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NEARLY SASAKIAN AND NEARLY COSYMPLECTIC MANIFOLDS

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Abstract: We prove that every nearly Sasakian manifold of dimension greater than five is Sasakian. This provides a new criterion for an almost contact metric manifold to be Sasakian. Moreover, we classify nearly cosymplectic manifolds of dimension greater than five.

Keywords: nearly Sasakian, nearly cosymplectic, Sasakian, contact manifold.
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1. Introduction

One of the most successful attempts to relax the definition of a Kähler manifold is provided by the notion of a nearly Kähler manifold. Namely, nearly Kähler manifolds are defined as almost Hermitian manifolds (M, J, g) such that the covariant derivative of the almost complex structure with respect to the Levi-Civita connection is skew-symmetric, that is

$$(\nabla_X J)X = 0,$$

for every vector field X on M. Thus, only the symmetric part of ∇J vanishes, in contrast to the Kähler case where $\nabla J = 0$. Nearly Sasakian and nearly cosymplectic manifolds were defined in the same spirit starting from Sasakian and coKähler (sometimes also called cosymplectic) manifolds, respectively.

A smooth manifold M endowed with an almost contact metric structure (ϕ, ξ, η, g) is said to be nearly Sasakian if

$$(\nabla_X \phi)X = g(X, X)\xi - \eta(X)X, \tag{1}$$

for every vector field X on M. Similarly, the condition for M to be nearly cosymplectic is given by

$$(\nabla_X \phi)X = 0, \tag{2}$$

for every vector field X on M.

The notion of a nearly Sasakian manifold was introduced by Blair and his collaborators in [4], while nearly cosymplectic manifolds were studied by Blair and Showers in [1, 3]. In the subsequent literature on the topic, quite

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important were the papers of Olszak [14, 15] for nearly Sasakian manifolds and those of Endo [9, 10] on nearly cosymplectic manifolds. Later on, these two classes have played a role in the Chinea-Gonzalez's classification of almost contact metric manifolds ([8]). They also appeared in the study of harmonic almost contact structures (cf. [11], [16]). In [13], Loubeau and Vergara-Diaz proved that a nearly cosymplectic structure, once identified with a section of a twistor bundle, always defines a harmonic map.

Recently, a systematic study of nearly Sasakian and nearly cosymplectic manifolds was carried forward in [7]. In that paper, the authors proved that any nearly Sasakian manifold is a contact manifold. In the 5-dimensional case, they showed that any nearly Sasakian manifold admits a nearly hypo $SU(2)$-structure that can be deformed to give a Sasaki-Einstein structure. Moreover, they proved that any nearly Sasakian manifold of dimension 5 has an associated nearly cosymplectic structure, thereby showing the close relation between these two notions. For 5-dimensional nearly cosymplectic manifolds, they proved that any such manifold is Einstein with positive scalar curvature. It is also worth remarking that (1-parameter families of) examples of both nearly Sasakian and nearly cosymplectic structures are provided by every Sasaki-Einstein 5-dimensional manifold.

While Sasakian manifolds are characterized by the equality

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

the defining condition (1) of a nearly Sasakian manifold gives a constraint only on the symmetric part of $\nabla\phi$. In this paper we show that, surprisingly, in dimension higher than five, condition (1) is enough for the manifold to be Sasakian.

Concerning nearly cosymplectic manifolds, we prove that a nearly cosymplectic non coKähler manifold M of dimension $2n+1 > 5$ is locally isometric to one of the following Riemannian products:

$$\mathbb{R} \times N^{2n}, \qquad M^5 \times N^{2n-4},$$

where N^{2n} is a nearly Kähler non Kähler manifold, N^{2n-4} is a nearly Kähler manifold, and M^5 is a nearly cosymplectic non coKähler manifold.

2. Definitions and known results

An almost contact metric manifold is a differentiable manifold M of odd dimension $2n+1$, endowed with a structure (ϕ, ξ, η, g), given by a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for every vector fields X, Y on M (see [2, 5] for further details). From the definition it follows that $\phi\xi = 0$ and $\eta \circ \phi = 0$. Moreover ϕ is skew-symmetric with respect to g, so that the bilinear form $\Phi := g(-, \phi-)$ defines a 2-form on M, called *fundamental 2-form*. An almost contact metric manifold such that $d\eta = 2\Phi$ is called a *contact metric manifold*. In this case η is a *contact form*, i.e. $\eta \wedge (d\eta)^n \neq 0$ everywhere on M.

A *Sasakian manifold* is defined as a contact metric manifold such that the tensor field $N_\phi := [\phi, \phi] + d\eta \otimes \xi$ vanishes identically. It is well known that an almost contact metric manifold is Sasakian if and only if the Levi-Civita connection satisfies:

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{3}$$

A *nearly Sasakian manifold* is an almost contact metric manifold (M, ϕ, ξ, η, g) such that

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X \tag{4}$$

for every vector fields X, Y on M, or, equivalently, (1) is satisfied.

