames, also for regularity, the relation $\prec\!\!\prec$ and complete regularity we will speak of below). A relation $\lhd \subseteq L \times L$ is said to be a $*$-*inclusion* if

($*1$) $a \lhd b \;\Rightarrow\; a \prec b$ (i.e. $a^* \vee b = 1$),
($*2$) $a \lhd b \;\Rightarrow\; b^* \lhd a^*$, and
($*3$) $\forall a \in L$, $a = \bigvee\{x \mid x \lhd a\}$.

Thus for instance the strong inclusions from Banaschewski [1] are $*$-inclusions. But also the relation $\prec$ itself is one in a regular L.

A subset $A \subseteq L$ is $\lhd$-*dense* if for each $x \lhd y$ there is an $a \in A$ such that $x \le a \lhd y$.

4.6. Proposition. *Let* $\lhd$ *be a* $*$-*inclusion in a regular* L *and let* A *be a* $\lhd$-*dense subset closed under finite joins. Then* $A \subseteq \mathsf{sat}(A)$.

Proof: Let $A \cap {\uparrow}u \subseteq {\uparrow}v$ and let $a \in A$. We need to show that $u \to a \le v \to a$.

Let $x, y \in L$ be such that $x \lhd y \lhd u \to a$. Then $y^* \lhd x^*$ and because of the $\lhd$-density we can find a $c \in A$ such that $y^* \le c \lhd x^*$. Then

$$x \wedge c = 0 \quad \text{and} \quad c \vee (u \to a) \ge y^* \vee (u \to a) = 1$$

so that

$$u = u \wedge (c \vee (u \to a)) = (u \wedge c) \vee (u \wedge (u \to a)) \le c \vee a \in A \cap {\uparrow}u$$

and $v \le c \vee a$. Now (as $x \wedge c = 0$)

$$x \wedge v \le x \wedge (c \vee a) = x \wedge a \le a \quad \text{and hence} \quad x \le v \to a.$$

Recalling the choice of x and y, using ($*3$) twice, we conclude that $u \to a \le v \to a$. ■

4.7. Recall that in classical topology, a cozero set in a space X is a preimage $f^{-1}[\mathbb{R}\smallsetminus\{0\}]$ where $f\colon X\to\mathbb{R}$ is a continuous map. Cozero sets, hence, are special open sets. The system of cozero sets is closed under countable unions and finite intersections.

All that can be precisely translated in the point-free setting, but it is much easier to work with the following equivalent definition. An element a of a frame L is *cozero* if

$$a=\bigvee\{a_n\mid a_n \prec\!\!\prec a,\ n=1,2,\dots\} \tag{4.7.1}$$

The set of all cozero elements will be denoted by $\operatorname{Coz} L$. It is obviously a σ-frame (that is, a lattice with countable joins and with the distributivity (1.3.1) assumed for such joins) and a sub-σ-frame of L (see e.g. [3, 5]).

In a completely regular L the formula (4.7.1) can be replaced by

$$a=\bigvee a_n,\ a_1\prec\!\!\prec a_2\prec\!\!\prec\cdots\prec\!\!\prec a_n\prec\!\!\prec\cdots \tag{4.7.2}$$

from which we easily infer that in a completely regular frame,

1. if $a\prec\!\!\prec b$ then there is a $c\in\operatorname{Coz}L$ such that $a\prec\!\!\prec c\prec\!\!\prec b$ (hence, it is $\prec\!\!\prec$-dense even in a stronger sense than required in 4.5), and
2. each $a\in L$ is a join of cozero elements.

Thus, the very important subset $A=\operatorname{Coz}L\subseteq L$ satisfies the conditions of 4.6 and hence $A\subseteq \mathsf{sat}(A)$. However, there are completely regular frames L such that $\operatorname{Coz}L$ does not satisfy (H) (see e.g. [4, 2]).

