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ASYMPTOTIC RESULTS FOR CERTAIN WEAK DEPENDENT VARIABLES

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Abstract: We consider a special class of weak dependent random variables with control on covariances of Lipschitz transformations. This class includes positively, negatively associated variables and a few other classes of weakly dependent structures. We prove a Strong Law of Large Numbers with a characterization of convergence rates which is almost optimal, in the sense that it is arbitrarily close to the optimal rate for independent variables. Moreover, we prove an inequality comparing the joint distributions with the product distributions of the margins, similar to the well known Newman inequality for characteristic functions of associated variables. As a consequence, we prove a Central Limit Theorem, together with its functional counterpart, and also the convergence of the empircal process for this class of weak dependent variables.

Keywords: L-weak dependence, strong law of large numbers, convergence rate, Central Limit Theorem.
AMS Subject Classification (2000): 60F10, 60F05, 60F17.

1. Introduction

Limit theorems, either with respect to almost sure convergence or convergence in distribution, are a central subject in statistics. In more recent years, many authors were interested on the asymptotics for dependent sequences of variables. Several forms of controlling the dependence have been proposed, many of them expressing a control on covariances of transformations of variables that may be thought of as representing the past with transformations of another set of variables representing the future. These dependence structures are commonly named weak dependence. Many of these notions stemmed from the positive dependence and association introduced by Lehmann [9] and Esary, Proschan and Walkup [5], respectively. Association was the first of these two notions to attract the interest of researchers, and as expected, Strong Law of Large Numbers and Central Limit Theorems were eventually proved. We refer the reader to the monographs by Bulinski

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and Shashkin [1], Oliveira [14] or Prakasa Rao [15] for an account of relevant literature. Inevitably, several variations and extensions of these dependence notions were introduced and limit theorems were proved. Among these, the negative association defined by Joag-Dev and Proschan [7] was one of the most popular, with various different extensions introduced in more recent years: extended negative dependent (END) introduced by Liu [10], widely orthant dependent (WOD) introduced by Wang, Wang and Gao [17] among other variations. The proof techniques for these dependence structures relied essentially on an adequate control of covariances between appropriate families of transformations of the random variables. Thus, it was natural to define the dependence control through some upper bound of a convenient family of covariances, leading to the weak dependence notions, as introduced by Doukhan and Louhichi [4]. Depending on the family and the control form, these authors introduced different dependence notions. For an account on these dependence structures and some relations, we refer the reader to the monograph by Dedecker et al. [3].

In this paper, we will be interested in a particular version of weak dependence, somewhat similar to the quasi-association introduced in Bulinski and Suquet [2], that includes the positive and negative dependence notions referred above. For this type of weak dependence, we will prove a Strong Law of Large Numbers with rates, a Central Limit Theorem and an invariance principle. Concerning weak dependence notions, some asymptotic results are proved in Doukhan and Louhichi [4]. However, the only inequality controlling tail probabilities, see Corollary 1 in [4], is a Bernstein type inequality, with a relatively weak form. Later, Corollary 4.1 and Theorem 4.5 in [3] and Kallabis and Neumann [8] also prove exponential inequalities that are analogous to the Bernstein inequalities, but again with weaker exponents in their upper bounds. This means that although Strong Law of Large Numbers may be derived, not only the assumptions will become stronger, but convergence rates that follow will not be almost optimal, in the sense that these rates may be arbitrarily close to the well known rates for independent variables. In the present paper, the version of weak dependence we will be studying allows for the adaptation of techniques used for associated variables (see, for example, Ioannides and Roussas [6], Oliveira [13], Sung [16]) providing stronger forms of the Bernstein-type inequality, meaning that we will obtain almost optimal convergence rates. As what concerns the asymptotic normality of sums of variables, some contributions for some variants of weakly dependent variable may be found in Doukhan and Louhichi [4].

The paper is organized as follows: Section 2 defines the framework, Section 3 proves some basic inequalities needed for the control of the almost sure convergence, which is the object of Section 4, where Strong Law of Large Numbers, with characterization of rates, are proved. Finally, in Section 5, we extend the Newman inequality for characteristic functions to the present dependence structure, from which a Central Limit Theorem, an invariance principle and the convergence of the empirical process follow.

2. Definitions and framework

Let X_n, $n \geq 1$, be centered random variables and define $S_n = X_1 + \cdots + X_n$. As mentioned before, we will be interested in a particular form of weak dependence, according to the following definition.

Definition 2.1. *A finite sequence of random variables X_i, $i = 1, \ldots, n$, is said to be L-weakly dependent if there exist nonnegative coefficients γ_ℓ, $\ell \geq 1$, such that for every disjoint subsets $I, J \subset \{1, \ldots, n\}$ and real valued Lipschitz functions f and g, defined on the appropriate Euclidean spaces, the following inequality is satisfied:*

$$\left|\mathrm{Cov}\left(f\left(X_i, i \in I\right), g\left(X_j, j \in J\right)\right)\right| \leq \|f\|_L \|g\|_L \sum_{i \in I} \sum_{j \in J} \gamma_{|j-i|},$$

where $\|f\|_L$ represents the Lipschitz norm of f:

$$\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

An infinite family of random variables is said to be L-weakly dependent if every finite subfamily is L-weakly dependent.

This is a form of weak dependence in the same spirit as in Doukhan and Louhichi [4] or Dedecker et al. [3]. With respect to the discussion in [3], this dependence follows from what these authors called the κ or the ζ coefficients. This means the examples of L-weakly dependent sequences include positively associated, negatively associated, Gaussian sequences or models for interacting particles systems (see Section 3.5.3 in [3] for details for this last example). Moreover, the notion of quasi-association, introduced by Bulinski and Suquet [2], is also included in the L-weak dependence structure by choosing $\gamma_k = \mathrm{Cov}(X_1, X_{k+1})$, of course assuming the stationarity of the random variables.

