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KNOT COMPLEMENT WITH ESSENTIAL SURFACES OF UNBOUNDED GENUS AND NUMBER OF PUNCTURES

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Abstract: We construct infinitely many knots, both hyperbolic and non-hyperbolic, where each complement contains meridional essential surfaces of simultaneously unbounded genus and number of boundary components. In particular, we construct examples of knot complements each of which having all possible compact surfaces embedded as meridional essential surfaces.

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1. Introduction

Surfaces have a preeminent presence on the understanding of 3-manifold topology. The prime decomposition theorem for 3-manifolds, by Kneser [13] in 1929, might be the first remarkable example of such role played by surfaces. Of similar statement, circa 1949, Schubert [22] proves the prime decomposition theorem for knots in S^3, and also introduces the concept of sattelite knot. However, it wasn't until the 1961 that the concept of incompressible embedded surface in a 3-manifold was formally introduced by Haken [9], with such manifolds being referred to since then as *Haken manifolds*. The work of Waldhausen brings essential surfaces to the mainstream of 3-manifold topology, as he solved several important questions for Haken manifolds: For instance, Waldhausen [24] proved that Haken 3-manifolds are determined by their fundamental groups. Confirming the importance of essential tori to the study of 3-manifolds, Jaco and Shalen [11] and Johannson [12] proved, independently, the JSJ decomposition of 3-manifolds, revealing itself as an important tool to study 3-manifolds. In a similar tradition, it is also noteworthy that most of the support for the Geometrization conjecture came from its proof by Thurston [23] for Haken manifolds.

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This paper concerns with the interesting phenomenon of certain knot complements having essential surfaces of arbitrarily large Euler characteristics. The first examples of knots with this property were given by Lyon [15], where he proves the existence of knot complements each of which with closed essential surfaces of arbitrarily high genus. Other examples were later obtained, for instance, by Oertel [20], and, more recently, by Li [14] or by Eudave-Muñoz and Neumann-Coto [6]. Similarly, the author proved in [18] the existence of knot complements such that each contains meridional essential surfaces with two boundary components and arbitrarily high genus. On the other hand, one might wonder if the unbounded Euler characteristics of essential surfaces in a knot complement can be from the number of boundaries instead of the genus. That is, if there is a knot complement with compact essential surfaces with infinitely many boundary components. This is in fact the case, as shown by the examples given by Eudave-Muñoz [5], with non-meridional non-separating essential surfaces, and also by the author [19], with meridional essential planar surfaces. The problem addressed in this paper is weather the arbitrarily large Euler characteristic can be obtained from simultaneously unbounded genus and number of boundary components. Theorem 1 answers affirmatively this question.

Theorem 1. *There are infinitely many knots each of which having in its exterior meridional essential surfaces of all genus and* $2n$ *boundary components for all* $n \geq 1$*. Moreover, the collection can be made of prime knots, naturally excluding the existence of meriodional essential annuli in their exteriors.*

In Theorem 2 we show that hyperbolic knots can also have a similar property.

Theorem 2. *There are infinitely many hyperbolic knots each of which having in its exterior meridional essential surfaces of simultaneously unbounded genus and number of boundary components.*

Each knot from Theorems 1 and 2 also has closed essential surfaces of unbounded genus. In fact, from [3], at least one swallow-follow surface obtained from each meridional essential surface in Theorems 1 and 2 is of higher genus and also essential in the exterior of the respective knot.
There are many examples of knots that don't have the properties as in the theorems above, with the most notorious among these being small or meridionally small knots. Well known examples of classes of small knots are the torus knots, the 2-bridge knots [8], Montesinos knots with lenght three [21],

among other examples. One particularly interesting result, in contrast with the theorems in this paper, is one by Menasco [16] stating that for a fixed number of boundaries there are finitely many meridional essential surfaces in the complement of a prime alternating link; in particular, for a fixed number of boundaries there is a bound on the genus for meridional essential surfaces. Hence, the knots from the Theorems 1 and 2 above are not alternating.

The paper is organized as follows: In section 2 of this paper we present a construction of knots used along the paper and prove some of their properties. For the construction we use sattelite knots together with handlebody-knots of genus two. In section 3 we show a process to obtain knot complements with meridional essential surfaces of arbitrarily many boundary components as in Proposition 3, and use the knots from the main theorem of [18] to prove Theorem 1. The main methods are classical in 3-manifold topology, as innermost curve arguments and branched surface theory. In section 4 we prove Theorem 2 using classical results in hyperbolic manifolds and degree-one maps. Throughout the paper we work in the smooth category, all knots are assumed to be in S^3, unless otherwise stated, and all submanifolds are assumed to be in general position.

