KNOT COMPLEMENT WITH ESSENTIAL SURFACES OF UNBOUNDED GENUS AND NUMBER OF PUNCTURES

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Abstract: We construct infinitely many knots, both hyperbolic and non-hyperbolic, where each complement contains meridional essential surfaces of simultaneously unbounded genus and number of boundary components. In particular, we construct examples of knot complements each of which having all possible compact surfaces embedded as meridional essential surfaces.

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1. Introduction

Surfaces have a preeminent presence on the understanding of 3-manifold topology. The prime decomposition theorem for 3-manifolds, by Kneser [13] in 1929, might be the first remarkable example of such role played by surfaces. Of similar statement, circa 1949, Schubert [22] proves the prime decomposition theorem for knots in $S^3$, and also introduces the concept of satellite knot. However, it wasn’t until the 1961 that the concept of incompressible embedded surface in a 3-manifold was formally introduced by Haken [9], with such manifolds being referred to since then as Haken manifolds. The work of Waldhausen brings essential surfaces to the mainstream of 3-manifold topology, as he solved several important questions for Haken manifolds: For instance, Waldhausen [24] proved that Haken 3-manifolds are determined by their fundamental groups. Confirming the importance of essential tori to the study of 3-manifolds, Jaco and Shalen [11] and Johannson [12] proved, independently, the JSJ decomposition of 3-manifolds, revealing itself as an important tool to study 3-manifolds. In a similar tradition, it is also noteworthy that most of the support for the Geometrization conjecture came from its proof by Thurston [23] for Haken manifolds.

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This paper concerns with the interesting phenomenon of certain knot comple-
ments having essential surfaces of arbitrarily large Euler characteristics. The
first examples of knots with this property were given by Lyon [15], where he
proves the existence of knot complements each of which with closed essential
surfaces of arbitrarily high genus. Other examples were later obtained, for
instance, by Oertel [20], and, more recently, by Li [14] or by Eudave-Muñoz
and Neumann-Coto [6]. Similarly, the author proved in [18] the existence
of knot complements such that each contains meridional essential surfaces
with two boundary components and arbitrarily high genus. On the other
hand, one might wonder if the unbounded Euler characteristics of essential
surfaces in a knot complement can be from the number of boundaries instead
of the genus. That is, if there is a knot complement with compact essential
surfaces with infinitely many boundary components. This is in fact the case,
as shown by the examples given by Eudave-Muñoz [5], with non-meridional
non-separating essential surfaces, and also by the author [19], with meridional
essential planar surfaces. The problem addressed in this paper is weather the
arbitrarily large Euler characteristic can be obtained from simultaneously un-
bounded genus and number of boundary components. Theorem 1 answers
affirmatively this question.

Theorem 1. There are infinitely many knots each of which having in its ex-
terior meridional essential surfaces of all genus and $2n$ boundary components
for all $n \geq 1$. Moreover, the collection can be made of prime knots, naturally
excluding the existence of meriodional essential annuli in their exteriors.

In Theorem 2 we show that hyperbolic knots can also have a similar property.

Theorem 2. There are infinitely many hyperbolic knots each of which hav-
ing in its exterior meridional essential surfaces of simultaneously unbounded
genus and number of boundary components.

Each knot from Theorems 1 and 2 also has closed essential surfaces of un-
bounded genus. In fact, from [3], at least one swallow-follow surface obtained
from each meridional essential surface in Theorems 1 and 2 is of higher genus
and also essential in the exterior of the respective knot.

There are many examples of knots that don’t have the properties as in the
theorems above, with the most notorious among these being small or merid-
ionally small knots. Well known examples of classes of small knots are the
torus knots, the 2-bridge knots [8], Montesinos knots with lenght three [21],
among other examples. One particularly interesting result, in contrast with the theorems in this paper, is one by Menasco [16] stating that for a fixed number of boundaries there are finitely many meridional essential surfaces in the complement of a prime alternating link; in particular, for a fixed number of boundaries there is a bound on the genus for meridional essential surfaces. Hence, the knots from the Theorems 1 and 2 above are not alternating.

The paper is organized as follows: In section 2 of this paper we present a construction of knots used along the paper and prove some of their properties. For the construction we use satellite knots together with handlebody-knots of genus two. In section 3 we show a process to obtain knot complements with meridional essential surfaces of arbitrarily many boundary components as in Proposition 3, and use the knots from the main theorem of [18] to prove Theorem 1. The main methods are classical in 3-manifold topology, as innermost curve arguments and branched surface theory. In section 4 we prove Theorem 2 using classical results in hyperbolic manifolds and degree-one maps. Throughout the paper we work in the smooth category, all knots are assumed to be in $S^3$, unless otherwise stated, and all submanifolds are assumed to be in general position.

