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DISCRETE PAINLEVÉ EQUATIONS FOR RECURRENCE COEFFICIENTS OF LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS OF CLASS ONE

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Abstract: In this paper we study recurrences for Laguerre-Hahn orthogonal polynomials of class one. It is shown for some families of such Laguerre-Hahn polynomials that the coefficients of the three term recurrence relation satisfy some forms of discrete Painlevé equations, namely, dP_I and dP_{IV}.

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1. Introduction

Laguerre-Hahn orthogonal polynomials are related to Stieltjes functions, S, that satisfy a Riccati differential equation with polynomial coefficients [11, 20, 23]

$$AS' = BS^2 + CS + D\,. \tag{1}$$

The study of recurrences for orthogonal polynomials related to the differential equations of general type (1) is connected to a wide range of subjects in mathematics, such as probability theory [16], differential equations [22, 25], constructive approximation [17], etc. In particular, the semiclassical case - $B \equiv 0$ in (1) - has been much studied in the literature due to the well-known connections with integrable systems and Painlevé equations (see, amongst many others, [19, 28]).

In the literature of orthogonal polynomials, the study of Laguerre-Hahn orthogonal polynomials proceeds in two directions. One of them is the study within the framework of modifications of measures and the analysis of the

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corresponding perturbations on the sequences of orthogonal polynomials. Indeed, Laguerre-Hahn families of orthogonal polynomials may appear as a result of modifications of semi-classical orthogonal polynomials, for example, through some perturbations of the coefficients of the three term recurrence relation [6, 10, 26], or through spectral rational modifications of the Stieltjes functions [7, 13, 29]. Another direction of study concerns the problem of classification of families of orthogonal polynomials in terms of classes of differential equations (1), which amounts to the classification in terms of classes of distributional equations. In more specific terms, given information on the polynomial coefficients of (1), the goal is to describe the systems of equations for the recurrence relation coefficients of the corresponding sequence of orthogonal polynomials, the so-called Laguerre-Freud equations [3, 4, 12, 15, 18]. Generally, such equations are given in terms of non-linear systems, whose complexity increases with the degrees of the polynomials in (1). In this topic one should emphasize the vast literature on the semi-classical case, showing that recurrence coefficients of semi-classical orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations (see, amongst many others, [14, 19, 28]).

In the present work we focus on the difference equations for the recurrence relation coefficients of Laguerre-Hahn orthogonal polynomials in class one (cf. Section 2), $\max\{\deg(A), \deg(B)\} \leq 3,\ \deg(C) = 2$ or $\max\{\deg(A), \deg(B)\} = 3,\ \deg(C) \leq 2$ in (1). The symmetric case has been analysed in [1], but the problem of a classification of such a class of orthogonal polynomials remains open. Our main results, contained in Section 3, show that the recurrence relation coefficients of some families of orthogonal polynomials belonging to Laguerre-Hahn class one in the the non-symmetric case are governed by difference equations of the Painlevé type, namely, dP_{IV} and dP_I. The main tool is the so-called structure relations for Laguerre-Hahn polynomials [11, Eqs. (3.9), (3.10)] which have been recently re-interpreted and studied within the theory of matrix Sylvester equations in [7, Theorem 1].

The rest of the paper is organized as follows. In Section 2 we give preliminary results on Laguerre-Hahn orthogonal polynomials. Section 3 is devoted to the derivation of recurrences for Laguerre-Hahn orthogonal polynomials and to show the connection to the discrete Painlevé equation dP_{IV} and dP_I. In Section 4 we present a few illustrative examples.

2. Preliminary Results

Let $\mathbb{P} = \operatorname{span}\{z^k : k \in \mathbb{N}_0\}$ be a linear space of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space. We will denote the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ by $\langle u, f\rangle$.

Given the moments of u, $u_n = \langle u, x^n\rangle$, $n \geq 0$, where we take $u_0 = 1$, the principal minors of the corresponding Hankel matrix are defined by $H_n = \det(u_{i+j})_{i,j=0}^{n}, n \geq 0$, where, by a convention, $H_{-1} = 1$. The functional u is said to be *quasi-definite* (respectively, *positive-definite*) if $H_n \neq 0$ (respectively, $H_n > 0$), for all $n \geq 0$.

Definition 1. (see [27]) Let $u \in \mathbb{P}'$ and let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials such that $\deg(P_n) = n\,,\ n \geq 0\,.$ $\{P_n\}_{n\geq 0}$ is said to be a sequence of orthogonal polynomials with respect to u if

$$\langle u, P_n P_m\rangle = h_n \delta_{n,m}\,,\ h_n \neq 0\,, \quad n, m \geq 0\,. \tag{2}$$

Throughout the paper we shall take each P_n monic, that is, $P_n(z) = z^n +$ lower degree terms, and we will abbreviate a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ by SMOP.

The equivalence between the quasi-definiteness of $u \in \mathbb{P}'$ and the existence of a SMOP with respect to u is well known in the literature on orthogonal polynomials (see [9, 27]). Furthermore, if u is positive-definite, then it has an integral representation in terms of a positive Borel measure μ supported on an infinite point set I of the real line such that

$$\langle u, x^n\rangle = \int_I x^n\, d\mu(x)\,,\ \ n \geq 0\,, \tag{3}$$

and the orthogonality condition (2) becomes

$$\int_I P_n(x) P_m(x) d\mu(x) = h_n \delta_{n,m}\,,\ \ h_n > 0\,, \quad n, m \geq 0\,.$$

If μ is an absolutely continuous measure supported on I, and w denotes its Radon-Nikodyn derivative with respect to the Lebesgue measure, i.e. $d\mu(x) = w(x)dx$, then we will also say that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to w.