2. A construction of knots.

A common method to construct knots is through the process defining satellite knots: We start considering a knot K_p in a solid torus T, that we refer to as the *pattern knot*. The solid torus T is embedded in S^3 by the map $\sigma : T \to S^3$ where the core of $\sigma(T)$ has image a knot K_c that is called the *companion knot*. The knot $\sigma(K_p)$ is called a *satellite knot* of K_c with pattern K_p. In this paper we consider the concept of satellite knot allowing the companion to be a handlebody-knot. Let us first define a handlebody-knot: A *handlebody-knot* of genus g in S^3 is an embedded handlebody of genus g in S^3. A spine γ of a handlebody-knot Γ is a graph embedded in S^3 with Γ a regular neighborhood.

In this section, we describe a method to construct knots with meridional essential surfaces of arbitrarily number of boundaries, that we will use to prove Theorem 1. The method consist on defining a specific knot used as the pattern on a sattelite operation function.

Let J be a prime knot as in the main theorem of [18], that is with meridional essential surfaces of any positive genus and two boundary components.

The knot J is obtained by identifying the boundaries of two particular solid tori, say H_1 and H_2, attaching meridian to longitude, and by identifying the boundaries of the respective essential arc each contains. Denote by X the torus obtained from the identified boundaries of these solid tori and by O a disk in X containing $X \cap J$. We isotope two copies of $X - O$ slightly to each side separated by X and denote by X_1 and X_2 the resulting copies of X. The tori X_1 and X_2 intersect at O and each bounds a solid torus, ambient isotopic to H_1 and H_2 respectively. The union of these solid tori along O defines a genus two handlebody-knot H with spine as in Figure 1.

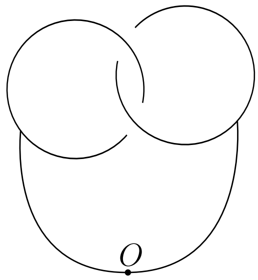

FIGURE 1: The spine of the handlebody-knot H, and the respective representation of O by a point.

Consider a ball B disjoint from O such that B^c intersects H at a cylinder containing O and J at two parallel trivial arcs. Note that the 2-string tangle $(B, B \cap J)$ is essential, otherwise the punctured torus obtained from X wouldn't be essential in $E(J)$. Denote by T the solid torus defined by $B \cup (B^c \cap H)$. Let J_1 and J_2 be two copies of J in the respective copies of T, say T_1 and T_2. We isotope the two arcs of $J_i \cap (T_i - B_i)$ into the boundary of T_i, where B_i is the copy of B with respect to T_i. For each knot J_i, we consider a segment of one of these arcs and a regular neighborhood R_i of it, disjoint from J_i otherwise. We proceed with a connect sum of J_1 and J_2 by removing the interior of R_1 and attaching the exterior of R_2, such that the disks $T_1 \cap \partial R_1$ and $T_2 \cap \partial R_2$ are identified. Hence, the knot $J_1 \# J_2$ is in a genus two handlebody G obtained by gluing T_1 and T_2 along a disk D in their boundaries. (See Figure 2.)

As the tangle $(B, B \cap J)$ is essential and $T \cap B^c$ is a regular neighborhood of each arc of $B^c \cap J$, we have that ∂T is essential in $T - J$. Moreover, from the construction of T and J, each meridian of T intersects J at least twice. Hence, ∂G is essential in $G - J_1 \# J_2$ and, similarly, each essential disk in G intersects $J_1 \# J_2$ at two points.

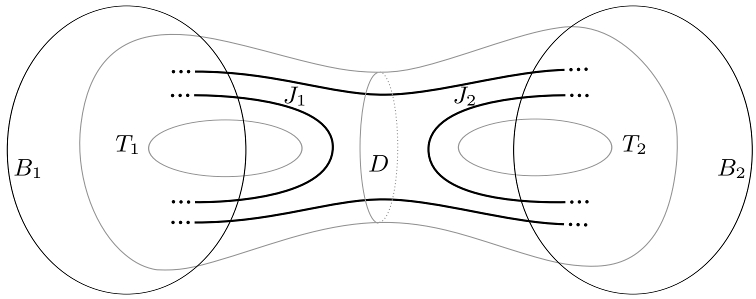

FIGURE 2: The handlebody G together with the knot $J_1 \# J_2$.

Let Γ be the genus 2 handlebody-knot 4_1, from the list in [10], with spine γ as in Figure 3.

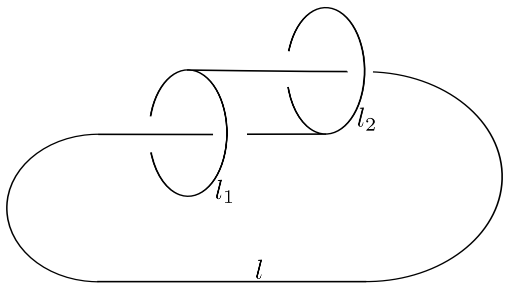

FIGURE 3: The spine of the handlebody-knot γ, defined by two loops l_1 and l_2 and the arc l.