2. A construction of knots.

A common method to construct knots is through the process defining satellite knots: We start considering a knot $K_p$ in a solid torus $T$, that we refer to as the pattern knot. The solid torus $T$ is embedded in $S^3$ by the map $\sigma: T \to S^3$ where the core of $\sigma(T)$ has image a knot $K_c$ that is called the companion knot. The knot $\sigma(K_p)$ is called a satellite knot of $K_c$ with pattern $K_p$. In this paper we consider the concept of satellite knot allowing the companion to be a handlebody-knot. Let us first define a handlebody-knot: A handlebody-knot of genus $g$ in $S^3$ is an embedded handlebody of genus $g$ in $S^3$. A spine $\gamma$ of a handlebody-knot $\Gamma$ is a graph embedded in $S^3$ with $\Gamma$ a regular neighborhood.

In this section, we describe a method to construct knots with meridional essential surfaces of arbitrarily number of boundaries, that we will use to prove Theorem 1. The method consist on defining a specific knot used as the pattern on a sattelite operation function.

Let $J$ be a prime knot as in the main theorem of [18], that is with meridional essential surfaces of any positive genus and two boundary components.
The knot $J$ is obtained by identifying the boundaries of two particular solid tori, say $H_1$ and $H_2$, attaching meridian to longitude, and by identifying the boundaries of the respective essential arc each contains. Denote by $X$ the torus obtained from the identified boundaries of these solid tori and by $O$ a disk in $X$ containing $X \cap J$. We isotope two copies of $X - O$ slightly to each side separated by $X$ and denote by $X_1$ and $X_2$ the resulting copies of $X$. The tori $X_1$ and $X_2$ intersect at $O$ and each bounds a solid torus, ambient isotopic to $H_1$ and $H_2$ respectively. The union of these solid tori along $O$ defines a genus two handlebody-knot $H$ with spine as in Figure 1.

Consider a ball $B$ disjoint from $O$ such that $B^c$ intersects $H$ at a cylinder containing $O$ and $J$ at two parallel trivial arcs. Note that the 2-string tangle $(B, B \cap J)$ is essential, otherwise the punctured torus obtained from $X$ wouldn’t be essential in $E(J)$. Denote by $T$ the solid torus defined by $B \cup (B^c \cap H)$. Let $J_1$ and $J_2$ be two copies of $J$ in the respective copies of $T$, say $T_1$ and $T_2$. We isotope the two arcs of $J_i \cap (T_i - B_i)$ into the boundary of $T_i$, where $B_i$ is the copy of $B$ with respect to $T_i$. For each knot $J_i$, we consider a segment of one of these arcs and a regular neighborhood $R_i$ of it, disjoint from $J_i$ otherwise. We proceed with a connect sum of $J_1$ and $J_2$ by removing the interior of $R_1$ and attaching the exterior of $R_2$, such that the disks $T_1 \cap \partial R_1$ and $T_2 \cap \partial R_2$ are identified. Hence, the knot $J_1\#J_2$ is in a genus two handlebody $G$ obtained by gluing $T_1$ and $T_2$ along a disk $D$ in their boundaries. (See Figure 2.)

As the tangle $(B, B \cap J)$ is essential and $T \cap B^c$ is a regular neighborhood of each arc of $B^c \cap J$, we have that $\partial T$ is essential in $T - J$. Moreover, from the construction of $T$ and $J$, each meridian of $T$ intersects $J$ at least twice. Hence, $\partial G$ is essential in $G - J_1\#J_2$ and, similarly, each essential disk in $G$ intersects $J_1\#J_2$ at two points.
Let $\Gamma$ be the genus 2 handlebody-knot $4_1$, from the list in [10], with spine $\gamma$ as in Figure 3.

Denote by $e : G \to S^3$ an embedding of $G$ into $S^3$ with image $\Gamma$, where $e(D)$ is an essential disk in a regular neighborhood $L$ of $l$. That is $e(D)$ is a disk that separates from $\Gamma$ two tori, $L_1$ and $L_2$, having cores $l_1$ and $l_2$, respectively, with $\Gamma = L \cup L_1 \cup L_2$. (See Figure 3.) We refer to $e(J_1 \# J_2)$ by $N$. The handlebody knot $\Gamma$ is embedded in a solid torus $P$ with core a trivial knot, such that there is a meridian disk of $P$ that intersects $\gamma$ at a single point in $l$. In the next definition we describe the operation used to prove Theorem 1.

**Definition 1.** Let $\mathcal{K}$ be the set of equivalence classes of knots in $S^3$ up to ambient isotopy. For a knot $K \in \mathcal{K}$ let $h_K : P \to S^3$ be an embedding of $S^3$ such that $h_K(P)$ is a solid torus with core $K$. We define the satellite operation function $h : \mathcal{K} \to \mathcal{K}$ such that for each $K \in \mathcal{K}$ we have $h(K) = h_K(N)$.