We recall some basic facts about nearly Sasakian manifolds. We refer to [4, 14, 15, 7] for the details.

In any nearly Sasakian manifold (M, ϕ, ξ, η, g), the characteristic vector field ξ is Killing and the Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$. One can define a tensor field h of type $(1,1)$ by putting

$$\nabla_X \xi = -\phi X + hX. \tag{5}$$

The operator h is skew-symmetric and anticommutes with ϕ. It satisfies $h\xi = 0$, $\eta \circ h = 0$ and

$$\nabla_\xi h = \nabla_\xi \phi = \phi h = \frac{1}{3}\mathcal{L}_\xi \phi,$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. The vanishing of h provides a necessary and sufficient condition for a nearly Sasakian manifold to be Sasakian ([15]). In [14] the following formulas are proved:

$$g((\nabla_X \phi)Y, hZ) = \eta(Y)g(h^2X, \phi Z) - \eta(X)g(h^2Y, \phi Z) + \eta(Y)g(hX, Z), \tag{6}$$

$$(\nabla_X h^2)Y = \eta(Y)(\phi - h)h^2X + g((\phi - h)h^2X, Y)\xi, \tag{7}$$

$$R(\xi, X)Y = (\nabla_X \phi)Y - (\nabla_X h)Y = g(X - h^2X, Y)\xi - \eta(Y)(X - h^2X), \quad (8)$$

where R is the Riemannian curvature of g.

A central role in the study of nearly Sasakian geometry is played by the symmetric operator h^2. We recall the fundamental result due to Olszak [14]:

Theorem 2.1. *If a nearly Sasakian non Sasakian manifold (M, ϕ, ξ, η, g) satisfies the condition*

$$h^2 = \lambda(I - \eta \otimes \xi)$$

for some real number λ, then $\dim(M) = 5$.

In [15] Olszak also proved that any 5-dimensional nearly Sasakian non Sasakian manifold is Einstein with scalar curvature > 20. In [7] it is proved that the eigenvalues of h^2 are constant. Being h skew-symmetric, the non-vanishing eigenvalues of h^2 are negative, so that the spectrum of h^2 is of type

$$\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_r^2\},$$

$\lambda_i \neq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Further, if X is an eigenvector of h^2 with eigenvalue $-\lambda_i^2$, then X, ϕX, hX, $h\phi X$ are orthogonal eigenvectors of h^2 with eigenvalue $-\lambda_i^2$. Hence the minimum dimension for a nearly Sasakian non Sasakian manifold is 5. In the following we denote by $[\xi]$ the 1-dimensional distribution generated by ξ, and by $\mathcal{D}(0)$ and $\mathcal{D}(-\lambda_i^2)$ the distributions of the eigenvectors 0 and $-\lambda_i^2$ respectively. We shall also denote by $\overline{\mathcal{D}}$ the distribution $[\xi] \oplus \mathcal{D}(-\lambda_1^2) \oplus \cdots \oplus \mathcal{D}(-\lambda_r^2)$, and by $\mathcal{D}_0$ the distribution orthogonal to $\overline{\mathcal{D}}$, so that $\mathcal{D}(0) = [\xi] \oplus \mathcal{D}_0$.

In [7] the following results are proved, concerning nearly Sasakian manifolds of dimension ≥ 5.

Theorem 2.2. *Let M be a nearly Sasakian manifold with structure (ϕ, ξ, η, g) and let* $\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_r^2\}$ *be the spectrum of h^2. Then the distributions $\mathcal{D}(0)$ and $[\xi] \oplus \mathcal{D}(-\lambda_i^2)$ are integrable with totally geodesic leaves. In particular,*

a) *the eigenvalue 0 has multiplicity $2p + 1$, $p \geq 0$. If $p > 0$, the leaves of $\mathcal{D}(0)$ are $(2p + 1)$-dimensional Sasakian manifolds;*
b) *each negative eigenvalue $-\lambda_i^2$ has multiplicity 4 and the leaves of the distribution $[\xi] \oplus \mathcal{D}(-\lambda_i^2)$ are 5-dimensional nearly Sasakian (non Sasakian) manifolds.*
c) *If $p > 0$, the distribution $\overline{\mathcal{D}} = [\xi] \oplus \mathcal{D}(-\lambda_1^2) \oplus \cdots \oplus \mathcal{D}(-\lambda_r^2)$ is integrable and defines a transversely Kähler foliation with totally geodesic leaves.*

Theorem 2.3. *For a nearly Sasakian manifold* (M, ϕ, ξ, η, g) *of dimension* $2n+1 \geq 5$ *the* 1*-form* η *is a contact form.*

Before listing some known results on nearly cosymplectic manifolds, we recall that an almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a *coKähler manifold* if $d\eta = 0$, $d\Phi = 0$ and $N_\phi \equiv 0$. Equivalently, one can require $\nabla\phi = 0$. It is known that a coKähler manifold is locally the Riemannian product of the real line and a Kähler manifold, which is an integral submanifold of the distribution $\mathcal{D} = \mathrm{Ker}(\eta)$. Note that some authors call cosymplectic the class of manifold that we denominate coKähler (see [6] for details).

