Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 15–55

BECK-CHEVALLEY CONDITION AND GOURSAT CATEGORIES

MARINO GRAN AND DIANA RODELO

Abstract: We characterise regular Goursat categories through a specific stability property of regular epimorphisms with respect to pullbacks. Under the assumption of the existence of some pushouts this property can be also expressed as a restricted Beck-Chevalley condition, with respect to the fibration of points, for a special class of commutative squares. In the case of varieties of universal algebras these results give, in particular, a structural explanation of the existence of the ternary operations characterising 3-permutable varieties of universal algebras. We then prove that the reflector to any (regular epi)-reflective subcategory of a regular Goursat category preserves pullbacks of split epimorphisms. This implies that the so-called internal Galois pregroupoid of an extension is an internal groupoid.

Keywords: Goursat categories, 3-permutable varieties, Shifting Lemma, Beck-Chevalley condition, reflective subcategories, Galois groupoid.
Math. Subject Classification (2010): 08C05, 08B05, 18C05, 18B99, 18E10.

Introduction

A variety of universal algebras is called a Mal'tsev variety [25] when any pair of congruences R and S on the same algebra 2-permute, meaning that $RS = SR$. The celebrated Mal'tsev theorem asserts that the algebraic theory of such a variety is characterised by the existence of a ternary term $p(x, y, z)$ such that the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ hold [22]. The weaker 3-permutability of congruences $RSR = SRS$, which defines 3-permutable varieties, is also equivalent to the existence of two ternary operations r and s such that the identities $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$ hold [17]. A nice feature of 3-permutable varieties is the fact that they are congruence modular, a condition that plays a crucial role in the development of commutator theory [12, 16].

Many interesting results have been discovered in regular Mal'tsev categories [11] and in regular Goursat categories [10], which can be seen as the

Received December 15, 2015.
This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

categorical extensions of Mal'tsev varieties and of 3-permutable varieties, respectively. The interested reader will find many properties of these categories in the references [2, 3, 5, 9, 10, 11, 13, 14, 19, 20, 21, 24], for instance.

For Mal'tsev categories, and many other algebraic categories, there are some elegant characterisations expressed in terms of the fibration of points [5, 3]. Recall that, given a category $\mathcal{C}$ with pullbacks, the fibration of points is the functor $\mathrm{Pt}(\mathcal{C}) \to \mathcal{C}$ associating the codomain W to any "point" in $\mathcal{C}$, i.e. to any split epimorphism $g\colon U \to W$ with a given splitting $j\colon W \to U$. For any morphism $\beta\colon Y \to W$ in $\mathcal{C}$, the change-of-base functor, defined by pulling back along β, is denoted by $\beta^*\colon \mathrm{Pt}_W(\mathcal{C}) \to \mathrm{Pt}_Y(\mathcal{C})$. One of the goals of this paper is to give a characterisation of regular Goursat categories by using this fibration. It turns out that such a characterisation not only involves the change-of-base functors β^* with respect to the fibration of points, but also their left adjoints $\beta_!$, which exist as soon as the category admits pushouts of split monomorphisms [4]. It is precisely the so-called Beck-Chevalley condition (Theorem 1.4) for the commutative squares of the following type

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, shift right, "f"'] & U \arrow[d, shift right, "g"'] \\ Y \arrow[r, two heads, "\beta"'] \arrow[u, shift right, "i"'] & W, \arrow[u, shift right, "j"'] \end{tikzcd}$$

where α and β are regular epimorphisms and f and g are split epimorphisms (i.e. the pair (α, β) is a regular epimorphism in the category $\mathrm{Pt}(\mathcal{C})$). In fact, the Goursat property can be also expressed in terms of the functors $\beta_!$ alone: $\beta_!$ preserves binary products, for any regular epimorphism β.

The proof of these results also relies on the fact that the Shifting Lemma [16] holds in any regular Goursat category [6], since the lattice of equivalence relations on any object is modular [10]. A more general characterisation of Goursat categories among regular categories is also obtained without requiring the existence of pushouts along split monomorphisms and involves a stability property for regular epimorphisms (Theorem 1.3). In the varietal case, the existence of the ternary operations characterising 3-permutable varieties mentioned above can be deduced from this theorem by applying it to a suitable diagram involving free algebras (Remark 1.5).

As a consequence of the results in this paper, we obtain an extension to regular Goursat categories of a known result in the regular Mal'tsev context (Proposition 3.6 in [9]): the reflector to a (regular epi)-reflective subcategory of a regular Goursat category always preserves pullbacks of split epimorphisms along split epimorphisms (Proposition 2.1). It then follows that the so-called internal Galois pregroupoid [18] associated to an extension is necessarily an internal groupoid (Corollary 2.3).

Let us finally mention that the possibility of formulating some exactness properties of Mal'tsev categories in terms of a suitable Beck-Chevalley condition has been first suggested to the authors by Zurab Janelidze during the *Workshop in Category Theory* at the University of Coimbra in 2012, where the work [14] was presented. We would like to warmly thank Zurab for this suggestion. The present paper shows that a suitable Beck-Chevalley condition characterises Goursat categories. Independently, Clemens Berger and Dominique Bourn have discovered a stronger condition that holds in the special case of exact Mal'tsev categories (Proposition 1.24 in [2]).

1. Goursat categories

In this section we give a new characterisation for a regular category [1] to be a Goursat category through a stability property of regular epimorphisms, similar to the known characterisation for regular Mal'tsev categories given in Proposition 3.6 in [14].