Monic orthogonal polynomials satisfy a three term recurrence relation [27]

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 1, 2, ... \tag{4}$$

with $P_0(x) = 1,\ P_1(x) = x - \beta_0$ and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = 1.$

Definition 2. (see [9]) Let $\{P_n\}_{n\geq 0}$ be a SMOP with respect to $u \in \mathbb{P}'$. A sequence of *associated polynomials of the first kind* is defined by

$$P_n^{(1)}(x) = \langle u_t, \frac{P_{n+1}(x) - P_{n+1}(t)}{x-t} \rangle , \quad n \geq 0 ,$$

where u_t denotes the action of u on the variable t.

The sequence $\{P_n^{(1)}\}_{n\geq 0}$ also satisfies a three term recurrence relation

$$P_n^{(1)}(x) = (x-\beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x) , \quad n = 1, 2, ... \tag{5}$$

with $P_{-1}^{(1)}(x) = 0$, $P_0^{(1)}(x) = 1$.

Definition 3. The *Stieltjes function* of $u \in \mathbb{P}'$ is defined by $S(x) = \displaystyle\sum_{n=0}^{+\infty} \frac{u_n}{x^{n+1}}$, where u_n are the moments of u.

The function S has a continued fraction expansion given by

$$S(x) = \cfrac{1}{x-\beta_0 - \cfrac{\gamma_1}{x-\beta_1 - \cfrac{\gamma_2}{\ddots}}} \tag{6}$$

where the γ's and the β's are the coefficients in the three term recurrence relation for the corresponding SMOP. Note that if u is positive-definite, defined by (3), then S is the so-called Borel or Stieltjes transform of the measure,

$$S(x) = \int_I \frac{d\mu(t)}{x-t} , \quad x \in \mathbb{C} \setminus I .$$

In account of (6), the Stieljtes functions corresponding to $\{P_n\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$, denoted by S and $S^{(1)}$, respectively, are related by [29]

$$\gamma_1 S^{(1)}(x) = -\frac{1}{S(x)} + (x-\beta_0) . \tag{7}$$

Definition 4. (see [23]) A Stieltjes function S and the corresponding linear functional u, are said to be *Laguerre-Hahn* if there exist polynomials A, B, C, D, such that S satisfies a Riccati differential equation

$$AS' = BS^2 + CS + D , \quad A \neq 0 . \tag{8}$$

A sequence of orthogonal polynomials related to S (or u) is then called Laguerre-Hahn. If $B = 0$, then S (or u) is said to be *semi-classical*.

Eq. (8) is equivalent to the distributional equation [11, Theorem 3.1]

$$\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)\,, \quad \psi = A' + C\,. \tag{9}$$

The polynomial D can be written in terms of A, B, C as

$$D = -(u\theta_0 A)'(x) + (u\theta_0(A' + C))(x) + (u^2\theta_0^2 B)(x)\,.$$

The definition of $\theta_0 p$, $p \in \mathbb{P}$, as well as the right product of a linear functional by a polynomial, and the product of two linear functionals, can by found in [11, 23].

Remark . Let $B \equiv 0$. If $\deg(\psi) = 1$, $\deg(A) \leq 2$, then u is called a *classical functional*, and the corresponding orthogonal polynomials are the so-called classical orthogonal polynomials, that is, Hermite, Laguerre and Jacobi polynomials. The case $\deg(\psi) \geq 1$ corresponds to the semi-classical case.

If u is positive-definite, defined in terms of a weight w, then the semi-classical character of u, that is, $\mathcal{D}(Au) = \psi u$ or $AS' = CS + D$, is equivalent to the Pearson equation $Aw' = Cw$, with w satisfying the boundary conditions [23]

$$x^n A(x)w(x)|_{a,b} = 0\,, \quad n \geq 0\,,$$

where a, b (eventually a or b infinite) are linked with the roots of A. In such a case, w is the weight function on the support $I = [a, b]$.
In the semi-classical case, integral representations for u are known [21]. Let us emphasize that, in general, the problem of representing Laguerre-Hahn linear functionals is an open problem.

Note that the triple (A, ψ, B) satisfying (9) is not unique, indeed, many triples of polynomials can be associated with such a distributional equation, but only one canonical set of minimal degree exists. To a Laguerre-Hahn linear functionals one associates the class, a non-negative integer, defined as follows.

Definition 5. [23] The class of the linear functional u satisfying (9) is the minimum value of $\max\{\deg(\psi) - 1, d - 2\}$, $d = \max\{\deg(A), \deg(B)\}$, for all triples of polynomials (A, ψ, B) satisfying (9).

The functional u related to the Riccati equation $AS' = BS^2 + CS + D$ is of class $s = \max\{\deg(C) - 1, d - 2\}$ if, and only if, the polynomials A, B, C and D have no common zeroes [1, Prop. 2.5].

The recurrence relations (4) and (5) can be written in the matrix form

$$Y_n = \mathcal{A}_n Y_{n-1}\,, \;\; Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}\,, \;\; \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}\,, \quad n \geq 1\,, \tag{10}$$

with initial conditions $Y_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}$. The matrix $\mathcal{A}_n$ is known as the transfer matrix.

