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TOWARDS THE $C^{p'}$-REGULARITY CONJECTURE

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ABSTRACT: We establish a new oscillation estimate for solutions of nonlinear partial differential equations of degenerate elliptic type, which yields a precise control on the growth rate of solutions near their set of critical points. We then apply this new tool in the investigation of a longstanding conjecture which inquires whether solutions of the degenerate p-Poisson equation with a bounded source are locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$.

KEYWORDS: Nonlinear pdes, regularity theory, sharp estimates.
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1. Introduction

In this paper we investigate sharp $C^{1,\alpha}$-regularity estimates for solutions of the degenerate elliptic equation, with a bounded source,

$$-\Delta_p u = f(x) \in L^\infty(B_1), \qquad p > 2. \tag{1.1}$$

Establishing optimal regularity estimates is quite often a delicate matter and, in particular, $f(x) \in L^\infty$ is known to be a borderline condition for regularity. In the linear, uniformly elliptic case $p = 2$, solutions of

$$-\Delta u = f(x) \in L^\infty(B_1)$$

are locally in $C^{1,\alpha}$, for every $\alpha \in (0,1)$, but may fail to be in $C^{1,1}$. Obtaining such an estimate in specific situations, like free boundary problems, often involves a deep and fine analysis.

In the degenerate setting $p > 2$, the smoothing effects of the operator are far less efficient. Nonetheless, it is well established, see for instance [4, 19], that a weak solution to (1.1) is locally of class $C^{1,\beta}$, for some exponent $\beta > 0$

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depending, in principle, on dimension and on p. If $p\prime$ denotes the conjugate of p, i.e.,

$$\frac{1}{p} + \frac{1}{p\prime} = 1,$$

the radial symmetric example

$$-\Delta_p \left(c_p |x|^{p\prime}\right) = 1 \tag{1.2}$$

sets the limits to the optimal regularity and gives rise to the following well known open problem among experts in the field.

Conjecture ($C^{p\prime}$-regularity conjecture). *Solutions to* (1.1) *are locally of class* $C^{1,\frac{1}{p-1}} = C^{p\prime}$, *and*

$$\|u\|_{C^{p\prime}(B_{1/2})} \leq M_{p,d} \left\{ \|f\|_{L^\infty(B_1)}^{\frac{1}{p-1}} + \|u\|_{L^\infty(B_1)} \right\},$$

for a constant $M_{p,d} > 1$ *independent of* u.

This problem touches very subtle issues in regularity theory. As mentioned above, the conjecture is not true in the linear, uniformly elliptic setting, $p = 2$, where merely $C^{1,\mathrm{LogLip}}$ estimates are possible. Notice further that a positive answer implies that $|x|^{p\prime}$ — a function whose p-laplacian is constant (real analytic) — is the least regular among all functions whose p-laplacian is bounded. This is, at first sight, counterintuitive.

While a full answer to this question still seems out of reach, in this article we provide a new oscillation estimate (Theorem 4.3) which reveals some essential nuances of the puzzle. This novel tool, which has some other far-reaching applications, gives a precise control on the oscillation of a solution to (1.1) in terms of the magnitude of its gradient,

$$\sup_{B_r} |u(x) - u(0)| \lesssim r^{1+\gamma} + |\nabla u(0)| r, \tag{1.3}$$

for a maximum exponent γ, to be better explained when time comes. Such an estimate allows us to bypass one of the key difficulties in the analysis of the $C^{p\prime}$-regularity conjecture, namely the fact that if u solves (1.1) and ℓ is an affine function, then no PDE is a priori satisfied by $(u - \ell)$. By means of geometric iteration, estimate (1.3) yields an improved $C^{1,\alpha}$-regularity for solutions to the p-Poisson equation (1.1) which is intimately related to the $C^{p\prime}$-regularity conjecture.

The general analysis we develop in this article confirms the conjecture in a number of meaningful cases. In particular, we prove the conjecture is true for the class of radially symmetric solutions of the p-Poisson equation. We also show that if u is a solution of (1.1) with no saddle critical points, then u is locally $C^{p'}$-smooth.

