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SINGULARLY PERTURBED FULLY NONLINEAR PARABOLIC PROBLEMS AND THEIR ASYMPTOTIC FREE BOUNDARIES

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ABSTRACT: We study fully nonlinear singularly perturbed parabolic equations and their limits. We show that solutions are uniformly Lipschitz continuous in space and Hölder continuous in time. For the limiting free boundary problem, we analyse the behaviour of solutions near the free boundary. We show, in particular, that, at each time level, the free boundary is a porous set and, consequently, is of Lebesgue measure zero.

KEYWORDS: Parabolic fully nonlinear equations, singularly perturbed problems, Lipschitz regularity, porosity of the free boundary.
AMS SUBJECT CLASSIFICATION (2010): Primary 35K55. Secondary 35D40, 35B65, 35R35.

1. Introduction

In this paper we study the following singular perturbation problem for a fully nonlinear parabolic equation

$$\left\{ \begin{array}{rcll} F(x,t,D^2u_\varepsilon) - \partial_t u_\varepsilon & = & \beta_\varepsilon(u_\varepsilon) + f_\varepsilon & \text{in} \quad \Omega_T \\ u_\varepsilon & = & \varphi & \text{on} \quad \partial_p \Omega_T, \end{array} \right. \tag{E_ε}$$

where $F(x,t,M)$ is a fully nonlinear uniformly elliptic operator, the Dirichlet data φ is nonnegative and the singularly perturbed potential $\beta_\varepsilon(\cdot)$ is a suitable approximation of a multiple of the Dirac mass δ_0. The problem appears, for example, in combustion theory and describes the propagation of curved, premixed deflagration flames. It is derived (cf. [3]) in the framework of the theory of equidiffusional premixed flames, analysed in the relevant limit of right activation energy for Lewis number equal to one, and the unknown u^ε represents the normalised temperature of the mixture.

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The study of the limit as $\varepsilon \to 0$ in (E_ε) (the high activation energy analysis) leads to a free boundary problem, and often provides an alternative way of approaching questions related to the existence and the regularity of solutions and the free boundary. For example, the one-phase elliptic problem

$$\begin{cases} \Delta u &= 0 \quad \text{in} \quad \{u>0\} \\ |\nabla u| &= C \quad \text{on} \quad \partial\{u>0\}, \end{cases} \tag{1.1}$$

studied by Alt and Caffarelli in [1], can be approached by taking $\varepsilon \to 0$ in

$$\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon).$$

In [1], it is shown that any minimiser u of the problem

$$\int_\Omega |\nabla v|^2 + \chi_{\{v>0\}} \to \min$$

is Lipschitz continuous and solves (1.1) with a nonnegative Dirichlet boundary condition. Alt and Caffarelli also proved that the free boundary condition holds in a weak sense, and that the free boundary $\partial\{u>0\}$ is a $C^{1,\alpha}$ surface except at a set of zero surface measure.

The idea of passing to the limit in a singular perturbation problem had been proposed in [21] but would only be treated rigorously in [2], in the one-phase case (that is, with $u \geq 0$), for general linear operators. The results in [2] include the Lipschitz continuity of the limit, the fact that it solves the free boundary problem in a weak sense and some geometric measure properties of particular level sets. The topic would become the object of intense research and we highlight the contributions of [5, 6, 9, 12, 14, 15], where, in particular, the two-phase problem (allowing u to change sign) was treated. The parabolic case

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$$

was studied in [7] for one phase and in [4, 5, 6] for the two-phase problem.

This alternative approach opens an avenue leading also to non-variational free boundary problems. Recently, the singular perturbation problem

$$F(x, D^2 u_\varepsilon) = \beta_\varepsilon(u_\varepsilon),$$

which is the elliptic counterpart of (E_ε), was studied in [16]; the authors obtain Lipschitz estimates and study the limiting free boundary problem. Our aim in this paper is to extend these results to the parabolic case. We consider a family of solutions of problem (E_ε) and show that, under suitable assumptions, the limit function u is a solution to the free boundary problem

$$\begin{cases} F(x,t,D^2u) - \partial_t u &= f \quad \text{in} \quad \{u>0\} \\ u &= \varphi \quad \text{on} \quad \partial_p\Omega_T \end{cases} \tag{1.2}$$

where $f = \lim f_\varepsilon$. We do not impose a free boundary condition and thus the limiting problem is not understood as overdetermined.

Unlike the elliptic case (see, for example, [16]), one can not apply the Harnack inequality in order to prove the (uniform) regularity of solutions. The reason is that we can only compare functions on parabolic boundaries, not on the *top* of a cylinder; we are thus unable to pass from one level to another. We overcome this difficulty by using a Bernstein type argument (see the proof of Proposition 4.1). For the same reason, the study of the free boundary of the limiting problem requires a totally different approach: in the elliptic case, using a covering argument, one can prove the finiteness of the $(n-1)$-dimensional Hausdorff measure of the free boundary (see [16]). In the parabolic case, what we are able to prove is that, at each time level, the n-dimensional Lebesgue measure of the free boundary is zero because it is porous. We prove this by obtaining a non-degeneracy result and by controlling the growth rate of the solution near the free boundary.

The paper is organised as follows. We first prove the existence of solutions to (E_ε) using Perron's method. We also show in Section 3 that solutions are uniformly bounded (Theorem 3.2). In Section 4, using a Bernstein type argument, we obtain a uniform gradient estimate for solutions (Proposition 4.1), which implies the uniform Hölder continuity in time with exponent 1/2 (Proposition 4.2), just as in the classical case of the heat equation. In Section 5, we pass to the limit in (E_ε) as $\varepsilon \to 0$. Invoking stability arguments, we show that the limit function is a solution of a free boundary problem (Theorem 5.1). The regularity of the free boundary is then studied in Section 6: we first prove the non-degeneracy of the solution of the limiting free boundary problem (Lemma 6.1) and next establish the growth rate of the solution near the free boundary (Lemma 6.3). These two results lead to the porosity of the free boundary at each time level (Theorem 6.1).

2. Mathematical set-up

Given a bounded domain $\Omega \subset \mathbb{R}^n$, with a smooth boundary $\partial\Omega$, we define, for $T > 0$, $\Omega_T = \Omega \times (0, T]$, its lateral boundary $\Sigma = \partial\Omega \times (0, T)$ and its parabolic boundary $\partial_p \Omega_T = \Sigma \cup (\Omega \times \{0\})$.