Theorem 1. *(see [7]) Let S be a Stieltjes function, let $\{Y_n\}_{n\geq 0}$ be the corresponding sequence defined in (10), and let β_n, γ_n be the coefficients in the recurrence relation. The following statements are equivalent:*
(a) S satisfies (8),

$$AS' = BS^2 + CS + D\,, \;\; A, B, C, D \in \mathbb{P}\,;$$

(b) Y_n satisfies the matrix Sylvester equation

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,, \quad n \geq 0\,, \tag{11}$$

where

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + (x-\beta_n)\Theta_{n-1}/\gamma_n \end{bmatrix}\,, \;\; \mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}$$

with l_n, Θ_n polynomials of uniformly bounded degrees, satisfying the initial conditions

$$A = (x-\beta_0)(l_0 - C/2) - B + \Theta_0\,, \; (x-\beta_0)D + l_0 + C/2 = 0\,,$$
$$\Theta_{-1} = D, \; l_{-1} = C/2\,; \tag{12}$$

(c) the transfer matrix $\mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}$ satisfies the matrix Sylvester equation

$$A\mathcal{A}_n' = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}\,, \quad n \geq 1\,. \tag{13}$$

Remark . If the class of the linear functional related to (8) is s, the polynomials l_n, Θ_n in $\mathcal{B}_n$ satisfy $\deg(l_n) = s+1, \deg(\Theta_n) = s$ [11].

Let us emphasize that Eq. (13) is obtained from the compatibility between the Lax pair

$$\begin{cases} Y_n = \mathcal{A}_n Y_{n-1} \\ AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,, \quad n \geq 1\,. \end{cases} \tag{14}$$

Furthermore, we have [7, Corollary 1]

$$\operatorname{tr}\mathcal{B}_n = 0\,, \quad n \geq 0\,, \tag{15}$$

$$\det\mathcal{B}_n = \det\mathcal{B}_0 + A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k}\,, \quad n \geq 1\,, \tag{16}$$

with $\det\mathcal{B}_0 = D(A+B) - (C/2)^2$.

Eq. (13) has two non-trivial equations, respectively, from positions $(1,1)$ and $(1,2)$:

$$A = (x-\beta_n)(l_n - l_{n-1}) + \Theta_n - \gamma_n\frac{\Theta_{n-2}}{\gamma_{n-1}}\,, \tag{17}$$

$$l_n + (x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n} = l_{n-2} + (x-\beta_{n-1})\frac{\Theta_{n-2}}{\gamma_{n-1}}\,. \tag{18}$$

Eq. (18) can be written as

$$l_n + l_{n-1} + (x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n} = l_{n-1} + l_{n-2} + (x-\beta_{n-1})\frac{\Theta_{n-2}}{\gamma_{n-1}}\,.$$

Thus, by denoting $L_n = l_n + l_{n-1} + (x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n}$, we have $L_n = L_{n-1},\ \ n \geq 1$, which yields $L_n = L_0,\ \ n \geq 1$. Using (12) we get $L_0 = 0$, hence,

$$l_n + l_{n-1} + (x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n} = 0\,, \quad n \geq 0\,, \tag{19}$$

which is another form of (15).

Let us now look at Eq. (17). By multiplying it by Θ_{n-1}/γ_n and using (19), we obtain

$$A\frac{\Theta_{n-1}}{\gamma_n} = -(l_n^2 - l_{n-1}^2) + \Theta_n\frac{\Theta_{n-1}}{\gamma_n} - \Theta_{n-1}\frac{\Theta_{n-2}}{\gamma_{n-1}}\,. \tag{20}$$

Note that (19) implies

$$\det\mathcal{B}_n = -l_n^2 + \Theta_n\frac{\Theta_{n-1}}{\gamma_n}\,, \tag{21}$$

therefore, Eq. (20) can be written as

$$A\frac{\Theta_{n-1}}{\gamma_n} = \det\mathcal{B}_n - \det\mathcal{B}_{n-1}\,.$$

By iterating, we get (16).

3. Recurrences for Laguerre-Hahn orthogonal polynomials of class $s = 1$

The Stieltjes functions S associated with the Laguerre-Hahn orthogonal polynomials of class $s = 1$ satisfy $AS' = BS^2 + CS + D$ with $\deg(D) = 1$ and

$$\max\{\deg(A), \deg(B)\} \leq 3\,,\ \deg(C) = 2\,,$$

or

$$\max\{\deg(A), \deg(B)\} = 3\,,\ \ \deg(C) \leq 2\,.$$

One of the key ingredients in the sequel is the differential system (11). This is the matrix form of the so-called structure relations

$$A\left(P_n^{(1)}\right)' = DP_{n+1} + (l_n + C/2)P_n^{(1)} + \Theta_n P_{n-1}^{(1)}\,, \quad n \geq 0\,, \tag{22}$$

$$AP'_{n+1} = (l_n - C/2)P_{n+1} - BP_n^{(1)} + \Theta_n P_n\,, \quad n \geq 0, \tag{23}$$

(see [11, Eqs. (3.9), (3.10)]) with the polynomials l_n, Θ_n satisfying $\deg(l_n) = 2, \deg(\Theta_n) = 1$.

Remark . In the semi-classical case, that is, $B \equiv 0$, Eq. (23) becomes $AP'_{n+1} = (l_n - C/2)P_{n+1} + \Theta_n P_n$, which can be comparable to [8, Eq. (1.5)].