The paper is organized as follows. In section 2 we describe the mathematical setup used in the paper and announce the main regularity theorem to be proven, Theorem 2.2. In section 3 we introduce C^1- small correctors that link the regularity theory for (1.1) to that of p-harmonic functions. The key, new oscillation estimate is delivered in section 4, and in section 5 we conclude the proof of the main theorem. Applications of the general analysis are discussed in section 6.

2. Mathematical setup

As usual, hereafter in this paper, $d \geq 1$ denotes the dimension of the Euclidian space $\mathbb{R}^d$. Given a real number $p > 2$, we consider the functional set

$$\Xi(p,d) := \left\{ u \in W^{1,p}(B_{1/2}) \mid \Delta_p u = 0 \ \text{ in } B_{1/2} \right\},$$

where Δ_p denotes the p-laplacian operator and the equation is interpreted in the weak sense. In order to announce our main result, we need a definition.

Definition 2.1. Given a number $0 < \alpha < 1$ and $t \in (0, 1/2)$, we define

$$\omega_\alpha(t) := \sup \left\{ \frac{|u(x) - [u(0) + \nabla u(0) \cdot x]|}{\|u\|_{L^\infty(B_1)} \cdot t^{1+\alpha}} \mid x \in B_t \text{ and } u \in \Xi(p,d) \right\}.$$

Finally, we set

$$\alpha_M := \sup \left\{ \alpha \in (0,1) \mid \inf_t \omega_\alpha(t) < 1 \right\}. \tag{2.1}$$

Note that the above definition does not restrict the analysis to the origin; it rather allows for a local inspection. It is well known, see for instance [20], that there exists an exponent $0 < \alpha(d,p) < 1$, and a constant $C = C(d,p) > 1$, such that

$$|u(x) - [u(0) + \nabla u(0) \cdot x]| \leq C \cdot \|u\|_{L^\infty(B_1)} \cdot |x|^{1+\alpha(d,p)},$$

for any $u \in \Xi(d,p)$ and all $x \in B_{1/4}$. Hence, it follows easily that

$$\alpha_M \geq \alpha(d,p) > 0.$$

The above setup, together with the new oscillation control to be proven in section 4, fosters a friendly platform to treat common issues related to sharp regularity estimates for the p-Poisson equation. In practical applications, it so often happens that further information is known about the solution. If such a property P is closed under the C^1 topology, the analysis we shall develop in this paper allows us to impose it to the *tangential* equation, restricting henceforth the tangential space,

$$\Xi_{\mathrm{P}}(p,d) := \left\{ u \in W^{1,p}(B_{1/2}) \;\middle|\; u \text{ satisfies P and } \Delta_p u = 0 \right\},$$

which ultimately increases the value of α_M. This powerful insight will be explored in section 6. If no further information is a priori known, our main result reads as follows.

Theorem 2.2. *Let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f(x)$, with $f \in L^\infty(B_1)$ and $\gamma \in (0, \frac{1}{p-1}] \cap (0, \alpha_M)$. Then $u \in C^{1+\gamma}(B_{1/2})$ and*

$$\|u\|_{C^{1+\gamma}(B_{1/2})} \leq C_{d,p,\gamma} \left(\|f\|_{L^\infty(B_1)}^{\frac{1}{p-1}} + \|u\|_{L^\infty(B_1)} \right),$$

where, $C_{d,p,\gamma} > 1$ depends only on dimension, p and γ.

3. Existence of C^1-small correctors

In this section, we show that if u is a normalized solution of

$$-\Delta_p u = f(x),$$

and $\|f\|_\infty \ll 1$, then we can find a C^1 corrector ξ, with $\|\xi\|_{C^1} \ll 1$, such that $u + \xi$ is p-harmonic. This will allow us to frame the $C^{p'}$ conjecture into the formalism of the so called geometric tangential analysis, e.g. [5], [1, 2] and [13, 14, 15, 16, 17, 18]. Here is the precise statement.