Denote by $e : G \to S^3$ an embedding of G into S^3 with image Γ, where $e(D)$ is an essential disk in a regular neighborhood L of l. That is $e(D)$ is a disk that separates from Γ two tori, L_1 and L_2, having cores l_1 and l_2, respectively, with $\Gamma = L \cup L_1 \cup L_2$. (See Figure 3.) We refer to $e(J_1 \# J_2)$ by N. The handlebody knot Γ is embedded in a solid torus P with core a trivial knot, such that there is a meridian disk of P that intersects γ at a single point in l. In the next definition we describe the operation used to prove Theorem 1.

Definition 1. Let $\mathcal{K}$ be the set of equivalence classes of knots in S^3 up to ambient isotopy. For a knot $K \in \mathcal{K}$ let $h_K : P \to S^3$ be an embedding of S^3 such that $h_K(P)$ is a solid torus with core K. We define the sattelite operation function $h : \mathcal{K} \to \mathcal{K}$ such that for each $K \in \mathcal{K}$ we have $h(K) = h_K(N)$.

Proposition 1. *For every knot K the knot $h(K)$ is prime.*

Proof: First observe that there is no local knot of $J_1\#J_2$ in G: As the knot J_i is prime and the tangle $(B_i, B_i\cap J_i)$ is essential, and $(B_i^c, B_i^c\cap J_i)$ is defined by two trivial arcs, we have necessarily that there is no local knot of J_i in T_i, and consequently there is no local knot of $J_1\#J_2$ in G.
Suppose $h(K)$ is a composite knot and consider a decomposing sphere S for $h(K)$. If S is disjoint from $\partial h_K(\Gamma)$ then we obtain a contradiction with the inexistence of local knots of $J_1\#J_2$ in G. Then consider the intersection of S with $\partial h_K(\Gamma)$ and assume that $|S\cap\partial h_K(\Gamma)|$ is minimal among all decomposing spheres for $h(K)$.
The sphere S intersects $\partial h_K(\Gamma)$ in a collection of simple closed curves. Let O be an innermost disk bounded by an innermost curve of $S\cap\partial h_K(\Gamma)$ in S. We have two possibilities: there is an innermost disk O disjoint from $h(K)$ or an innermost disk O that intersects $h(K)$ at a single point. If O is disjoint from $h(K)$, as ∂G is essential in $G-J_1\#J_2$ and $\partial\Gamma$ is essential in the exterior of Γ, then ∂O bounds a disk in $\partial h_K(\Gamma)$. Using a ball bounded by this disk and O we can isotope S through $\partial h_K(\Gamma)$ in $E(h(K))$ reducing $|S\cap\partial h_K(\Gamma)|$, contradicting its minimality. If O intersects $h(K)$ at a single point then O is an essential disk in $h_K(\Gamma)$ intersecting $h(K)$ at a single point, which contradicts the fact that every essential disk of G intersects $J_1\#J_2$ at least in two points (as observed before). Therefore, the knot $h(K)$ is prime. ■

Besides being prime, the knots $h(K)$ can be decomposed into two essential arcs by surfaces of genus higher than zero, keeping the properties of Theorem 1 in [**?**] used in their construction.

Proposition 2. *For every knot K the exterior of $h(K)$ has meridional essential surfaces of any positive genus and two boundary components.*

Proof: First we note that the meridional essential surfaces of any positive genus and two boundary components $S_{g;2}$ in $E(J)$ are in the solid torus T. The surface $S_{g;2}$ intersects the cylinder $B^c\cap T$ at $g-2$ annuli parallel to one string of $B^c\cap T\cap J$ and g annuli parallel to the other string of $B^c\cap T\cap J$, with the latter denoted by s. For each copy of T, T_1 and T_2, denote the respective copies of s by s_1 and s_2. We isotope the arc s_i into the boundary of T_i, and consider a regular neighborhood R_i of this segment, disjoint from J_i otherwise. We consider the surfaces $S_{g;2}$ in T_1 and assume that the annuli of $S_{g;2}\cap T_1\cap B_1^c$ parallel to s_1 are in R_1. After the connected sum between J_1 and J_2, assumed along the arcs s_1 and s_2 as described before in this section, we replace these annuli in R_1 by g annuli in the exterior of R_2 parallel to the