Lemma 1. *Let S be a Stieltjes function of class $s = 1$. Let $\{P_n\}_{n\geq 0}$ be the SMOP associated with S, satisfying the recurrence relation (4),*

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots.$$

Set

$$A(x) = a_3x^3 + a_2x^2 + a_1x + a_0\,,\ \ B(x) = b_3x^3 + b_2x^2 + b_1x + b_0\,,$$
$$C(x) = c_2x^2 + c_1x + c_0\,,\ \ D(x) = d_1x + d_0$$
$$l_n(x) = \ell_{n,2}x^2 + \ell_{n,1}x + \ell_{n,0}\,,\ \ \Theta_n(x) = \theta_{n,1}x + \theta_{n,0}\,.$$

With the functions defined above, we have

$$d_1 = -a_3 - b_3 - c_2\,, \tag{24}$$

$$d_0 = -(2a_3 + 2b_3 + c_2)\beta_0 - a_2 - b_2 - c_1\,, \tag{25}$$

$$\ell_{n,2} = (n+1)a_3 + b_3 + c_2/2\,, \tag{26}$$

$$\ell_{n,1} = a_3(\eta_n + \beta_0) + (n+1)a_2 + b_3\beta_0 + b_2 + c_1/2\,, \tag{27}$$

$$\begin{aligned}
\ell_{n,0} &= \lambda_{n,0} - \theta_{n,1}\,, && (28)\\
\lambda_{n,0} &= -2a_3(\nu_n + \beta_0\eta_n - \gamma_1) + (n+1)a_1 + b_3(\gamma_1 + \beta_0^2) + b_2\beta_0 + b_1 + c_0/2 \\
&\quad +(\eta_n + \beta_0)(a_3(\eta_n + \beta_0) + a_2)\,, && (29)\\
\theta_{n,1} &= -\gamma_{n+1}((2n+3)a_3 + 2b_3 + c_2)\,, && (30)\\
\theta_{n,0} &= -\gamma_{n+1}\{2a_3(\eta_n + (n+2)\beta_{n+1} + \beta_0) + a_2(2n+3) + 2b_3(\beta_0 + \beta_{n+1}) \\
&\quad +2b_2 + c_2\beta_{n+1} + c_1\} && (31)
\end{aligned}$$

with

$$\eta_n = \sum_{k=1}^{n} \beta_k\,, \quad \nu_n = \sum_{1\le i<j\le n}^{n} \beta_i\beta_j - \sum_{k=2}^{n}\gamma_k\,, \qquad n \ge 1\,. \tag{32}$$

Also, we have

$$a_1+b_1+c_0+(2a_2+2b_2+c_1)\beta_0+(3a_3+3b_3+c_2)\beta_0^2+(3a_3+2b_3+c_2)\gamma_1 = 0\,. \tag{33}$$

Proof: In account of (4) and (5),

$$\begin{aligned}
P_n^{(1)}(x) &= x^n - \eta_n x^{n-1} + \nu_n x^{n-2} + \cdots\,,\\
P_{n+1}(x) &= x^{n+1} - (\eta_n + \beta_0)x^n + (\nu_n + \beta_0\eta_n - \gamma_1)x^{n-1} + \cdots\,.
\end{aligned}$$

Eqs. (24) and (25) follow by equating the coefficients of x^{n+2} and x^{n+1}, respectively, in (22). Eqs. (26) – (28) follow by equating the coefficients of x^{n+3}, x^{n+2}, and x^{n+1}, respectively, in (23). Eqs. (30) – (31) follow by equating the coefficients of x^2 and x, respectively, in (15) (written as Eq. (19)). Eq. (33) follows by equating the coefficients of x^n in (22). ∎

3.1. Discrete Painlevé equations for some Laguerre-Hahn Stieltjes functions. We will use two key ingredients to deduce recurrences for the Laguerre-Hahn polynomials: the formula for the trace and the determinant of the matrix $\mathcal{B}_n$, Eqs. (15) and (16), defining the Lax pair (14).

From (15), equivalently, $l_n(x) + l_{n-1}(x) + (x-\beta_n)\frac{\Theta_{n-1}(x)}{\gamma_n} = 0\,, \quad n \ge 0$, we get

$$l_n(\beta_n) + l_{n-1}(\beta_n) = 0\,, \tag{34}$$

$$\frac{l_n(0) + l_{n-1}(0)}{\beta_n} = \frac{\Theta_{n-1}(0)}{\gamma_n}\,, \qquad \beta_n \ne 0\,. \tag{35}$$

From (16) we get, for every α such that $A(\alpha) = 0$,

$$\det \mathcal{B}_{n-1}(\alpha) = \det \mathcal{B}_0(\alpha)\,,$$

hence, taking into account (21), we have

$$\Theta_{n-1}(\alpha)\frac{\Theta_{n-2}(\alpha)}{\gamma_{n-1}} = \det \mathcal{B}_0(\alpha) + l_{n-1}^2(\alpha)\,.$$

Therefore, for α such that $A(\alpha)=0$,

$$\frac{\Theta_{n-1}(\alpha)}{\gamma_n}\frac{\Theta_{n-2}(\alpha)}{\gamma_{n-1}} = \frac{\det(\mathcal{B}_0)(\alpha) + (l_{n-1}(\alpha))^2}{\gamma_n}\,. \tag{36}$$

Eqs. (34) – (36) will be used in the proof of the next Theorem.