**Proposition 1.** For every knot $K$ the knot $h(K)$ is prime.
Proof: First observe that there is no local knot of $J_1 \# J_2$ in $G$: As the knot $J_i$ is prime and the tangle $(B_i, B_i \cap J_i)$ is essential, and $(B^e_i, B^e_i \cap J_i)$ is defined by two trivial arcs, we have necessarily that there is no local knot of $J_i$ in $T_i$, and consequently there is no local knot of $J_1 \# J_2$ in $G$.

Suppose $h(K)$ is a composite knot and consider a decomposing sphere $S$ for $h(K)$. If $S$ is disjoint from $\partial h(K)(\Gamma)$ then we obtain a contradiction with the inexistence of local knots of $J_1 \# J_2$ in $G$. Then consider the intersection of $S$ with $\partial h(K)(\Gamma)$ and assume that $|S \cap \partial h(K)(\Gamma)|$ is minimal among all decomposing spheres for $h(K)$.

The sphere $S$ intersects $\partial h(K)(\Gamma)$ in a collection of simple closed curves. Let $O$ be an innermost disk bounded by an innermost curve of $S \cap \partial h(K)(\Gamma)$ in $S$. We have two possibilities: there is an innermost disk $O$ disjoint from $h(K)$ or an innermost disk $O$ that intersects $h(K)$ at a single point. If $O$ is disjoint from $h(K)$, as $\partial G$ is essential in $G - J_1 \# J_2$ and $\partial \Gamma$ is essential in the exterior of $\Gamma$, then $\partial O$ bounds a disk in $\partial h(K)(\Gamma)$. Using a ball bounded by this disk and $O$ we can isotope $S$ through $\partial h(K)(\Gamma)$ in $E(h(K))$ reducing $|S \cap \partial h(K)(\Gamma)|$, contradicting its minimality. If $O$ intersects $h(K)$ at a single point then $O$ is an essential disk in $h(K)(\Gamma)$ intersecting $h(K)$ at a single point, which contradicts the fact that every essential disk of $G$ intersects $J_1 \# J_2$ at least in two points (as observed before). Therefore, the knot $h(K)$ is prime.

Besides being prime, the knots $h(K)$ can be decomposed into two essential arcs by surfaces of genus higher than zero, keeping the properties of Theorem 1 in [?] used in their construction.

**Proposition 2.** For every knot $K$ the exterior of $h(K)$ has meridional essential surfaces of any positive genus and two boundary components.

**Proof:** First we note that the meridional essential surfaces of any positive genus and two boundary components $S_{g;2}$ in $E(J)$ are in the solid torus $T$. The surface $S_{g;2}$ intersects the cylinder $B^e \cap T$ at $g - 2$ annuli parallel to one string of $B^e \cap T \cap J$ and $g$ annuli parallel to the other string of $B^e \cap T \cap J$, with the latter denoted by $s$. For each copy of $T$, $T_1$ and $T_2$, denote the respective copies of $s$ by $s_1$ and $s_2$. We isotope the arc $s_i$ into the boundary of $T_i$, and consider a regular neighborhood $R_i$ of this segment, disjoint from $J_i$ otherwise. We consider the surfaces $S_{g;2}$ in $T_1$ and assume that the annuli of $S_{g;2} \cap T_1 \cap B^e_i$ parallel to $s_1$ are in $R_1$. After the connected sum between $J_1$ and $J_2$, assumed along the arcs $s_1$ and $s_2$ as described before in this section, we replace these annuli in $R_1$ by $g$ annuli in the exterior of $R_2$ parallel to the
resulting arc of \( J_2 \) in \( T_2 \) (that we also denote by \( J_2 \)). In this way, we define a new surface \( S'_{g;2} \), in the handlebody \( G \), obtained from \( S_{g;2} \) and also with genus \( g \) and two boundary components.

As \( S'_{g;2} \cap T_2 \) is a collection of annuli in \( T_2 \), cutting a regular neighborhood of \( J_2 \) in \( T_2 \), there is no compressing or boundary compressing disk for \( S'_{g;2} \) in \( T_2 \). As \( S_{g;2} \) is essential in \( T_1 - J_1 \) there is no compressing or boundary compressing disk of \( S'_{g;2} \) in \( T_1 \). Hence, if there is a compressing or boundary compressing disk of \( S'_{g;2} \) in \( G \) it intersects \( D \). By an outermost disk type of argument in the compressing disk with respect to its intersection with \( D \) we obtain a contradiction with the essentiality of the annuli \( S'_{g;2} \cap T_2 \) in \( T_2 \) or the essentiality of \( S_{g;2} \) in \( T_1 - J_1 \). Hence, \( S'_{g;2} \) is essential in the complement of \( J_1 \# J_2 \) in \( G \).