An operator $F : \Omega_T \times \mathbb{R} \times \mathrm{Sym}(n) \to \mathbb{R}$ is uniformly elliptic if there exist two positive constants $\lambda \leq \Lambda$ (the ellipticity constants) such that, for any $M \in \mathrm{Sym}(n)$ and $(x, t) \in \Omega_T$,

$$\lambda \|P\| \leq F(x, t, M + P) - F(x, t, M) \leq \Lambda \|P\|, \tag{2.1}$$

for every non-negative definite symmetric matrix P. Here, $\mathrm{Sym}(n)$ is the space of real $n \times n$ symmetric matrices and $\|P\|$ equals the maximum eigenvalue of P.

We let $\mathcal{P}^-_{\lambda,\Lambda}$ and $\mathcal{P}^+_{\lambda,\Lambda}$ denote the minimal and maximal Pucci extremal operators corresponding to λ, Λ, that is, for $M \in \mathrm{Sym}(n)$,

$$\mathcal{P}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i \quad \text{and} \quad \mathcal{P}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $\{e_i = e_i(M), 1 \leq i \leq n\}$ is the set of eigenvalues of M. We recall also that

$$\mathcal{P}^-_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AM) \quad \text{and} \quad \mathcal{P}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AM),$$

where $\mathcal{A}_{\lambda,\Lambda} = \left\{ A \in \mathrm{Sym}(n) : \lambda |\xi|^2 \leq A_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2, \, \forall\, \xi \in \mathbb{R}^n \right\}$. Note that uniform ellipticity implies that, for $A, B \in \mathrm{Sym}(n)$,

$$\mathcal{P}^-_{\frac{\lambda}{n},\Lambda}(A - B) \leq F(x, t, A) - F(x, t, B) \leq \mathcal{P}^+_{\frac{\lambda}{n},\Lambda}(A - B). \tag{2.2}$$

Any operator F which satisfies condition (2.1) will be referred to as a (λ, Λ)-elliptic operator.

We now define, following [10, 18], the notion of viscosity solution for a fully nonlinear parabolic equation.

Definition 2.1. *A function $u \in C(\Omega_T)$ is a viscosity sub-solution (resp. super-solution) of*

$$F(x, t, D^2 u) - \partial_t u = g(x, t, u) \quad in \quad \Omega_T$$

if, whenever $\phi \in C^2(\Omega_T)$ and $u - \phi$ has a local maximum (resp. minimum) at $(x_0, t_0) \in \Omega_T$, there holds

$$F(x_0, t_0, D^2 \phi(x_0, t_0)) - \partial_t \phi(x_0, t_0) \geq g(x_0, t_0, \phi(x_0, t_0)). \quad (resp. \leq)$$

A function u is a viscosity solution if it is both a viscosity sub-solution and a viscosity super-solution.

We also define the class of functions, that will be useful in the sequel,

$$\mathcal{S}(\lambda, \Lambda, f) := \overline{\mathcal{S}}(\lambda, \Lambda, f) \cap \underline{\mathcal{S}}(\lambda, \Lambda, f).$$

where

$$\begin{aligned}\overline{\mathcal{S}}(\lambda, \Lambda, f) &:= \left\{u \in C(\Omega_T) : \mathcal{P}^-(D^2u) - \partial_t u \le f \text{ in } \Omega_T\right\}\\ \underline{\mathcal{S}}(\lambda, \Lambda, f) &:= \left\{u \in C(\Omega_T) : \mathcal{P}^+(D^2u) - \partial_t u \ge f \text{ in } \Omega_T\right\},\end{aligned}$$

the inequalities taken in the viscosity sense.

We need to clarify what is a Lipschitz function defined in a space-time domain.

Definition 2.2. *Let $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}$. We say that $v \in Lip_{\text{loc}}(1, 1/2)(\mathcal{D})$ if, for every compact $K \Subset \mathcal{D}$, there exists a constant $C = C(K)$ such that*

$$|v(x,t) - v(y,s)| \le C\left(|x-y| + |t-s|^{\frac{1}{2}}\right),$$

for every $(x,t), (y,s) \in K$. If the constant C does not depend on the set K we say $v \in Lip(1, 1/2)(\mathcal{D})$.

We also define the $\text{Lip}(1, 1/2)(\mathcal{D})$ seminorm in $\mathcal{D}$

$$[v]_{\text{Lip}(1,1/2)(\mathcal{D})} := \sup_{(x,t),(y,s)\in\mathcal{D}} \frac{|v(x,t) - v(y,s)|}{|x-y| + |t-s|^{1/2}}$$

and the $\text{Lip}(1, 1/2)(\mathcal{D})$ norm in $\mathcal{D}$

$$\|v\|_{\text{Lip}(1,1/2)(\mathcal{D})} := \|v\|_{L^\infty(\mathcal{D})} + [v]_{\text{Lip}(1,1/2)(\mathcal{D})}.$$

For future reference and further clarity, we gather next the set of assumptions concerning the data in (E_ε).

Assumptions on the data for (E_ε).

(A1): $F = F(x,t,M)$ is uniformly elliptic, concave and of class $C^{1,\alpha}$ in M and of class $C^{1,\alpha}_{\text{loc}}$ in (x,t), for some $\alpha > 0$, and $F(\cdot, \cdot, 0) = 0$.

(A2): The singular reaction term $\beta_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$0 \le \beta_\varepsilon(s) \le \frac{1}{\varepsilon}\chi_{(0,\varepsilon)}(s), \quad \forall\, s \in \mathbb{R}_+.$$

For example, it can be built as an approximation of unity

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right),$$

where β is a nonnegative smooth real function with supp $\beta = [0,1]$, such that

$$\|\beta\|_\infty \le 1 \quad \text{and} \quad \int_{\mathbb{R}} \beta(s)\, ds < \infty.$$

Such a sequence of potentials converges, in the distributional sense, to $\int \beta$ times the Dirac measure δ_0.

(A3): $f_\varepsilon(x,t) \in C^{1,\alpha}(\overline{\Omega_T})$, is non-increasing in t and satisfies

$$0 < c_0 \le f_\varepsilon(x,t) \le c_1 < \infty \quad \text{in} \quad \Omega_T$$

and

$$\|\nabla f_\varepsilon\|_\infty \le C.$$

(A4): The Dirichlet data $0 \le \varphi(x,t) \in C^{1,\alpha}(\partial_p \Omega_T)$, is non-decreasing in t and satisfies $\varphi(x,0) = 0$.

Finally, we introduce some further notation.