4.8. The mapping $\mathsf{sl}\colon \mathscr{M}(L)\to\mathcal{S}(L)$ (1.5.1) is a right adjoint of the inclusion $j\colon\mathcal{S}(L)\to\mathscr{M}(L)$. But j has also a left adjoint $\mathsf{ls}\colon\mathscr{M}(L)\to\mathcal{S}(L)$ where $\mathsf{ls}(A)$ is the *largest* sublocale contained in A. This construction plays a role e.g. in the image-preimage adjunction for a localic map (see [10]). Note that $A\in\mathscr{M}(L)$ is essential; ls does not work for the embedding $\mathcal{S}(L)\subseteq\mathscr{P}(L)$. Here it will help us to understand better the general relationship between A and $\mathsf{sat}(A)$.

4.8.1. Lemma. $\mathsf{sat}(A)=\mathsf{sat}(\mathsf{m}(A))$.

Proof: If $M\subseteq A$ then $\bigwedge M\ge u$ iff $a\ge u$ for each $a\in M$. Consequently, $\mathsf{m}(A)\cap{\uparrow}u\subseteq{\uparrow}v$ iff $A\cap{\uparrow}u\subseteq v$. ∎

4.8.2. Proposition. *We have*

$$\mathsf{sat}(A)=\mathsf{ls}(\mathsf{m}(A)).$$

Proof: By 4.2, $\mathsf{sat}(A) \subseteq \mathsf{m}(A)$ and by 4.1 it is a sublocale; hence $\mathsf{sat}(A) \subseteq \mathsf{ls}(\mathsf{m}(A))$. The operation sat is obviously monotone, and by propositions 4.4 and 4.3, if S is a sublocale then $\mathsf{sat}(S) = S$. Thus, if S is a sublocale and $S \subseteq \mathsf{m}(A)$ then $S = \mathsf{sat}(S) \subseteq \mathsf{sat}(\mathsf{m}(A))$ and hence, by the lemma, $S \subseteq \mathsf{sat}(A)$ so that $\mathsf{sat}(A)$ is the largest sublocale contained in $\mathsf{m}(A)$. ■

5. More about the inclusion $A \subseteq \mathsf{sat}(A)$

5.1. Consider an element $b \in L$ and the closed sublocale $\mathfrak{c}(b) = \uparrow b$. In $\mathfrak{c}(b)$, the element b is the zero, and we have the pseudocomplement

$$x^{*b} = x \to b$$

and the *relative rather below* relation

$$x \prec_b y \quad \text{iff} \quad x^{*b} \vee y = 1.$$

Now if L is regular, $\mathfrak{c}(b)$ is regular, as every sublocale of L, and we have for $x \geq b$, $x = \bigvee \{y \mid y \prec_b x\}$.

In the sequel, b, x^{*b} and $\prec_b$ will be always used in the sense just indicated.

5.2. Theorem. *Let A be a subset of a regular L. Set $b = \bigwedge A$ and consider the following statements.*

(1) *A is $\prec_b$-dense in $\mathfrak{c}(b)$ and closed under finite joins.*
(2) *$A \subseteq \mathsf{sat}(A)$.*
(3) *$\mathsf{m}(A)$ is $\prec_b$-dense in $\mathfrak{c}(b)$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof: (The proof of the first implication is in fact a repetition of the proof of 4.6 but we do it in detail, because the circumstances are changed.)

(1)$\Rightarrow$(2): Let A be $\prec_b$-dense in $\mathfrak{c}(b)$ and let $A \cap \uparrow u \subseteq \uparrow v$. Pick an $a \in A$ and x, y with $x \prec_b y \prec_b u \to a$. Then

$$y^{*b} \prec_b x^{*b} \quad \text{and} \quad y^{*b} \vee (u \to a) = 1.$$

Using the $\prec_b$-density choose a $c \in A$ with $y^{*b} \leq c \prec_b x^{*b}$. Then

$$x \wedge c = b \quad \text{and} \quad c \vee (u \to a) \geq y^{*b} \vee (u \to a) = 1$$

and hence (use the closedness of A under $\vee$)

$$u = u \wedge (c \vee (u \to a)) = (u \wedge c) \vee (u \wedge (u \to a)) \leq c \vee a \in A \cap \uparrow u.$$

Thus, $v \leq c \vee a$ and since $a \in A$ and hence $a \geq b$,

$$x \wedge v \leq x \wedge (c \vee a) = (x \wedge c) \vee (x \wedge a) = b \vee (x \wedge a) \leq a$$

and we conclude that $x \leq v \to a$, and since $x \prec_b y \prec_b u \to a$ were otherwise arbitrary, finally $u \to a \leq v \to a$.