We will assume throughout this paper that

$$\frac{1}{n}\mathbb{E}S_n^2 \longrightarrow \sigma^2 < \infty. \tag{1}$$

This, obviously, implies that for n large enough, we have $\mathbb{E}S_n^2 \leq 2\sigma^2 n$. Besides, we will need to decompose S_n into an appropriate sum of blocks. For this purpose, consider an increasing sequence of integers $p_n \leq \frac{n}{2}$ such that $p_n \longrightarrow +\infty$, put $r_n = \lfloor \frac{n}{2p_n} \rfloor$, where $\lfloor x \rfloor$ represents the integer part of x, and define the blocks:

$$Y_{j,n} = \sum_{k=(j-1)p_n+1}^{jp_n} X_k, \quad j = 1, \ldots, 2r_n. \tag{2}$$

Notice that, if the random variables are bounded by $c > 0$, then $|Y_{j,n}| \leq cp_n$. Moreover, define the alternate sums:

$$Z_{n,od} = \sum_{j=1}^{r_n} Y_{2j-1,n} \qquad \text{and} \qquad Z_{n,ev} = \sum_{j=1}^{r_n} Y_{2j,n}.$$

Finally, we introduce the generalized Cox-Grimmett coefficients adapted to the L-weak dependence structure,

$$v(n) = \sum_{j=n}^{\infty} \gamma_j. \tag{3}$$

3. Inequalities for bounded variables

This section establishes a few inequalities that are the basic tools for proving the almost sure convergence results. The inequalities below are extensions of analogous results for associated random variables. We start by proving a bound for the Laplace transform of the blocks $Y_{j,n}$.

Lemma 3.1. *Assume that the random variables X_n, $n \geq 1$, are stationary, there exists some $c > 0$ such that for every $n \geq 1$, $|X_n| \leq c$ almost surely and that (1) holds. Let $d_n > 1$, $n \geq 1$, be a sequence of real numbers. Then, for every $t \leq \frac{d_n-1}{d_n}\frac{1}{cp_n}$ and n large enough,*

$$\mathbb{E}e^{tY_{j,n}} \leq \exp\left(2t^2\sigma^2 p_n d_n\right).$$

Proof. Using a Taylor expansion and taking into account the boundedness of the random variables, we have

$$\mathbb{E}e^{tY_{j,n}} = 1+\sum_{k=2}^{\infty}\frac{t^k\mathbb{E}Y_{j,n}^k}{k!} \le 1+\sum_{k=2}^{\infty}\frac{t^kc^{k-2}p_n^{k-2}\mathbb{E}Y_{j,n}^2}{k!} \le 1+t^2\mathbb{E}Y_{j,n}^2\sum_{k=2}^{\infty}(tcp_n)^{k-2}.$$

It follows from the assumption on t that $tcp_n \le \frac{d_n-1}{d_n} < 1$, thus, as the variables are stationary, we may write

$$\mathbb{E}e^{tY_{j,n}} \le 1+\frac{t^2\mathbb{E}S_{p_n}^2}{1-tcp_n}.$$

Now as $p_n \longrightarrow +\infty$, we have that for n large enough, $\mathbb{E}S_{p_n}^2 \le 2\sigma^2 p_n$. Moreover, we have $\frac{1}{1-tcp_n} < d_n$, so $\mathbb{E}e^{tY_{j,n}} \le 1+2t^2\sigma^2p_nd_n \le \exp\left(2t^2\sigma^2p_nd_n\right)$. ■

Considering now L-weakly dependent variables, we prove an upper bound for $\mathbb{E}e^{tZ_{n,od}}$.

Lemma 3.2. *Assume the conditions of Lemma 3.1 are satisfied and the variables X_n, $n \ge 1$, are L-weakly dependent. Then, for every $t \le \frac{d_n-1}{d_n}\frac{1}{cp_n}$ and n large enough,*

$$\mathbb{E}e^{tZ_{n,od}} \le t^2 e^{\frac{tcn}{2}} p_n v(p_n) \sum_{j=0}^{r_n-2} \exp\left(jtp_n(2t\sigma^2 d_n - c)\right) + \exp\left(t^2\sigma^2 n d_n\right). \qquad (4)$$

Proof. Remark first that $\mathbb{E}e^{tZ_{n,od}} = \mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right)$. Now, by adding and subtracting appropriate terms, we find that

$$\begin{aligned}
&\mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right)\\
&\quad= \mathrm{Cov}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}, e^{tY_{2r_n-1,n}}\right) + \mathbb{E}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}\right)\mathbb{E}e^{tY_{2r_n-1,n}}\\
&\quad= \mathrm{Cov}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}, e^{tY_{2r_n-1,n}}\right) + \mathrm{Cov}\left(\prod_{j=1}^{r_n-2} e^{tY_{2j-1,n}}, e^{tY_{2r_n-3,n}}\right)\mathbb{E}e^{tY_{2r_n-1,n}}\\
&\qquad+\mathbb{E}\left(\prod_{j=1}^{r_n-3} e^{tY_{2j-1,n}}\right)\mathbb{E}e^{tY_{2r_n-3,n}}\mathbb{E}e^{tY_{2r_n-1,n}}.
\end{aligned}$$

Before iterating this procedure, remark that due to the stationarity of the random variables, $\mathbb{E}e^{tY_{2r_n-3,n}}=\mathbb{E}e^{tY_{2r_n-1,n}}=\mathbb{E}e^{tY_{1,n}}$, so the previous expression may be rewritten as

$$\begin{aligned}
&\mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right)\\
&\quad=\mathrm{Cov}\left(\prod_{j=1}^{r_n-1} e^{tY_{2j-1,n}}, e^{tY_{2r_n-1,n}}\right)+\mathrm{Cov}\left(\prod_{j=1}^{r_n-2} e^{tY_{2j-1,n}}, e^{tY_{2r_n-3,n}}\right)\mathbb{E}e^{tY_{1,n}}\\
&\qquad+\mathbb{E}\left(\prod_{j=1}^{r_n-3} e^{tY_{2j-1,n}}\right)\left(\mathbb{E}e^{tY_{1,n}}\right)^2.
\end{aligned}$$