Recall that a regular category $\mathcal{C}$ is called a *Goursat* category [10] when the composition of (effective) equivalence relations R and S on a same object in $\mathcal{C}$ is 3-permutable: $RSR = SRS$. Given a relation $R = (R, r_1, r_2)$ from an object X to an object Y, we write R^o for the opposite relation (R, r_2, r_1) from Y to X.

From [10] and [20] we have:

Theorem 1.1. *Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathcal{C}$ is a Goursat category;*
(ii) *$E^\circ \leqslant EE$, for any reflexive relation E;*
(iii) *$(1_X \wedge T)T^\circ(1_X \wedge T) \leqslant TT$, for any relation T on an object X;*
(iv) *$PP^\circ PP^\circ \leqslant PP^\circ$, for any relation P.*

Proof: (i) $\Leftrightarrow$ (ii). By Theorem 1 in [20].
(i) $\Leftrightarrow$ (iv). By Theorem 3.5 of [10].
(i) $\Leftrightarrow$ (iii). This type of equivalence was mentioned at the end of [20]. Note that condition (iii) is a stronger version of condition (ii), so that it will suffice to prove that (iv) $\Rightarrow$ (iii). In a regular context, it suffices to give a proof in

set-theoretical terms (see Metatheorem $A.5.7$ in [3], for instance). Suppose that $(x,x) \in T, (y,x) \in T$ and $(y,y) \in T$. We want to prove that $(x,y) \in TT$, i.e. that $(x,\alpha) \in T$ and $(\alpha,y) \in T$, for some $\alpha \in X$. We define a relation P from $X \times X$ to X by: $((a,b),c) \in P$ if and only if $(a,c) \in T$ and $(c,b) \in T$. One then sees that

$$(x,x)Px, (y,x)Px, (y,x)Py, (y,y)Py$$

implying that

$$(x,x)PP^{\circ}PP^{\circ}(y,y).$$

By condition (iv) it follows that

$$(x,x)PP^{\circ}(y,y)$$

and there is an $\alpha \in X$ such that

$$(x,x)P\alpha, (y,y)P\alpha.$$

One concludes then that

$$xT\alpha Ty. \quad \blacksquare$$

Before proving the main results of this section we need to recall a useful property of regular Goursat categories, namely the validity of the so-called *Shifting Lemma* [16]. In the context of varieties of universal algebras this property is equivalent to the modularity of the lattice of congruences on any of its algebras. The modularity of the lattice $(L_X, \vee, \wedge)$ of equivalence relations on any object X also holds in any regular Goursat category, as shown in [10]. More precisely, given equivalence relations R, S and T in L_X,

$$R \leqslant T \Rightarrow (\ R \vee (S \wedge T) = (R \vee S) \wedge T\ ).$$

In this context, the supremum $R \vee S$ of two equivalence relations in L_X is given by the triple relational composite $R \vee S = RSR$. By using generalized elements the validity of the Shifting Lemma can be expressed as follows:

Shifting Lemma

Given equivalence relations R, S and T on the same object X such that $R \wedge S \leqslant T$, whenever x, y, t, z are elements in X with $(x,y) \in R \wedge T$, $(x,t) \in S$, $(y,z) \in S$ and $(t,z) \in R$, it then follows that $(t,z) \in T$:

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T"'] & t \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ y \arrow[r, dash, "S"'] & z \end{tikzcd} \tag{1}$$

The Shifting Lemma holds in any regular Goursat category [6], as we are now going to recall by using the internal logic of a regular category. Given a diagram (1), the 3-permutability of the equivalence relations implies that

$$(t, z) \in S(R \wedge T)S = (R \wedge T)S(R \wedge T).$$

Accordingly, there exist a and b such that $(t, a) \in R \wedge T$, $(a, b) \in S$, and $(b, z) \in R \wedge T$. Then (a, b) is also in R, thus in T, since $R \wedge S \leqslant T$; it follows that $(t, z) \in T$, as desired.

The property expressed by the Shifting Lemma has been extended to a categorical context in [7], giving rise to the notion of Gumm category. Indeed, the Shifting Lemma can be equivalently reformulated in any finitely complete category $\mathcal{C}$ by asking that a specific class of internal functors are discrete fibrations (see [6] and [7] for more details).

One of the fundamental results in this paper is given in Theorem 1.3 below, where regular Goursat categories are characterised by a stability property of regular epimorphisms with respect to pullbacks. Such a stability condition is an extension of the following one were regular epimorphisms are stable with respect to kernel pairs:

Theorem 1.2. [13] *Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

(*i*) *$\mathcal{C}$ is a Goursat category;*
(*ii*) *any pushout*

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, shift right, "f"'] & U \arrow[d, shift right, "g"'] \\ Y \arrow[r, two heads, "\beta"'] \arrow[u, shift right, "i"'] & W, \arrow[u, shift right, "j"'] \end{tikzcd}$$

where f and g are split epimorphisms and α and β are regular epimorphisms (commuting also with the splittings), is a Goursat pushout*: the comparison morphism $\lambda\colon \mathrm{Eq}(f) \to \mathrm{Eq}(g)$ induced by the universal property of the kernel pair $\mathrm{Eq}(g)$ of g is also a regular epimorphism.*

Theorem 1.3. *Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathcal{C}$ is a Goursat category;*