Notation. For $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $\tau > 0$, we denote

$$\begin{aligned}
B_\tau(x_0) &:= \{x \in \mathbb{R}^n : |x - x_0| < \tau\},\\
Q_\tau(x_0,t_0) &:= B_\tau(x_0) \times (t_0 - \tau^2, t_0 + \tau^2),\\
Q_\tau^-(x_0,t_0) &:= B_\tau(x_0) \times (t_0 - \tau^2, t_0],
\end{aligned}$$

and, for a set $K \subset \mathbb{R}^{n+1}$ and $\tau > 0$,

$$\mathcal{N}_\tau(K) := \bigcup_{(x_0,t_0)\in K} Q_\tau(x_0,t_0) \quad \text{and} \quad \mathcal{N}_\tau^-(K) := \bigcup_{(x_0,t_0)\in K} Q_\tau^-(x_0,t_0).$$

3. Existence of viscosity solutions

Our first goal is to show that (E_ε) has at least one viscosity solution. Because of the lack of monotonicity of equation (E_ε) with respect to the variable u, the classical Perron's method can not be applied directly. The following result is a suitable adaptation, stated in a more general form, since we feel it may be of independent interest.

Theorem 3.1. *Let F satisfy (A1), $g \in C^{0,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f \in C(\Omega_T)$ and $\varphi \in C(\partial_p \Omega_T)$. If $u_\star$, $u^\star$ are, correspondingly, a viscosity sub-solution and a viscosity super-solution of*

$$F(x,t,D^2u) - \partial_t u = g(u) + f \quad in \quad \Omega_T, \tag{3.1}$$

with $u_\star = u^\star = \varphi$ on $\partial_p \Omega_T$, then

$$u := \inf_{v \in \mathcal{S}} v$$

is a viscosity solution of (3.1)*, where*

$$\mathcal{S} := \{v \in C(\overline{\Omega_T});\ u_\star \le v \le u^\star \text{ and } v \text{ is a super-solution of } (3.1)\}.$$

Proof: Let $\mu > 0$ be such that $|g'| < \mu/2$ and let $h(z) := \mu z - g(z)$, which is then increasing. For $\psi \in C^{0,1}(\overline{\Omega_T})$ we define the following (uniformly elliptic) operator

$$G_\psi[u] := G_\psi(x,t,u,D^2u) := F(x,t,D^2u) - \mu u - f + \psi.$$

Next, set $u_0 := u_\star$ and let u_{k+1} be a solution of

$$\begin{cases} G_{\psi_k}[u] - \partial_t u = 0 & \text{in} \quad \Omega_T \\ \qquad\qquad\quad u = \varphi & \text{on} \quad \partial_p \Omega_T, \end{cases} \tag{3.2}$$

where $\psi_k = h(u_k)$. The existence of a solution to (3.2) is assured by the classical Perron's method (see [8, 11]), since $G_\psi(x,t,r,M)$ is now non-increasing in r. We claim that

$$u_\star = u_0 \le u_1 \le \cdots \le u_k \le u_{k+1} \le \cdots \le u^\star \text{ in } \Omega_T. \tag{3.3}$$

Indeed, since u_0 is a viscosity sub-solution of (3.1) and u_1 solves (3.2) with $k=0$, we have

$$G_{\psi_0}[u_1] - \partial_t u_1 = 0 \le G_{\psi_0}[u_0] - \partial_t u_0$$

in the viscosity sense. Moreover, $u_1 = u_0 = \varphi$ on $\partial_p \Omega_T$, so the comparison principle (see [10]) gives $u_0 \le u_1$ in Ω_T. Assume inductively that we have verified that $u_{k-1} \leq u_k$ in Ω_T. Since h is increasing, having in mind the inductive assumption and the fact that u_{k+1} is a solution of (3.2), we conclude

$$G_{\psi_k}[u_{k+1}] - \partial_t u_{k+1} = 0 \leq G_{\psi_k}[u_k] - \partial_t u_k$$

in the viscosity sense. Also $u_{k+1} = u_k = \varphi$ on $\partial_p \Omega_T$. Applying once more the comparison principle, we get $u_k \leq u_{k+1}$. Analogously, one can also show that $u_k \leq u^\star$, $\forall k \geq 0$.

Using (3.3), we define the pointwise limit

$$u := \lim_{k \to \infty} u_k.$$

For any $Q \Subset \Omega_T$, there exists a constant C (depending only on μ, $\|u_\star\|_{L^\infty(Q)}$, $\|u^\star\|_{L^\infty(Q)}$ and $\|f\|_{L^\infty(Q)}$) such that

$$|F(x,t,D^2 u_k) - \partial_t u_k| \leq C \quad \text{in } Q$$

in the viscosity sense, $\forall k \geq 0$. Therefore, u_k is locally uniformly Hölder continuous (see [10]). By the Arzelà–Ascoli Theorem, it converges, up to a subsequence, locally uniformly in Ω_T. Invoking stability arguments (see [10, 19]) and passing to the limit as $k \to \infty$, we conclude that u is a viscosity solution of

$$F(x,t,D^2 u) - \partial_t u = g(u) + f.$$

To conclude the proof, it remains to check that $u = \inf_{v \in \mathcal{S}} v$. Obviously, $u \in \mathcal{S}$. Let $v \in \mathcal{S}$; since

$$G_{\psi_k}[u_{k+1}] - \partial_t u_{k+1} = 0 \geq G_{\psi_k}[v] - \partial_t v$$

in the viscosity sense, arguing as above, we get $v \geq u_{k+1}$, $\forall k \geq 0$. Passing to the limit as $k \to \infty$ we conclude that $u = \inf_{v \in \mathcal{S}} v$. ■

As a consequence of this result, we get the existence of solutions of (E_ε). The Alexandrov-Bakelman-Pucci (ABP) estimate then implies their uniform boundedness.

Theorem 3.2. *If (A1)-(A4) hold, then the problem (E_ε) has a solution and*

$$0 \leq u_\varepsilon \leq \Upsilon \quad in \quad \Omega_T, \tag{3.4}$$

where $\Upsilon = \Upsilon(\lambda, \Lambda, n, \|\varphi\|_\infty, c_0)$.

Proof: The existence of a solution follows from Theorem 3.1, with $g = \beta_\varepsilon$, $f = f_\varepsilon$. To prove (3.4), let $v_\varepsilon := u_\varepsilon - \|\varphi\|_\infty$. Note that $v_\varepsilon \le 0$ on $\partial_p\Omega_T$ and from (2.2) one has

$$\mathcal{P}^+_{\frac{\lambda}{n},\Lambda}(D^2 v_\varepsilon) - \partial_t v_\varepsilon \ \ \ge \ \ F(x,t,D^2 v_\varepsilon) - \partial_t v_\varepsilon = F(x,t,D^2 u_\varepsilon) - \partial_t u_\varepsilon \ge c_0.$$

This means that $v_\varepsilon \in \underline{\mathcal{S}}(\frac{\lambda}{n},\Lambda,c_0)$. The ABP estimate ([18, Theorem 3.14]) then implies

$$\sup_{\Omega_T}(v_\varepsilon)^+ \le C(\lambda,\Lambda,n,c_0).$$

Thus, $u_\varepsilon \le \|\varphi\|_\infty + C(\lambda,\Lambda,n,c_0) =: \Upsilon$.