A *nearly cosymplectic manifold* is an almost contact metric manifold (M, ϕ, ξ, η, g) such that

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0 \tag{9}$$

for every vector fields X, Y. Clearly, this condition is equivalent to (2). It is known that in a nearly cosymplectic manifold the Reeb vector field ξ is Killing and satisfies $\nabla_\xi\xi = 0$ and $\nabla_\xi\eta = 0$. The tensor field h of type $(1,1)$ defined by

$$\nabla_X\xi = hX \tag{10}$$

is skew-symmetric and anticommutes with ϕ. It satisfies $h\xi = 0$, $\eta \circ h = 0$ and

$$\nabla_\xi\phi = \phi h = \frac{1}{3}\mathcal{L}_\xi\phi.$$

The following formulas hold ([9, 10]):

$$g((\nabla_X\phi)Y, hZ) = \eta(Y)g(h^2X, \phi Z) - \eta(X)g(h^2Y, \phi Z), \tag{11}$$

$$(\nabla_X h)Y = g(h^2X, Y)\xi - \eta(Y)h^2X, \tag{12}$$

$$\mathrm{tr}(h^2) = \text{constant}. \tag{13}$$

3. Nearly Sasakian manifolds

Proposition 3.1. *Let* (M, ϕ, ξ, η, g) *be a nearly Sasakian manifold of dimension* $2n+1 \geq 5$. *Then for all vector fields* X, Y *on* M *one has*

$$(\nabla_X\phi)Y = \eta(X)\phi hY - \eta(Y)(X + \phi hX) + g(X + \phi hX, Y)\xi, \tag{14}$$

$$(\nabla_X h)Y = \eta(X)\phi hY - \eta(Y)(h^2X + \phi hX) + g(h^2X + \phi hX, Y)\xi, \tag{15}$$

$$(\nabla_X\phi h)Y = g(\phi h^2X - hX, Y)\xi + \eta(X)(\phi h^2Y - hY) - \eta(Y)(\phi h^2X - hX). \tag{16}$$

Proof: From (6), for every vector fields X, Y, Z we have

$$g((\nabla_X \phi)Y, hZ) = -\eta(Y)g(\phi hX, hZ) + \eta(X)g(\phi hY, hZ) - \eta(Y)g(X, hZ),$$

which is coherent with (14). On the other hand,

$$\begin{aligned} g((\nabla_X \phi)Y, \xi) &= -g(Y, (\nabla_X \phi)\xi) = g(Y, \phi\nabla_X \xi) = g(Y, -\phi^2 X + \phi hX) \\ &= g(X + \phi hX, Y) - \eta(X)\eta(Y). \end{aligned}$$

Now, assume that $\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \dots, -\lambda_r^2\}$ and consider the distribution $\overline{\mathcal{D}} = [\xi] \oplus \mathcal{D}(-\lambda_1^2) \oplus \cdots \oplus \mathcal{D}(-\lambda_r^2)$. In order to complete the proof of (14), it remains to show that

$$g((\nabla_X \phi)Y, V) = -\eta(Y)g(X, V) \tag{17}$$

for every $X, Y \in \mathfrak{X}(M)$ and $V \in \mathcal{D}_0$. Since the distribution $\overline{\mathcal{D}}$ is integrable with totally geodesic leaves, if $X, Y \in \overline{\mathcal{D}}$ then $(\nabla_X \phi)Y \in \overline{\mathcal{D}}$ and both sides in (17) vanish. Now consider $X \in \mathcal{D}_0$ and $Y \in \overline{\mathcal{D}}$. Then

$$g((\nabla_X \phi)Y, V) = -g(Y, (\nabla_X \phi)V) = -\eta(Y)g(X, V),$$

where we applied the fact that the distribution $\mathcal{D}(0) = [\xi] \oplus \mathcal{D}_0$ is integrable with totally geodesic leaves, and the induced almost contact metric structure on each leaf is Sasakian, so that $(\nabla_X \phi)V = g(X, V)\xi - \eta(V)X$. On the other hand, if we take $X \in \overline{\mathcal{D}}$ and $Y \in \mathcal{D}_0$, then $g((\nabla_Y \phi)X, V) = -\eta(X)g(Y, V)$, and applying (4), we have

$$g((\nabla_X \phi)Y, V) = -g((\nabla_Y \phi)X + \eta(X)Y, V) = 0,$$

which is again coherent with (17). Finally, taking $X, Y \in \mathcal{D}_0$, (17) is verified because of (3) and the fact that the vector fields X, Y, V are orthogonal to ξ.