(ii) *for any commutative cube*

$$
\begin{tikzcd}
X \times_Y Z \arrow[rrr, dotted, two heads, "\lambda"] \arrow[dd, shift right] \arrow[dr, shift left] & & & U \times_W V \arrow[dd, dashed, shift right] \arrow[dr, shift left] & \\
& Z \arrow[rrr, two heads, "\gamma"] \arrow[dd, shift right, "l"'] \arrow[ul, shift left] & & & V \arrow[dd, shift right, "h"'] \arrow[ul, shift left] \\
X \arrow[rrr, dashed, two heads, "\alpha"] \arrow[uu, shift right] \arrow[dr, shift right, "f"'] & & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "g"'] & \\
& Y \arrow[rrr, two heads, "\beta"'] \arrow[uu, shift right, "k"'] \arrow[ul, shift right, "i"'] & & & W, \arrow[uu, shift right] \arrow[ul, dashed, shift right, "j"']
\end{tikzcd}
\tag{2}
$$

where the left and right faces are pullbacks of split epimorphisms and α, β and γ are regular epimorphisms (commuting also with the splittings), then the comparison morphism $\lambda \colon X \times_Y Z \to U \times_W V$ is also a regular epimorphism;

(iii) *for any commutative cube*

$$
\begin{tikzcd}
X \times_Y Z \arrow[rrr, two heads, "\delta"] \arrow[dd, shift right, "\pi_X"'] \arrow[dr, shift left, "\pi_Z"'] & & & A \arrow[dd, dashed, shift right] \arrow[dr, shift left] & \\
& Z \arrow[rrr, two heads, "\gamma"] \arrow[dd, shift right, "l"'] \arrow[ul, shift left, "\langle il{,}1_Z \rangle"'] & & & V \arrow[dd, shift right, "h"'] \arrow[ul, shift left] \\
X \arrow[rrr, dashed, two heads, "\alpha"] \arrow[uu, shift right, "\langle 1_X{,}kf \rangle"'] \arrow[dr, shift right, "f"'] & & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "g"'] & \\
& Y \arrow[rrr, two heads, "\beta"'] \arrow[uu, shift right, "k"'] \arrow[ul, shift right, "i"'] & & & W, \arrow[uu, shift right] \arrow[ul, dashed, shift right, "j"']
\end{tikzcd}
\tag{3}
$$

where the left face is a pullback of split epimorphisms, the right face is a commutative diagram of split epimorphisms and the horizontal arrows $\alpha, \beta, \gamma, \delta$ are regular epimorphisms (commuting also with the splittings), then the right face is a pullback.

Proof: (i) $\Rightarrow$ (ii). Again, it suffices to give a proof in set-theoretical terms. Let $(u, v) \in U \times_W V$. Then there exist $x \in X$ and $z \in Z$ such that $\alpha(x) = u$ and $\gamma(z) = v$; thus $g\alpha(x) = h\gamma(z)$. We define a binary relation R on $Y \times U \times V$ by:

$$((y_1, u_1, v_1), (y_2, u_2, v_2)) \in R$$

if $y_1 = l(\bar{z}), u_1 = \alpha(\bar{x}), y_2 = f(\bar{x})$ and $v_2 = \gamma(\bar{z})$, for some $\bar{x}$ and $\bar{z}$. We have the following relations:

• $(f(x), \alpha(x), \gamma kf(x))\, R\, (f(x), \alpha(x), \gamma kf(x))$, for $\bar{x} = x$ and $\bar{z} = kf(x)$;
• $(l(z), \alpha if(x), \gamma(z))\, R\, (f(x), \alpha(x), \gamma kl(z))$, for $\bar{x} = if(x)$ and $\bar{z} = kl(z)$;
• $(l(z), \alpha il(z), \gamma(z))\, R\, (l(z), \alpha il(z), \gamma(z))$, for $\bar{x} = il(z)$ and $\bar{z} = z$.

From $g\alpha(x) = h\gamma(z)$, we get that $\gamma kf(x) = \gamma kl(z)$ and $\alpha if(x) = \alpha il(z)$. So:

$$(f(x), \alpha(x), \gamma kf(x))(1_{Y\times U\times V} \wedge R)R^{\circ}(1_{Y\times U\times V} \wedge R)(l(z), \alpha il(z), \gamma(z)).$$

From Theorem 1.1(iii), we can conclude that

$$(f(x), \alpha(x), \gamma kf(x))\, RR\, (l(z), \alpha il(z), \gamma(z)).$$

So, $(f(x), \alpha(x), \gamma kf(x))R(y', u', v')R(l(z), \alpha il(z), \gamma(z))$, for some (y', u', v'). From the definition of R we can conclude that: $l(\bar{z}) = f(x)$, $\alpha(\bar{x}) = \alpha(x)$, $f(\bar{x}) = y'$, $\gamma(\bar{z}) = v'$ and $l(\bar{\bar{z}}) = y'$, $\alpha(\bar{\bar{x}}) = u'$, $f(\bar{\bar{x}}) = l(z)$, $\gamma(\bar{\bar{z}}) = \gamma(z)$, for some $\bar{x}, \bar{z}, \bar{\bar{x}}, \bar{\bar{z}}$. Then, there exists $(\bar{x}, \bar{\bar{z}}) \in X \times_Y Z$, since $f(\bar{x}) = y' = l(\bar{\bar{z}})$, such that $\lambda(\bar{x}, \bar{\bar{z}}) = (\alpha(\bar{x}), \gamma(\bar{\bar{z}})) = (\alpha(x), \gamma(z)) = (u, v)$.

(ii) $\Rightarrow$ (i). We can consider the left and right faces in the cube (2) to be kernel pairs of split epimorphisms. Then condition (ii) translates into the statement of Theorem 1.2(ii).

(ii) $\Rightarrow$ (iii). By assumption we know that the induced arrow $\lambda\colon X \times_Y Z \to U \times_W V$ is a regular epimorphism, and this implies that the unique induced arrow $c\colon A \to U \times_W V$ such that $c\delta = \lambda$ is a regular epimorphism as well. To show that c is also a monomorphism it suffices to show that $\mathrm{Eq}(\lambda) \leqslant \mathrm{Eq}(\delta)$, since one always has $\mathrm{Eq}(\delta) \leqslant \mathrm{Eq}(\lambda)$.