Theorem 2. *Let S be a Stieltjes function satisfying $AS' = BS^2 + CS + D$ with*

$$A(x) = a_1 x\,,\ B(x) = b_3x^3 + b_2x^2 + b_1x + b_0\,, \tag{37}$$
$$C(x) = c_2x^2 + c_1x + c_0\,,\ D(x) = d_1x + d_0\,,$$

where d_1 and d_0 are given by (24) and (25), respectively. Let $\{P_n\}_{n\geq 0}$ be a SMOP associated with S, satisfying the recurrence relation (4),

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0,1,2,\ldots\,,$$

with $\beta_n \neq 0\,,\ \ n = 0,1,2,\ldots$.
Set $\eta = 2b_3 + c_2,\ \mu = b_3(\gamma_1 + \beta_0^2) + b_2\beta_0 + b_1 + c_0/2\,,\ \lambda = 2b_3\beta_0 + 2b_2 + c_1$.
The following statements hold:
(a) Let $\eta < 0$. The sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ expressed in terms of the coefficients in the three term recurrence relation by

$$x_n = \frac{\sqrt{-\eta}}{\eta\beta_n + \lambda}\,, \quad y_n = -(\eta\gamma_n + na_1 + \mu) \tag{38}$$

satisfy the following discrete system of Painlevé equations dP_{IV}:

$$x_{n-1}x_n = \frac{y_n + na_1 + \mu}{y_n^2 - \xi_0}\,, \quad y_n + y_{n+1} = \frac{1}{x_n}\left(\frac{\lambda}{\sqrt{-\eta}} - \frac{1}{x_n}\right)\,, \tag{39}$$

with the initial conditions $x_0 = \dfrac{\sqrt{-\eta}}{\eta\beta_0 + \lambda}$, $y_0 = -(\eta + \mu)$.
(b) Let $\eta > 0$. The sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ expressed in terms of the coefficients in the three term recurrence relation by

$$x_n = \frac{\sqrt{\eta}}{\eta\beta_n + \lambda}\,, \quad y_n = \eta\gamma_n + na_1 + \mu \tag{40}$$

satisfy the following discrete system of Painlevé equations dP_{IV}:

$$x_{n-1}x_n = \frac{y_n - na_1 - \mu}{y_n^2 - \xi_0}\,, \quad y_n + y_{n+1} = \frac{1}{x_n}\left(\frac{\lambda}{\sqrt{\eta}} - \frac{1}{x_n}\right)\,, \tag{41}$$

with the initial conditions $x_0 = \dfrac{\sqrt{\eta}}{\eta\beta_0 + \lambda}$, $y_0 = \eta + \mu$.

Proof: Let us show how to deduce equations in case (a). Case (b) is similar.

Eq. (34) gives us, under the notation of Lemma 1,

$$(\ell_{n,2}\beta_n^2 + \ell_{n,1}\beta_n + \ell_{n,0}) + (\ell_{n-1,2}\beta_n^2 + \ell_{n-1,1}\beta_n + \ell_{n-1,0}) = 0\,.$$

Taking into account the formulae for $\ell_{n,2}, \ell_{n,1}$ given in (26) and (27), the above equation yields

$$\eta\beta_n^2 + \lambda\beta_n = -\ell_{n-1,0} - \ell_{n,0}, \quad \eta = 2b_3 + c_2,\ \lambda = 2b_3\beta_0 + 2b_2 + c_1\,,$$

thus,

$$\beta_n\left(\eta\beta_n + \lambda\right) = -l_{n-1}(0) - l_n(0)\,. \tag{42}$$

Therefore, we obtain, for $\beta_n \neq 0$,

$$\left(\eta\beta_n + \lambda\right)\left(\eta\beta_{n-1} + \lambda\right) = \frac{\left(l_{n-1}(0) + l_n(0)\right)}{\beta_n}\frac{\left(l_{n-2}(0) + l_{n-1}(0)\right)}{\beta_{n-1}}\,.$$

In account of (35), the equation above can be written as

$$\left(\eta\beta_n + \lambda\right)\left(\eta\beta_{n-1} + \lambda\right) = \frac{\Theta_{n-1}(0)}{\gamma_n}\frac{\Theta_{n-2}(0)}{\gamma_{n-1}}\,. \tag{43}$$

In account of (36) with $\alpha = 0$ (as $A(x) = a_1 x$), Eq. (43) can be written as

$$\left(\eta\beta_n + \lambda\right)\left(\eta\beta_{n-1} + \lambda\right) = \frac{\left(l_{n-1}(0)\right)^2 - \xi_0}{\gamma_n} \tag{44}$$

with $\xi_0 = -\det \mathcal{B}_0(0)$.

Note that, as $l_n(0) = \eta\gamma_{n+1} + (n+1)a_1 + \mu$, $\mu = b_3(\gamma_1 + \beta_0^2) + b_2\beta_0 + b_1 + c_0/2$, then (42) yields

$$\beta_n(\eta\beta_n + \lambda) = -(\eta\gamma_n + na_1 + \mu) - (\eta\gamma_{n+1} + (n+1)a_1 + \mu)\,. \tag{45}$$

By defining x_n, y_n as in (38),

$$x_n = \frac{\sqrt{-\eta}}{\eta\beta_n + \lambda}\,, \quad y_n = -(\eta\gamma_n + na_1 + \mu)\,,$$

we obtain that Eqs. (44) and (45) are, respectively, the first and second equations given in (39). Indeed, let us write Eq. (44) in its equivalent form

$$\frac{\sqrt{-\eta}}{(\eta\beta_n+\lambda)}\frac{\sqrt{-\eta}}{(\eta\beta_{n-1}+\lambda)}=\frac{-\eta\gamma_n}{(l_{n-1}(0))^2-\xi_0}\,,$$

that is, we have $x_{n-1}x_n = \dfrac{y_n+na_1+\mu}{y_n^2-\xi_0}$. Also, let us write the left-hand side of (45) as

$$\frac{\eta\beta_n+\lambda}{\sqrt{-\eta}}\left(\frac{\lambda}{\sqrt{-\eta}}-\frac{\eta\beta_n+\lambda}{\sqrt{-\eta}}\right).$$

Then, (45) reads as $\dfrac{1}{x_n}\left(\dfrac{\lambda}{\sqrt{-\eta}}-\dfrac{1}{x_n}\right)=y_n+y_{n+1}$. ■

Remark . The case $A(x)\equiv 0$, that is, $a_1=0$ in (37), concerns a second degree equation $BS^2+CS+D=0$. In such a case, S is related to the so-called family of second degree forms [24].