In order to prove the nonnegativity of u_ε we assume the contrary, i.e. that $A_\varepsilon := \{(x,t) \in \Omega_T : u_\varepsilon(x,t) < 0\} \ne \emptyset$. Since β_ε is supported in $[0,\varepsilon]$, then

$$\mathcal{P}^-_{\frac{\lambda}{n},\Lambda}(D^2 u_\varepsilon) - \partial_t u_\varepsilon \le F(x,t,D^2 u_\varepsilon) - \partial_t u_\varepsilon = f_\varepsilon \le c_1 \ \text{ in } \ A_\varepsilon,$$

which means that $u_\varepsilon \in \overline{\mathcal{S}}(\frac{\lambda}{n},\Lambda,c_1)$. Another application of the ABP estimate provides that $u_\varepsilon \ge 0$ in A_ε, which is a contradiction. ■

4. Uniform Lipschitz regularity in space-time

In this section we show that the family $\{u_\varepsilon\}_{\varepsilon>0}$ of solutions of (E_ε) is locally uniformly bounded in the $\mathrm{Lip}_{\mathrm{loc}}(1,1/2)$-norm. As a consequence, we show that the limit function u is a solution of the free boundary problem (1.2). The main result of this section is the following theorem.

Theorem 4.1. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of (E_ε). Let $K \subset \Omega_T$ be compact and $\tau > 0$ be such that $\mathcal{N}_{2\tau}(K) \subset \Omega_T$. If (A1)-(A4) hold, then there exists a constant $L = L(\tau, \|\varphi\|_\infty)$ such that*

$$\|u_\varepsilon\|_{Lip(1,1/2)(K)} \le L.$$

Theorem 4.1 will be an immediate consequence of the following two results. First, using a Bernstein type argument, we obtain the uniform boundedness of the gradients of solutions (Proposition 4.1). Next, we show that uniform spatial Lipschitz continuity implies uniform Hölder continuity in time with exponent 1/2 (Proposition 4.2).

4.1. Uniform spatial regularity.

We start with the uniform Lipschitz regularity in the spatial variables.

Proposition 4.1. *If $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of solutions of (E_ε), and $(A1)$-$(A4)$ hold, then there exists a constant $L>0$, independent of $\varepsilon\in(0,1)$, such that*

$$|\nabla u_\varepsilon(x,t)|\le L,\ \forall (x,t)\in\Omega_T.$$

Proof: Note that the regularity assumptions on F, f_ε and φ guarantee that solutions are locally of class C^3 ([20, Theorem 2]).

Now, since $\beta_\varepsilon=0$ in $\{u_\varepsilon\ge\varepsilon\}$, we conclude from up to the boundary parabolic regularity theory (see [18, Theorem 4.19] and [19, Theorem 2.5]) that

$$|\nabla u_\varepsilon|\le C(\|u_\varepsilon\|_\infty+\|f_\varepsilon\|_{n+1}+\|\varphi\|_\infty),$$

in this region, where C does not depend on ε. The result then follows from $(A3)$ and (3.4) with $L=L(\Upsilon,c_1,C)$.

To prove the uniform Lipschitz regularity in $\{u_\varepsilon\le\varepsilon\}$, it is enough to show that at the maximum point of

$$v_\varepsilon:=\frac{1}{2}|\nabla u_\varepsilon|^2+\frac{\Gamma}{2\varepsilon^2}u_\varepsilon^2,$$

where $\Gamma>0$ is a constant (independent of ε) to be chosen later, $|\nabla u_\varepsilon|$ can be controlled by a universal constant C, since then one can write

$$|\nabla u_\varepsilon|^2\le 2v_\varepsilon\le C^2+\Gamma=:L^2.$$

Let (x_0,t_0) be a maximum point of v_ε in $\{u_\varepsilon\le\varepsilon\}$. From the uniform gradient estimate in $\{u_\varepsilon\ge\varepsilon\}$, we may assume that it is an interior point. We drop the subscript ε in v_ε, u_ε and f_ε for convenience. Direct computation shows that

$$\begin{aligned}
D_iv &= \sum_k D_kuD_{ik}u+\Gamma\varepsilon^{-2}uD_iu,\\
D_{ij}v &= \sum_k (D_{kj}uD_{ki}u+D_kuD_{ijk}u)+\Gamma\varepsilon^{-2}(D_iuD_ju+uD_{ij}u),\\
\partial_tv &= \sum_k D_kuD_k\partial_tu+\Gamma\varepsilon^{-2}u\partial_tu,
\end{aligned}$$

where $D_ku=\partial u/\partial x_k$. Differentiating (E_ε) in the k-th direction one gets

$$\sum_{i,j}F_{ij}(x,t,D^2u)D_{ijk}u-D_k\partial_tu=\varepsilon^{-2}\beta'D_ku+D_kf, \tag{4.1}$$

where $F_{ij}(\cdot, M) := \partial F/\partial m_{ij}$, $M = (m_{ij})$. The uniform ellipticity of F implies that $A_{ij} := F_{ij}(x_0, t_0, D^2u(x_0, t_0))$ is a positive matrix, therefore at (x_0, t_0) we have

$$\begin{aligned}
0 \;&\geq\; \sum_{i,j} A_{ij} D_{ij} v - \partial_t v = \mathrm{tr}(D^2 u (A_{ij} D^2 u)) \\
&+\; \sum_k D_k u \left(\sum_{i,j} A_{ij} D_{ijk} u \right) + \Gamma \varepsilon^{-2} \sum_{i,j} A_{ij} D_i u D_j u \\
&+\; \Gamma \varepsilon^{-2} u \sum_{i,j} A_{ij} D_{ij} u - \sum_k D_k u D_k \partial_t u - \Gamma \varepsilon^{-2} u \partial_t u,
\end{aligned}$$

which, together with (4.1), provides

$$\begin{aligned}
0 \;&\geq\; \mathrm{tr}(D^2 u (A_{ij} D^2 u)) + \sum_k D_k u \left(\sum_{i,j} A_{ij} D_{ijk} u \right) \\
&+\; \Gamma \varepsilon^{-2} \sum_{i,j} A_{ij} D_i u D_j u + \Gamma \varepsilon^{-2} u \sum_{i,j} A_{ij} D_{ij} u \\
&-\; \sum_k D_k u D_k \partial_t u - \Gamma \varepsilon^{-2} u \partial_t u \\
&\geq\; \sum_k D_k u \left(D_k \partial_t u + \varepsilon^{-2} \beta' D_k u + D_k f \right) + \Gamma \varepsilon^{-2} \lambda |\nabla u|^2 \\
&-\; \sum_k D_k u D_k \partial_t u + \Gamma \varepsilon^{-2} u \left(\sum_{i,j} A_{ij} D_{ij} u - \partial_t u \right) \\
&\geq\; \varepsilon^{-2} \beta' |\nabla u|^2 + \sum_k D_k u D_k f + \Gamma \varepsilon^{-2} \lambda |\nabla u|^2 - \lambda |u| \varepsilon^{-2} \varepsilon^{-1} \beta \\
&=\; \varepsilon^{-2} \left(\beta' |\nabla u|^2 - \varepsilon^2 |\nabla u| |\nabla f| + \Gamma \lambda |\nabla u|^2 - \Gamma \varepsilon^{-1} \lambda |u| \beta \right).
\end{aligned}$$