Let now \( F_{g;2} \) be \( e(S'_{g;2}) \). We will show that \( F_{g;2} \) is essential in \( E(N) \). Note that \( N \) is, in particular, \( h(K) \) for \( K \) unknotted. Suppose there is a compressing or boundary compressing disk \( Q \) for \( F_{g;2} \) in \( E(N) \). If \( Q \) is disjoint from \( \partial \Gamma \) we get a contradiction with \( S'_{g;2} \) being essential in the exterior of \( J_1 \# J_2 \) in \( G \).

Hence, \( Q \) intersects \( \partial \Gamma \). Suppose \( |Q \cap \partial \Gamma| \) is minimal between all compressing or boundary compressing disks of \( F_{g;2} \) in \( E(N) \). As \( F_{g;2} \) is disjoint from \( \partial \Gamma \), the disk \( Q \) intersects \( \partial \Gamma \) at simple closed curves. Denote by \( O \) an innermost disk defined by the curves of \( Q \cap \partial \Gamma \) in \( D \). As \( \partial \Gamma \) is irreducible in \( E(\Gamma) \), the disk \( O \) cannot be essential in \( E(\Gamma) \). As \( J_1 \# J_2 \) is essential in \( G \), the disk \( O \) cannot be essential in \( \Gamma - N \). Therefore, \( \partial O \) bounds a disk in \( \partial \Gamma \) which, after an isotopy of \( O \) through this disk, contradicts the minimality of \( |Q \cap \partial \Gamma| \).

Then \( F_{g;2} \) is essential in \( E(N) \).

For a given non-trivial knot \( K \) consider \( h_K(P) \) and the knot \( h(K) \). Denote by \( F'_{g;2} \) the surface \( h_K(F_{g;2}) \) in \( h_K(\Gamma) \). Assume there is a compressing or boundary compressing disk \( Q' \) for \( F'_{g;2} \) in \( E(h(K)) \). In case \( Q' \) is disjoint from \( \partial h_K(P) \) we get a contradiction with \( F_{g;2} \) being essential in \( E(N) \). Then, \( Q' \) intersects \( \partial h_K(P) \). Suppose that \( |Q' \cap \partial h_K(P)| \) is minimal between all compressing or boundary compressing disks of \( F'_{g;2} \). Denote also by \( O' \) an innermost disk defined by \( Q' \cap \partial h_K(P) \) in \( Q' \). As \( \partial h_K(P) \) is essential in \( E(h(K)) \), the disk \( O' \) cannot be essential in \( E(h(K)) \). As \( N \) is essential in \( P \) (from the construction of \( P \)), the disk \( O' \) also cannot be essential in \( P - h(K) \). Then, \( \partial O' \) bounds a disk in \( \partial h_K(P) \) and, as before, we get a contradiction with the minimality of \( |Q' \cap \partial h_K(P)| \).

In conclusion, for any knot \( K \) the knot \( h(K) \) has a meridional essential surface of any positive genus and two boundary components.
3. Proof of Theorem \[1\]

In this section we use the sattelite operation described on Definition \[1\] and the knots from the main theorem of \[18\] to prove Theorem \[1\]. First, we start with the following proposition where we show that for any knot with a meridional essential surface in its exterior there is a knot with meridional essential surfaces of the same genus and unlimited number of boundaries.

**Proposition 3.** Let $K$ be a knot with a meridional essential surface of genus $g$ and $n$ boundary components. Then, the knot $h(K)$ has a meridional essential surface of genus $g$ and $b$ boundary components for all even $b \geq 2n$.

**Proof:** Let $S$ be a closed surface of genus $g$ which $K$ intersects at $n$ points, corresponding to a meridional essential surface of genus $g$ and $n$ boundary components in $E(K)$, as in the statement. With the association of $h_K(P)$ with a regular neighborhood of $K$, we denote by $S'$ the meridional essential surface obtained from $S$ in the complement of $h_K(P)$. Each boundary component of $S'$ bounds a meridian disk in $h_K(P)$. We consider two types of meridian disks of $h_K(P)$, with respect to $h_K(\Gamma)$, as it intersects $h_K(\Gamma)$ at one separating disk or at two separating disks in $h_K(L)$. We refer to a meridian disk of $h_K(P)$ as type-$a$ in case it intersects $h_K(\Gamma)$ at one disk, separating $h_K(\Gamma)$ into two components. We refer to a meridian disk as of type-$b$ in case it intersects $h_K(\Gamma)$ at two disks, separating $h_K(\Gamma)$ into three components: one being a cylinder in $h_K(L)$, and each of the other two containing either $h_K(L_1)$ or $h_K(L_2)$.

In what follows we construct the surfaces used to prove the statement of this proposition. We define $S_{n+i}$, $i = 0, 1, \ldots, n$, as a surface obtained from $S'$ by capping off its boundaries with $i$ meridians of type-$b$ and $n-i$ meridians of type-$a$. Hence, $S_{n+i}$ has genus $g$ and, in the exterior of $h_K(N)$, has $2 \times (n+i)$ boundary components.