(2)⇒(3): By 4.3, $\mathsf{m}(A)$ is a sublocale, and since $b \in \mathsf{m}(A)$ we have $x^{*b} = x \to b \in \mathsf{m}(A)$ for every x. In particular, $x^{*b*b} \in \mathsf{m}(A)$ and hence, if $x \prec_b y$ we can insert $x \leq x^{*b*b} \prec_b y$. ■

5.2.1. Note. The previous statement is, of course, still far from a necessary and sufficient condition (in particular, the requirement of the finite joins in (1) is very strong: for instance, (2) is trivial for A a sublocale, and a sublocale typically is not closed under joins). On the other hand, in view of the fact that $A \subseteq \mathsf{sat}(A)$ iff $\mathsf{m}(A) = \mathsf{sat}(A) = \mathsf{sat}(\mathsf{m}(A))$, the discrepancy in the density condition in (1) and (2) does not seem to be quite so bad.

5.3. From 4.4 we immediately obtain

Fact. *For every up-set A, that is, every $A = {\uparrow}A = \{x \mid \exists a \in A,\ x \geq a\}$, we have $A \subseteq \mathsf{sat}(A)$.*

(This is also obvious from the following observation: if $A = {\uparrow}A$ then $A = \bigcup_{a\in A} {\uparrow}a = \bigcup_{a\in A} \mathfrak{c}(a)$ and hence $\mathsf{m}(A) = \bigvee_{a\in A} \mathfrak{c}(a)$ in the coframe of sublocales.)

5.4. On the other hand, for down-sets, that is, the $A = {\downarrow}A = \{x \mid \exists a \in A,\ x \leq a\}$, the inclusion $A \subseteq \mathsf{sat}(A)$ is rare. We will analyze the case of the $A = {\downarrow}u$ generated by a single element u.

Obviously

$$\mathsf{m}({\downarrow}u) = {\downarrow}u \cup \{1\}.$$

5.4.1. Lemma. ${\downarrow}u \subseteq \mathsf{sat}({\downarrow}u)$ *iff*

$$\forall a \leq u,\ x \wedge y \leq a \ \Rightarrow\ (x \leq a \text{ or } y \leq u). \tag{$*$}$$

Notes. 1. The implication $(*)$ is to be taken literally, that is, with the order of x, y as indicated: if $x \not\leq a$ then y has to be $\leq u$ even if $x \leq u$.

2. Note that $(*)$ is a stronger form of primeness; in particular, applying the implication for $a = u$ we see that u has to be prime.

Proof: By 4.3, ${\downarrow}u \subseteq \mathsf{sat}({\downarrow}u)$ iff $\mathsf{m}({\downarrow}u) = {\downarrow}u \cup \{1\}$ is a sublocale.

Let $\mathsf{m}({\downarrow}u)$ be a sublocale, let $a \le u$ and let $x \wedge y \le a$, that is, $y \le x \to a$. We have $x \to a \in \mathsf{m}({\downarrow}u)$ and hence if $x \nleq a$, that is, $x \to a \ne 1$, $y \le x \to a \le u$.

On the other hand, let $(*)$ hold and let $a \in \mathsf{m}({\downarrow}u)$. If $a = 1$, $x \to a = 1 \in \mathsf{m}({\downarrow}u)$ trivially. Else, $a \le u$ and since $x \wedge (x \to a) \le a$ we have either $x \le a$ and $x \to a = 1$ or $x \to a \le u$. ■

5.4.2. Proposition. *We have* ${\downarrow}u \subseteq \mathsf{sat}({\downarrow}u)$ *in a frame* L *if and only if* u *is a prime and* $L = {\downarrow}u \cup {\uparrow}u$.