Therefore,

$$(\beta'(u/\varepsilon) + \Gamma\lambda)|\nabla u|^2 - \varepsilon^2 |\nabla u||\nabla f| \;\leq\; \Gamma \lambda |u| \varepsilon^{-1} \beta(u/\varepsilon). \tag{4.2}$$

By choosing $\Gamma := \frac{2}{\lambda} \max|\beta'|$, from (4.2) we get

$$C_1 |\nabla u|^2 - C_2 \varepsilon^2 |\nabla u| \leq C_1 |u| \varepsilon^{-1} \beta(u/\varepsilon) \leq C_1 C_3,$$

with $C_1 = \max|\beta'|$, $C_2 = \|\nabla f\|_\infty$ and $C_3 = \max|\beta|$, which leads to

$$|\nabla u(x_0, t_0)| \leq C,$$

where C depends only on dimension, ellipticity, $\|\beta\|_{C^1}$ and $\|\nabla f\|_\infty$, thus being independent of ε. ∎

As an immediate consequence we have the following result.

Corollary 4.1. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of (E_ε). Let $K \subset \Omega_T$ be a compact set and $\tau > 0$ be such that $\mathcal{N}_\tau^-(K) \subset \Omega_T$. If (A1)-(A4) hold, then there exists a constant $L = L(\tau)$ such that*

$$|\nabla u_\varepsilon(x,t)| \leq L, \quad \forall (x,t) \in K.$$

Proof: For $(x_0,t_0) \in K$, consider the function

$$w_{\varepsilon,r}(x,t) := \frac{1}{r} u_\varepsilon(x_0 + rx, t_0 + r^2 t).$$

For $r \in (0,\tau)$ we have that $w_{\varepsilon,r}$ is a solution of

$$F_r(x,t,D^2 w_{\varepsilon,r}) - \partial_t w_{\varepsilon,r} = \beta_{\varepsilon/r}(w_{\varepsilon,r}) + r f_\varepsilon =: g_\varepsilon(x,t)$$

in $B_1 \times (-1,0)$, where $F_r(x,t,M) \colon = rF\left(x_0 + rx, t_0 + r^2 t, \frac{1}{r}M\right)$. The result now follows from Proposition 4.1. ∎

4.2. Uniform regularity in time. Next, as was mentioned above, using the uniform Lipschitz continuity in the space variables, we obtain the uniform Hölder continuity in time. First, we need the following lemma.

Lemma 4.1. *Let $u \in C(\overline{B}_1(0) \times [0, 1/(4n + M_0)])$ be such that*

$$|F(x,t,D^2u) - \partial_t u| \leq M_0 \quad \textit{in} \quad \{u > 1\},$$

for some $M_0 > 0$, and $|\nabla u| \leq L$, for some $L > 0$. Then there exists a constant $C = C(L)$ such that

$$|u(0,t) - u(0,0)| \leq C, \quad \textit{if} \quad 0 \leq t \leq \frac{1}{4n + M_0}.$$

Proof: Without loss of generality we may assume that $L > 1$. We divide the proof into two steps.

Step 1. First we claim that, if

$$Q_{t_0,t_1} := B_1(0) \times (t_0,t_1) \subset \{u > 1\} \quad \text{for} \quad t_1 - t_0 \leq \frac{1}{4n + M_0},$$

then

$$|u(0,t_1) - u(0,t_0)| \leq 2L.$$

In fact, let

$$h^{\pm}(x,t) := u(0,t_0) \pm L \pm \frac{2L}{\Lambda}|x|^2 \pm (4nL + M_0)(t - t_0).$$

By (2.2) one has

$$\begin{aligned} \partial_t h^+ - F(x,t,D^2h^+) &\geq \partial_t h^+ - \mathcal{P}^+_{\frac{\lambda}{n},\Lambda}(D^2h^+) \\ &= \partial_t h^+ - \left(\Lambda \sum_{e_i>0} e_i + \frac{\lambda}{n} \sum_{e_i<0} e_i\right) \\ &= (4nL + M_0) - \Lambda \frac{4Ln}{\Lambda} = M_0, \end{aligned}$$

and

$$\begin{aligned} \partial_t h^- - F(x,t,D^2h^-) &\leq \partial_t h^- - \mathcal{P}^-_{\frac{\lambda}{n},\Lambda}(D^2h^-) \\ &= \partial_t h^- - \left(\frac{\lambda}{n} \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i\right) \\ &= -(4nL + M_0) - \Lambda \frac{-4Ln}{\Lambda} = -M_0. \end{aligned}$$

Set

$$t_2 := \sup_{t_0 \leq \bar{t} \leq t_1} \{\bar{t} : |u(0,t) - u(0,t_0)| \leq 2L, \ \forall\, t_0 \leq t \leq \bar{t}\}.$$

So $t_0 < t_2 \leq t_1$ is such that

$$|u(0,t) - u(0,t_0)| \leq 2L, \quad \text{for} \quad t \in [t_0, t_2).$$

Moreover, from the Lipschitz continuity in space, one has

$$h^- \leq u \leq h^+ \quad \text{on} \quad \partial_p Q_{t_0,t_2}.$$

On the other hand,

$$\begin{aligned} \partial_t h^- - F(x,t,D^2h^-) &\leq -M_0 \leq \partial_t u - F(x,t,D^2u) \\ &\leq M_0 \leq \partial_t h^+ - F(x,t,D^2h^+). \end{aligned}$$

Therefore,

$$h^- \leq u \leq h^+ \quad \text{in} \quad Q_{t_0,t_2}.$$

In particular, since $t_2 - t_0 \leq t_1 - t_0 \leq \frac{1}{4n+M_0}$ and $L > 1$ one has

$$|u(0,t_2) - u(0,t_0)| < 2L.$$

Because of the strict inequality above, we may take $t_2 = t_1$ and therefore the claim is proved.

Step 2. Let us consider now the cylinder $Q_{0,t}$ with $0 < t \leq \frac{1}{4n+M_0}$.