As regards (15), it follows from (8) and (14). Finally, a straightforward computation using (14) and (15) gives (16). ■

Given a k-form σ on a manifold M, we denote by ϵ_σ the operator on $\Omega^*(M)$ defined by $\epsilon_\sigma \theta = \sigma \wedge \theta$, for every $\theta \in \Omega^l(M)$. Given two linear operators A, B on $\Omega^*(M)$ of degrees a, b, respectively, we denote their graded commutator by

$$[A, B] := AB - (-1)^{ab} BA.$$

Let (M, η) be a contact manifold with Reeb vector field ξ, so that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. By Darboux theorem, around each point $p \in M$ there exists a local basis of vector fields $(\xi, X_1, ..., X_{2n})$ and a dual basis of 1-forms $(\eta, \alpha_1, ..., \alpha_{2n})$, such that

$$d\eta = \sum_{l=1}^{n} \alpha_{2l-1} \wedge \alpha_{2l}.$$

We introduce the operator $\Lambda : \Omega^p(M) \to \Omega^{p-2}(M)$ locally defined by

$$\omega \mapsto \sum_{k=1}^{n} i_{X_{2k-1}} i_{X_{2k}} \omega.$$

One can prove that Λ is well defined.

Lemma 3.2. *Let (M, η) be a contact manifold of dimension $2n + 1$. Then, the operator Λ satisfies the following identity for any p-form ω*

$$[\Lambda, \epsilon_{d\eta}]\omega = (p - n)\omega - \epsilon_\eta i_\xi \omega.$$

Proof: Recall that, in general, for a vector field X and a k-form σ on manifold M, one has $[i_X, \epsilon_\sigma] = \epsilon_{i_X\sigma}$. Hence one gets

$$[i_X i_Y, \epsilon_\sigma] = (-1)^k \epsilon_{i_X\sigma} i_Y + i_X \epsilon_{i_Y\sigma},$$

for any vector fields X, Y and a k-form σ. Thus,

$$\begin{aligned}[i_{X_{2k-1}} i_{X_{2k}}, \epsilon_{\alpha_{2l-1}\wedge\alpha_{2l}}] &= \delta_{kl}(\epsilon_{\alpha_{2k}} i_{X_{2k}} - i_{X_{2k-1}} \epsilon_{\alpha_{2k-1}}) \\ &= \delta_{kl}(\epsilon_{\alpha_{2k}} i_{X_{2k}} + \epsilon_{\alpha_{2k-1}} i_{X_{2k-1}} - \mathrm{Id}).\end{aligned}$$

We obtain

$$\left[\sum_{k=1}^{n} i_{X_{2k-1}} i_{X_{2k}}, \sum_{l=1}^{n} \epsilon_{\alpha_{2l-1}\wedge\alpha_{2l}}\right] \omega = \sum_{h=1}^{2n} \epsilon_{\alpha_h} i_{X_h} \omega - n\omega = (p - n)\omega - \epsilon_\eta i_\xi \omega.$$

This completes the proof. ∎

Proposition 3.3. *Let (M, η) be a contact manifold of dimension $2n + 1$. Then, the operator*

$$\begin{aligned}\epsilon_{d\eta} : \Omega^2(M) &\to \Omega^4(M) \\ \beta &\mapsto d\eta \wedge \beta\end{aligned}$$

is injective for $n \geq 3$.

Proof: From Lemma 3.2 we have for $\beta \in \Omega^2(M)$

$$\Lambda\epsilon_{d\eta}\beta = \epsilon_{d\eta}(\Lambda\beta) + (2-n)\beta - \epsilon_\eta i_\xi \beta. \tag{18}$$

By applying $\epsilon_\eta i_\xi$ we obtain

$$\epsilon_\eta i_\xi \Lambda \epsilon_{d\eta}\beta = (1-n)\epsilon_\eta i_\xi \beta. \tag{19}$$

Moreover, by applying Λ to (18) we obtain

$$\Lambda^2\epsilon_{d\eta}\beta = (\Lambda\beta)(\Lambda d\eta) + (2-n)\Lambda\beta - \Lambda\epsilon_\eta i_\xi\beta.$$

If we keep in account that $\Lambda\epsilon_\eta i_\xi\beta = 0$, and that $\Lambda d\eta = -n$ (e. g. by using Lemma 3.2 applied to the constant function 1) we get

$$\Lambda^2\epsilon_{d\eta}\beta = 2(1-n)\Lambda\beta.$$

Then, by applying $\epsilon_{d\eta}$ we obtain

$$\epsilon_{d\eta}\Lambda^2\epsilon_{d\eta}\beta = 2(1-n)\epsilon_{d\eta}\Lambda\beta. \tag{20}$$

Thus, from (18), (19) and (20) we have

$$\left(\Lambda + \frac{\epsilon_\eta i_\xi \Lambda}{1-n} - \frac{\epsilon_{d\eta}\Lambda^2}{2(1-n)}\right)\epsilon_{d\eta}\beta = (2-n)\beta.$$

Hence $\epsilon_{d\eta}$ has a left inverse for $n \geq 3$. ■

Now we are able to prove our main result.