As a preliminary step, we first show that $\mathrm{Eq}(\pi_X) \wedge \mathrm{Eq}(\lambda) \leqslant \mathrm{Eq}(\delta)$. Consider an element $((x, z), (x, w)) \in \mathrm{Eq}(\pi_X) \wedge \mathrm{Eq}(\lambda)$. This implies that $f(x) = l(z) = l(w)$, and $(z, w) \in \mathrm{Eq}(\gamma)$; accordingly, $((il(z), z), (il(w), w)) \in \mathrm{Eq}(\delta)$. Define then the relation P from $X \times_Y Z$ to Z as follows:

$$((a, b), c) \in P \text{ if and only if } ((a, b), (a, c)) \in \mathrm{Eq}(\delta),$$

so that also $(a, c) \in X \times_Y Z$ by definition. Then:

• $((x, z), z) \in P$;
• $((il(z), w), z) \in P$, since $l(z) = l(w)$ and $((il(z), z), (il(w), w)) \in \mathrm{Eq}(\delta)$, as observed above;
• $((il(w), w), w) \in P$;
• $((x, w), w) \in P$.

It follows that $((x, z), (x, w)) \in PP^{\circ}PP^{\circ}$ and, from Theorem 1.1, we get $((x, z), (x, w)) \in PP^{\circ}$. There is then a θ such that $((x, z), \theta) \in P$ and $((x, w), \theta) \in P$. The fact that $\mathrm{Eq}(\delta)$ is an equivalence relation implies that

$((x, z), (x, w)) \in \mathrm{Eq}(\delta)$, since $((x, z), (x, \theta)) \in \mathrm{Eq}(\delta)$ and $((x, w), (x, \theta)) \in \mathrm{Eq}(\delta)$. It follows that $\mathrm{Eq}(\pi_X) \wedge \mathrm{Eq}(\lambda) \leq \mathrm{Eq}(\delta)$.

Consider then an element $((x_1, z_1), (x_2, z_2)) \in \mathrm{Eq}(\lambda)$: this means that $f(x_1) = l(z_1)$, $f(x_2) = l(z_2)$ and $\lambda(x_1, z_1) = \lambda(x_2, z_2)$ (thus, in particular, $\alpha(x_1) = \alpha(x_2)$). We are going to show that $\delta(x_1, z_1) = \delta(x_2, z_2)$. To do so, we apply the Shifting Lemma to the following situation

$$
\begin{tikzcd}
(x_1, kf(x_1)) \arrow[rr, dash, "\mathrm{Eq}(\pi_X)"] \arrow[d, dash, "\mathrm{Eq}(\lambda)"] \arrow[d, dash, bend right, "\mathrm{Eq}(\delta)"'] & & (x_1, z_1) \arrow[d, dash, "\mathrm{Eq}(\lambda)"'] \arrow[d, dash, dashed, bend left, "\mathrm{Eq}(\delta)"] \\
(x_2, kf(x_2)) \arrow[rr, dash, "\mathrm{Eq}(\pi_X)"'] & & (x_2, z_2),
\end{tikzcd}
$$

where the solid lines represent relations holding by assumption. Note that all the elements $(x_1, z_1), (x_2, z_2), (x_1, kf(x_1)), (x_2, kf(x_2))$ are in $X \times_Y Z$ and, moreover, $((x_1, kf(x_1)), (x_2, kf(x_2))) \in \mathrm{Eq}(\lambda)$: this follows from the fact that $((x_1, kf(x_1)), (x_2, kf(x_2))) \in \mathrm{Eq}(\delta)$ (since $\alpha(x_1) = \alpha(x_2)$). The inequality $\mathrm{Eq}(\pi_X) \wedge \mathrm{Eq}(\lambda) \leqslant \mathrm{Eq}(\delta)$ allows one to apply the Shifting Lemma to the diagram above and to conclude that also the relation in the dashed line holds, i.e. $((x_1, z_1), (x_2, z_2)) \in \mathrm{Eq}(\delta)$.

(iii) $\Rightarrow$ (ii). This implication easily follows by taking the (regular epimorphism, monomorphism) factorisation of the comparison morphism λ given in diagram (2), say $\lambda = m\delta$. One then obtains a cube of the type (3) which, by assumption, is such that the right face is a pullback. Consequently, λ is isomorphic to the regular epimorphism δ. ∎

In the last part of this section we give a characterisation of regular Goursat categories through the fibration of points. Thus, we add Goursat categories to the list of (many) algebraic categories characterised in these terms (see [5, 3]).

A *point* in a category $\mathcal{C}$ is a split epimorphism $f\colon X \to Y$ together with a fixed splitting $i\colon Y \to X$, usually depicted as

$$
\begin{tikzcd}
X \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "i"'] .
\end{tikzcd}
$$