Now, we iterate the procedure above to decompose the mathematical expectation of the product to find

$$\mathbb{E}\left(\prod_{j=1}^{r_n} e^{tY_{2j-1,n}}\right)=\sum_{j=1}^{r_n-1}\left(\mathbb{E}e^{tY_{1,n}}\right)^{j-1}\mathrm{Cov}\left(\prod_{k=1}^{r_n-j} e^{tY_{2k-1,n}}, e^{tY_{2(r_n-j)+1,n}}\right)+\left(\mathbb{E}e^{tY_{1,n}}\right)^{r_n}.$$

The L-weak dependence of the variables implies that

$$\begin{aligned}
&\left|\mathrm{Cov}\left(\prod_{k=1}^{r_n-j} e^{tY_{2k-1,n}}, e^{tY_{2(r_n-j)+1,n}}\right)\right|\\
&\qquad\leq t^2 e^{tcp_n(r_n-j+1)}\sum_{k=1}^{r_n-j}\sum_{\ell=2(k-2)p_n+1}^{(2k-1)p_n}\sum_{\ell'=2(r_n-j)p_n+1}^{(2(r_n-j)+1)p_n}\gamma_{\ell'-\ell}.
\end{aligned}\tag{5}$$

The summation above is similar to the one treated in course of proof of Lemma 3.1 in [6]. Adapting their arguments, one easily finds that

$$\begin{aligned}
\sum_{\ell=2(k-2)p_n+1}^{(2k-1)p_n}\sum_{\ell'=2(r_n-j)p_n+1}^{(2(r_n-j)+1)p_n}\gamma_{\ell'-\ell} &= \sum_{\ell=0}^{p_n-1}(p_n-\ell)\gamma_{2kp_n+\ell}+\sum_{\ell=1}^{p_n-1}(p_n-\ell)\gamma_{2kp_n-\ell}\\
&\leq p_n\sum_{\ell=(2k-1)p_n+1}^{(2k+1)p_n-1}\gamma_\ell,
\end{aligned}$$

thus,

$$\sum_{k=1}^{r_n-j}\sum_{\ell=2(k-2)p_n+1}^{(2k-1)p_n}\sum_{\ell'=2(r_n-j)p_n+1}^{(2(r_n-j)+1)p_n}\gamma_{\ell'-\ell}\leq\sum_{k=1}^{r_n-j}p_n\sum_{\ell=(2k-1)p_n+1}^{(2k+1)p_n-1}\gamma_\ell\leq p_nv(p_n).$$

Plug this into (5) and use the inequality proved in Lemma 3.1 to obtain upper bounds for $\left(\mathbb{E}e^{tY_{1,n}}\right)^{j-1}$ and $\left(\mathbb{E}e^{tY_{1,n}}\right)^{r_n}$. Finally, remember that $2p_nr_n\leq n$ to conclude the proof. ■

Lemma 3.3. *Assume the conditions of Lemma 3.2 are satisfied. Then, there exists a constant $c_1>0$ such that, for each fixed $x>0$ and n large enough,*

$$\mathrm{P}(Z_{n,od}>x)\leq\left(\frac{c_1x^2}{4\sigma^4n^2d_n^2}e^{\frac{tcn}{2}}p_nv(p_n)+1\right)\exp\left(-\frac{x^2}{4\sigma^2nd_n}\right)\tag{6}$$

Proof. Using Markov's inequality and taking into account (4), it follows that

$$\begin{aligned}\mathrm{P}(Z_{n,od}>x)\ &\leq\ t^2e^{\frac{tcn}{2}}p_nv(p_n)e^{-tx}\sum_{j=0}^{r_n-2}\exp\left(jtp_n(2t\sigma^2d_n-c)\right)\\&\qquad+\exp\left(t^2\sigma^2nd_n-tx\right).\end{aligned}\tag{7}$$

Minimizing the exponent on the second term above leads to the choice $t=\frac{x}{2\sigma^2nd_n}$, which implies that

$$t^2\sigma^2nd_n-tx=-\frac{x^2}{4\sigma^2nd_n}.$$

We still have to control the summation on the first term. For this purpose, remark that for the choice of t as above, $2t\sigma^2d_n-c=\frac{x}{n}-c$. Thus, for n large enough $2t\sigma^2d_n-c<0$, so the series corresponding to this sum is convergent. Finally, remark that, again for the choice made for t, we have $tx=\frac{x^2}{2\sigma^2nd_n}$, so $e^{-tx}\leq c'\exp\left(-\frac{x^2}{4\sigma^2nd_n}\right)$, and the proof is concluded. ■

4. Strong laws and convergence rates

With the tools proved in the previous section, we may now find conditions for the Strong Law of Large Numbers and characterize its convergence rate. The first subsection will deal with bounded random variables, using directly the inequalities of Section 3, while on the second subsection we will extend these results to arbitrary (unbounded) L-weakly dependent variables by using a truncation technique.

4.1. The case of bounded variables.

Theorem 4.1. *Assume that the variables X_n, $n \geq 1$, are stationary and L-weakly dependent, there exists some $c > 0$ such that for every $n \geq 1$, $|X_n| \leq c$ almost surely and that (1) holds. Assume that the generalized Cox-Grimmett coefficients (3) satisfy $v(n) = \rho^n$, for some $\rho \in (0,1)$. Then $\frac{1}{n} Z_{n,od} \longrightarrow 0$ almost surely.*

Proof. Let $\varepsilon > 0$ be arbitrarily chosen, and apply (6) with $x = n\varepsilon$ to find the upper bound

$$\mathrm{P}(Z_{n,od} > n\varepsilon) \leq \left(\frac{c_1 \varepsilon^2}{4\sigma^4 d_n^2} e^{\frac{tcn}{2}} p_n v(p_n) + 1 \right) \exp\left(-\frac{n\varepsilon^2}{4\sigma^2 d_n} \right). \tag{8}$$