If $Q_{0,t} \subset \{u > 1\}$, we apply Step 1 to get

$$|u(0,t) - u(0,0)| \leq 2L.$$

If $Q_{0,t} \not\subseteq \{u > 1\}$, let $0 \leq t_1 \leq t_2 \leq t$ and $x_1, x_2 \in \overline{B}_1(0)$ be such that

$$0 \leq u(x_1, t_1) \leq 1, \quad 0 \leq u(x_2, t_2) \leq 1$$

and

$$(\overline{B}_1(0) \times (0, t_1)) \cup (\overline{B}_1(0) \times (t_2, t)) \subset \{u > 1\}.$$

Then, Step 1 and the Lipschitz continuity in space provide

$$\begin{aligned} |u(0,t) - u(0,0)| &\leq |u(0,t) - u(0,t_2)| + |u(0,t_2) - u(x_2,t_2)| + |u(x_2,t_2)| \\ &+ |u(x_1,t_1)| + |u(x_1,t_1) - u(0,t_1)| + |u(0,t_1) - u(0,0)| \\ &\leq 2(2L + L + 1), \end{aligned}$$

which completes the proof. ■

We are now ready to prove uniform Hölder continuity of solutions in time.

Proposition 4.2. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of (E_ε). Let $K \subset \Omega_T$ be compact and $\tau > 0$ be such that $\mathcal{N}_{2\tau}(K) \subset \Omega_T$. If (A1)-(A4) hold, then there exists a constant $C > 0$, independent of ε, such that*

$$|u_\varepsilon(x, t+\Delta t) - u_\varepsilon(x,t)| \leq C|\Delta t|^{1/2}, \quad \textit{for } (x,t), (x, t+\Delta t) \in K.$$

Proof: Let $r \in (0, \tau)$, $(x_0, t_0) \in K$ and $w_{\varepsilon,r}(x,t)$, $g_\varepsilon(x,t)$ be as in the proof of Corollary 4.1. From $(A2)$ and $(A3)$ we get, in the set $\{w_{\varepsilon,r} > 1\}$,

$$0 \leq g_\varepsilon(x,t) \leq (1 + rc_1) \leq (1 + \tau c_1) =: C_\star.$$

Also $|\nabla w_{\varepsilon,r}(x,t)| \leq L$. Therefore, we may apply Lemma 4.1, with $M_0 = C_\star$, to obtain

$$|w_{\varepsilon,r}(0,t) - w_{\varepsilon,r}(0,0)| \leq C, \quad \text{for} \quad 0 \leq t \leq \frac{1}{4n + C_\star},$$

or in other terms

$$|u_\varepsilon(x_0, t_0 + r^2 t) - u_\varepsilon(x_0, t_0)| \leq Cr, \quad \text{for } 0 \leq t \leq \frac{1}{4n + C_\star}.$$

In particular, for $r \in (0, \tau)$, one has

$$\left| u_\varepsilon\left(x_0, t_0 + \frac{r^2}{4n + C_\star}\right) - u_\varepsilon(x_0, t_0)\right| \leq Cr. \tag{4.3}$$

Now if $(x_0, t_0 + \Delta t) \in K$ and $0 < \Delta t < \frac{r^2}{4n+C_\star}$, taking $r = \Delta t^{1/2}\sqrt{4n + C_\star}$ in (4.3) leads to

$$|u_\varepsilon(x_0, t_0 + \Delta t) - u_\varepsilon(x_0, t_0)| \leq C\sqrt{4n + C_\star}\Delta t^{1/2}.$$

On the order hand, if $\Delta t \geq \frac{r^2}{4n+C_\star}$, from (3.4) we get

$$|u_\varepsilon(x_0, t_0 + \Delta t) - u_\varepsilon(x_0, t_0)| \leq 2\Upsilon \leq \frac{2\Upsilon}{\tau}\sqrt{4n + C_\star}\Delta t^{1/2},$$

which completes the proof. ∎

5. The limiting free boundary problem

We start this section by letting $\varepsilon \to 0$ in (E_ε). Recalling Theorem 4.1, we know that up to a subsequence, there exists a limiting function u, obtained as the uniform limit of u_ε as $\varepsilon \to 0$. We now show that u is a viscosity solution of (1.2), where f is the uniform limit of f_ε.

Theorem 5.1. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of solution of (E_ε). If (A1)-(A4) hold then, up to a subsequence,*

(1) *$u_\varepsilon \to u$ locally uniformly in Ω_T and $u \in Lip_{\text{loc}}(1, 1/2)(\Omega_T)$;*
(2) *u is a solution of (1.2), where f is the uniform limit of f_ε;*
(3) *the function $t \mapsto u(x, t)$ is non-decreasing in time.*

Proof: Parts (1) and (2) follow from Theorem 4.1 and the Arzelà–Ascoli Theorem. In fact, since $u_\varepsilon \in \text{Lip}_{\text{loc}}(1, 1/2)(\Omega_T)$, with a uniform estimate, we can pass to the limit (up to a subsequence) and obtain a function

$$u(x, t) = \lim_{\varepsilon \to 0} u_\varepsilon(x, t),$$

with the convergence being uniform on compact subsets of $\overline{\Omega_T}$. Hence, $u \in \text{Lip}_{\text{loc}}(1, 1/2)(\Omega_T)$. Moreover, u is a viscosity solution of (1.2). Indeed, if $u(x_0, t_0) = c > 0$, then using the uniform convergence $u_\varepsilon \to u$ and the equicontinuity of u_ε, we conclude that for every small ε one has, in a small neighbourhood of (x_0, t_0), that $u_\varepsilon \geq \frac{c}{2} > \varepsilon$. So $\beta_\varepsilon(u_\varepsilon) = 0$. Since $f_\varepsilon \to f$, invoking stability arguments ([10, 19]) and passing to the limit in (E_ε), we conclude that u is a solution of (1.2).

In order to check (3), we define, for $t > 0$ and $h > 0$,

$$u_h(\cdot, t) := u(\cdot, t+h); \quad f_h(\cdot, t) := f(\cdot, t+h); \quad \varphi_h(\cdot, t) := \varphi(\cdot, t+h)$$

and $F_h(\cdot, t, \cdot) := F(\cdot, t+h, \cdot)$. Set also $\varphi_h(x, 0) := \varphi(x, 0) = 0$. Since u is a solution of (1.2), then u_h is a solution of the same problem with $F = F_h$, $f = f_h$ and $\varphi = \varphi_h$. From $(A4)$ we know that φ is non-decreasing in t and $\varphi(x, 0) = 0$, therefore $u_h \geq u$ on $\partial_p \Omega_T$. Observe that $(A3)$ provides $f_h(x, t) \leq f(x, t)$. Since also $u \geq 0$, we can apply a comparison argument to verify that $u_h \geq u$ in Ω_T, so the function $t \mapsto u(x, t)$ is non-decreasing. ■

6. Porosity of the free boundary

In this section we establish the exact growth of the solution near the free boundary, from which we deduce the porosity of its time level sets.