Theorem 3.4. *Every nearly Sasakian manifold of dimension* $2n+1 > 5$ *is Sasakian.*

Proof: Let M be a nearly Sasakian manifold with structure (ϕ, ξ, η, g), of dimension $2n+1$. We consider the two forms H and Φ_k, $k = 1, 2$, defined by

$$H(X,Y) = g(hX,Y), \qquad \Phi_k(X,Y) = g(\phi h^k X, Y).$$

We shall prove that

$$dH = 3\eta \wedge \Phi_1, \tag{21}$$

$$d\Phi_1 = 3\eta \wedge (\Phi_2 - H). \tag{22}$$

From (15), we have that for every vector fields X, Y, Z,

$$\begin{aligned} g((\nabla_X h)Y, Z) &= \eta(X)g(\phi hY, Z) - \eta(Y)g(h^2X + \phi hX, Z) + \eta(Z)g(h^2X + \phi hX, Y) \\ &= \eta(X)g(\phi hY, Z) + \eta(Y)g(\phi hZ, X) + \eta(Z)g(\phi hX, Y) \\ &\quad - \eta(Y)g(h^2Z, X) + \eta(Z)g(h^2X, Y). \end{aligned}$$

Therefore,

$$\begin{aligned} dH(X,Y,Z) &= g((\nabla_X h)Y,Z) + g((\nabla_Y h)Z,X) + g((\nabla_Z h)X,Y) \\ &= 3\,(\eta(X)g(\phi hY,Z) + \eta(Y)g(\phi hZ,X) + \eta(Z)g(\phi hX,Y)) \\ &= 3\eta\wedge\Phi_1(X,Y,Z). \end{aligned}$$

Analogously, from (16), we have

$$\begin{aligned} g((\nabla_X \phi h)Y,Z) &= \eta(X)g(\phi h^2 Y - hY,Z) - \eta(Y)g(\phi h^2 X - hX,Z) \\ &\quad + \eta(Z)g(\phi h^2 X - hX,Y) \\ &= \eta(X)g(\phi h^2 Y,Z) + \eta(Y)g(\phi h^2 Z,X) + \eta(Z)g(\phi h^2 X,Y) \\ &\quad - \eta(X)g(hY,Z) - \eta(Y)g(hZ,X) - \eta(Z)g(hX,Y). \end{aligned}$$

Hence,

$$\begin{aligned} d\Phi_1(X,Y,Z) &= g((\nabla_X \phi h)Y,Z) + g((\nabla_Y \phi h)Z,X) + g((\nabla_Z \phi h)X,Y) \\ &= 3\,\big(\eta(X)g(\phi h^2 Y,Z) + \eta(Y)g(\phi h^2 Z,X) + \eta(Z)g(\phi h^2 X,Y)\big) \\ &\quad - 3\,(\eta(X)g(hY,Z) + \eta(Y)g(hZ,X) + \eta(Z)g(hX,Y)) \\ &= 3\,\eta\wedge\Phi_2(X,Y,Z) - 3\,\eta\wedge H(X,Y,Z). \end{aligned}$$

Now, from (21) and (22), we have

$$0 = d^2 H = 3\,d\eta\wedge\Phi_1 - 3\eta\wedge d\Phi_1 = 3\,d\eta\wedge\Phi_1.$$

If we assume that the dimension of M is $2n+1>5$, being η a contact form, the fact that $d\eta\wedge\Phi_1 = 0$ implies $\Phi_1 = 0$, by Proposition 3.3. Therefore $h=0$, and the structure is Sasakian. ■

4. Nearly cosymplectic manifolds

In this section we will classify nearly cosymplectic manifolds of dimension higher than five. In the following, given a nearly cosymplectic manifold (M,ϕ,ξ,η,g), we shall denote by h the operator defined in (10).

Proposition 4.1. *Let (M,ϕ,ξ,η,g) be a nearly cosymplectic manifold. Then $h=0$ if and only if M is locally isometric to the Riemannian product $\mathbb{R}\times N$, where N is a nearly Kähler manifold.*

Proof: For every vector fields X,Y we have

$$d\eta(X,Y) = g(\nabla_X\xi,Y) - g(\nabla_Y\xi,X) = 2g(hX,Y). \tag{23}$$

Therefore, if $h = 0$ the distribution $\mathcal{D} = \mathrm{Ker}(\eta)$ is integrable. Denoting by N an integral submanifold of $\mathcal{D}$, it is a totally geodesic hypersurface of M. Indeed, for every $X, Y \in \mathcal{D}$, we have $g(\nabla_X Y, \xi) = -g(Y, hX) = 0$. Being also $\nabla_\xi \xi = 0$, M turns out to be locally isometric to the Riemannian product $\mathbb{R} \times N$. Further, the almost contact metric structure induces on N an almost Hermitian structure which is nearly Kähler.