We have now to reverify the conditions of Lemma 3.1 and the convergence of the series that appeared in course of proof of Lemma 3.3. Indeed, when proving this lemma, we verified the assumptions of Lemma 3.1 with x fixed, while the present choice considers x growing with n. First, we need to verify that $t = \frac{\varepsilon}{2\sigma^2 d_n} < \frac{d_n - 1}{d_n} \frac{1}{c p_n}$, as required to use Lemma 3.1. This inequality is equivalent to

$$\varepsilon < \frac{\sigma^2}{c} \frac{(d_n - 1)}{p_n}. \tag{9}$$

Thus, we need to choose the sequences such that $\frac{d_n}{p_n}$ is bounded away from 0. Secondly, for the control of the series appearing in (7), we need that $2t\sigma^2 d_n - c < 0$. Now, as $x = n\varepsilon$, we have $2t\sigma^2 d_n - c = \varepsilon - c$, so, choosing ε small enough this will be negative. Choose the tuning sequence as $d_n = a p_n$, for some $a > 0$, and $p_n = n^\theta$, for some $\theta \in (\frac{1}{2}, 1)$. Let us now look at the term inside the large parenthesis in (8). The growth rate of this term is dominated by the exponential factors, $e^{\frac{tcn}{2}} v(p_n)$, as the remaining terms have polynomial behavior. Taking into account the choice for t, it follows easily that $e^{\frac{tcn}{2}} v(p_n) \sim \exp\left(\frac{\varepsilon n}{4\sigma^2 d_n} + n^\theta \log \rho \right) \sim \exp\left(n^{1-\theta} + n^\theta \log \rho \right)$ and this is bounded as $\theta > \frac{1}{2}$ and $\rho \in (0,1)$. Finally, choose ε such that $\frac{n\varepsilon^2}{4\sigma^2 d_n} = \alpha \log n$, for some $\alpha > 1$. We have just proved that

$$\mathrm{P}(Z_{n,od} > n\varepsilon) \leq C \exp\left(-\alpha \log n \right) = \frac{C}{n^\alpha},$$

which define a convergence series, thus concluding the proof. ■

Note that it follows from the arguments of this proof that under the assumptions of Theorem 4.1, there exists $C>0$ such that

$$\mathrm{P}\left(|Z_{n,od}|>n\varepsilon\right)\leq C\exp\left(-\frac{n\varepsilon^2}{4\sigma^2 d_n}\right). \tag{10}$$

This remark will be useful later while extending these results to unbounded variables.

It is obvious that the result just proved also holds if we replace $Z_{n,od}$ by $Z_{n,ev}$, thus we have the almost sure convergence of $\frac{1}{n}S_n$. For sake of completeness, we state this result.

Theorem 4.2. *Assume that the conditions of Theorem 4.1 are satisfied. Then* $\frac{1}{n}S_n \longrightarrow 0$ *almost surely.*

We may further identify a convergence rate for the almost sure convergence above.

Theorem 4.3. *Assume that the conditions of Theorem 4.1 are satisfied. Then* $\frac{1}{n}Z_{n,od}\longrightarrow 0$ *almost surely with convergence rate* $\frac{\log n}{n^{1/2-\delta}}$*, where* $\delta>0$ *is arbitrarily small.*

Proof. We follow the proof of Theorem 4.1, allowing now ε to depend on n, that is, considering ε_n such that

$$\varepsilon_n^2=\frac{4\sigma^2\alpha d_n\log n}{n},\qquad \alpha>1.$$

We need to verify that the condition on t of Lemma 3.1 is satisfied for an appropriate choice of the sequences p_n and d_n, that is, that it holds $t=\frac{\varepsilon_n}{2\sigma^2 d_n}<\frac{d_n-1}{d_n}\frac{1}{cp_n}$. Note that once this is checked, the final arguments of the proof of Theorem 4.1 follow. So, choose $p_n=n^\theta$, for some $\theta>\frac{1}{2}$. Remember that we have $t=\frac{\varepsilon_n}{2\sigma^2 d_n}$. We need to choose $d_n\longrightarrow+\infty$ such that

$$\begin{aligned} tcp_n=\frac{\alpha^{1/2}c}{\sigma}p_n\left(\frac{d_n\log n}{n}\right)^{1/2} &\leq d_n-1\leq 2d_n\\ \Leftrightarrow\qquad \frac{\alpha^{1/2}c}{2\sigma}\frac{p_n(\log n)^{1/2}}{n^{1/2}} &\leq d_n^{1/2}, \end{aligned}$$

leading to $d_n\sim n^{2\theta-1}\log n$. The analysis of the exponential terms follows analogously as in the proof of Theorem 4.1. Indeed, taking into account the choices made for t, ε_n and d_n,

$$
\begin{aligned}
e^{\frac{tcn}{2}} v(p_n) &\sim \exp\left(\frac{\varepsilon_n n}{4\sigma^2 d_n} + n^\theta \log \rho\right) \\
&\sim \exp\left(\frac{\alpha^{1/2} c}{2\sigma}\left(\frac{n \log n}{d_n}\right)^{1/2} + n^\theta \log \rho\right) \\
&\sim \exp\left(n^{1-\theta} + n^\theta \log \rho\right),
\end{aligned}
$$

which is bounded as $\theta \in \left(\frac{1}{2}, 1\right)$ and $\rho \in (0,1)$. The convergence rate that follows from the above construction is then of order $\varepsilon_n \sim \frac{\log n}{n^{1-\theta}}$. To conclude the proof, just rewrite $\theta = \frac{1}{2} + \delta$. ■

The previous result was proved for $\frac{1}{n} Z_{n,od}$ for convenience of the exposition. An analogous version obviously holds for $\frac{1}{n} Z_{n,ev}$, thus implying the same result for $\frac{1}{n} S_n$. Again, for sake of completeness, we state the final result.