Lemma 3.1. *Let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$. Given $\epsilon > 0$, there exists $\delta = \delta(p,d,\epsilon) > 0$ such that if $\|f\|_\infty \leq \delta$ then we can find a corrector $\xi \in C^1(B_{1/2})$, with*

$$|\xi(x)| \leq \epsilon \quad \text{and} \quad |\nabla \xi(x)| \leq \epsilon, \quad \text{in } B_{1/2} \tag{3.1}$$

such that

$$-\Delta_p(u+\xi) = 0 \quad \text{in } B_{1/2}. \tag{3.2}$$

Proof: Suppose the result does not hold. We can then find $\epsilon_0 > 0$ and sequences of functions (u_j) and (f_j) in $W^{1,p}(B_1)$ and $L^\infty(B_1)$, respectively, such that

$$-\Delta_p u_j = f_j \quad \text{in } B_1; \qquad \|u_j\|_\infty \leq 1; \qquad \|f_j\|_\infty \leq 1/j$$

but, nonetheless, for every $\xi \in C^1(B_{1/2})$ such that

$$-\Delta_p(u_j + \xi) = 0 \quad \text{in } B_{1/2},$$

we have either $|\xi(x_0)| > \epsilon_0$ or $|\nabla \xi(x_0)| > \epsilon_0$, for a certain $x_0 \in B_{1/2}$.

From classical estimates for the p-Poisson equation, we can extract a subsequence, such that, upon relabelling,

$$u_j \longrightarrow u_\infty$$

in $C^1(B_{1/2})$ as $j \to \infty$. Passing to the limit in the pde, we obtain

$$-\Delta_p u_\infty = 0 \quad \text{in } B_{1/2}, \qquad \text{with} \quad \|u_\infty\|_\infty \leq 1.$$

Now, let $\xi_j := u_\infty - u_j$. For $j_* \gg 1$, we have

$$-\Delta_p(u_{j_*} + \xi_{j_*}) = -\Delta_p u_\infty = 0 \quad \text{in } B_{1/2}$$

and

$$|\xi_{j_*}(x)| \leq \epsilon_0 \quad \text{and} \quad |\nabla \xi_{j_*}(x)| \leq \epsilon_0, \quad \forall x \in B_{1/2},$$

thus reaching a contradiction. ■

We conclude this section by commenting that in order to prove Theorem 2.2 it is enough to establish it for normalized solutions with small RHS, i.e., with $\|f\|_\infty \leq \delta_0$. Indeed, if u verifies $-\Delta_p u = f(x)$, with $f \in L^\infty$, then the function

$$v(x) := \frac{u(\theta x)}{\|u\|_\infty}$$

is obviously normalized and

$$-\Delta_p v = \frac{\theta^p}{\|u\|_\infty^{p-1}} f(\theta x).$$

Thus, choosing

$$\theta := \sqrt[p]{\frac{\delta_0 \|u\|_\infty^{p-1}}{\|f\|_\infty}},$$

v satisfies (1.1), with small RHS. Once Theorem 2.2 is proven for v, it immediately gives the corresponding estimate for u.

4. Analysis on the critical set

In this section, based on an iterative reasoning, we establish a key tool that allows us to prove the main result of this work.

Hereafter we fix a number

$$\gamma \in \left(0, \frac{1}{p-1}\right] \cap (0, \alpha_M)$$

and denote by μ_γ the average between $\inf\limits_{t \in (0, \frac{1}{2})} \omega_\gamma(t)$ and 1, that is:

$$\mu_\gamma := \frac{1 + \inf\limits_{t \in (0, \frac{1}{2})} \omega_\gamma(t)}{2} < 1. \tag{4.1}$$

The following result is the first step in the iteration.

Lemma 4.1. *There exists $0 < \lambda_0 < 1/2$ and $\delta_0 > 0$ such that if $\|f\|_\infty \leq \delta_0$ and $u \in W^{1,p}(B_1)$ is a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$, then*

$$\sup_{x \in B_{\lambda_0}} \Big| u(x) - [u(0) + \nabla u(0) \cdot x] \Big| \leq {\lambda_0}^{1+\gamma}.$$

Proof: Take $\epsilon > 0$ to be fixed later, apply the previous lemma to find δ_0 and, under the smallness assumption on f, a respective corrector ξ satisfying (3.1) and (3.2). From construction, there exists $\lambda_0 < 1/2$, such that $\omega_\gamma(\lambda_0) < \mu_\gamma$, and, since $(u + \xi)$ is p-harmonic in $B_{1/2}$, we can estimate

$$\sup_{B_{\lambda_0}} |u + \xi| \leq \mu_\gamma (1 + \epsilon) \lambda_0^{1+\gamma}.$$