Conversely, if M is locally isometric to the Riemannian product $\mathbb{R} \times N$, where N is a nearly Kähler manifold, then $d\eta(X, Y) = 0$ for every vector fields X, Y orthogonal to ξ. By (23) and $h\xi = 0$, we deduce that $h = 0$. ■

As a consequence of the above proposition, a nearly cosymplectic manifold (M, ϕ, ξ, η, g) is coKähler if and only if $h = 0$ and the leaves of the distribution $\mathcal{D}$ are Kähler manifolds. Recall that 4-dimensional nearly Kähler manifolds are Kähler (see [12, Theorem 5.1]), and this implies that if M is a 5-dimensional nearly cosymplectic manifold with $h = 0$, then it is a coKähler manifold.

We shall now study the spectrum of the symmetric operator h^2.

Proposition 4.2. *The eigenvalues of the symmetric operator h^2 are constant.*

Proof: From (12) it follows that

$$(\nabla_X h^2)Y = g(X, h^3 Y)\xi - \eta(Y)h^3 X. \tag{24}$$

Let us consider an eigenvalue μ of h^2 and a local unit vector field Y, orthogonal to ξ, such that $h^2 Y = \mu Y$. Applying (24) for any vector field X, we have

$$\begin{aligned} 0 &= g((\nabla_X h^2)Y, Y) \\ &= g(\nabla_X (h^2 Y), Y) - g(h^2(\nabla_X Y), Y) \\ &= X(\mu)g(Y, Y) + \mu g(\nabla_X Y, Y) - g(\nabla_X Y, h^2 Y) \\ &= X(\mu)g(Y, Y) \end{aligned}$$

which implies that $X(\mu) = 0$. ■

Since h is skew-symmetric, the non-vanishing eigenvalues of h^2 are negative. Therefore, the spectrum of h^2 is of type

$$\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_r^2\},$$

where we can assume that each λ_i is a positive real number and $\lambda_i \neq \lambda_j$ for $i \neq j$. Notice that if X is an eigenvector of h^2 with eigenvalue $-\lambda_i^2$, then X, ϕX, hX, $h\phi X$ are orthogonal eigenvectors of h^2 with eigenvalue $-\lambda_i^2$. Since $h(\xi) = 0$, we get the eigenvalue 0 has multiplicity $2p+1$ for some integer $p \geq 0$.

We denote by $\mathcal{D}(0)$ the distribution of the eigenvectors with eigenvalue 0, and by $\mathcal{D}_0$ the distribution of the eigenvectors in $\mathcal{D}(0)$ orthogonal to ξ, so that $\mathcal{D}(0) = [\xi] \oplus \mathcal{D}_0$. Let $\mathcal{D}(-\lambda_i^2)$ be the distribution of the eigenvectors with eigenvalue $-\lambda_i^2$. We remark that the distributions $\mathcal{D}_0$ and $\mathcal{D}(-\lambda_i^2)$ are ϕ-invariant and h-invariant.

Proposition 4.3. *Let (M, ϕ, ξ, η, g) be a nearly cosymplectic manifold and let* $\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \dots, -\lambda_r^2\}$ *be the spectrum of h^2. Then,*

a) *for each $i = 1, \dots, r$, the distribution $[\xi] \oplus \mathcal{D}(-\lambda_i^2)$ is integrable with totally geodesic leaves.*

Assuming that the eigenvalue 0 *is not simple,*

b) *the distribution $\mathcal{D}_0$ is integrable with totally geodesic leaves, and each leaf of $\mathcal{D}_0$ is endowed with a nearly Kähler structure;*

c) *the distribution $[\xi] \oplus \mathcal{D}(-\lambda_1^2) \oplus \dots \oplus \mathcal{D}(-\lambda_r^2)$ is integrable with totally geodesic leaves.*

Proof: Consider an eigenvector X of h^2 with eigenvalue $-\lambda_i^2$. Then $\nabla_X \xi = hX \in \mathcal{D}(-\lambda_i^2)$. On the other hand, (24) implies that $\nabla_\xi h^2 = 0$, and thus $\nabla_\xi X$ is also an eigenvector with eigenvalue $-\lambda_i^2$. Now, taking $X, Y \in \mathcal{D}(-\lambda_i^2)$ and applying (24), we get

$$h^2(\nabla_X Y) = -\lambda_i^2 \nabla_X Y - (\nabla_X h^2) Y = -\lambda_i^2 \nabla_X Y + \lambda_i^2 g(X, hY)\xi.$$

Therefore,

$$h^2(\phi^2 \nabla_X Y) = \phi^2(h^2 \nabla_X Y) = -\lambda_i^2 \phi^2(\nabla_X Y).$$

Thus $\phi^2 \nabla_X Y \in \mathcal{D}(-\lambda_i^2)$. It follows that $\nabla_X Y = -\phi^2 \nabla_X Y + \eta(\nabla_X Y)\xi$ belongs to the distribution $[\xi] \oplus \mathcal{D}(-\lambda_i^2)$. This proves a).