Remark . Equations (39) and (41) can be obtained as a limiting case of the asymmetric Painlevé equation dP_{IV} [5, Eq. (1.2)] given by

$$u_nu_{n-1}=\frac{a(v_n+z_n-b)}{v_n^2-\gamma^2},\quad v_n+v_{n+1}=\frac{c}{u_n}+\frac{z_{n+1/2}+d}{u_n-1} \tag{46}$$

with $z_n=\alpha_1 n+\beta_1$. Indeed, by taking $b=0$, $u_n=x_n/\varepsilon$, $v_n=\varepsilon y_n$, $z_n=\varepsilon(\mu+na_1)$, $a=1/\varepsilon$, $\gamma=\varepsilon/\sqrt{\xi_0}$, $c=1/\varepsilon+\lambda/\sqrt{-\eta}$, $d=-1/\varepsilon$ and letting ε tend to zero we obtain (39). For (41) the calculations are similar.

3.1.1. *The case $A(x)=a_0\neq 0$.*. Let us emphasize that the technique used in Theorem 2 to deduce the difference equations for the coefficients β_n,γ_n cannot be applied to the case $A(x)=a_0\neq 0$. Indeed, (34) yields (42), with $l_n(0)=\eta\gamma_{n+1}+\mu$, and such an equation implies (43). However, in the present case, (43) does not give new identities and (44) does not hold since $A(0)\neq 0$.

The difference equations concerning the case $A(x)=a_0\neq 0$ are given in the theorem that follows.

Theorem 3. *Let S be a Stieltjes function satisfying $AS'=BS^2+CS+D$ with*

$$A(x)=a_0\neq 0\,,\ B(x)=b_3x^3+b_2x^2+b_1x+b_0\,,$$
$$C(x)=c_2x^2+c_1x+c_0\,,\ D(x)=d_1x+d_0\,,$$

where d_1 and d_0 are given by (24) and (25), respectively. Let $\{P_n\}_{n\geq 0}$ be a SMOP associated with S, satisfying the recurrence relation (4),

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0,1,2,\dots\,,$$

with $\beta_n \neq 0\,,\;\; n = 0,1,2,\dots$.
Set $\eta = 2b_3 + c_2,\;\mu = b_3(\gamma_1+\beta_0^2) + b_2\beta_0 + b_1 + c_0/2,\;\lambda = 2b_3\beta_0 + 2b_2 + c_1$. The recurrence relation coefficients β_n, γ_n are related through the following discrete system:

$$\beta_n(\eta\beta_n+\lambda) = -(\eta\gamma_n+\mu) - (\eta\gamma_{n+1}+\mu)\,, \quad n\geq 0\,, \tag{47}$$

$$\eta(\beta_n+\beta_{n+1}) + \lambda = \frac{-na_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)}{\gamma_{n+1}}\,, \quad n\geq 1\,. \tag{48}$$

Moreover, the sequences

$$x_n = \eta\beta_n + \lambda/2\,, \quad z_n = \eta^2\left(-na_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)\right) \tag{49}$$

satisfy the alternative discrete Painlevé equation dP_I

$$\frac{z_{n-1}}{x_{n-1}+x_n} + \frac{z_n}{x_n+x_{n+1}} = -x_n^2 + \gamma\,, \quad \gamma = (\lambda/2)^2 - 2\mu\eta \tag{50}$$

with the initial conditions $x_0 = \eta\beta_0 + \lambda/2,\;\; x_1 = \eta\beta_1 + \lambda/2\,.$

Proof: Set $A(x) = a_0$, where a_0 is a non-zero constant. Eq. (34) yields (42), with $l_n(0) = \eta\gamma_{n+1} + \mu$, thus, we have (47).

Let us now look at Eq. (17), $A = (x-\beta_n)(l_n - l_{n-1}) + \Theta_n - \gamma_n \frac{\Theta_{n-2}}{\gamma_{n-1}}$, evaluated at $x = \beta_n$,

$$a_0 = \Theta_n(\beta_n) - \gamma_n \frac{\Theta_{n-2}(\beta_n)}{\gamma_{n-1}}\,.$$

Using the formulae in Lemma 1 we get, after simplifications,

$$a_0 = -\gamma_{n+1}(\eta(\beta_n+\beta_{n+1})+\lambda) + \gamma_n(\eta(\beta_{n-1}+\beta_n)+\lambda)\,, \quad n\geq 1\,, \tag{51}$$

where $\eta = 2b_3 + c_2,\;\lambda = 2b_3\beta_0 + 2b_2 + c_1$. By writing (51) as

$$T_{n+1} = T_n - a_0, \quad T_n = \gamma_n(\eta(\beta_{n-1}+\beta_n)+\lambda)\,,$$

iteration yields $T_{n+1} = -na_0 + T_1$, thus we obtain

$$\gamma_{n+1}(\eta(\beta_n+\beta_{n+1})+\lambda) = -na_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)\,,$$

hence, we obtain (48).