We further estimate in B_{λ_0}:

$$\begin{aligned} |u(x) - [u(0) + \nabla u(0) \cdot x]| &\leq |(u + \xi)(x) - [(u + \xi)(0) + \nabla (u + \xi)(0) \cdot x]| \\ &\quad + |\xi(x)| + |\xi(0)| + |\nabla \xi(0) \cdot x| \\ &\leq \mu_\gamma (1 + \epsilon) \lambda_0^{1+\gamma} + 3\epsilon. \end{aligned}$$

Finally, by continuity, we can choose ϵ universally small such that

$$\mu_\gamma (1 + \epsilon) \lambda_0^{1+\gamma} + 3\epsilon = \lambda_0^{1+\gamma},$$

which determines the smallness assumption on $\|f\|_\infty$ – the constant $\delta_0 > 0$ in the statement of the current lemma – through the conclusion of Lemma 3.1, and the proof is complete. ∎

The conclusion of Lemma 4.1 does not, *per se*, allow an iteration since no obvious PDE is satisfied by $u + \ell$, when ℓ is an affine function. Nonetheless, it provides the following information on the oscillation of u in B_{λ_0}.

Corollary 4.2. *Under the assumptions of the previous lemma,*

$$\sup_{x \in B_{\lambda_0}} |u(x) - u(0)| \leq {\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0.$$

Proof: This is an immediate application of the triangular inequality. ■

The idea is now to iterate Corollary 4.2 in dyadic balls, keeping a precise track on the magnitude of the influence of $|\nabla u(0)|$.

Theorem 4.3. *Under the same assumptions of Lemma 4.1, there exists a constant $C > 1$, depending only on p and γ, such that*

$$\sup_{x \in B_r} |u(x) - u(0)| \leq C r^{1+\gamma} \left(1 + |\nabla u(0)|\, r^{-\gamma}\right),$$

holds for all $r > 0$.

Proof: We proceed by geometric iteration. Consider the universal constants λ_0 and δ_0 obtained in Lemma 4.1 and let

$$v(x) = \frac{u(\lambda_0 x) - u(0)}{{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0}, \qquad x \in B_1. \tag{4.2}$$

We have $\|v\|_\infty \leq 1$, $v(0) = 0$, and

$$\nabla v(0) = \frac{\lambda_0}{{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0}\, \nabla u(0).$$

Also, one easily estimates

$$|\Delta_p v| = \frac{\lambda_0^p}{\left({\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0\right)^{p-1}}\, |f(\lambda_0 x)| \leq \frac{\lambda_0^p}{{\lambda_0}^{(1+\gamma)(p-1)}}\, |f(\lambda_0 x)| \leq \delta_0,$$

which entitles v to Lemma 4.1. Thus

$$\sup_{x \in B_{\lambda_0}} |v(x) - v(0)| \leq {\lambda_0}^{1+\gamma} + |\nabla v(0)|\lambda_0,$$

which reads

$$\sup_{x \in B_{\lambda_0}} \left| \frac{u(\lambda_0 x) - u(0)}{{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0} \right| \leq {\lambda_0}^{1+\gamma} + \left| \frac{\lambda_0}{{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0}\, \nabla u(0) \right| \lambda_0,$$

and hence

$$\sup_{x\in B_{\lambda_0^2}} |u(x)-u(0)| \le {\lambda_0}^{1+\gamma}\Big[{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0\Big] + |\nabla u(0)|\,\lambda_0^2.$$

In the sequel, we define

$$a_k := \sup_{x\in B_{\lambda_0^k}} |u(x)-u(0)|,$$

and set

$$b_k := \frac{a_k}{\lambda_0^{k(1+\gamma)}}.$$

Iterating the previous reasoning, we obtain the recurrence law

$$a_{k+1} \le {\lambda_0}^{1+\gamma} a_k + |\nabla u(0)|\,\lambda_0^{k+1}.$$

Consequently, we estimate

$$b_{k+1} = \frac{a_{k+1}}{\lambda_0^{(k+1)(1+\gamma)}} \le \frac{{\lambda_0}^{1+\gamma} a_k + |\nabla u(0)|\,\lambda_0^{k+1}}{\lambda_0^{(k+1)(1+\gamma)}} = b_k + |\nabla u(0)|\,\lambda_0^{-(k+1)\gamma}.$$