resulting arc of J_2 in T_2 (that we also denote by J_2). In this way, we define a new surface $S'_{g;2}$, in the handlebody G, obtained from $S_{g;2}$ and also with genus g and two boundary components.
As $S'_{g;2} \cap T_2$ is a collection of annuli in T_2, cutting a regular neighborhood of J_2 in T_2, there is no compressing or boundary compressing disk for $S'_{g;2}$ in T_2. As $S_{g;2}$ is essential in $T_1 - J_1$ there is no compressing or boundary compressing disk of $S'_{g;2}$ in T_1. Hence, if there is a compressing or boundary compressing disk of $S'_{g;2}$ in G it intersects D. By an outermost disk type of argument in the compressing disk with respect to its intersection with D we obtain a contradiction with the essentiality of the annuli $S'_{g;2} \cap T_2$ in T_2 or the essentiality of $S_{g;2}$ in $T_1 - J_1$. Hence, $S'_{g;2}$ is essential in the complement of $J_1 \# J_2$ in G.
Let now $F_{g;2}$ be $e(S'_{g;2})$. We will show that $F_{g;2}$ is essential in $E(N)$. Note that N is, in particular, $h(K)$ for K unknotted. Suppose there is a compressing or boundary compressing disk Q for $F_{g;2}$ in $E(N)$. If Q is disjoint from $\partial\Gamma$ we get a contradiction with $S'_{g;2}$ being essential in the exterior of $J_1 \# J_2$ in G. Hence, Q intersects $\partial\Gamma$. Suppose $|Q \cap \partial\Gamma|$ is minimal between all compressing or boundary compressing disks of $F_{g;2}$ in $E(N)$. As $F_{g;2}$ is disjoint from $\partial\Gamma$, the disk Q intersects $\partial\Gamma$ at simple closed curves. Denote by O an innermost disk defined by the curves of $Q \cap \partial\Gamma$ in D. As $\partial\Gamma$ is irreducible in $E(\Gamma)$, the disk O cannot be essential in $E(\Gamma)$. As $J_1 \# J_2$ is essential in G, the disk O cannot be essential in $\Gamma - N$. Therefore, ∂O bounds a disk in $\partial\Gamma$ which, after an isotopy of O through this disk, contradicts the minimality of $|Q \cap \partial\Gamma|$. Then $F_{g;2}$ is essential in $E(N)$.
For a given non-trivial knot K consider $h_K(P)$ and the knot $h(K)$. Denote by $F'_{g;2}$ the surface $h_K(F_{g;2})$ in $h_K(\Gamma)$. Assume there is a compressing or boundary compressing disk Q' for $F'_{g;2}$ in $E(h(K))$. In case Q' is disjoint from $\partial h_K(P)$ we get a contradiction with $F_{g;2}$ being essential in $E(N)$. Then, Q' intersects $\partial h_K(P)$. Suppose that $|Q' \cap \partial h_K(P)|$ is minimal between all compressing or boundary compressing disks of $F'_{g;2}$. Denote also by O' an innermost disk defined by $Q' \cap \partial h_K(P)$ in Q'. As $\partial h_K(P)$ is essential in $E(h_K(P))$, the disk O' cannot be essential in $E(h_K(P))$. As N is essential in P (from the construction of P), the disk O' also cannot be essential in $P - h(K)$. Then, $\partial O'$ bounds a disk in $\partial h_K(P)$ and, as before, we get a contradiction with the minimality of $|Q' \cap \partial h_K(P)|$.
In conclusion, for any knot K the knot $h(K)$ has a meridional essential surface of any positive genus and two boundary components. ∎

3. Proof of Theorem 1

In this section we use the sattelite operation described on Definition 1 and the knots from the main theorem of [18] to prove Theorem 1. First, we start with the following proposition where we show that for any knot with a meridional essential surface in its exterior there is a knot with meridional essential surfaces of the same genus and unlimited number of boundaries.

Proposition 3. *Let K be a knot with a meridional essential surface of genus g and n boundary components.*
Then, the knot $h(K)$ has a meridional essential surface of genus g and b boundary components for all even $b \geq 2n$.

Proof: Let S be a closed surface of genus g which K intersects at n points, corresponding to a meridional essential surface of genus g and n boundary components in $E(K)$, as in the statement. With the association of $h_K(P)$ with a regular neighborhood of K, we denote by S' the meridional essential surface obatined from S in the complement of $h_K(P)$. Each boundary component of S' bounds a meridian disk in $h_K(P)$. We consider two types of meridian disks of $h_K(P)$, with respect to $h_K(\Gamma)$, as it intersects $h_K(\Gamma)$ at one separating disk or at two separating disks in $h_K(L)$. We refer to a meridian disk of $h_K(P)$ as type-a in case it intersects $h_K(\Gamma)$ at one disk, separating $h_K(\Gamma)$ into two components. We refer to a meridian disk as of type-b in case it intersects $h_K(\Gamma)$ at two disks, separating $h_K(\Gamma)$ into three components: one being a cylinder in $h_K(L)$, and each of the other two containing either $h_K(L_1)$ or $h_K(L_2)$.