Proof: Let $(*)$ hold and let $x \nleq u$. Suppose that also $x \ngeq u$ so that $a = u \wedge x \ne x, u$. As $u \nleq a$ we have to have $x \le u$, a contradiction. The primeness follows applying $(*)$ for $a = u$.

On the other hand, let $L = {\downarrow}u \cup {\uparrow}u$ and let u be prime. Then $(*)$ holds for $a = u$. Thus, let $a < u$ and $x \wedge y \le a$, that is, $y \le x \to a$. If $y \nleq u$ then $u \le y \le x \to a$ and $u \wedge x \le a$. Assuming $u \le x$ leads to the excluded $u = u \wedge x \le a$ so that $x \le u$ and $x = u \wedge x \le a$. ■

6. The ρ-ε adjunction for frames

6.1. An H*-subset* of a frame is an $A \subseteq L$ satisfying the property (H) from 4.4. Obviously an intersection of H-sets is an H-set and hence we have the smallest H-set $\mathsf{h}(M)$ containing an arbitrary subset $M \subseteq L$, resulting in a map

$$\mathsf{h}\colon \mathscr{P}(L) \to \mathscr{H}(L)$$

left adjoint to the embedding $\mathsf{j}\colon \mathscr{H}(L) \to \mathscr{P}(L)$.

6.1.1. Note. Hence, $\mathscr{H}(L)$ is a complete lattice with meets coinciding with the intersections. Note that the joins are easily seen to be again the

$$\bigvee H_i = \{\bigwedge M \mid M \subseteq \bigcup H_i\}$$

(since $x \to \bigwedge M = \bigwedge\{x \to m \mid m \in M\}$). Hence $\mathcal{S}(L)$ is a complete sublattice of $\mathscr{H}(L)$.

6.2. Since every sublocale is an H-set we have that $\mathsf{h}(A) \subseteq \mathsf{sl}(A)$. Obviously $\mathsf{sl}(A) = \mathsf{sl}(\mathsf{h}(A))$, and by 4.4. $\mathsf{h}(A) \subseteq \mathsf{sat}(\mathsf{h}(A))$. Hence, by 4.3

$$\mathsf{sl}(A) = \mathsf{sl}(\mathsf{h}(A)) = \mathsf{sat}(\mathsf{h}(A)) = \mathsf{m}(\mathsf{h}(A))$$

resulting in the commutative triangle

$$\begin{tikzcd}
\mathscr{P}(L) \arrow[d, "\mathsf{h}"'] \arrow[dr, "\mathsf{sl}"] & \\
\mathscr{H}(L) \arrow[r, "\mathsf{m}"'] & \mathcal{S}(L)
\end{tikzcd}$$

Thus, recalling the facts from 4.3 we can describe the situation in the following tangle of Galois adjunctions

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathscr{P}(L) \arrow[r, shift right=2, "\mathsf{m}"'] \arrow[r, phantom, "\top"] \arrow[d, shift right=2, "\mathsf{h}"'] \arrow[d, phantom, "\dashv"] \arrow[dr, shift right=1, "\mathsf{sl}"'] \arrow[dr, phantom, "\top"] \arrow[dr, bend left, "\mathsf{sat}"] & \mathscr{M}(L) \arrow[l, shift right=2, "\mathsf{j}"'] \arrow[d, bend right, "\mathsf{sl}_M"'] \arrow[d, bend left, "\mathsf{ls}"] \\
\mathscr{H}(L) \arrow[u, shift right=2, "\mathsf{j}"'] \arrow[r, shift right=2, "\mathsf{m}_H"'] \arrow[r, phantom, "\top"] & \mathcal{S}(L) \arrow[l, shift right=2, "\mathsf{j}"'] \arrow[ul, shift right=1, "\mathsf{j}"'] \arrow[u, "\mathsf{j}" description]
\end{tikzcd}$$

Note that the only arrow in this diagram which is not adjoint to anything is sat.