Theorem 4.4. *Assume that the conditions of Theorem 4.1 are satisfied. Then* $\frac{1}{n} S_n \longrightarrow 0$ *almost surely with convergence rate* $\frac{\log n}{n^{1/2-\delta}}$, *where* $\delta > 0$ *is arbitrarily small.*

4.2. General random variables. We now want to drop the boundedness assumption. To extend the results just proved, we will use a truncation technique together with a control on the tails of the distributions. Define, for a given fixed $c > 0$, the nondecreasing function $g_c(x) = \max(\min(x, c), -c)$, performing a truncation at level c. Remark that, for every $c > 0$, g_c is Lipschtizian with $\|g_c\|_L = 1$. Choose some sequence $c_n \longrightarrow +\infty$, to be made precise later, and define, for $j, n \geq 1$, the random variables

$$
X_{1,j,n} = g_{c_n}(X_j), \qquad X_{2,j,n} = X_j - X_{1,j,n},
$$

and the partial summations

$$
S_{1,n} = \sum_{j=1}^{n} (X_{1,j,n} - \mathbb{E} X_{1,j,n}), \qquad S_{2,n} = \sum_{j=1}^{n} (X_{2,j,n} - \mathbb{E} X_{2,j,n}).
$$

Theorem 4.5. *Assume that the L-weakly dependent variables* X_n, $n \geq 1$, *are stationary, (1) holds, and the generalized Cox-Grimmett coefficients (3) satisfy* $v(n) = \rho^n$, *for some* $\rho \in (0,1)$. *Assume further that,*

$$
\exists \tau > 3, U > 0 : \sup_{|t| \leq \tau} \mathbb{E} e^{t|X|} \leq U. \tag{11}
$$

Then $\frac{1}{n}S_n \longrightarrow 0$ *almost surely with convergence rate* $\frac{(\log n)^{3/2}}{n^{1/2-\delta}}$*, where* $\delta > 0$ *is arbitrarily small.*

Proof. It is obvious that $\mathrm{P}(|S_n| > 2n\varepsilon) \leq \mathrm{P}(|S_{1,n}| > n\varepsilon) + \mathrm{P}(|S_{2,n}| > n\varepsilon)$. As Theorem 4.1 applies it follows, taking into account (10), that,

$$\mathrm{P}\left(|S_{1,n}| > n\varepsilon_n\right) \leq 2C \exp\left(-\frac{n\varepsilon_n^2}{4\sigma^2 d_n}\right).$$

As in the proof of Theorem 4.3 choose $\varepsilon_n^2 = \frac{4\sigma^2\alpha d_n \log n}{n}$, for some $\alpha > 1$. This means that $\mathrm{P}\left(|S_{1,n}| > n\varepsilon_n\right) \leq 2Cn^{-\alpha}$, thus defining a convergent series. As before, choose $p_n = n^\theta$, for some $\theta \in \left(\frac{1}{2}, 1\right)$. As in the proof of Theorem 4.3, we need to verify that the assumptions of Lemma 3.1 are satisfied. Taking into account the bounding value for the truncated variables, the assumption of Lemma 3.1 is now written as $t = \frac{\varepsilon_n}{2\sigma^2 d_n} \leq \frac{d_n - 1}{d_n}\frac{1}{\ell_n p_n}$, which is equivalent to

$$\begin{aligned} tc_np_n = \frac{\alpha^{1/2}}{\sigma}\left(\frac{d_n \log n}{n}\right)^{1/2} c_np_n \leq d_n - 1 \leq d_n \\ \Leftrightarrow \qquad \frac{\alpha^{1/2}}{\sigma}\frac{(\log n)^{1/2}}{n^{1/2}} c_np_n \leq d_n^{1/2}. \end{aligned}$$

Using now the choice for p_n, this means we may choose $d_n = \frac{\alpha}{\sigma^2} n^{2\theta-1} c_n^2 \log n$, thus obtaining

$$\varepsilon_n^2 = 4\alpha^2 n^{2\theta-2} c_n^2 \log n.$$

We need now to control $\mathrm{P}\left(|S_{2,n}| > n\varepsilon_n\right)$. Note first that, taking into account the stationarity,

$$\mathrm{P}\left(|S_{2,n}| > n\varepsilon_n\right) \leq n\mathrm{P}\left(|X_{2,1,n} - \mathbb{E}X_{2,1,n}| > \varepsilon\right) \leq \frac{n}{\varepsilon}\mathbb{E}X_{2,1,n}^2.$$

Denoting $\bar{F}(x) = \mathrm{P}(|X_1| > x)$, we have that

$$\mathbb{E}X_{2,1,n}^2 = \int_{(c_n,+\infty)} (x - c_n)^2\, \bar{F}(dx) = \int_{c_n}^{+\infty} 2(x - c_n)\bar{F}(x)\, dx.$$

Now, using Markov's inequality, it follows that $\bar{F}(x) \leq e^{-tx}\mathbb{E}e^{t|X_1|} \leq Ue^{-tx}$, if $t \in (0, \tau)$. Thus, for $t \in (0, \tau)$, by integrating the expression above it follows that

$$\mathbb{E}X_{2,1,n}^2 \leq \frac{2U}{t^2} e^{-tc_n},$$

so finally,

$$\mathrm{P}\left(|S_{2,n}| > n\varepsilon_n\right) \leq \frac{2nU}{t^2\varepsilon_n^2} e^{-tc_n}.$$

If we now choose $c_n = \log n$ and $t = \alpha + 2(1-\theta)$, this upper bound behaves like $n^{-\alpha}$, as the upper bound for $\mathrm{P}\left(|S_{2,n}| > n\varepsilon_n\right)$. Finally, plug these choices into the expression of ε_n to explicitly identify the convergence rate, finding

$$\varepsilon_n = 4\alpha^2 \frac{(\log n)^{3/2}}{n^{1-\theta}},$$

and write $\theta = \frac{1}{2} + \delta$. ∎

Note that the convergence rate proved in Theorem 4.5 is close to the optimal convergence rate for the Strong Law of Large Numbers for associated random variables which is of order $\frac{(\log n)^{1/2}(\log\log n)^{\eta/2}}{n^{1/2}}$ for arbitrarily small $\eta > 0$, as proved by Yang, Su and Yu [18].