The category of points in $\mathcal{C}$ is denoted by $\mathrm{Pt}(\mathcal{C})$. When $\mathcal{C}$ has pullbacks of split epimorphisms, the functor sending a point to its codomain

$$\begin{array}{ccc} \mathrm{Pt}(\mathcal{C}) & \to & \mathcal{C} \\ U \overset{j}{\underset{g}{\leftrightarrows}} W & \mapsto & W \end{array}$$

is a fibration, called the *fibration of points* [4]. Given a morphism $\beta \colon Y \to W$, the change-of-base functor with respect to this fibration is denoted by $\beta^* \colon \mathrm{Pt}_W(\mathcal{C}) \to \mathrm{Pt}_Y(\mathcal{C})$. If $\mathcal{C}$ has, moreover, pushouts along split monomorphisms, then any pullback functor β^* has a left adjoint

$$\begin{array}{cccc} \beta_! : & \mathrm{Pt}_Y(\mathcal{C}) & \to & \mathrm{Pt}_W(\mathcal{C}), \\ & X \overset{i}{\underset{f}{\leftrightarrows}} Y & \mapsto & \beta_!(X) \overset{\beta_!(i)}{\underset{\beta_!(f)}{\leftrightarrows}} W \end{array}$$

where $(\beta_!(X), \beta_!(f), \beta_!(i)) \in \mathrm{Pt}_W(\mathcal{C})$ is determined by the right hand part of the following pushout:

$$\begin{tikzcd} X \arrow[r, "\overline{\beta}"] & \beta_!(X) \\ Y \arrow[u, "i"] \arrow[r, "\beta"'] & W. \arrow[u, "\beta_!(i)"'] \end{tikzcd}$$

Observe that the arrow $\overline{\beta}$ in the diagram above is a regular epimorphism whenever β is a regular epimorphism.

Theorem 1.4. *Let $\mathcal{C}$ be a regular category with pushouts along split monomorphisms. Then the following conditions are equivalent:*

(i) *$\mathcal{C}$ is a Goursat category;*

(ii) *for any regular epimorphism $\beta \colon Y \to W$ in $\mathcal{C}$ the functor $\beta_! \colon \mathsf{Pt}_Y(\mathcal{C}) \to \mathsf{Pt}_W(\mathcal{C})$ preserves binary products;*

(iii) *for any commutative square*

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, shift right, "f"'] & U \arrow[d, shift right, "g"'] \\ Y \arrow[u, shift right, "i"'] \arrow[r, two heads, "\beta"'] & W \arrow[u, shift right, "j"'] \end{tikzcd}$$

where f and g are split epimorphisms and α and β are regular epimorphisms (commuting also with the splittings), the Beck-Chevalley condition holds: there is a functor isomorphism $\alpha_! f^* \cong g^* \beta_!$.

Proof: (i) ⇒ (iii). Given a point (Z, l, k) over Y consider the pullback defining $f^*(Z, l, k)$:

$$
\begin{tikzcd}
X \times_Y Z \arrow[r, shift left, "\pi_Z"'] \arrow[d, shift right, "\pi_X"'] & Z \arrow[l, shift left] \arrow[d, shift right, "l"'] \\
X \arrow[u, shift right] \arrow[r, shift right, "f"'] & Y. \arrow[l, shift right, "i"'] \arrow[u, shift right, "k"']
\end{tikzcd}
\tag{4}
$$

The following commutative cube is obtained by applying the functor $\alpha_!$ and $\beta_!$ to the points $f^*(Z, l, k)$ (over X) and (Z, l, k) (over Y), respectively:

$$
\begin{tikzcd}
X \times_Y Z \arrow[rr, two heads, "\overline{\alpha}"] \arrow[dd, shift right, "\pi_X"'] \arrow[dr, shift left, "\pi_Z"'] & & \alpha_!(X \times_Y Z) \arrow[dd, dashed, shift right] \arrow[dr, shift left] & \\
& Z \arrow[ul, shift left] \arrow[rr, two heads, "\overline{\beta}"] \arrow[dd, shift right, "l"'] & & \beta_!(Z) \arrow[ul, shift left] \arrow[dd, shift right] \\
X \arrow[uu, shift right] \arrow[rr, dashed, two heads, "\alpha"] \arrow[dr, shift right, "f"'] & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "g"'] & \\
& Y \arrow[ul, shift right, "i"'] \arrow[uu, shift right, "k"'] \arrow[rr, two heads, "\beta"'] & & W. \arrow[ul, dashed, shift right, "j"'] \arrow[uu, shift right]
\end{tikzcd}
$$

This diagram is of the form (3). Consequently, the right face is a pullback by Theorem 1.3(iii), so that $g^*\beta_! \cong \alpha_! f^*$.

(iii) ⇒ (ii). Let (X, f, i) and (Z, l, k) be points over Y and consider their product in $\mathrm{Pt}_Y(\mathcal{C})$, which is given by the pullback (4). We take its image through the functor $\beta_! \colon \mathrm{Pt}_Y(\mathcal{C}) \to \mathrm{Pt}_W(\mathcal{C})$:

$$
\begin{tikzcd}
X \times_Y Z \arrow[rr, two heads, "\overline{\beta}"] \arrow[dd, shift right, "\pi_X"'] \arrow[dr, shift left, "\pi_Z"'] & & \beta_!(X \times_Y Z) \arrow[dd, dashed, shift right] \arrow[dr, shift left] & \\
& Z \arrow[ul, shift left] \arrow[rr, two heads] \arrow[dd, shift right, "l"'] & & \beta_!(Z) \arrow[ul, shift left] \arrow[dd, shift right] \\
X \arrow[uu, shift right] \arrow[rr, dashed, two heads] \arrow[dr, shift right, "f"'] & & \beta_!(X) \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right] & \\
& Y \arrow[ul, shift right, "i"'] \arrow[uu, shift right, "k"'] \arrow[rr, two heads, "\beta"'] & & W. \arrow[ul, dashed, shift right] \arrow[uu, shift right]
\end{tikzcd}
\tag{5}
$$

By applying the assumption to the bottom commutative face, we conclude that the right face is a pullback, i.e. $\beta_!$ preserves binary products.