Using the equivalent form of (48),

$$\gamma_{n+1} = \frac{-na_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)}{\eta\beta_n + \eta\beta_{n+1} + \lambda}\,,$$

into (47), we obtain the second order difference equation

$$\eta\frac{-(n-1)a_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)}{\eta\beta_{n-1} + \eta\beta_n + \lambda} + \eta\frac{-na_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)}{\eta\beta_n + \eta\beta_{n+1} + \lambda} = -\eta\beta_n^2 - \lambda\beta_n - 2\mu\,.$$

By multiplying the above equation by η we get

$$\eta^2\frac{-(n-1)a_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)}{\eta\beta_{n-1} + \eta\beta_n + \lambda} + \eta^2\frac{-na_0 + \gamma_1(\eta(\beta_0+\beta_1)+\lambda)}{\eta\beta_n + \eta\beta_{n+1} + \lambda} = -\left(\eta^2\beta_n^2 + \lambda\eta\beta_n + (\lambda/2)^2\right) + (\lambda/2)^2 - 2\mu\eta\,.$$

Thus, by defining x_n, z_n as in (49), the above equation reads as (50). ∎

Remark . Moreover, using (47) with $n = 0$ and (33), we get the following condition on β_0:

$$4b_3^2 + c_2(b_1 + c_2 + 2b_2\beta_0) + b_3(c_2(4+\beta_0^2) - 2c_1\beta_0 - 2c_0) = 0.$$

3.2. Further results and generalizations. Another way to prove Theorem 2 and to generalize to the case $A(x) = x - \alpha$, where α is an arbitrary parameter (recall that $\alpha = 0$ in Theorem 2) is by symbolic computations, which can be performed in any computer algebra system, for instance, Mathematica*.

Indeed, assuming the form

$$x_n = \frac{k_1}{k_2\beta_n + k_3}, \quad y_n = k_4\gamma_n + k_5(n)$$

and

$$x_{n-1}x_n = \frac{y_n - k_6(n)}{y_n^2 - \xi_0}, \quad y_n + y_{n+1} = \frac{1}{x_n}\left(k_7 - \frac{1}{x_n}\right),$$

where only k_5 and k_6 are the functions of n and others are constants, and substituting into (13) and (15), we obtain the following theorem.

*www.wolfram.com

Theorem 4. *Let S be a Stieltjes function satisfying $AS' = BS^2 + CS + D$ with*

$$A(x) = x - \alpha\,,\ B(x) = b_3x^3 + b_2x^2 + b_1x + b_0\,,$$
$$C(x) = c_2x^2 + c_1x + c_0\,,\ D(x) = d_1x + d_0\,,$$

where d_1 and d_0 are given by (24) and (25), respectively. Let $\{P_n\}_{n\geq 0}$ be a SMOP associated with S, satisfying the recurrence relation (4),

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots.$$

The corresponding equations (13) and (15) are solved by

$$x_n = \frac{k_1(2b_3 + c_2)}{k_2(2b_2 + 2\alpha b_3 + c_1 + \alpha c_2 + 2b_3\beta_0) + \beta_n(2b_3 + c_2)},$$

$$y_n = \frac{k_2^2(2n + 2b_1 + c_0 + 2b_2\beta_0 + 2b_3(\beta_0^2 + \gamma_1) + 2(2b_3 + c_2)\gamma_n + g_1}{2(2b_3 + c_2)k_1^2}\,,$$

satisfying

$$x_{n-1}x_n = \frac{2(k_2^2(2n + 2b_1 + c_0 + 2b_3\beta_0 + 2b_3(\beta_0^2 + \gamma_1) + g_2) - 2(2b_3 + c_2)k_1^2y_n)}{(2b_3 + c_2)k_1^2(c_0^2 + 4b_0(b_2 + c_1 + (2b_3 + c_2)\beta_0) - 4y_n^2)}\,,$$

and

$$y_n + y_{n-1} = \frac{k_2x_n(2b_2 + c_1 + 2b_3\beta_0 + 2\alpha(2b_3 + c_2)) - (2b_3 + c_2)k_1}{(2b_3 + c_2)k_1x_n^2}.$$

Here, $g_1 = \alpha^2(2b_3 + c_2) + \alpha(2b_2 + c_1 + 2b_3\beta_0))$, $g_2 = \alpha^2(2b_3 + c_2) + \alpha(2b_2 + c_1 + 2b_3\beta_0)$, and k_1, k_2 are constants. The initial conditions are given by

$$x_0 = \frac{(2b_3 + c_2)k_1}{k_2(2b_2 + c_1 + 4b_3\beta_0 + c_2\beta_0 + \alpha(2b_3 + c_2))}$$

and y_0, which is obtained from the formulae above with $n = 0$.

Note that by taking $k_2 = \eta k_1/\sqrt{-\eta}$, where $\eta = 2b_3 + c_2$ and $\alpha = 0$, we obtain case (a) in Theorem 2, and similarly for case (b).

Remark . The technique used in Theorem 2 can not be extended to the case $\deg(A) \geq 2$. Note that if $\deg(A) \geq 2$, then in the right-hand side of (42) there appear additional sequences η_n and ν_n (see (32) for definitions).

4. Examples

There are several families of orthogonal polynomials related to the Stieltjes function satisfying $AS' = BS^2 + CS + D$ such that the previous Theorems apply. Some of them are as follows.