To proceed, we consider the surface $S_{2n}$ that intersects $h_K(P)$ at $n$ meridian disks of type-$b$, $D_1, \ldots, D_n$, ordered by index, such that $D_1 \cup D_n$ cut a cylinder from $h_K(P)$ containing all the other disks $D_i$. Let $D$ be a meridian of type-$a$. We assume that $D \cup D_1$ cut a cylinder $Q_{L_1}$ from $h_K(P)$ containing $h_K(L_1)$, and $D_n \cup D$ cut a cylinder $Q_{L_2}$ from $h_K(P)$ containing $h_K(L_2)$. Let $\partial Q_{L_2}$ be the annulus of intersection of $\partial Q_{L_2}$ with $\partial h_K(P)$. Let $O$ be the annulus of intersection of $Q_{L_1}$ with $\partial h_K(\Gamma)$ (the component that is disjoint from
$h_K(L_1))$. Let $A$ be the annulus obtained by the union of $O$ with $D \cap E(h_K(\Gamma))$ with $\partial^*Q_{L_2}$. We define the surface $S_{2n+j}$, $j \geq 1$, as follows: Start with the surface $S_{2n}$. We consider $j$ meridians of type-b in sequence after $D_n$ and denoted $D_{n+1}, D_{n+2}, \ldots, D_{n+j}$. We consider also $j$ copies of $A$ in $h_K(P)$ denoted, from the outside to the inside, by $A_1, A_2, \ldots, A_j$. First we consider the disk $O_p$ obtained by capping off the annulus $A_p$ by the disk one of its boundary components bounds in $D_{n+p}$, for all $p = 1, \ldots, j$. We continue by extending, parallely, the boundary of $O_p$ until it reaches $D_p$, and we surger the disk by $O_p \cap D_p$ in $D_p$ by $O_p$. The resulting surface is the surface $S_{2n+j}$, which has genus $g$ and $2 \times (2n + j)$ boundary components.

**Lemma 1.** The surfaces $S_{n+i}$, for $i \in \{0, 1, \ldots, n\}$, are essential in $E(h(K))$.

**Proof of Lemma 1:** Suppose a surface $S_{n+i}$, for some $i \in \{0, 1, \ldots, n\}$, is not essential and denote by $D$ a compressing disk or boundary compressing disk of $S_{n+i}$ in $E(h(K))$. Assume that $|D \cap \partial h_K(P)|$ is minimal between all compressing and boundary compressing disks of $S_{n+i}$.

If $D$ intersects $\partial h_K(P)$ at some simple closed curve let $\delta$ be an innermost one in $D$ bounding an innermost disk $\Delta$. As $N$ is essential in $P$ the disk $\Delta$ cannot be essential in $h_K(P)$. As a non-trivial knot complement in $S^3$ is boundary irreducible, the disk $\Delta$ cannot be essential in the complement of $h_K(P)$. Hence, $\delta$ bounds a disk in $\partial h_K(P)$ and by an isotopy of $\Delta$ through this disk we can reduce $|D \cap \partial h_K(P)|$, contradicting its minimality. Then, $D$ doesn’t intersect $\partial h_K(P)$ at simple closed curves.

Assume now that $D$ intersects $\partial h_K(P)$ at some arc. As $h(K)$ is disjoint from $\partial h_K(P)$ the arcs of $D \cap \partial h_K(P)$ have both ends in $D \cap S_{n+i}$, even when $D$ is a boundary compressing disk. Denote by $\delta$ an outermost arc of $D \cap \partial h_K(P)$ in $D$, cutting an outermost disk $\Delta$ from $D$ with boundary $\delta$ union with an arc in $D \cap S_{n+i}$. (This latter condition is not always true for all outermost disks as $\Delta$. In fact, when $D$ is a boundary compressing disk, an outermost disk $\Delta$ might include in its boundary an arc in $\partial E(h(K))$; but an outermost disk in its complement in $D$ has the desired property.) If $\delta$ has both ends in the same disk component $D_j$ of $h_K(P) \cap S_{n+i}$, by cutting and pasting along the disk cut by $\delta$ and $\partial D_j$ from $\partial h_K(P)$, we can reduce $|D \cap \partial h_K(P)|$, contradicting its minimality. Hence, $\delta$ has ends in different components of $h_K(P) \cap S_{n+i}$. As $h_K(P) \cap S_{n+i}$ is a collection of disks, $\Delta$ cannot be in $h_K(P)$. Consequently, $\Delta$ is in the complement of $h_K(P)$ implying that $S$ is boundary.
compressible in \( E(K) \), which contradicts its essentiality. Hence, the surfaces \( S_{n+i} \), for \( i \in \{0, \ldots, n\} \), are essential in \( E(h(K)) \).

\[ \text{Lemma 2.} \, \text{The surfaces } S_{n+i}, \text{ for } i > n, \text{ are essential in } E(h(K)). \]

\textit{Proof of Lemma 2}\: For the proof of this lemma we use branched surface theory based on work of Oertel [20] and Floyd and Oertel [7] that we revise concisely over the next paragraphs.