(ii) $\Rightarrow$ (i). Consider the diagram (2), where α, β, γ are assumed to be regular epimorphisms, and let us show that the induced arrow λ is also a regular epimorphism. The image of the points over Y of the left face of (2) by the functor $\beta_!$ determines the commutative diagram (5). One then obtains the following commutative diagram

$$
\begin{tikzcd}
\beta_!(X \times_Y Z) \arrow[rr, dashed, "\phi"] \arrow[dd, shift right] \arrow[dr, shift right] & & U \times_W V \arrow[dd, dashed, shift right] \arrow[dr, shift right] & \\
& \beta_!(Z) \arrow[rr, two heads, "\tau"] \arrow[ul, shift right] \arrow[dd, shift right] & & V \arrow[ul, shift right] \arrow[dd, shift right] \\
\beta_!(X) \arrow[rr, dashed, two heads, "\sigma"] \arrow[uu, shift right] \arrow[dr, shift right] & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right] & \\
& W \arrow[rr, equal, "1_W"'] \arrow[ul, shift right] \arrow[uu, shift right] & & W, \arrow[ul, dashed, shift right] \arrow[uu, shift right]
\end{tikzcd}
\tag{6}
$$

where the arrows σ, τ and ϕ are induced by the universal properties of the pushouts defining $\beta_!(X), \beta_!(Z)$ and $\beta_!(X \times_Y Z)$, respectively, and $\phi\overline{\beta} = \lambda$. The fact that α and γ are regular epimorphisms implies that σ and τ are regular epimorphisms, while the assumption guarantees that the left face in the diagram (6) is a pullback in $\mathcal{C}$. It then follows that the induced arrow ϕ is a regular epimorphism as well, and so is then the arrow

$$\lambda \colon X \times_Y Z \to U \times_W V.$$

This shows that condition (ii) in Theorem 1.3 is satisfied, and $\mathcal{C}$ is then a Goursat category, as desired. ∎

Remark 1.5. A variety of universal algebras is 3-permutable when its algebraic theory has two ternary operations r and s such that the identities

$$r(x,y,y) = x, \quad r(x,x,y) = s(x,y,y) \quad \text{and} \quad s(x,x,y) = y$$

hold [17]. We can prove the existence of such ternary operations by applying the property stated in Theorem 1.3(ii) to the following commutative cube

$$
\begin{tikzcd}
P \arrow[rrr, dotted, "\lambda"] \arrow[dr, shift left] \arrow[dd, shift right] & & & \mathrm{Eq}(\nabla) \arrow[dr, shift left] \arrow[d, dashed, shift right] & \\
& 3X \arrow[rrr, two heads, "\nabla+1"] \arrow[ul, shift left] \arrow[dd, shift right, "1+\nabla"'] & & \arrow[u, dashed, shift right] & 2X \arrow[ul, shift left] \arrow[dd, shift right, "\nabla"'] \\
3X \arrow[rrr, dashed, two heads, "1+\nabla"'] \arrow[uu, shift right] \arrow[dr, shift right, "\nabla+1"'] & & & 2X \arrow[dr, dashed, shift right, "\nabla"'] \arrow[uu, dashed, shift right] & \\
& 2X \arrow[rrr, two heads, "\nabla"'] \arrow[ul, shift right, "\begin{pmatrix} i_2\ i_3 \end{pmatrix}"'] \arrow[uu, shift right, "\begin{pmatrix} i_1\ i_2 \end{pmatrix}"'] & & & X, \arrow[ul, dashed, shift right, "i_2"'] \arrow[uu, shift right, "i_1"']
\end{tikzcd}
$$

where X is the free algebra on one element, kX denotes a k-indexed copower of X and i_j the canonical j-th injection, P denotes the object part of the pullback defining the left face and

$$\nabla = \begin{pmatrix} 1_X & 1_X \end{pmatrix} : 2X \to X$$

is the codiagonal. By Theorem 1.3(ii), the comparison morphism λ is a surjective homomorphism. The terms $p_1(x, y) = x$ and $p_2(x, y) = y$ are such that $(p_1, p_2) \in \mathrm{Eq}(\nabla)$. Since λ is surjective, there exist ternary terms, say r and s, such that:
$(r, s) \in P$, from which we deduce that $r(x, x, y) = s(x, y, y)$, and
$\lambda(r, s) = (p_1, p_2)$, which gives $r(x, y, y) = x$ and $s(x, x, y) = y$.

2. Reflections for Goursat categories

In this section we show that any (regular epi)-reflective subcategory $\mathcal{X}$ of a regular Goursat category $\mathcal{C}$ has the property that the reflector $I \colon \mathcal{C} \to \mathcal{X}$ preserves pullbacks of split epimorphisms along split epimorphisms, extending the same result which is known for regular Ma'tsev categories (Proposition 3.6 in [9]). Consequently, every internal Galois pregroupoid of an extension is an internal groupoid.

Proposition 2.1. *Consider a (regular epi)-reflective subcategory $\mathcal{X}$ of a regular Goursat category $\mathcal{C}$*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{X}, \arrow[l, shift left, hook, "U"] \end{tikzcd}$$

where $U\colon \mathcal{X} \to \mathcal{C}$ is a full inclusion. Then the reflector $I\colon \mathcal{C} \to \mathcal{X}$ preserves pullbacks of pairs of split epimorphisms.