4.1. Example 1: the modified semi-classical Laguerre polynomials. Consider a SMOP $\{P_n\}_{n\geq 0}$ with respect to the modified Laguerre weight

$$w(x) = |x|^{\alpha} e^{-x^2+tx}, \quad x \in \mathbb{R},$$

where $\alpha > -1$ and $t \in \mathbb{R}$ is some *time* parameter (see [5]). As w satisfies the Pearson equation $xw' = (-2x^2+tx+\alpha)w$, the Stieltjes function of w satisfies $AS' = CS + D$ with $A(x) = x,\ C(x) = -2x^2 + tx + \alpha,\ D = 2x + 2\beta_0 - t$, $\beta_0 = \left(\int_{\mathbb{R}} xw(x)dx\right) / \left(\int_{\mathbb{R}} w(x)dx\right)$.

Then (38) is

$$x_n = \frac{\sqrt{2}}{-2\beta_n + t}, \quad y_n = 2\gamma_n - n - \alpha/2$$

and system (39) is

$$x_{n-1}x_n = \frac{y_n + n + \alpha/2}{y_n^2 - \frac{1}{4}\alpha^2}, \quad y_n + y_{n+1} = \frac{1}{x_n}\left(\frac{t}{\sqrt{2}} - \frac{1}{x_n}\right).$$

4.2. Example 2: the associated semi-classical Laguerre polynomials. Let us start by considering the SMOP $\{P_n\}_{n\geq 0}$ with respect to a modified Laguerre weight

$$w(x) = |x|^{\alpha} e^{\alpha_2 x^2 + \alpha_1 x + \alpha_0}, \quad x \in \mathbb{R}, \tag{52}$$

where, for integrability reasons, $\alpha > -1,\ \alpha_2 < 0$. The Stieltjes function related to $\{P_n\}_{n\geq 0}$ satisfies

$$x\tilde{S}' = \tilde{C}S + \tilde{D}, \quad \tilde{C}(x) = 2\alpha_2 x^2 + \alpha_1 x + \alpha, \ \tilde{D}(x) = -2\alpha_2 x - 2\alpha_2\beta_0 - \alpha_1. \tag{53}$$

Let us now consider the sequence of associated polynomials of the first kind $\{P_n^{(1)}\}_{n\geq 0}$, and let $S^{(1)}$ denote its Stieltjes function. In account of (53) and (7), $S^{(1)}$ is Laguerre-Hahn of class $s = 1$, as it satisfies

$$x\left(S^{(1)}\right)' = B\left(S^{(1)}\right)^2 + CS^{(1)} + D \tag{54}$$

with

$$B = \gamma_1 \tilde{D}\,, \quad C = -\tilde{C} - 2(x-\beta_0)\tilde{D}\,, \quad D = \frac{1}{\gamma_1}\left(x + (x-\beta_0)\tilde{C} + (x-\beta_0)^2\tilde{D}\right)\,,$$

that is,

$$\begin{aligned} &B(x) = b_1 x + b_0\,, \quad b_1 = -2\gamma_1\alpha_2\,,\ b_0 = -\gamma_1(2\alpha_2\beta_0 + \alpha_1)\,,\\ &C(x) = c_2x^2 + c_1x + c_0\,, \quad c_2 = 2\alpha_2\,,\ c_1 = \alpha_1\,,\ c_0 = -\alpha - 2\alpha_1\beta_0 - 4\alpha_2\beta_0^2\,,\\ &D(x) = d_1x + d_0\,, \quad d_1 = \frac{1}{\gamma_1}(1 + \alpha + \alpha_1\beta_0 + 2\alpha_2\beta_0^2)\,,\\ &\qquad\qquad d_0 = -\frac{\beta_0}{\gamma_1}(\alpha + \alpha_1\beta_0 + 2\alpha_2\beta_0^2)\,. \end{aligned}$$

Then, (38) is

$$x_n = \frac{\sqrt{-2\alpha_2}}{2\alpha_2\beta_n + c_1}\,, \quad y_n = -(2\alpha_2\gamma_n + n + \mu)\,, \quad \mu = -2\alpha_2\gamma_1 - \alpha/2 - \alpha_1\beta_0 - 2\alpha_2\beta_0^2$$

and (39) is

$$x_{n-1}x_n = \frac{y_n + n + \mu}{y_n^2 - \xi_0}\,, \quad y_n + y_{n+1} = \frac{1}{x_n}\left(\frac{\alpha_1}{\sqrt{-2\alpha_2}} - \frac{1}{x_n}\right)$$

with $\xi_0 = \frac{1}{4}C^2(0) - B(0)D(0)$.

4.2.1. *Remarks.* Let us remark that when (52) reduces to $w(x) = x^\alpha e^{-x}$, $\alpha > 2$, $x > 0$, an integral representation for the linear functional corresponding to $S^{(1)}$ is known. In such case, $\{P_n^{(1)}\}_{n\geq 0}$ is orthogonal with respect to the linear functional defined by [2]

$$\langle u, p(x)\rangle = \int_0^\infty \frac{p(x)x^\alpha e^{-x}\,dx}{|\psi(\alpha, 1-\alpha, xe^{-\pi i})|^2}\,, \quad \alpha > 2\,,$$

where

$$\psi(a,b,x) = 1 + \sum_{n=1}^{+\infty}\frac{(a)_n}{(b)_n}x^n\,,$$
$$(a)_n = a(a+1)\cdots(a+n-1)\,, \quad (a)_0 = 1\,.$$

In general, the integral representation of the Laguerre-Hahn Stieltjes functions is an open problem. Another open problem (see [29]) is to determine all the Stieltjes functions, say S^1, satisfying a Riccati equation such as (54), which are obtained by a rational spectral transformation of S satisfying (53), that is, $S^1 = \dfrac{aS+b}{cS+d}$, where a, b, c, d are polynomials.

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