A \textit{branched surface} \( B \) with generic branched locus in a 3-manifold \( M \) is a compact space locally modeled on Figure 4(a).

![Figure 4: Local model for a branched surface, in (a), its regular neighborhood, in (b), and branch equations, in (c).](image)

We denote by \( N(B) \) a fibered regular neighborhood of \( B \) embedded in \( M \), locally modelled on Figure 4(b). The boundary of \( N(B) \) is the union of three compact surfaces \( \partial_h N(B) \), \( \partial_v N(B) \) and \( \partial M \cap \partial N(B) \), where a fiber of \( N(B) \) meets \( \partial_h N(B) \) transversely at its endpoints and either is disjoint from \( \partial_v N(B) \) or meets \( \partial_v N(B) \) in a closed interval in its interior. We say that a surface \( S \) is \textit{carried} by \( B \) if it can be isotoped into \( N(B) \) so that it is transverse to the fibers. If we associate a weight \( w_i \geq 0 \) to each component on the complement of the branch locus in \( B \) we say that we have an invariant measure provided that the weights satisfy branch equations as in Figure 4(c).

Given an invariant measure on \( B \) we can define a surface carried by \( B \), with respect to the number of intersections between the fibers and surface. We also note that if all weights are positive we say that \( S \) is carried with \textit{positive weights} by \( B \), which is equivalent to \( S \) being transverse to all fibers of \( N(B) \).

A \textit{disk of contact} is a disk \( D \) embedded in \( N(B) \) transverse to fibers and with \( \partial D \) in \( \partial_v N(B) \). A \textit{half-disk of contact} is a disk \( D \) embedded in \( N(B) \) transverse to fibers with \( \partial D \) being the union of an arc in \( \partial M \cap \partial N(B) \) and an arc in \( \partial_v N(B) \). A \textit{monogon} in the closure of \( M - N(B) \) is a disk \( D \) with \( D \cap \partial N(B) = \partial D \) which intersects \( \partial_v N(B) \) in a single fiber. (See Figure 5.)

A branched surface \( B \) embedded in \( M \) is said \textit{incompressible} if it satisfies the following three properties:
Figure 5: Illustration of a monogon and a disk of contact on a branched surface.

(i) $B$ has no disk of contact or half-disk of contact;
(ii) $\partial hN(B)$ is incompressible and boundary incompressible in the closure of $M - N(B)$;
(iii) there are no monogons in the closure of $M - N(B)$.

With following theorem, by Floyd and Oertel in [7], we can determine if a surface carried by a branched surface is essential.

**Theorem 3** (Floyd and Oertel, [7]). A surface carried with positive weights by an incompressible branched surface is essential.

Now we prove that the surfaces $S_{n+i}$, for $i > n$, are essential in $E(h(K))$ by showing that these surfaces are carried with positive weights by an incompressible branched surface. Let us consider the surface $S'$ and denote by $b_1, b_2, \ldots, b_n$ its boundary components in consecutive order in $\partial h_K(P)$. Denote by $Q_j$ the annulus component of $\partial h_K(P) - b_1 \cup \cdots \cup b_n$ bounded by $b_j \cup b_{j+1}$. We consider the union of $S'$, the annuli $Q_j$, $j = 1, \ldots, n - 1$, the annulus $A$ and a type-$b$ meridian $D_1$ of $h_K(P)$ with boundary $b_1$, and denote the resulting space by $B$. We smooth the space $B$ on the intersection of the surface $S'$, $Q_j$, $A$ and $D_1$ as explained next. For each annulus $Q_j$: isotope the boundary in $b_j$ into the exterior of $h_K(P)$ and smooth it towards $b_j$; also, smooth the boundary in $b_{j+1}$ towards the exterior of $h_K(P)$. We also smooth the boundary of $D_1$, that is $b_1$, with $S'$. With respect to the annulus $A$, we smooth its boundary in $D_1$ towards $b_1$ and its boundary in $b_n$ we isotope it into the exterior of $h_K(P)$ and smooth it towards $b_n$. In Figure 5 we have a schematic representation of the branched surface $B$.