5. A Central Limit Theorem

We now look at the convergence in distribution of sums of L-weakly dependent variables, extending a Central Limit Theorem (CLT) for associated random variables by Newman [11, 12] to the L-weak dependence structure. The proof of Newman's result (see Theorem 2 in [11] or Theorem 12 in [12]) relies on an inequality for characteristic functions, the Newman inequality for characteristic functions (Theorem 1 in Newman [11] or Theorem 10 in Newman [12]) that controls the approximation between the joint distribution and the product of the marginal distributions. So, we start by proving a version of this inequality for the present dependence structure.

Theorem 5.1. *(Newman's inequality for L-weakly dependent random variables) Let $X_1, X_2, \ldots, X_n$ be L-weakly dependent random variables. Then, for every $t \in \mathbb{R}$,*

$$\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n} \mathbb{E}\left(e^{itX_j}\right)\right| \leq 4t^2 \sum_{j=1}^{n-2} (n-j-1)\gamma_j. \tag{12}$$

Proof. We start by adding and subtracting the appropriate terms to the left side of (12) to find

$$\begin{aligned}
&\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n}\mathbb{E}\left(e^{itX_j}\right)\right| \\
&\quad \leq \left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \mathbb{E}\left(e^{itX_n}\right)\mathbb{E}\left(\prod_{j=1}^{n-1} e^{itX_j}\right)\right| \\
&\qquad + \left|\mathbb{E}\left(e^{itX_n}\right)\mathbb{E}\left(\prod_{j=1}^{n-1} e^{itX_j}\right) - \prod_{j=1}^{n}\mathbb{E}\left(e^{itX_j}\right)\right| \\
&\quad \leq \left|\operatorname{Cov}\left(\prod_{j=1}^{n-1} e^{itX_j}, e^{itX_n}\right)\right| + \left|\mathbb{E}\left(\prod_{j=1}^{n-1} e^{itX_j}\right) - \prod_{j=1}^{n-1}\mathbb{E}\left(e^{itX_j}\right)\right|.
\end{aligned}$$

Iterating now this procedure, we find that

$$\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n}\mathbb{E}\left(e^{itX_j}\right)\right| \leq \sum_{m=2}^{n-1}\left|\operatorname{Cov}\left(\prod_{j=1}^{m-1} e^{itX_j}, e^{itX_m}\right)\right|.$$

To bound the covariance terms above, expand this covariance using the trigonometric representation of the complex exponential to find four terms involving cosinus or sinus functions. Now, for example,

$$\left|\operatorname{Cov}\left(\cos\left(t\sum_{j=1}^{m-1} X_j\right), \cos\left(tX_m\right)\right)\right| \leq t^2\sum_{j=1}^{m-1}\gamma_{m-j},$$

taking into account that $\|\cos(tx)\|_L = t$ and using the L-weakly dependence of the variables. Obviously, the same upper bound applies to the remaining terms, so we finally have

$$\left|\mathbb{E}\left(\prod_{j=1}^{n} e^{itX_j}\right) - \prod_{j=1}^{n}\mathbb{E}\left(e^{itX_j}\right)\right| \leq 4t^2\sum_{m=2}^{n-1}\sum_{j=1}^{m-1}\gamma_{m-j} = 4t^2\sum_{j=1}^{n-2}(n-j-1)\gamma_j.$$

■

This inequality is the main tool for proving a Central Limit Theorem for associated variables (see, for example, Theorem 4.1 in Oliveira [14]). So, having extended Newman's inequality to L-weakly dependent variables, we immediately may state the corresponding CLT. The arguments for the proof are similar to those of Theorem 5 in Newman [11], except on what regards the control of the approximation to independence.

Theorem 5.2. *Let X_n, $n \geq 1$, be centered, L-weakly dependent, strictly stationary and square integrable random variables satisfying (1) and*

$$D = \sum_{\ell=1}^{\infty} \gamma_\ell < \infty. \tag{13}$$

Then, $\frac{1}{\sqrt{n}} S_n$ converges in distribution to a centered normal random variable with variance σ^2.

Proof. The proof is based on a decomposition S_n similar to (2), into the sum of blocks of size $p \in \mathbb{N}$, now not depending n as before, and using (12). So, given $p \in \mathbb{N}$, put $m = \lfloor \frac{n}{p} \rfloor$, and redefine the blocks

$$Y_{j,p} = \sum_{k=(j-1)p+1}^{jp} X_k, \; j = 1, \ldots, m, \qquad \text{and} \qquad Y_{m+1,p} = \sum_{k=mp+1}^{n} X_k.$$

Let $\varphi_n(t)$ represent the characteristic function of $\frac{1}{\sqrt{n}} S_n$. We will establish that $\left| \varphi_n(t) - e^{-t^2\sigma^2/2} \right| \longrightarrow 0$. Let us start by writing

$$\begin{aligned} \left| \varphi_n(t) - e^{-\frac{t^2\sigma^2}{2}} \right| \leq\; & |\varphi_n(t) - \varphi_{mp}(t)| + \left| \varphi_{mp}(t) - \varphi_p^m(t) \right| \\ & + \left| \varphi_p^m(t) - e^{-\frac{t^2\sigma_p^2}{2}} \right| + \left| e^{-\frac{t^2\sigma_p^2}{2}} - e^{-\frac{t^2\sigma^2}{2}} \right|, \end{aligned} \tag{14}$$

where $\sigma_p^2 = \frac{1}{\sqrt{p}} \mathrm{Var}(S_p)$, and prove that each term of the right hand side goes to zero. Let p be fixed for the time being. As what concerns the first term of the upper bound in (14), we have, using Cauchy's inequality,

$$\begin{aligned} |\varphi_n(t) - \varphi_{mp}(t)| &\leq \mathbb{E} \left| \exp\left(\frac{it}{\sqrt{n}} S_n \right) - \exp\left(\frac{it}{\sqrt{mp}} S_{mp} \right) \right| \\ &\leq |t| \, \mathbb{E} \left| \frac{S_n}{\sqrt{n}} - \frac{S_{mp}}{\sqrt{mp}} \right| \leq |t| \left(\mathbb{E} \left(\frac{S_n}{\sqrt{n}} - \frac{S_{mp}}{\sqrt{mp}} \right)^2 \right)^{1/2} \\ &\leq |t| \left(\frac{1}{\sqrt{mp}} - \frac{1}{\sqrt{n}} \right) \left(\mathbb{E} S_{mp}^2 \right)^{1/2} + \frac{|t|}{\sqrt{n}} \left(\mathbb{E} Y_{m+1,p}^2 \right)^{1/2}. \end{aligned}$$