As regards b), applying again (24), we have $(\nabla_X h^2) Y = 0$ for every $X, Y \in \mathcal{D}_0$, so that $h^2(\nabla_X Y) = 0$. Moreover,

$$g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, hX) = 0.$$

Hence, $\mathcal{D}_0$ is integrable with totally geodesic leaves. Since the leaves of $\mathcal{D}_0$ are ϕ-invariant, the nearly cosymplectic structure induces a nearly Kähler structure on each integral submanifold of $\mathcal{D}_0$.

Finally, in order to prove c), owing to a), we only have to show that

$$g(\nabla_X Y, Z) = 0$$

for every $X \in \mathcal{D}(-\lambda_i^2)$, $Y \in \mathcal{D}(-\lambda_j^2)$, $i \neq j$, and $Z \in \mathcal{D}_0$. In fact, from (24), we have

$$\begin{aligned} g(\nabla_X Y, Z) &= -\frac{1}{\lambda_j^2} g(\nabla_X (h^2 Y), Z) \\ &= -\frac{1}{\lambda_j^2} g((\nabla_X h^2)Y + h^2(\nabla_X Y), Z) \\ &= -\frac{1}{\lambda_j^2}\eta(Z) g(X, h^3 Y) - \frac{1}{\lambda_j^2} g(\nabla_X Y, h^2 Z) \end{aligned}$$

which vanishes since $\eta(Z) = 0$ and $h^2 Z = 0$. ■

Theorem 4.4. *Let (M, ϕ, ξ, η, g) be a nearly cosymplectic manifold such that 0 is a simple eigenvalue of h^2. Then M is a 5-dimensional manifold.*

Proof: First we show that

$$(\nabla_X \phi)Y = g(\phi h X, Y)\xi + \eta(X)\phi h Y - \eta(Y)\phi h X, \tag{25}$$

$$(\nabla_X \phi h)Y = g(\phi h^2 X, Y)\xi + \eta(X)\phi h^2 Y - \eta(Y)\phi h^2 X \tag{26}$$

for every vector fields X and Y. Applying (10) we have

$$g((\nabla_X \phi)Y, \xi) = -g(Y, (\nabla_X \phi)\xi) = g(Y, \phi \nabla_X \xi) = g(Y, \phi h X).$$

Taking a vector field U orthogonal to ξ, then $U = hZ$ for some vector field Z. Then, by applying (11) and recalling that ϕ anticommutes with h, we get

$$\begin{aligned} g((\nabla_X \phi)Y, U) &= \eta(Y) g(h^2 X, \phi Z) - \eta(X) g(h^2 Y, \phi Z) \\ &= \eta(Y) g(hX, \phi h Z) - \eta(X) g(hY, \phi h Z) \\ &= -\eta(Y) g(\phi h X, U) + \eta(X) g(\phi h Y, U) \end{aligned}$$

which completes the proof of (25). From (12) and (25) we easily get (26).

We consider now the 2-forms Φ_k, $k = 0, 1, 2$, defined by

$$\Phi_k(X, Y) = g(\phi h^k X, Y).$$

In particular, $\Phi_0 = -\Phi$. We prove that

$$d\Phi_0 = 3\eta \wedge \Phi_1, \qquad d\Phi_1 = 3\eta \wedge \Phi_2. \tag{27}$$

From (25), for every vector fields X, Y, Z we have

$$g((\nabla_X \phi)Y, Z) = \eta(X)g(\phi hY, Z) + \eta(Y)g(\phi hZ, X) + \eta(Z)g(\phi hX, Y),$$

which implies that $d\Phi_0 = 3\eta \wedge \Phi_1$. Analogously, from (26), we have

$$g((\nabla_X \phi h)Y, Z) = \eta(X)g(\phi h^2 Y, Z) + \eta(Y)g(\phi h^2 Z, X) + \eta(Z)g(\phi h^2 X, Y),$$

so that $d\Phi_1 = 3\eta \wedge \Phi_2$. From (27),

$$0 = d^2\Phi_0 = 3\, d\eta \wedge \Phi_1 - 3\eta \wedge d\Phi_1 = 3\, d\eta \wedge \Phi_1.$$

Next we show that if 0 is a simple eigenvalue, then η is a contact form. This, by an argument similar to the one in the proof of Theorem 3.4 will imply that $\dim M = 5$.