6.3. A relation $R \subseteq L \times L$ is $\wedge$-*stable* if

$$(a_i, b_i) \in R, i = 1,2 \quad \Rightarrow \quad (a_1 \wedge a_2, b_1 \wedge b_2) \in R$$

(thus, if L is a frame, the frame congruences are precisely the $\wedge$-stable $\bigvee$-congruences). The set of all $\wedge$-stable relations will be denoted by

$$\mathrm{Rel}_\wedge(L).$$

6.3.1. Proposition. *Let A be an H-subset of a frame. Then $\rho(A)$ is $\wedge$-stable.*

Proof: Let $\uparrow a \cap A = \uparrow b \cap A$ and let $x \in \uparrow (a \wedge c) \cap A$. Then $a \wedge c \leq x$, hence $a \leq c \to x \in A$ and $(c \to x) \in \uparrow a \cap A = \uparrow b \cap A$, so that $b \leq c \to x$ and $b \wedge c \leq x$. ∎

6.3.2. Corollary. *If A is an H-subset of a frame then $\rho(A)$ is a frame congruence.*

6.4. Proposition. *Let A be a meet-subset of L. Then the following statements are equivalent:*
(1) *A is a sublocale (that is, an H-set).*
(2) *$\rho(A)$ is $\wedge$-stable.*
(3) *$\rho(A)$ is a frame congruence.*

Proof: (1)⇒(2) is in 6.3.1 and by 3.3, (2)≡(3).
(2)⇒(1): Let $a \in A$ and let $x \in L$ be arbitrary. Set $c = x \to a$ and $\kappa(c) = \bigwedge\{y \in A \mid c \leq y\}$. Then obviously $(c, \kappa(c)) \in \rho(A)$ and by the $\wedge$-stability $(c \wedge x, \kappa(c) \wedge x) \in \rho(A)$. We have $c \wedge x \leq a \in A$ and hence $\kappa(c) \wedge x \leq a$ so that $\kappa(c) = \kappa(x \to a) \leq x \to a$ and hence $x \to a = \kappa(x \to a) \in A$. ■

6.4.1. Since $\varepsilon(R)$ is always a meet-set we obtain

Corollary. *If $\rho\varepsilon(R)$ is $\wedge$-stable then $\varepsilon(R)$ is a sublocale.*

6.4.2. Proposition. *If R is $\wedge$-stable then $\varepsilon(R)$ is a sublocale.*

Proof: Suppose $u \in \varepsilon(R)$ and $x \to u \notin \varepsilon(R)$ for some x. Then there is an $(a, b) \in R$ such that $x \to u \in Q(a, b)$ and hence, say, $a \leq x \to u$ and $b \nleq x \to u$. But then $a \wedge x \leq u$ and $b \wedge x \nleq u$, while $(a \wedge x, b \wedge x) \in R$. ■

6.5. Recall the contravariant Galois adjunction

$$\mathscr{P}(L) \underset{\varepsilon}{\overset{\rho}{\rightleftarrows}} \mathrm{Rel}(L)$$

from 3.2. By 6.3.2 and 6.4.2 this now restricts, for a frame L, to an adjunction

$$\mathscr{H}(L) \underset{\varepsilon'}{\overset{\rho'}{\rightleftarrows}} \mathrm{Rel}_\wedge(L)$$

Consider the compositions

$$\mathscr{P}(L) \underset{\mathsf{j}\varepsilon'=\overline{\varepsilon}}{\overset{\rho'\mathsf{h}=\overline{\rho}}{\rightleftarrows}} \mathrm{Rel}_\wedge(L)$$

We have here again a contravariant adjunction on the right. Indeed

$$R \subseteq \rho\,\mathsf{h}(A) \quad \text{iff} \quad \mathsf{h}(A) \subseteq \varepsilon'(R) \quad \text{iff} \quad A \subseteq \mathsf{j}\,\varepsilon'(R).$$

Now, finally, we have a counterpart to the identity $\mathsf{m} = \varepsilon\rho$ from 3.4.