Now, given $0<r\ll\lambda_0$, let $k\in\mathbb{N}$ be such that $\lambda_0^{k+1}<r\le\lambda_0^k$. Then

$$\begin{aligned}
\sup_{x\in B_r}\frac{|u(x)-u(0)|}{r^{1+\gamma}} &\le \sup_{x\in B_{\lambda_0^k}}\frac{|u(x)-u(0)|}{(\lambda_0^{k+1})^{1+\gamma}} = \frac{b_k}{\lambda_0^{1+\gamma}}\\
&\le \frac{b_0 + |\nabla u(0)|\displaystyle\sum_{i=1}^{k}\lambda_0^{-\gamma i}}{\lambda_0^{1+\gamma}}\\
&= \frac{a_0 + |\nabla u(0)|\,\lambda_0^{-\gamma}\,\frac{{\lambda_0}^{-\gamma k}-1}{{\lambda_0}^{-\gamma}-1}}{\lambda_0^{1+\gamma}}\\
&\le 2 + C(\lambda_0,\gamma)\,|\nabla u(0)|\,r^{-\gamma}\\
&\le C\left(1+|\nabla u(0)|\,r^{-\gamma}\right),
\end{aligned}$$

as desired. Observe that, γ being fixed, the constant $\lambda_0>0$ is universal. ∎

In accordance to [15], Theorem 4.3 provides the aimed regularity along the set of critical points of u, $|\nabla u|^{-1}(0)$. In fact, when $|\nabla u(0)|\le r^\gamma$, Theorem

4.3 gives

$$\begin{aligned}\sup_{x\in B_r}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big| &\le \sup_{x\in B_r}|u(x)-u(0)|+|\nabla u(0)|\,r\\ &\le (C+1)r^{1+\gamma}.\end{aligned}$$

In the next section we show how Theorem 4.3 can be used in its full strength to yield the aimed regularity at any point, regardless of the value of $|\nabla u|$; it will be a softer analysis.

5. Analysis on the set of non-degenerate points

We now analyze the oscillation decay around points where the gradient is large. Recall our ultimate goal is to show that

$$\sup_{x\in B_r}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big|\le C\,r^{1+\gamma},\quad \forall\, 0<r\ll 1.$$

For large values of $|\nabla u|$, the operator is uniformly elliptic and hence stronger estimates are available. Assume then $|\nabla u(0)|>r^\gamma$, define $\mu:=|\nabla u(0)|^{1/\gamma}$ and take

$$w(x):=\frac{u(\mu x)-u(0)}{\mu^{1+\gamma}}.$$

Clearly

$$w(0)=0,\quad |\nabla w(0)|=1\quad\text{and}\quad |\Delta_p w|\le |f(\mu x)|\,.$$

Moreover, from Theorem 4.3, it follows that

$$\sup_{x\in B_1}|w(x)|=\sup_{x\in B_\mu}\frac{|u(x)-u(0)|}{\mu^{1+\gamma}}\le C,$$

since $\mu^\gamma=|\nabla u(0)|$. From classical $C^{1,\beta}$ regularity estimates, there exists a radius ρ_0, depending only on the data, such that

$$|\nabla w(x)|\ge\frac{1}{2},\quad \forall\, x\in B_{\rho_0}.$$

This implies that, in B_{ρ_0}, w solves a uniformly elliptic equation. In particular, we have

$$w\in C^{1,\beta}(B_{\rho_0}),\quad\text{for some}\ \ \gamma\le\beta<1.$$

As an immediate consequence,

$$\sup_{x\in B_r}\Big|w(x)-\nabla w(0)\cdot x\Big|\le C\,r^{1+\beta},\quad \forall\, 0<r<\frac{\rho_0}{2}$$

which, in terms of u, reads

$$\sup_{x\in B_r}\left|\frac{u(\mu x)-u(0)}{\mu^{1+\gamma}}-\mu^{-\gamma}\nabla u(0)\cdot x\right|\le C\,r^{1+\beta}.$$

Since $\gamma\le\beta$, we conclude

$$\sup_{x\in B_r}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big|\le C\,r^{1+\gamma},\quad \forall\, 0<r<\mu\frac{\rho_0}{2}.$$