Definition 6.1. *A set $E \subset \mathbb{R}^n$ is called porous with porosity $\delta > 0$, if there exists $R > 0$ such that*

$$\forall x \in E, \ \ \forall r \in (0, R), \ \ \exists y \in \mathbb{R}^n \ \text{ such that } \ B_{\delta r}(y) \subset B_r(x) \setminus E.$$

A porous set of porosity δ has Hausdorff dimension not exceeding $n - c\delta^n$, where $c = c(n) > 0$ is a constant depending only on n. In particular, a porous set has Lebesgue measure zero.

The following theorem is the main result of this section.

Theorem 6.1. *Let u be a solution of* (1.2)*. If (A1) holds and f satisfies (A3) then, for every compact set $K \subset \Omega_T$ and every $t_0 \in (0, T)$, the set*

$$\partial\{u > 0\} \cap K \cap \{t = t_0\}$$

is porous in $\mathbb{R}^n$, with porosity depending only on Υ and $dist(K, \partial_p \Omega_T)$.

To prove the theorem we need to prove some auxiliary results.

6.1. Non-degeneracy. We start by proving a non-degeneracy result. Let us remark that, without loss of generality, we may consider in what follows the domain $Q_1 = Q_1(0, 0)$ instead of $Q_1(z, s)$.

Lemma 6.1. *Let $u \in C(Q_1)$ be a solution of*

$$F(x, t, D^2 u) - \partial_t u = f \quad in \quad \{u > 0\},$$

with f satisfying the lower bound in (A3). Then for every $(z,s) \in \overline{\{u>0\}}$ and $r>0$ with $Q_r(z,s) \subset Q_1$ we have

$$\sup_{(x,t)\in\partial_p Q_r^-(z,s)} u(x,t) \geq \mu_0 r^2 + u(z,s),$$

where $\mu_0 = \min\left(\frac{c_0}{2}, \frac{c_0}{4n\Lambda}\right)$.

Proof: Suppose that $(z,s) \in \{u>0\}$, and, for small $\delta>0$, set

$$\omega_\delta(x,t) := u(x,t) - (1-\delta)u(z,s) \quad \text{and} \quad \psi(x,t) := \frac{c_0}{4n\Lambda}|x-z|^2 - \frac{c_0}{2}(t-s).$$

Since $D_{ij}\psi = \frac{c_0}{2n\Lambda}\delta_{ij}$ then, from (2.2), one has

$$\begin{aligned} F(x,t,D^2\psi) - \partial_t\psi &\leq \mathcal{P}^+_{\frac{\lambda}{n},\Lambda}(D^2\psi) - \partial_t\psi \\ &= \Lambda\sum_{e_i>0} e_i + \frac{\lambda}{n}\sum_{e_i<0} e_i + \frac{c_0}{2} \\ &= \Lambda\frac{nc_0}{2n\Lambda} + \frac{c_0}{2} = c_0 \\ &\leq f(x,t) = F(x,t,D^2u) - \partial_t u \\ &= F(x,t,D^2\omega_\delta) - \partial_t\omega_\delta. \end{aligned}$$

Moreover, $\omega_\delta \leq \psi$ on $\partial\{u>0\} \cap Q_r^-(z,s)$. Note that we can not have

$$\omega_\delta \leq \psi \quad \text{on} \quad \partial_p Q_r^-(z,s) \cap \{u>0\},$$

because otherwise we could apply the comparison principle to obtain

$$\omega_\delta \leq \psi \quad \text{in} \quad Q_r^-(z,s) \cap \{u>0\},$$

which contradicts the fact that $\omega_\delta(z,s) = \delta u(z,s) > 0 = \psi(z,s)$. Hence, for $(y,\tau) \in \partial_p Q_r^-(z,s)$ we must have

$$\omega_\delta(y,\tau) > \psi(y,\tau) = \mu_0 r^2.$$

Letting $\delta \to 0$ in the last inequality we conclude the proof. ■

6.2. A class of functions in the unit cylinder. Next, we establish the growth rate of the solution near the free boundary, which is known for p-parabolic variational problems (see [17]) but is new in the fully nonlinear framework. We start by introducing a class of functions.

Definition 6.2. *We say that a function* $u \in C(Q_1)$ *is in the class* Θ *if* $0 \le u \le 1$ *in* Q_1*,* $\|F(x,t,D^2u) - \partial_t u\|_\infty \le 1$ *in* Q_1*, in the viscosity sense and, moreover,* $\partial_t u \ge 0$ *and* $u(0,0) = 0$*.*

Note that the last two conditions make sense due to the regularity of u guaranteed by the first two ([18, 19]).

In order to proceed, we need to introduce some notation. Set

$$S(r,u,z,s) := \sup_{Q_r^-(z,s)} u.$$

For $u \in \Theta$, we define

$$H(u,z,s) := \left\{ j \in \mathbb{N} \cup \{0\} : S(2^{-j},u,z,s) \le M S(2^{-j-1},u,z,s) \right\},$$

where $M := 4\max(1, \frac{1}{\mu_0})$, with μ_0 as in Lemma 6.1. When (z,s) is the origin, we suppress the point dependence.

The following lemma is the main step towards the growth control of the solution near the free boundary.

Lemma 6.2. *If* $u \in \Theta$*, then there is a constant* $C_1 = C_1(n,c_1) > 0$ *such that*

$$S(2^{-j-1},u) \le C_1 2^{-2j}, \quad \forall j \in H(u).$$

Proof: First, note that $H(u) \neq \emptyset$ because $0 \in H(u)$. Indeed, using Lemma 6.1, we have

$$S(1,u) \le 1 = 4\left(\frac{1}{\mu_0}\right)\mu_0 2^{-2} \le 4\left(\frac{1}{\mu_0}\right) S(2^{-1},u) \le MS(2^{-1},u).$$

Next, suppose the conclusion of the lemma fails. Then, for every $k \in \mathbb{N}$, there is $u_k \in \Theta$ and $j_k \in H(u_k)$ such that

$$S(2^{-j_k-1},u_k) \ge k 2^{-2j_k}.$$

Define $v_k : Q_1 \to \mathbb{R}$ by

$$v_k(x,t) := \frac{u(2^{-j_k}x, 2^{-2j_k}t)}{S(2^{-j_k-1},u_k)}.$$

One easily verifies that

$$0 \le v_k \le 1 \quad \text{in } Q_1^-; \qquad \|F_k(x,t,D^2v_k) - \partial_t v_k\|_\infty \le \frac{c_1}{k};$$

$$\sup_{Q_{1/2}^-} v_k = 1; \qquad v_k(0,0) = 0; \qquad \partial_t v_k \ge 0 \quad \text{in } Q_1^-,$$

where

$$F_k(x,t,M) := \frac{2^{-2j_k}}{S(2^{-j_k-1},u_k)} F\left(2^{-j_k}x, (2^{-j_k})^2 t, \frac{S(2^{-j_k-1},u_k)}{2^{-2j_k}} M\right)$$

is a uniform (λ,Λ)-elliptic operator. Using compactness arguments (see [18, 19]), we infer that there is a subsequence of v_k converging locally uniformly in Q_1^- to a function v. Moreover,

$$\mathcal{F}(x,t,D^2 v) - \partial_t v = 0, \quad v(0,0) = 0, \quad v \geq 0, \quad \partial_t v \geq 0$$

in Q_1^- for some (λ,Λ)-elliptic operator $\mathcal{F}$. The strong maximum principle (see [13]) then implies that $v \equiv 0$, which contradicts the fact that

$$\sup_{Q_{1/2}^-} v = 1.$$

■

We are now ready to prove the growth control of the solution near the free boundary.