It follows from (13) and the stationarity of the variables that $\mathbb{E}S_{mp}^2 \leq 2\sigma^2 mp$ and $\mathbb{E}Y_{m+1,p}^2 \leq 2\sigma^2(n-mp) < 2\sigma^2 p$. Thus, as $n \longrightarrow +\infty$, which implies that $m \longrightarrow +\infty$, it follows

$$|\varphi_n(t) - \varphi_{mp}(t)| \leq \sqrt{2}\,|t|\,\sigma\left(1 - \frac{\sqrt{mp}}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right) \longrightarrow 0.$$

The second term in (14) represents the difference between the joint distribution of the blocks and what we would find if they were independent. To control this term, define $W_{j,p} = \frac{1}{\sqrt{p}} Y_{j,p}$. Taking into account the stationarity of the variables, the characteristic function of $W_{j,p}$ is $\varphi_p(t)$. As the variables $W_{j,p}$ are transformations of $X_{(j-1)p+1}, \ldots, X_{jp}$, it follows from the definition of L-weak dependence, representing the exponential with the trigonometric functions as done for the proof of Theorem 5.1, that

$$\begin{aligned}
&\left|\varphi_n(t) - \varphi_p^m(t)\right| \\
&\quad = \left|\mathbb{E}\left(\exp\left(\frac{it}{\sqrt{m}}\sum_{k=1}^m W_{k,p}\right)\right) - \prod_{k=1}^m \mathbb{E}\exp\left(\frac{it}{\sqrt{m}} W_{k,p}\right)\right| \\
&\quad \leq \frac{4t^2}{mp}\sum_{\ell=2}^{m-1}\sum_{j=1}^{(\ell-1)p}\sum_{j'=(\ell-1)p+1}^{\ell p}\gamma_{j'-j} \qquad (15) \\
&\quad = \frac{2t^2}{mp}\left(\sum_{j,j'=1}^{mp}\gamma_{|j'-j|} - m\sum_{j,j'=1}^{p}\gamma_{|j'-j|}\right).
\end{aligned}$$

It is easy to verify that $\frac{1}{mp}\sum_{j,j'=1}^{mp}\gamma_{|j'-j|} = \sum_{j=1}^{mp-1}(1-\frac{j}{mp})\gamma_j \longrightarrow D$, so

$$\limsup_{n\to+\infty}\left|\varphi_n(t) - \varphi_p^m(t)\right| \leq 2t^2\left(D - \frac{1}{p}\sum_{j,j'=1}^{p}\gamma_{|j'-j|}\right).$$

For the third term in (14), the classical Central Limit Theorem for independent random variables implies that

$$\lim_{m\to+\infty}\left|\varphi_p^m(t) - e^{-\frac{t^2\sigma_p^2}{2}}\right| \longrightarrow 0.$$

For the last term in (14), we have $\left|e^{-t^2\sigma_p^2/2} - e^{-t^2\sigma^2/2}\right| \leq \frac{t^2}{2}\left|\sigma_p^2 - \sigma^2\right|$. So, finally we obtain,

$$\limsup_{n\to+\infty}\left|\varphi_n(t) - e^{-\frac{t^2\sigma^2}{2}}\right| \leq \frac{t^2}{2}\left|\sigma_p^2 - \sigma^2\right| + 2t^2\left(D - \frac{1}{p}\sum_{j,j'=1}^{p}\gamma_{|j'-j|}\right).$$

Note that the left hand side above does not depend on p. Allowing now $p \longrightarrow +\infty$ and taking into account that $\lim_{p\to+\infty}\sigma_p^2 = \sigma^2$, it follows that

$$\limsup_{n\to+\infty}\left|\varphi_n(t) - e^{-\frac{t^2\sigma^2}{2}}\right| = 0.$$

■

We now prove a functional version of Theorem 5.2, giving sufficient conditions for the convergence in distribution of the partial sums process:

$$\xi_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor} X_j, \qquad 0 \leq t \leq 1. \tag{16}$$

Theorem 5.3. *Let X_n, $n \geq 1$, be centered, L-weakly dependent, strictly stationary random variables satisfying $\mathbb{E}\left|X_1\right|^{4+\delta} < \infty$, for some $\delta > 0$, and (1). If the L-weak dependence coefficients γ_k, $k \geq 1$, are decreasing such that $\gamma_k = O(k^{-2-8/\delta})$, then $\xi_n(t)$, $n \geq 1$, converges in distribution to σW, where W is a standard Brownian motion, in the Skhorohod space $D[0,1]$.*

Proof. The proof follows the usual arguments to prove the convergence with respect to the Skhorohod topology: prove the convergence of the finite dimensional distributions and the tightness of the sequence. The one dimensional distributions follows directly from Theorem 5.2. Choose now k points such that $0 = u_0 \leq u_1 < u_2 < \cdots < u_k \leq 1$. We shall prove the asymptotic normality of the random vector

$$H(u_1,\ldots,u_k) = \frac{1}{\sqrt{n}}\left(\xi_n(u_1), \xi_n(u_2) - \xi_n(u_1), \ldots, \xi_n(u_k) - \xi_n(u_{k-1})\right).$$