In what follows we construct the surfaces used to prove the statement of this proposition. We define S_{n+i}, $i = 0, 1, \ldots, n$, as a surface obtained from S' by capping off its boundaries with i meridians of type-b and $n-i$ meridians of type-a. Hence, S_{n+i} has genus g and, in the exterior of $h_K(N)$, has $2 \times (n+i)$ boundary components.
To proceed, we consider the surface S_{2n}, that intersects $h_K(P)$ at n meridian disks of type-b, $D_1, \ldots, D_n$, ordered by index, such that $D_1 \cup D_n$ cut a cylinder from $h_K(P)$ containing all the other disks D_i. Let D be a meridian of type-a. We assume that $D \cup D_1$ cut a cylinder Q_{L_1} from $h_K(P)$ containing $h_K(L_1)$, and $D_n \cup D$ cut a cylinder Q_{L_2} from $h_K(P)$ containing $h_K(L_2)$. Let $\partial^* Q_{L_2}$ be the annulus of intersection of ∂Q_{L_2} with $\partial h_K(P)$. Let O be the annulus of intersection of Q_{L_1} with $\partial h_K(\Gamma)$ (the component that is disjoint from

$h_K(L_1))$. Let A be the annulus obtained by the union of O with $D \cap E(h_K(\Gamma))$ with $\partial^* Q_{L_2}$. We define the surface S_{2n+j}, $j \geq 1$, as follows: Start with the surface S_{2n}. We consider j meridians of type-b in sequence after D_n and denoted $D_{n+1}, D_{n+2}, \dots, D_{n+j}$. We consider also j copies of A in $h_K(P)$ denoted, from the outside to the inside, by $A_1, A_2, \dots, A_j$. First we consider the disk O_p obtained by capping off the annulus A_p by the disk one of its boundary components bounds in D_{n+p}, for all $p = 1, \dots, j$. We continue by extending, parallely, the boundary of O_p until it reaches D_p, and we surger the disk by $O_p \cap D_p$ in D_p by O_p. The resulting surface is the surface S_{2n+j}, which has genus g and $2 \times (2n + j)$ boundary components.

Lemma 1. *The surfaces S_{n+i}, for $i \in \{0, 1, \dots, n\}$, are essential in $E(h(K))$.*

Proof of Lemma 1: Suppose a surface S_{n+i}, for some $i \in \{0, 1, \dots, n\}$, is not essential and denote by D a compressing disk or boundary compressing disk of S_{n+i} in $E(h(K))$. Assume that $|D \cap \partial h_K(P)|$ is minimal between all compressing and boundary compressing disks of S_{n+i}.
If D intersects $\partial h_K(P)$ at some simple closed curve let δ be an innermost one in D bounding an innermost disk Δ. As N is essential in P the disk Δ cannot be essential in $h_K(P)$. As a non-trivial knot complement in S^3 is boundary irreducible, the disk Δ cannot be essential in the complement of $h_K(P)$. Hence, δ bounds a disk in $\partial h_K(P)$ and by an isotopy of Δ through this disk we can reduce $|D \cap \partial h_K(P)|$, contradicting its minimality. Then, D doesn't intersect $\partial h_K(P)$ at simple closed curves.
Assume now that D intersects $\partial h_K(P)$ at some arc. As $h(K)$ is disjoint from $\partial h_K(P)$ the arcs of $D \cap \partial h_K(P)$ have both ends in $D \cap S_{n+i}$, even when D is a boundary compressing disk. Denote by δ an outermost arc of $D \cap \partial h_K(P)$ in D, cutting an outermost disk Δ from D with boundary δ union with an arc in $D \cap S_{n+i}$. (This latter condition is not always true for all outermost disks as Δ. In fact, when D is a boundary compressing disk, an outermost disk Δ might include in its boundary an arc in $\partial E(h(K))$; but an outermost disk in its complement in D has the desired property.) If δ has both ends in the same disk component D_j of $h_K(P) \cap S_{n+i}$, by cutting and pasting along the disk cut by δ and ∂D_j from $\partial h_K(P)$, we can reduce $|D \cap \partial h_K(P)|$, contradicting its minimality. Hence, δ has ends in different components of $h_K(P) \cap S_{n+i}$. As $h_K(P) \cap S_{n+i}$ is a collection of disks, Δ cannot be in $h_K(P)$. Consequently, Δ is in the complement of $h_K(P)$ implying that S is boundary

compressible in $E(K)$, which contradicts its essentiality. Hence, the surfaces S_{n+i}, for $i \in \{0, \dots, n\}$, are essential in $E(h(K))$. ■

Lemma 2. *The surfaces S_{n+i}, for $i > n$, are essential in $E(h(K))$.*

Proof of Lemma 2: For the proof of this lemma we use branched surface theory based on work of Oertel [20] and Floyd and Oertel [7] that we revise concisely over the next paragraphs.
A *branched surface* B with generic branched locus in a 3-manifold M is a compact space locally modeled on Figure 4(a).

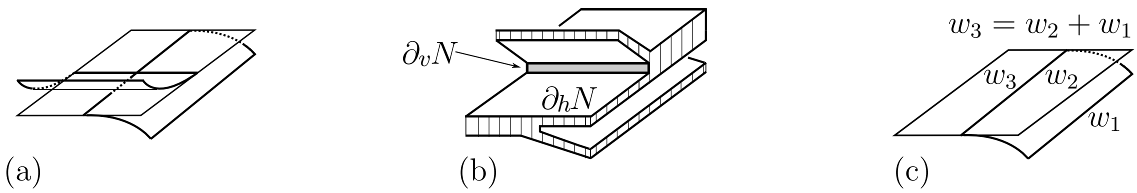

FIGURE 4: Local model for a branched surface, in (a), its regular neighborhood, in (b), and branch equations, in (c).