Lemma 6.3. *If $u \in \Theta$, then there is a constant $C_0 = C_0(n,L,c_1) > 0$ such that*

$$|u(x,t)| \leq C_0 (d(x,t))^2, \quad \forall\ (x,t) \in Q_{1/2},$$

where

$$d(x,t) := \begin{cases} \sup\{r : Q_r(x,t) \subset \{u > 0\}\}, & \text{if } (x,t) \in \{u > 0\} \\ 0, & \text{otherwise.} \end{cases}$$

Proof: It suffices to show that

$$S(2^{-j},u) \leq 4C_1 2^{-2j}, \ \ \forall j \in \mathbb{N}. \tag{6.1}$$

In fact, for a fixed $r \in (0,1)$, by choosing $j \in \mathbb{N}$ such that $2^{-j-1} \leq r \leq 2^{-j}$, one has

$$\sup_{Q_r^-(0,0)} u \leq \sup_{Q_{2^{-j}}^-} u \leq 4C_1 2^{-2j} = 16 C_1 2^{-2j-2} \leq 16 C_1 r^2. \tag{6.2}$$

In order to prove (6.1), let us take the first j for which it fails (if there is no such j, we are done). Then

$$S(2^{-(j-1)},u) \leq 4C_1 2^{-2(j-1)} < 4S(2^{-j},u) \leq M S(2^{-j},u),$$

so $j-1 \in H(u)$, and we can apply Lemma 6.2 to reach the contradiction

$$S(2^{-j},u) \leq C_1 2^{-2(j-1)} = 4C_1 2^{-2j}.$$

To obtain a similar estimate for u over the whole cylinder (and not only over its lower half) we use a barrier from above. Set

$$\omega(x,t) := A_1|x|^2 + A_2 t,$$

where $A_2 = 2\Lambda n A_1$ and $A_1 > 0$. Then in $Q_1^+ = B_1(0) \times (0,1)$ one gets from (2.2) that

$$\begin{aligned} F(x,t,D^2\omega) - \partial_t \omega &\leq \mathcal{P}^+_{\frac{\lambda}{n},\Lambda}(D^2\omega) - \partial_t\omega \\ &= \Lambda \sum_{e_i>0} e_i + \frac{\lambda}{n}\sum_{e_i<0} e_i - A_2 \\ &= 2n\Lambda A_1 - A_2 = 0 \leq F(x,t,D^2u) - \partial_t u. \end{aligned}$$

If A_1 is large enough, then $\omega \geq u$ on $\partial_p Q_1^+$, where for the estimate on $\{t=0\}$ we used $S(r,u) \leq 16C_1 r^2$ from (6.2). Hence, by the comparison principle one has $\omega \geq u$ in Q_1^+. Therefore

$$\sup_{Q_r(0,0)} u \leq C_0 r^2,$$

for a constant $C_0 > 0$. ■

6.3. Porosity of the free boundary in time levels. We close the paper by proving Theorem 6.1.

Proof: Without loss of generality, we assume that K is the closed unit cylinder $\overline{Q}_1$, and $\overline{Q}_2 \subset \Omega_T$. For $(x,t) \in \{u>0\} \cap \overline{Q}_1$, let $d(x,t)$ be as in Lemma 6.3 and take $(x_0,t_0) \in \partial\{u>0\} \cap \overline{Q}_1$ to be the point where the distance is attained. Define

$$v(y,s) := u(x_0+y, t_0+s), \quad \text{for } (y,s) \in Q_1.$$

We have

$$\|F(x,t,D^2v) - \partial_t v\|_\infty \leq c_1, \quad 0 \leq v \leq \Upsilon, \quad v(0,0) = 0,$$

hence, if $\kappa = \max\{1, c_1, \Upsilon\}$, then $(1/\kappa)v(y,s) \in \Theta$. Lemma 6.3 then provides

$$u(x,t) = v(x-x_0, t-t_0) \leq \kappa C_0 (d(x,t))^2. \tag{6.3}$$

Now if $(z,\tau) \in \partial\{u>0\} \cap \overline{Q}_1$, then for $r \in (0,1)$, using Lemma 6.1, and the fact that $\partial_t u \geq 0$ in Q_1, one concludes that there exists $x_1 \in \partial B_r(z)$, such that

$$u(x_1,\tau) \geq \mu_0 r^2.$$

Together with (6.3), we have

$$\mu_0 r^2 \leq u(x_1,\tau) \leq \kappa C_0 (d(x_1,\tau))^2,$$

which implies that

$$d(x_1,\tau) \geq \delta r, \quad \delta = \sqrt{\frac{\mu_0}{\kappa C_0}}$$

and hence

$$B_{\delta r}(x_1) \subset B_{d(x_1,\tau)}(x_1) \subset \{u>0\}.$$

Note that $\delta \leq 1$. We claim now that there is a ball

$$B_{\frac{\delta}{2}r}(y) \subset B_{\delta r}(x_1) \cap B_r(z) \subset B_r(z) \setminus \partial\{u>0\}, \tag{6.4}$$

which means that the set $\partial\{u>0\} \cap \{t=\tau\} \cap \overline{B}_1$ is porous with porosity constant $\delta/2$.

To check (6.4) we choose $y \in [z, x_1]$ such that $|y - x_1| = \delta r/2$. For each $\xi \in B_{\frac{\delta}{2}r}(y)$ one has

$$|\xi - x_1| \leq |\xi - y| + |y - x_1| \leq \frac{\delta}{2}r + \frac{\delta}{2}r = \delta r$$

and, since $x_1 \in \partial B_r(z)$, also

$$|\xi - z| \leq |\xi - y| + |z - x_1| - |y - x_1| \leq \frac{\delta}{2}r + r - \frac{\delta}{2}r = r,$$

and therefore (6.4) is true. ■

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