Finally, for $\mu\frac{\rho_0}{2}\le r<\mu$, we have

$$\begin{aligned}\sup_{x\in B_r}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big| &\le \sup_{x\in B_\mu}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big|\\ &\le \sup_{x\in B_\mu}|u(x)-u(0)|+|\nabla u(0)|\,\mu\\ &\le (C+1)\mu^{1+\gamma}\\ &\le C\left(\frac{2r}{\rho_0}\right)^{1+\gamma}\\ &= Cr^{1+\gamma}.\end{aligned}$$

In view of the reduction discussed at the end of section 3, the proof of Theorem 2.2 is complete. □

6. Special scenarios

Theorem 2.2 is linked to the $C^{p'}$-regularity conjecture in the following way: "if $\alpha_M>\frac{1}{p-1}$, then any function whose p-laplacian is bounded is of class $C^{p'}$, and the conjecture is verified." While it seems hard to verify that the strict inequality holds in general, in this section we explore some particular scenarios in which further information can be inferred.

6.1. Low dimensions. We start off by observing that p-harmonic functions in the real line are affine functions, thus in $1d$, $\alpha_M=1$ and hence Theorem 2.2 provides a proof of the $C^{p'}$-regularity conjecture in the line — a result that could perhaps be established by softer tools. In any case, for the sake of completeness, we write this conclusion as a proposition.

Proposition 6.1. *The $C^{p'}$-regularity conjecture holds true in the real line.*

Such result becomes more appealing when applied to the analysis of d-dimensional problems which carry some sort of symmetry. By way of example, we mention here the theory of (degenerate) phase transition problems, namely entire solutions of

$$\Delta_p u = (1-u^2)^p, \tag{6.1}$$

satisfying $\partial_{x_d} u > 0$ and $\lim\limits_{x_d \to \pm\infty} u(\cdot, x_d) = \pm 1$. Clearly, by monotonicity in the x_d-variable, any solution to (6.1) is smooth; however, as no uniform lower bound on $\partial_{x_d} u$ is granted, such smoothness could deteriorate. Notwithstanding, by a striking result from [12], if $d \le 8$ or $\{u = 0\}$ has at most linear growth at infinity, then level sets are hyperplanes and thus it follows by the results established in the current paper that we can locally bound the $C^{p'}$-norm of u uniformly, i.e., independently of the inf $\partial_{x_d} u$.

Next, for flatland problems, $d = 2$, more can be said about the underlying regularity theory for the p-Poisson equation. It is well known, see [3], that the complex gradient of a p-harmonic function in $2d$ is K-quasiregular, for

$$K = K(p) = \frac{1}{2}\left[(p-1) + \frac{1}{p-1}\right].$$

In particular, if u is p-harmonic in the unit disk $\mathbb{D}_1 \subset \mathbb{R}^2$, there exists a constant $C = C(p)$ such that

$$\|u\|_{C^{1+\frac{1}{p-1}}(\mathbb{D}_{1/2})} \le C\|u\|_{L^\infty(\mathbb{D}_1)}.$$

Hence, in dimension two, $\alpha_M \ge \frac{1}{p-1}$ and thus a direct application of Theorem 2.2 yields an asymptotic version of the $C^{p'}$ regularity conjecture, namely a solution to the p-Poisson equation (1.1) is locally of class $C^{p'^-}$ (i.e., belongs to $C^{p'-\epsilon}$, for every $\epsilon > 0$). We write up this conclusion as a proposition – compare with the result from [9] and see also [8].

Proposition 6.2. *Let $p > 2$ and $u \in W^{1,p}(\mathbb{D}_1)$ be a two-dimensional weak solution of $-\Delta_p u = f(x)$, with $f \in L^\infty(\mathbb{D}_1)$. Given any number $0 < \gamma < \frac{1}{p-1}$, there exists a constant $C = C(p,\gamma) > 1$ such that*

$$\|u\|_{C^{1+\gamma}(\mathbb{D}_{1/2})} \le C\left(\|f\|_{L^\infty(\mathbb{D}_1)}^{\frac{1}{p-1}} + \|u\|_{L^\infty(\mathbb{D}_1)}\right).$$

By means of a hodograph transformation, Iwaniec and Manfredi in [7] give explicit estimates for the Hölder continuity exponent of the gradient of p-harmonic functions in the plane which are beyond the threshold $\frac{1}{p-1}$. Unfortunately, the estimates in [7] are merely qualitative, i.e., no universal control is provided through their analysis.