We denote by $N(B)$ a fibered regular neighborhood of B embedded in M, locally modelled on Figure 4(b). The boundary of $N(B)$ is the union of three compact surfaces $\partial_h N(B)$, $\partial_v N(B)$ and $\partial M \cap \partial N(B)$, where a fiber of $N(B)$ meets $\partial_h N(B)$ transversely at its endpoints and either is disjoint from $\partial_v N(B)$ or meets $\partial_v N(B)$ in a closed interval in its interior. We say that a surface S is *carried* by B if it can be isotoped into $N(B)$ so that it is transverse to the fibers. If we associate a weight $w_i \geq 0$ to each component on the complement of the branch locus in B we say that we have an invariant measure provided that the weights satisfy branch equations as in Figure 4(c). Given an invariant measure on B we can define a surface carried by B, with respect to the number of intersections between the fibers and surface. We also note that if all weights are positive we say that S is carried with *positive weights* by B, which is equivalent to S being transverse to all fibers of $N(B)$.
A *disk of contact* is a disk D embedded in $N(B)$ transverse to fibers and with ∂D in $\partial_v N(B)$. A *half-disk of contact* is a disk D embedded in $N(B)$ transverse to fibers with ∂D being the union of an arc in $\partial M \cap \partial N(B)$ and an arc in $\partial_v N(B)$. A *monogon* in the closure of $M - N(B)$ is a disk D with $D \cap \partial N(B) = \partial D$ which intersects $\partial_v N(B)$ in a single fiber. (See Figure 5.)

A branched surface B embedded in M is said *incompressible* if it satisfies the following three properties:

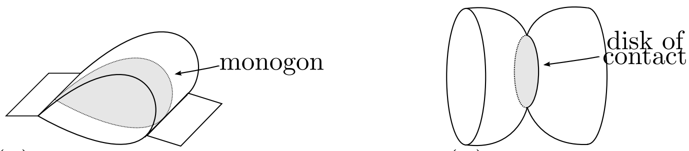

FIGURE 5: Illustration of a monogon and a disk of contact on a branched surface.

(i) B has no disk of contact or half-disk of contact;
(ii) $\partial_h N(B)$ is incompressible and boundary incompressible in the closure of $M - N(B)$;
(iii) there are no monogons in the closure of $M - N(B)$.

With following theorem, by Floyd and Oertel in [7], we can determine if a surface carried by a branched surface is essential.

Theorem 3 (Floyd and Oertel, [7]). *A surface carried with positive weights by an incompressible branched surface is essential.*

Now we prove that the surfaces S_{n+i}, for $i > n$, are essential in $E(h(K))$ by showing that these surfaces are carried with positive weights by an incompressible branched surface. Let us consider the surface S' and denote by $b_1, b_2, \dots, b_n$ its boundary components in consecutive order in $\partial h_K(P)$. Denote by Q_j the annulus component of $\partial h_K(P) - b_1 \cup \dots \cup b_n$ bounded by $b_j \cup b_{j+1}$. We consider the union of S', the annuli Q_j, $j = 1, \dots, n-1$, the annulus A and a type-b meridian D_1 of $h_K(P)$ with boundary b_1, and denote the resulting space by B. We smooth the space B on the intersection of the surface S', Q_j, A and D_1 as explained next. For each annulus Q_j: isotope the boundary in b_j into the exterior of $h_K(P)$ and smooth it towards b_j; also, smooth the boundary in b_{j+1} towards the exterior of $h_K(P)$. We also smooth the boundary of D_1, that is b_1, with S'. With respect to the annulus A, we smooth its boundary in D_1 towards b_1 and its boundary in b_n we isotope it into the exterior of $h_K(P)$ and smooth it towards b_n. In Figure 5 we have a schematic representation of the branched surface B.

Let us denote by D_1^a the disk bounded by $D_1 \cap A$ in D_1 and by D_1^b the annulus defined by $D_1 - D_1^a$. From the construction, the space B is a branched surface with sections denoted naturally by S', Q_j for $j = 1, \dots, n-1$, A, D_1^a and D_1^b, as illustrated in Figure 5. We denote a regular neighborhood of B by $N(B)$. The surface S_{n+i} is carried with positive weights by B together with