First we assume that $\mathrm{Spec}(h^2) = \{0, -\lambda^2\}$, with $\lambda > 0$, 0 being a simple eigenvalue. This is equivalent to require that

$$h^2 = -\lambda^2 (I - \eta \otimes \xi).$$

Let us take the tensor fields

$$\tilde{\phi} = -\frac{1}{\lambda} h, \quad \tilde{\xi} = \frac{1}{\lambda}\xi, \quad \tilde{\eta} = \lambda\eta, \quad \tilde{g} = \lambda^2 g.$$

One can verify that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric structure. Moreover, from (23) we have

$$d\tilde{\eta}(X, Y) = 2\lambda g(hX, Y) = \frac{2}{\lambda}\, \tilde{g}(hX, Y) = 2\, \tilde{g}(X, -\frac{1}{\lambda} hY) = 2\, \tilde{g}(X, \tilde{\phi} Y).$$

Therefore $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a contact metric structure. In particular, both the forms $\tilde{\eta}$ and η are contact forms. Hence, in this case M is a 5-dimensional manifold and the multiplicity of the eigenvalue $-\lambda^2$ is 4.

We assume now that

$$\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \ldots, -\lambda_r^2\},$$

where λ_i is a positive real number and $\lambda_i \neq \lambda_j$ for $i \neq j$. From Proposition 4.3, we know that for each $i = 1, \ldots, r$, the distribution $[\xi] \oplus \mathcal{D}(-\lambda_i^2)$ is integrable with totally geodesic leaves. Each integral submanifold of this distribution is endowed with an induced almost contact metric structure, here again denoted by (ϕ, ξ, η, g), whose structure tensor field h satisfies

$$h^2 = -\lambda_i^2 (I - \eta \otimes \xi).$$

We deduce that η is a contact form on the leaves of the distribution. In particular, each eigenvalue $-\lambda_i^2$ of h^2 has multiplicity 4.

Notice that, taking two distinct eigenvalues $-\lambda_i^2$ and $-\lambda_j^2$, for every $X \in \mathcal{D}(-\lambda_i^2)$ and $Y \in \mathcal{D}(-\lambda_j^2)$, we have

$$d\eta(X,Y) = 2g(hX,Y) = 0, \tag{28}$$

since the operator h preserves the distributions $\mathcal{D}(-\lambda_i^2)$ and $\mathcal{D}(-\lambda_j^2)$, which are mutually orthogonal.

Now, fix a point $x \in M$. Since η is a contact form on the leaves of each distribution $[\xi] \oplus \mathcal{D}(-\lambda_i^2)$, for any $i \in \{1,\dots,r\}$ one can find a basis $(v_1^i, v_2^i, v_3^i, v_4^i)$ of $\mathcal{D}_x(-\lambda_i^2)$ such that

$$\eta \wedge (d\eta)^2(\xi_x, v_1^i, v_2^i, v_3^i, v_4^i) \neq 0. \tag{29}$$

Therefore, putting $n = 2r$, the dimension of M is $2n+1$ and

$$\begin{aligned}\eta \wedge (d\eta)^n &\left(\xi_x, v_1^1, v_2^1, v_3^1, v_4^1, \dots, v_1^r, v_2^r, v_3^r, v_4^r\right) \\ &= \eta(\xi_x)(d\eta)^2(v_1^1, v_2^1, v_3^1, v_4^1) \dots (d\eta)^2(v_1^r, v_2^r, v_3^r, v_4^r) \neq 0.\end{aligned}$$

This proves that η is a contact form. ■

Theorem 4.5. *Let (M,ϕ,ξ,η,g) be a nearly cosymplectic non coKähler manifold of dimension $2n+1 > 5$. Then M is locally isometric to one of the following Riemannian products:*

$$\mathbb{R} \times N^{2n}, \qquad M^5 \times N^{2n-4},$$

where N^{2n} is a nearly Kähler non Kähler manifold, N^{2n-4} is a nearly Kähler manifold, and M^5 is a nearly cosymplectic non coKähler manifold.

Proof: If $h = 0$, then M is locally isometric to the Riemannian product $\mathbb{R} \times N^{2n}$, where N^{2n} is a nearly Kähler non Kähler manifold.

If $h \neq 0$, then h^2 admits non vanishing eigenvalues and we can assume $\mathrm{Spec}(h^2) = \{0, -\lambda_1^2, \dots, -\lambda_r^2\}$, where each λ_i is a positive real number. Since $\dim M > 5$, owing to Theorem 4.4, the eigenvalue 0 is not a simple eigenvalue. From b) and c) of Proposition 4.3, M is locally isometric to the Riemannian product $M' \times N$, where M' is an integral submanifold of the distribution $[\xi] \oplus \mathcal{D}(-\lambda_1^2) \oplus \dots \oplus \mathcal{D}(-\lambda_r^2)$, and N is an integral submanifold of $\mathcal{D}_0$, which is endowed with a nearly Kähler structure. Now, M' is endowed with an induced nearly cosymplectic structure for which 0 is a simple eigenvalue of the operator h^2. Therefore, by Theorem 4.4, we have that $\lambda_1 = \dots = \lambda_r$ and M' is a 5-dimensional nearly cosymplectic non coKähler manifold. Consequently, the dimension of N is $2n-4$. ■

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