Proof: Consider the following commutative diagram where the left face is a pullback of split epimorphisms, the right face is its image through the functor $UI\colon \mathcal{C} \to \mathcal{C}$ and η denotes the unit of the adjunction:

$$\begin{tikzcd}
X \times_Y Z \arrow[rr, two heads, "\eta_{X \times_Y Z}"] \arrow[dd, shift right, "\pi_X"'] \arrow[dr, shift left, "\pi_Z"] & & UI(X \times_Y Z) \arrow[dd, dashed, shift right] \arrow[dr, shift left] & \\
& Z \arrow[rr, two heads, "\eta_Z"] \arrow[dd, shift right, "l"'] \arrow[ul, shift left] & & UI(Z) \arrow[dd, shift right] \arrow[ul, shift left] \\
X \arrow[rr, dashed, two heads, "\eta_X"] \arrow[dr, shift right, "f"'] \arrow[uu, shift right] & & UI(X) \arrow[dr, dashed, shift right] \arrow[uu, dashed, shift right] & \\
& Y \arrow[rr, two heads, "\eta_Y"'] \arrow[uu, shift right, "k"'] \arrow[ul, shift right, "i"'] & & UI(Y). \arrow[uu, shift right] \arrow[ul, dashed, shift right]
\end{tikzcd}$$

This commutative diagram verifies the conditions of the commutative diagram (3) in Theorem 1.3. It follows that the right face is a pullback, since $\mathcal{C}$ is a Goursat category. But $U\colon \mathcal{X} \to \mathcal{C}$ is a full inclusion that preserves and reflects pullbacks, and this completes the proof. ∎

We follow [19] in calling a (regular epi)-reflective subcategory $\mathcal{X}$ of an exact category $\mathcal{C}$ a *Birkhoff* subcategory if, moreover, $\mathcal{X}$ is closed in $\mathcal{C}$ under regular quotients.

Corollary 2.2. *Consider a Birkhoff subcategory $\mathcal{X}$ of an exact Goursat category $\mathcal{C}$*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{X}, \arrow[l, shift left, hook, "U"] \end{tikzcd}$$

where $U\colon \mathcal{X} \to \mathcal{C}$ is a full inclusion. Then the reflector $I\colon \mathcal{C} \to \mathcal{X}$ preserves pullbacks of split epimorphisms along regular epimorphisms.

Proof: Let

$$\begin{tikzcd}
P \arrow[r, two heads, "p_2"] \arrow[d, shift right, "p_1"'] & C \arrow[d, shift right, "g"'] \\
A \arrow[u, shift right] \arrow[r, two heads, "p"'] & B \arrow[u, shift right]
\end{tikzcd} \tag{7}$$

be a pullback of a split epimorphism g along a regular epimorphism p. By taking the kernel pairs $\mathrm{Eq}(p_2)$ of p_2 and $\mathrm{Eq}(p)$ of p and then applying UI one gets the commutative diagram

$$\begin{tikzcd}
UI(\mathrm{Eq}(p_2)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] & UI(P) \arrow[r, two heads, "UI(p_2)"] \arrow[d, shift right, "UI(p_1)"'] \arrow[dr, phantom, "\text{(A)}"] & UI(C) \arrow[d, shift right, "UI(g)"'] \\
UI(\mathrm{Eq}(p)) \arrow[u, shift right] \arrow[r, shift left] \arrow[r, shift right] & UI(A) \arrow[u, shift right] \arrow[r, two heads, "UI(p)"'] & UI(B). \arrow[u, shift right]
\end{tikzcd} \tag{8}$$

The two (downward oriented) commutative squares on the left are pullbacks, since UI preserves pullbacks of split epimorphisms along split epimorphisms. It suffices to prove that (A) is a pullback, since U reflects pullbacks. Now, by taking the regular image of $\mathrm{Eq}(p_2)$ along the regular epimorphism η_P one gets an effective equivalence relation $\eta_P(\mathrm{Eq}(p_2))$ on $UI(P)$, since $\mathcal{C}$ is an exact Goursat category. Moreover, $UI(p_2)$ is its coequaliser in $\mathcal{C}$, since $\mathcal{X}$ is stable in $\mathcal{C}$ under regular quotients. Accordingly, $\eta_P(\mathrm{Eq}(p_2)) \cong \mathrm{Eq}(UI(p_2))$. Similarly, $\eta_A(\mathrm{Eq}(p)) \cong \mathrm{Eq}(UI(p))$ yielding the following commutative diagram:

$$\begin{tikzcd}
UI(\mathrm{Eq}(p_2)) \arrow[r, dotted, two heads] \arrow[d, shift right] & \mathrm{Eq}(UI(p_2)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[dr, phantom, "\text{(B)}"] & UI(P) \arrow[d, shift right, "UI(p_1)"'] \\
UI(\mathrm{Eq}(p)) \arrow[u, shift right] \arrow[r, dotted, two heads] & \mathrm{Eq}(UI(p)) \arrow[u, shift right] \arrow[r, shift left] \arrow[r, shift right] & UI(A). \arrow[u, shift right]
\end{tikzcd}$$

The exterior rectangles are the left pullbacks in diagram (8) and the dotted arrows are regular epimorphisms by construction. By applying Proposition 4.1 in [15] we conclude that the squares (B) are pullbacks. By the usually called Barr-Kock theorem [1] we conclude that (A) is a pullback, as desired.

$\blacksquare$

We finally observe that the so-called internal Galois pregroupoids associated to an extension are always internal groupoids in the Goursat context.

First recall from [18] that an *internal precategory* in a category $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} P_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & P_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & P_0 \arrow[l, "s" description] \end{tikzcd}$$

with

(1) $d_1 s = 1_{P_0} = d_2 s$;
(2) $d_2 p_1 = d_1 p_2$;
(3) $d_1 p_1 = d_1 m$, $d_2 p_2 = d_2 m$.

In other words, an internal precategory in $\mathcal{C}$ can be seen as what remains of the definition of an internal category in $\mathcal{C}$ when one cancels all references to pullbacks.