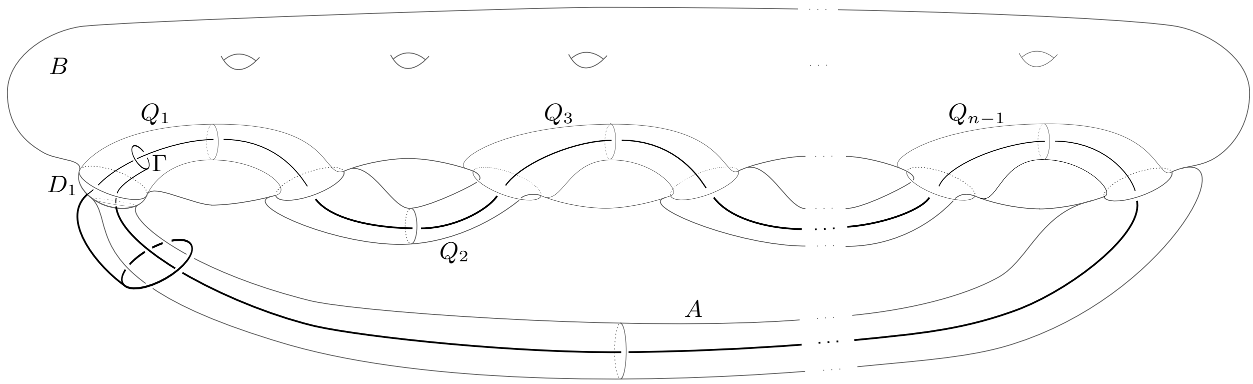

FIGURE 6: Schematic representation of the branched surface B.

the invariant measure on B defined by $W_{S'} = 1$, $W_{Q_j} = n - j + i$, $W_A = i$, $W_{D_1^a} = n$ and $W_{D_1^b} = n + i$, on the sections S?, Q_j, A, D_1^a, D_1^b, respectively. To prove that S_{n+i}, $i > n$, is essential in the complement of $h(K)$ we show that B is an incompressible branched surface and use Theorem 3.

The space $N(B)$ decomposes $E(h(K))$ into three components: a component cut from $E(h(K))$ by S' and the annuli Q_j with odd index that we denote by E_1; a component cut from $E(h(K))$ by S', the annuli Q_j with even index and the annulus A that we denote by E_2; a component cut from $E(h(K))$ by S', D_1^a, D_1^b and the annuli Q_j, $j = 1, \dots, n-1$, and A, that we denote by E_p. As E_1 is disjoint from $\partial E(h(K))$ there are boundary compressing disks for $\partial_h N(B)$ in E_1. In ∂E_1, the components of $\partial_v N(B)$ correspond to annuli associated to the boundary of Q_j in b_j, for j odd. Hence, if $\partial_h N(B)$ has a compressing disk in E_1, as it is disjoint from $\partial_v N(B)$, we can isotope its boundary into S', contradicting S' being essential in $E(K)$. On the other hand, a monogon disk in E_1 would have boundary defined by an arc in some Q_j and an arc in S', being a boundary compressing disk for S' in $E(K)$, contradicting again S' being essential.

In ∂E_2, the components of $\partial_v N(B)$ correspond to annuli associated to the boundary of Q_j in b_j, for j even, and to the annulus associated to the boundary of A in b_n. As in the case for E_1, if there is a compressing disk for $\partial_h N(B)$ in E_2 then we get a contradiction with S' being essential in $E(K)$. If there is a monogon disk in E_2 it would have boundary defined by an arc in some Q_j or in A and an arc in S', defining a boundary compressing disk for S' in $E(K)$, and contradicting again S' being essential in $E(K)$. If there is a boundary compressing disk for $\partial_h N(B)$ in E_2 then an arc of such a disk boundary can be assumed to be the arc s_1 defined by $h_K(L_1) \cap h(K)$. The solid torus $h_K(L_1)$ is in a ball in E_2 intersecting S_{n+i} in D_1, and s_1 has the isotopy type of the knotted arc J_1 in this ball. The existence of a boundary compressing disk of $\partial_h N(B)$ in E_2 with s_1 in the boundary contradicts J_1 being a knotted arc. The component E_p defines together with $E_p \cap h(K)$ a 3-string tangle defined by a knotted arc s_2, with the pattern of J_2 in $h_K(L_2)$ and two parallel trivial arcs in $h_K(L)$, denoted by t_1 and t_2. There is only one component of $\partial_v N(B)$ in ∂E_p and it corresponds to the boundary of A in D_1, denoted by a. The end points of each t_i, $i = 1, 2$, in E_p are separated by a in ∂E_p, and the ends of s_2 are in the same disk bounded by a in ∂E_p, say D_a. We denote the other disk bounded by a in ∂E_p by D'_a. As a is separating in ∂E_p there are no monogons of $\partial N(B)$ in E_p, and a the boundary of a compressing or boundary compressing disk for $\partial_h N(B)$ in E_p intersects $\partial_h N(B)$ only at D_a. If there is a boundary compressing disk then we can assume the arc s_2 is on its boundary, contradicting s_2 being knotted. If there is a compressing disk then it separates s_2 from $t_1 \cup t_2$ implying that Γ is trivial in P, which is a contradiction with Γ being knotted (more exactly, the handlebody knot 4_1 as in [10]). This finishes the proof that B is an incompressible branched surface and, consequently, from Theorem 3 we also have that S_{n+i} is essential in $E(h(K))$, for $i > n$. ■

From Lemmas 1 and 2 we obtain the statement of the proposition as the surfaces S_{n+i}, $i \geq 0$, are essential in $E(h(K))$ and S_{n+i} has genus g and $2n + 2i$ boundary components. ■

This proposition offers a base for the proof of Theorem 1 as follows.