Let us denote by $D_1^a$ the disk bounded by $D_1 \cap A$ in $D_1$ and by $D_1^b$ the annulus defined by $D_1 - D_1^a$. From the construction, the space $B$ is a branched surface with sections denoted naturally by $S'$, $Q_j$ for $j = 1, \ldots, n - 1$, $A$, $D_1^a$ and $D_1^b$, as illustrated in Figure 5. We denote a regular neighborhood of $B$ by $N(B)$. The surface $S_{n+i}$ is carried with positive weights by $B$ together with
the invariant measure on $B$ defined by $W_1 = 1$, $W_{Q_j} = n - j + i$, $W_A = i$, $W_{D_1^a} = n$ and $W_{D_1^b} = n + i$, on the sections $S_i$, $Q_j$, $A$, $D_1^a$, $D_1^b$, respectively.

To prove that $S_{n+i}$, $i > n$, is essential in the complement of $h(K)$ we show that $B$ is an incompressible branched surface and use Theorem 3.

The space $N(B)$ decomposes $E(h(K))$ into three components: a component cut from $E(h(K))$ by $S'$ and the annuli $Q_j$ with odd index that we denote by $E_1$; a component cut from $E(h(K))$ by $S'$, the annuli $Q_j$ with even index and the annulus $A$ that we denote by $E_2$; a component cut from $E(h(K))$ by $S'$, $D_1^a$, $D_1^b$ and the annuli $Q_j$, $j = 1, \ldots, n-1$, and $A$, that we denote by $E_p$.

As $E_1$ is disjoint from $\partial E(h(K))$ there are boundary compressing disks for $\partial hN(B)$ in $E_1$. In $\partial E_1$, the components of $\partial_v N(B)$ correspond to annuli associated to the boundary of $Q_j$ in $b_j$, for $j$ odd. Hence, if $\partial hN(B)$ has a compressing disk in $E_1$, as it is disjoint from $\partial_v N(B)$, we can isotope its boundary into $S'$, contradicting $S'$ being essential in $E(K)$. On the other hand, a monogon disk in $E_1$ would have boundary defined by an arc in some $Q_j$ and an arc in $S'$, being a boundary compressing disk for $S'$ in $E(K)$, contradicting again $S'$ being essential.

In $\partial E_2$, the components of $\partial_v N(B)$ correspond to annuli associated to the boundary of $Q_j$ in $b_j$, for $j$ even, and to the annulus associated to the boundary of $A$ in $b_n$. As in the case for $E_1$, if there is a compressing disk for $\partial hN(B)$ in $E_2$ then we get a contradiction with $S'$ being essential in $E(K)$.

If there is a monogon disk in $E_2$ it would have boundary defined by an arc in some $Q_j$ or in $A$ and an arc in $S'$, defining a boundary compressing disk for $S'$ in $E(K)$, and contradicting again $S'$ being essential in $E(K)$. If there is a boundary compressing disk for $\partial hN(B)$ in $E_2$ then an arc of such a disk boundary can be assumed to be the arc $s_1$ defined by $h_K(L_1) \cap h(K)$. The
solid torus $h_K(L_1)$ is in a ball in $E_2$ intersecting $S_{n+i}$ in $D_1$, and $s_1$ has the isotopy type of the knotted arc $J_1$ in this ball. The existence of a boundary compressing disk of $\partial_h N(B)$ in $E_2$ with $s_1$ in the boundary contradicts $J_1$ being a knotted arc. The component $E_p$ defines together with $E_p \cap h(K)$ a 3-string tangle defined by a knotted arc $s_2$, with the pattern of $J_2$ in $h_K(L_2)$ and two parallel trivial arcs in $h_K(L)$, denoted by $t_1$ and $t_2$. There is only one component of $\partial_h N(B)$ in $\partial E_p$ and it corresponds to the boundary of $A$ in $D_1$, denoted by $a$. The end points of each $t_i$, $i = 1, 2$, in $E_p$ are separated by $a$ in $\partial E_p$, and the ends of $s_2$ are in the same disk bounded by $a$ in $\partial E_p$, say $D_a$. We denote the other disk bounded by $a$ in $\partial E_p$ by $D'_a$. As $a$ is separating in $\partial E_p$ there are no monogons of $\partial N(B)$ in $E_p$, and a the boundary of a compressing or boundary compressing disk for $\partial_h N(B)$ in $E_p$ intersects $\partial_h N(B)$ only at $D_a$. If there is a boundary compressing disk then we can assume the arc $s_2$ is on its boundary, contradicting $s_2$ being knotted. If there is a compressing disk then it separates $s_2$ from $t_1 \cup t_2$ implying that $\Gamma$ is trivial in $P$, which is a contradiction with $\Gamma$ being knotted (more exactly, the handlebody knot $4_1$ as in [10]). This finishes the proof that $B$ is an incompressible branched surface and, consequently, from Theorem 3 we also have that $S_{n+i}$ is essential in $E(h(K))$, for $i > n$. }

From Lemmas 1 and 2 we obtain the statement of the proposition as the surfaces $S_{n+i}$, $i \geq 0$, are essential in $E(h(K))$ and $S_{n+i}$ has genus $g$ and $2n + 2i$ boundary components.