Still in the plane, Evans and Savin proved in [6] (see also [11]) that infinity harmonic functions, i.e., viscosity solutions of

$$\Delta_\infty u := u_{x_i x_j} u_{x_i} u_{x_j} = 0,$$

are locally of class $C^{1,\gamma}$ for some $0 < \gamma < 1/3$. This result also connects to the $C^{p'}$-regularity conjecture. Indeed, within the framework setup in section 2, the Evans-Savin Theorem suggests the possibility of proving that α_M is bounded below, uniformly with respect to p. Once this is confirmed, the $C^{p'}$-regularity conjecture is solved for $p \gg 1$. We plan to come back to this issue in a forthcoming paper.

6.2. Problems with symmetry. Continuing our analysis, in view of the extremal example mentioned in (1.2), it is only natural to inquire about problems having radial symmetry. Taking full advantage of our general setup, the key observation is that the functional set

$$\Xi_{\rm rad}(p,d) := \left\{ u \in W^{1,p}(B_{1/2}) \;\middle|\; \Delta_p u = 0 \text{ and } u \text{ is bounded and radial} \right\}$$

contains only constants. Indeed, if $u(x) = \varphi(r)$, then

$$\Delta_p u = |\varphi'(r)|^{p-2} \left\{ (p-1)\varphi''(r) + \frac{d-1}{r}\varphi'(r) \right\}.$$

Solving the homogeneous ODE, we obtain

$$\varphi(r) = \begin{cases} a + b \cdot r^{\frac{1-d}{p-1}+1} & \text{if } p \neq d, \\ a + b \cdot \ln r & \text{if } p = d, \end{cases}$$

for constants $a, b \in \mathbb{R}$. For $d \geq 2$, φ is C^1 at the origin if, and only if, $b = 0$. As a consequence, when restricted to the set of radially symmetric functions, one has

$$\alpha_M^{\rm rad} = 1,$$

and therefore we are able to establish the following version of the $C^{p'}$-regularity conjecture for radially symmetric functions.

Theorem 6.3. *Let $u \in W^{1,p}(B_1)$ be a radially symmetric function whose p-laplacian is bounded. Then $u \in C^{p'}(B_{1/2})$, with universal estimates.*

Proof: We start off by revisiting the proof of the existence of C^1-small correctors, Lemma 3.1. If we add the extra information that u is radially symmetric and carry out the compactness argument in the proof, we end up with the existence of a radially symmetric corrector, ξ_{rad}, for which $u + \xi_{\mathrm{rad}}$ is p-harmonic, and $\|\xi_{\mathrm{rad}}\|_{C^1} \leq \epsilon$, provided $\|f\|_\infty$ is small enough. As commented above, $u+\xi_{\mathrm{rad}}$ must be constant. Next, we prove the radial version of Lemma 4.1.

Lemma 6.4. *Let $u \in W^{1,p}(B_1)$ be a radially symmetric weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$. There exists $\delta_0 > 0$ such that if $\|f\|_\infty \leq \delta_0$ then*

$$\sup_{x \in B_{\frac{1}{7}}} \Big| u(x) - [u(0) + \nabla u(0) \cdot x] \Big| \leq \left(\frac{1}{7}\right)^{p'}.$$

Proof: For $\epsilon > 0$ to be fixed later, let ξ_{rad} be a radially symmetric corrector for u with $\|\xi_{\mathrm{rad}}\|_{C^1} \leq \epsilon$. Since $(u + \xi_{\mathrm{rad}})$ is constant, we can estimate in $B_{1/7}$,

$$\begin{aligned} |u(x) - [u(0) + \nabla u(0) \cdot x]| &\leq |(u + \xi_{\mathrm{rad}})(x) - [(u + \xi_{\mathrm{rad}})(0) \\ &\quad + \nabla (u + \xi_{\mathrm{rad}})(0) \cdot x]| \\ &\quad + |\xi_{\mathrm{rad}}(x)| + |\xi_{\mathrm{rad}}(0)| + |\nabla \xi_{\mathrm{rad}}(0) \cdot x| \\ &\leq 3\epsilon. \end{aligned}$$