6.5.1. Proposition. *For any subset $A \subseteq L$, $\mathsf{sl}(A) = \overline{\varepsilon}\,\overline{\rho}(A)$.*

Proof: By 6.2 $\overline{\varepsilon}\,\overline{\rho}(A) = \mathsf{j}\,\varepsilon'\rho'\mathsf{h}(A) = \mathsf{j}\,\mathsf{mh}(A) = \mathsf{j}(\mathsf{sl}(A))$. ■

7. The quotient theorems in view of the adjunctions

7.1. For a relation R on a sup-lattice L and a subset $A \subseteq L$ we will write

$$R \dashv\vdash A$$

if $\varepsilon(R) = \varepsilon\rho(A)$ or, equivalently, $\rho(A) = \rho\varepsilon(R)$.

7.2. Observations. 1. If $R = \rho(M)$ or $A = \varepsilon(R)$ then $R \dashv\vdash M$.

2. If $R \dashv\vdash A$ then $\rho\varepsilon(R) \dashv\vdash A$ and $R \dashv\vdash \varepsilon\rho(A)$.

3. More generally, if $R \dashv\vdash A$, and if $R \subseteq R' \subseteq \rho\varepsilon(R)$ and $A \subseteq A' \subseteq \varepsilon\rho(A)$ then $R' \dashv\vdash A'$.

(For 2 use the identities $\varepsilon\rho\varepsilon = \varepsilon$ and $\rho\varepsilon\rho = \rho$, and for 3 the obvious fact that if $R_i \dashv\vdash A$ for $i = 1, 2$ and $R_1 \subseteq R \subseteq R_2$ then $R \dashv\vdash A$, and similarly for A.)

7.3. Proposition. *Let $R \dashv\vdash A$ and let $f\colon K \to L$ be the right adjoint of a $\bigvee$-homomorphism $h\colon L \to K$. Then*

$$\forall (a, b) \in R,\ h(a) = h(b) \quad \textit{iff} \quad \rho(A) \subseteq \rho(f[K]).$$

Hence, if $h(a) = h(b)$ for all $(a, b) \in R$, then $f[K] \subseteq \mathsf{m}(A)$.

Proof: $\Leftarrow$: Let $\rho(A) \subseteq \rho f[K]$, that is, $f[K] \subseteq \varepsilon\rho(A)$. Then, as $R \dashv\vdash A$, $f[K] \subseteq \varepsilon(R)$ and by (3.1.1), $f[K] \subseteq \bigcap\{Q(a, b) \mid (a, b) \in R\}$, and we have

$$\forall x \in K\ \forall (a, b) \in R, \quad a \le f(x) \text{ iff } b \le f(x),$$

that is,

$$\forall x \in K\ \forall (a, b) \in R, \quad h(a) \le x \text{ iff } h(b) \le x.$$

Hence $(a, b) \in R$ implies that $h(a) = h(b)$.

$\Rightarrow$: If $(a, b) \in R$ implies $h(a) = h(b)$, that is, $h(a) \le x$ iff $h(b) \le x$, we have for all $x \in K$ that $(a, b) \in R$ implies $a \le f(x)$ iff $b \le f(x)$. In other words, $(a, b) \in R$ implies that $f(x) \in Q(a, b)$, and $f(x) \in \bigcap\{Q(a, b) \mid (a, b) \in R\} = \varepsilon(R)$, so that $f[K] \subseteq \varepsilon\rho(A)$. ■

7.4. Modifying the definition of $R \dashv\vdash A$ to $\wedge$-stable relations and $\overline{\varepsilon}$, $\overline{\rho}$ as in 6.5 we obtain by the same procedure

Theorem. *Let $f\colon K \to L$ be a localic map and let $h\colon L \to K$ be its left adjoint. Let $R \dashv\vdash A$. Then*

$$\forall (a,b) \in R,\ h(a) = h(b) \quad \textit{iff} \quad f[K] \subseteq \mathsf{sl}(A).$$

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M. Andrew Moshier
Faculty of Mathematics, Chapman University, Orange, CA 92866, USA
E-mail address: `moshier@chapman.edu`

Jorge Picado
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `picado@mat.uc.pt`

Aleš Pultr
Department of Applied Mathematics and ITI, MFF, Charles University, Malostranské nám. 24, 11800 Praha 1, Czech Republic
E-mail address: `pultr@kam.ms.mff.cuni.cz`