This proposition offers a base for the proof of Theorem 1 as follows.

Proof of Theorem 1: Consider an infinite collection of knots $C_i$, $i \in \mathbb{N}$, as in the main theorem in [13], that is each of which having in their exterior a meridional essential surface $S_g$ for every genus $g \geq 0$ and two boundary components. Using the satellite operation defined in the previous section, for each knot $C_i$ we define the knot $h(C_i)$ that we denote by $K_i$. Hence, each knot $K_i$ has in its exterior, from Proposition 2, a meridional essential surface of any positive genus and two boundary components and, from Proposition 3, a meridional essential surface $S_g;2n$ of any genus $g$ and $2n$ boundary components for all $n \geq 2$. From Proposition 1, the knots $h(K_i)$ are prime, and together with examples obtained after their connected sum with a non-trivial knot we complete the proof of the statement of the theorem.
4. Meridional essential surfaces on hyperbolic knot exteriors

In this section we extend the work in the previous sections to hyperbolic knots and prove Theorem 2.

**Proof of Theorem 2**: Consider a knot $K$ as in the statement of Theorem 1. That is, $E(K)$ contains meridional essential surfaces $S_{g,b}$ of any genus $g$ and (even) number $b$ of boundary components. We will show that for $K$ there is a hyperbolic knot whose exterior contains meridional essential surfaces $F_{g,b}$ of genus and number of boundary components greater than or equal to $g$ and $b$, respectively, for each surface $S_{g,b}$.

From Myers [17], let $J \subset E(K)$ be a null-homotopic knot with hyperbolic complement. Consider $E(J \cup K)$ and do $\frac{1}{r}$-Dehn filling on $J$ to produce a hyperbolic knot $K_r \subset S^3$. As in the proof of Proposition 3.2 of [2] by Boileau-Wang [2], there is a degree-one map $f: E(K_r) \to E(K)$. From the construction of the degree-one map $f$, as on the proof of Proposition 3.2 on [2], we can homotope $f$ to be transverse to $S_{g,b}$ by making the immersed disk bounded by $J$ in $E(K)$, used to define $f$, transverse to $S_{g,b}$. After this homotopy of $f$, if necessary, we have that $F_{g,b} = f^{-1}(S_{g,b})$ is a 2-dimensional submanifold of $E(K_r)$. The restriction map $f: F_{g,b} \to S_{g,b}$ is a degree-one map, by definition of degree-one map. Hence, by the work of Edmonds [4], it is a pinch map: there is a compact connected submanifold $F \subset F_{g,b}$ with no more than one component of its boundary being a simple closed curve in the interior of $F_{g,b}$, such that $f$, restricted to $F_{g,b}$, is homotopic to the quotient map $F_{g,b} \to F_{g,b}/F$. In particular, this means that the genus of $F_{g,b}$ is higher than the one of $S_{g,b}$. On top of this, as $f$ is the identity near $\partial E(K)$, the number of boundary components of $F_{g,b}$ is the same as $S_{g,b}$.

In case $F_{g,b}$ is incompressible, we have completed the proof. Otherwise, let $D$ be a compressing disk of $F_{g,b}$ in $E(K_r)$. As $f: F_{g,b} \to S_{g,b}$ is a pinch map as described above, $f(D)$ is an immersed disk in $E(K)$. In case $\partial D$ is in the pinched region of $F_{g,b}$ by $f$, then $\partial D$ is mapped to a point of $S_{g,b}$. Hence, we can compress $F_{g,b}$ by $D$ obtaining a surface that is still mapped by a degree-one map into $S_{g,b}$. We keep compressing until there are no more compressing disks with boundary in the pinched region. In case the induced homomorphism $f_*: \pi_1(F_{g,b}) \to \pi_1(S_{g,b})$ takes the class of $\partial D$ to the identity of $\pi_1(S_{g,b})$, we can homotope $\partial D$ into the pinched region of $F_{g,b}$ by $f$, and repeat the previous argument. In case the class of $\partial D$ is not in the kernel of
f_*, from the commutativity of the diagram

\[ \begin{array}{ccc}
\pi_1(F_{g;b}) & \xrightarrow{f_*} & \pi_1(S_{g;b}) \\
\downarrow{j_*} & & \downarrow{i_*} \\
\pi_1(E(K_r)) & \xrightarrow{f_*} & \pi_1(E(K))
\end{array} \]

the kernel of \( i_* \) contains the class of \( f_*([\partial D]) \) and is non-trivial. By the Loop Theorem, there is a compressing disk of \( S_{g;b} \) in \( E(K) \), which is a contradiction. Hence, \( F_{g;b} \) is essential in \( E(K_r) \).

\[ \square \]

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