Every extension (= regular epimorphism) $f\colon A \to B$ in $\mathcal{C}$ induces the internal groupoid given by the equivalence relation determined by the kernel pair of f:

$$\begin{tikzcd} \mathrm{Eq}(f) \times_A \mathrm{Eq}(f) \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & \mathrm{Eq}(f) \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & A. \arrow[l, "s" description] \end{tikzcd}$$

By applying the left adjoint $I\colon \mathcal{C} \to \mathcal{X}$ of the inclusion functor $U\colon \mathcal{X} \to \mathcal{C}$ to this equivalence relation one always obtains an internal precategory in $\mathcal{X}$:

$$\begin{tikzcd} I(\mathrm{Eq}(f) \times_A \mathrm{Eq}(f)) \arrow[r, shift left=2, "I(p_1)"] \arrow[r, "I(m)" description] \arrow[r, shift right=2, "I(p_2)"'] & I(\mathrm{Eq}(f)) \arrow[r, shift left=2, "I(d_1)"] \arrow[r, shift right=2, "I(d_2)"'] & I(A). \arrow[l, "I(s)" description] \end{tikzcd}$$

This special kind of internal precategory is called the *internal Galois pregroupoid of f* (see [18] for more details), and is denoted by $\mathsf{Gal}(f)$. It turns out to always be an internal groupoid in the Goursat context:

Corollary 2.3. *Consider a (regular epi)-reflective subcategory $\mathcal{X}$ of a regular Goursat category $\mathcal{C}$*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=1.5, "I"] \arrow[r, phantom, "\perp"] & \mathcal{X}, \arrow[l, shift left=1.5, hook, "U"] \end{tikzcd}$$

where $U\colon \mathcal{X} \to \mathcal{C}$ is a full inclusion. Given any extension $f\colon A \to B$ in $\mathcal{C}$, then the internal Galois pregroupoid $\mathsf{Gal}(f)$ is an internal groupoid.

Proof: An internal precategory is an internal groupoid when certain commutative squares of split epimorphisms are pullbacks (see [3], for instance). The result follows immediately from the fact that those pullbacks are preserved by I (Proposition 2.1). ■

References

[1] M. Barr, Exact Categories, in: *Lecture Notes in Math.* 236, Springer 1-120 (1971).

[2] C. Berger and D. Bourn, *Central reflections and nilpotency in exact Mal'cev categories*, Preprint, 2015, arXiv:1511.00824.

[3] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Kluwer, (2004).

[4] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Springer Lecture Notes in Math. 1488 (1991) 43-62.

[5] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Applied Categorical Structures 4 (1996) 307-327.

[6] D. Bourn and M. Gran, *Categorical Aspects of Modularity.* Galois Theory, Hopf Algebras and Semiabelian Categories, Fields Instit. Commun., 43, Amer. Math. Soc., Providence RI (2004) 77-100.

[7] D. Bourn and M. Gran, *Normal sections and direct product decompositions* Comm. Algebra. 32 (2004) no. 10, 3825-3842.

[8] D. Bourn and G. Janelidze, *Protomodularity, descent and semidirect products*, Theory Appl. Categ. Vol. 4 (2) (1998) 37-46.

[9] D. Bourn and D. Rodelo, *Comprehensive factorization and I-central extensions*, J. Pure Appl. Algebra 216 (2012) 598-617.

[10] A. Carboni, G.M. Kelly, M.C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Structures 1, no.4 (1993) 385-421.

[11] A. Carboni, J. Lambek, M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra 69, no. 3 (1991) 271-284.

[12] R. Freese and R. McKenzie, Commutator theory for congruence modular varieties, Cambridge University Press, Cambridge (1987).

[13] M. Gran and D. Rodelo, *A new characterisation of Goursat categories*, Appl. Categ. Structures Vol. 20 (2012) 229-238.

[14] M. Gran and D. Rodelo, *The Cuboid Lemma and Mal'tsev categories*, Applied Categorical Structures 22(5) (2014) 805-816.

[15] M. Gran and D. Rodelo, *Some remarks on pullbacks in Gumm categories*, Texts on Mathematics, 46, University of Coimbra (2014) 125-136.

[16] H.P. Gumm, *Geometrical methods in congruence modular algebras*, Mem. Amer. Math. Soc. 45 (1983), no. 286.

[17] J. Hagemann and A. Mitschke, *On n-permutable congruences*, Algebra Universalis 3 (1973), 8–12.

[18] G. Janelidze, *Precategories and Galois theory*, Lecture Notes Math. 1488, Springer (1991) 157-173.

[19] G. Janelidze and G.M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra 97 (1994), no. 2, 135-161.

[20] Z. Janelidze, D. Rodelo, T. Van der Linden, *Hagemann's theorem for regular categories*, Journal of Homotopy and Related Structures 9(1) (2014) 55-66.

[21] S. Lack, *The 3-by-3 lemma for regular Goursat categories*, Homology, Homotopy Appl., 6(1), (2004) 1-3.

[22] A. I. Mal'cev, *On the general theory of algebraic systems*, Mat. Sbornik N. S. 35 (1954), no. 6, 3–20.
[23] A. Mitschke, *Implication algebras are 3-permutable and 3-distributive*, Algebra Universalis 1 (1971), 182–186.
[24] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra, 177 (1995) 647-657.
[25] J.D.H. Smith, *Mal'cev Varieties*, Lecture Notes in Math. 554 (1976).

Marino Gran
Insitut de Recherche en Mathématiques et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: Marino.Gran@uclouvain.be

Diana Rodelo
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro and Centre for Mathematics of the University of Coimbra, 3001-454 Coimbra, Portugal

E-mail address: drodelo@ualg.pt