Note that, due to the stationarity, it follows again from Theorem 5.2 that each coordinate of $H(u_1,\ldots,u_k)$ is asymptotically centered normal with variance $(u_s - u_{s-1})\sigma^2$, $s = 1,\ldots,k$. We now compare the characteristic function of the random vector with the product of the characteristic functions of its margins. Denote on the sequel $T = \max_{s=1,\ldots,k}\left|t_s\right|$. From the definition of L-weak dependence, reasoning as for the decomposition (15), taking into account that $\|\cos(\sum_j t_j X_j)\|_L = \max_{j=1,\ldots,k} |t_j|$, it follows that, for every $t_1, \ldots, t_k \in \mathbb{R}$,

$$\left| \mathbb{E} \exp \left(\frac{i}{\sqrt{n}} \sum_{s=1}^{k} t_s \left(\xi_n(u_s) - \xi_n(u_{s-1}) \right) \right) - \prod_{s=1}^{k} \mathbb{E} \exp \left(\frac{it_s}{\sqrt{n}} \left(\xi_n(u_s) - \xi_n(u_{s-1}) \right) \right) \right|$$
$$\leq \frac{4kT^2}{n} \sum_{s=2}^{k-1} \sum_{j=1}^{\lfloor nu_{s-1} \rfloor} \sum_{j'=\lfloor nu_{s-1} \rfloor+1}^{\lfloor nu_s \rfloor} \gamma_{j'-j}$$
$$= \frac{2T^2}{n} \left(\sum_{j,j'=1}^{\lfloor nu_k \rfloor} \gamma_{j'-j} - \sum_{s=1}^{k} \sum_{j,j'=\lfloor nu_{s-1} \rfloor+1}^{\lfloor nu_s \rfloor} \gamma_{j'-j} \right).$$

Note that our assumption on the decrease rate of the γ_j coefficients implies that (13) holds. So, the above expression is easily seen to converge to $2T^2 D(u_k - u_1 - (u_2 - u_1) - \cdots - (u_k - u_{k-1})) = 0$, hence the asymptotic normality of $H(u_1, \ldots, u_k)$ follows.

To complete the proof, we still have to prove the tightness. We follow the arguments in the proof of Theorem 5 in Doukhan and Louhichi [4], thus needing to prove that

$$\begin{aligned} &\sum_{j=1}^{\infty} j \left| \mathbb{E}(X_1 X_{j+1}) \right| < \infty, \\ &\operatorname{Cov}(X_i X_j, X_k X_\ell) = O((k-j)^{-2}), \quad 1 \leq i \leq j < k \leq \ell. \end{aligned} \tag{17}$$

As what concerns the first condition, as the variables are centered and taking into account the assumption on the decrease rate of the γ_ℓ coefficients:

$$\sum_{j=1}^{\infty} j \left| \mathbb{E}(X_1 X_{j+1}) \right| \leq \sum_{j=1}^{\infty} j \gamma_j < \infty.$$

Concerning the second condition in (17), write first, for some $c > 0$ and for each $k \geq 1$, $V_k = X_k - (g_c(X_k) - \mathbb{E} g_c(X_k))$, using the function $g_c(\cdot)$ introduced in Subsection 4.2. Using this representation the covariance $\operatorname{Cov}(X_i X_j, X_k X_\ell)$ is written as a sum of terms of the form $\operatorname{Cov}(U_1 U_2, U_3 U_4)$ where each U_j is either bounded by $2c$ or chosen among V_i, V_j, V_k or V_ℓ. If all the U_j's are bounded by $2c$, from the definition of L-weak dependence and the assumption that coefficients are decreasing, it follows that

$$|\mathrm{Cov}(U_iU_j, U_kU_\ell)| \leq c^4(\gamma_{k-i} + \gamma_{k-j} + \gamma_{\ell-i} + \gamma_{\ell-k}) \leq 4c^4\gamma_{k-j}.$$

If exactly one of the U_j's is not bounded, say $U_i = Y_i$, we have that, using Hölder inequality followed by Markov inequality,

$$|\mathrm{Cov}(Y_iU_j, U_kU_\ell)| \leq 2c^3\mathbb{E}\,|Y_i| = 2c^3\mathbb{E}(|X_1|\,\mathbb{I}_{|X_1|>c}) \leq 2c^{-\delta}\mathbb{E}\,|X_1|^{4+\delta}\,.$$

For the remaining terms, we may reason in the same way, always finding an upper bound that, up to multiplication by a constant, is $c^{-\delta}\mathbb{E}\,|X_1|^{4+\delta}$. Thus, summing all the terms, we have that $\mathrm{Cov}(X_iX_j, X_kX_\ell) \sim c^{-\delta} + c^4\gamma_{k-j}$. Choose now $c = \gamma_{k-j}^{-1/(4+\delta)}$ to find $\mathrm{Cov}(X_iX_j, X_kX_\ell) \sim \gamma_{k-j}^{\delta/(4+\delta)} = (k-j)^{-2}$, taking into account the decrease rate for the dependence coefficients. So, the tightness follows, which concludes the proof of the theorem. ∎

This result complements Theorem 5 in Doukhan and Louhichi [4]. Indeed, these authors proved a similar result but considering different forms of weak dependence, as expressed by their ψ coefficients which involved the sum of the Lipschitz norms of the transformations instead of the product as we considered in Definition 2.1. It is still possible to prove a result concerning the convergence of the empirical process, again in a similar way as done in Doukhan and Louhichi [4]. For this later result, in [4] a different dependence coefficient was considered, so that their result implies directly the corresponding one for L-weak dependent variables. We state the result here, without proof, for easier reference on asymptotic results on L-weak dependent variables.

Theorem 5.4. *Let X_n, $n \geq 1$, be centered, L-weakly dependent, strictly stationary random variables uniformly distributed on $[0,1]$. If the L-weak dependence coefficients γ_k, $k \geq 1$, are such that $\gamma_k = O(k^{-15/2-\delta})$, for some $\delta > 0$, then $\zeta_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^n \mathbb{I}_{[0,t]}(X_j) - t\right)$, $t \in [0,1]$, $n \geq 1$, converges in distribution in the Skhorohod space $D[0,1]$ to a centered Gaussian process indexed by $[0,1]$ with covariance operator*

$$\Gamma(s,t) = \sum_{k=-\infty}^{+\infty} \mathrm{Cov}\left(\mathbb{I}_{[0,s]}(X_0), \mathbb{I}_{[0,t]}(X_{|k|})\right).$$

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