Proof of Theorem 1: Consider an infinite collection of knots C_i, $i \in \mathbb{N}$, as in the main theorem in [18], that is each of which having in their exterior a meridional essential surface S_g for every genus $g \geq 0$ and two boundary components. Using the sattelite operation defined in the previous section, for each knot C_i we define the knot $h(C_i)$ that we denote by K_i. Hence, each knot K_i has in its exterior, from Proposition 2, a meridional essential surface of any positive genus and two boundary components and, from Proposition 3, a meridional essential surface $S_{g;2n}$ of any genus g and $2n$ boundary components for all $n \geq 2$. From Proposition 1, the knots $h(K_i)$ are prime, and together with examples obtained after their connected sum with a non-trivial knot we complete the proof of the statement of the theorem. ■

4. Meridional essential surfaces on hyperbolic knot exteriors

In this section we extend the work in the previous sections to hyperbolic knots and prove Theorem 2.

Proof of Theorem 2: Consider a knot K as in the statement of Theorem 1. That is, $E(K)$ contains meridional essential surfaces $S_{g;b}$ of any genus g and (even) number b of boundary components. We will show that for K there is a hyperbolic knot whose exterior contains meridional essential surfaces $F_{g;b}$ of genus and number of boundary components greater than or equal to g and b, respectively, for each surface $S_{g;b}$.
From Myers [17], let $J \subset E(K)$ be a null-homotopic knot with hyperbolic complement. Consider $E(J \cup K)$ and do $\frac{1}{r}$-Dehn filling on J to produce a hyperbolic knot $K_r \subset S^3$. As in the proof of Proposition 3.2 of [2] by Boileau-Wang [2], there is a degree-one map $f : E(K_r) \to E(K)$. From the construction of the degree-one map f, as on the proof of Proposition 3.2 on [2], we can homotope f to be transverse to $S_{g;b}$ by making the immersed disk bounded by J in $E(K)$, used to define f, transverse to $S_{g;b}$. After this homotopy of f, if necessary, we have that $F_{g;b} = f^{-1}(S_{g;b})$ is a 2-dimensional submanifold of $E(K_r)$. The restriction map $f : F_{g;b} \to S_{g;b}$ is a degree-one map, by definition of degree-one map. Hence, by the work of Edmonds [4], it is a pinch map: there is a compact connected submanifold $F \subset F_{g;b}$ with no more than one component of its boundary being a simple closed curve in the interior of $F_{g;b}$, such that f, restricted to $F_{g;b}$, is homotopic to the quotient map $F_{g;b} \to F_{g;b}/F$. In particular, this means that the genus of $F_{g;b}$ is higher than the one of $S_{g;b}$. On top of this, as f is the identity near $\partial E(K)$, the number of boundary components of $F_{g;b}$ is the same as $S_{g;b}$.
In case $F_{g;b}$ is incompressible, we have completed the proof. Otherwise, let D be a compressing disk of $F_{g;b}$ in $E(K_r)$. As $f : F_{g;b} \to S_{g;b}$ is a pinch map as described above, $f(D)$ is an immersed disk in $E(K)$. In case ∂D is in the pinched region of $F_{g;b}$ by f, then ∂D is mapped to a point of $S_{g;b}$. Hence, we can compress $F_{g;b}$ by D obtaining a surface that is still mapped by a degree-one map into $S_{g;b}$. We keep compressing until there are no more compressing disks with boundary in the pinched region. In case the induced homomorphism $f_* : \pi_1(F_{g;b}) \to \pi_1(S_{g;b})$ takes the class of ∂D to the identity of $\pi_1(S_{g;b})$, we can homotope ∂D into the pinched region of $F_{g;b}$ by f, and repeat the previous argument. In case the class of ∂D is not in the kernel of

f_*, from the commutativity of the diagram

$$\begin{tikzcd} \pi_1(F_{g;b}) \arrow[r, "f_*"] \arrow[d, "j_*"'] & \pi_1(S_{g;b}) \arrow[d, "i_*"] \\ \pi_1(E(K_r)) \arrow[r, "f_*"] & \pi_1(E(K)) \end{tikzcd}$$

the kernel of i_* contains the class of $f_*([\partial D])$ and is non-trivial. By the Loop Theorem, there is a compressing disk of $S_{g;b}$ in $E(K)$, which is a contradiction. Hence, $F_{g;b}$ is essential in $E(K_r)$. ■

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