Finally, we choose $\epsilon = \frac{1}{3 \cdot 7^{p'}}$, which determines $\delta_0 > 0$ – the smallness condition on $\|f\|_\infty$ –, and the proof is concluded. ■

We continue the proof of Theorem 6.3 under the assumptions of Lemma 6.4. Note that when u is radially symmetric, then so is

$$v(x) = \frac{u(\frac{1}{7}x) - u(0)}{\frac{1}{7^{p'}} + |\nabla u(0)| \frac{1}{7}}, \qquad x \in B_1.$$

The triangular inequality applied to Lemma 6.4 assures $|v| \leq 1$. As before, one verifies that

$$|\Delta_p v| \leq \delta_0,$$

and thus v is also entitled to the conclusion of Lemma 6.4. In summary, we can carry on with the proof of Theorem 4.3, which ultimately provides the following oscillation control for u:

$$\sup_{x \in B_r} |u(x) - u(0)| \leq C r^{p'} \left(1 + |\nabla u(0)| \, r^{\frac{1}{1-p}}\right), \quad \forall r > 0. \tag{6.2}$$

In particular, this key estimate implies the aimed $C^{p'}$ regularity estimate at critical points and allows for the analysis carried out in section 5, which finally completes the proof of Theorem 6.3. ■

6.3. Problems with controlled singular set. In many applications, the aimed sharp estimate expected from the $C^{p'}$-regularity conjecture needs only to be verified along the set of critical points. This is particularly meaningful in the theory of free boundary problems. In this section we pursue a general analysis which in particular gives pointwise $C^{p'}$ estimates at local maxima or local minima of solutions of (1.1).

Definition 6.5. Given a positive number $\sigma > 0$, a function $\varphi\colon B_1 \to \mathbb{R}$ is said to be a σ-cap at 0 if

$$\sup_{B_r} |\varphi(x) - \varphi(0)| \leq C|x|^{\sigma},$$

for a constant $C > 0$. The infimum among all constants is denoted by $[\varphi]_\sigma$.

Theorem 6.6. *Let u be a bounded solution of $-\Delta_p u = f(x)$ in B_1, with $f \in L^\infty(B_1)$. Let ξ_0 be an interior point and suppose u can be touched from below at ξ_0 by a $p\prime$-cap φ. Then u is precisely $C^{p'}$ continuous at ξ_0, that is,*

$$|u(x) - u(\xi_0)| \leq C|x - \xi_0|^{p'},$$

for a positive constant $C > 0$ that depends only on dimension, p, $\|u\|_{L^\infty(B_1)}$, $\|f\|_{L^\infty(B_1)}$ and $[\varphi]_{p\prime}$.

Theorem 6.6 has an analogous version for points that one can touch from above by a $p\prime$-cap. An immediate consequence is that solutions to p-Poisson equations are $C^{p\prime}$-regular at any local extremal point, where in fact one can touch u by a hyperplane. The proof of Theorem 6.6 is based on a flattening argument, which goes along the lines of the arguments developed in section 4.

Lemma 6.7. *Given $\eta > 0$, there exists $\delta > 0$, depending only on η and universal parameters, such that if $|v| \leq 1$ in B_1 and $-\Delta_p v = f(x)$ in B_1, then*

$$\|f\|_{L^\infty(B_1)} + \left(v(0) - \inf_{B_1} v \right) \leq \delta,$$

implies

$$\operatorname*{osc}_{B_{1/2}} v \leq \eta.$$

Proof: Suppose, for the sake of contradiction, that the thesis of the lemma fails to hold. This means we can find $\eta_0 > 0$, a sequence of positive numbers $\kappa_j \to 0$, and sequences of functions $(v_j)_{j\in\mathbb{N}}$, $(f_j)_{j\in\mathbb{N}}$ satisfying

a) $|v_j| \le 1$, and $\|f_j\|_{L^\infty(B_1)} + \left(v_j(0) - \inf\limits_{B_1} v_j\right) \le \kappa_j$;
b) $-\Delta_p v_j = f_j(x)$ in B_1;
c) $\operatorname*{osc}\limits_{B_{1/2}} v_j \ge \eta_0$.

By compactness, v_j converges in the C^1-topology to a function v_∞ in $B_{1/2}$ and from stability we conclude that v_∞ solves the homogeneous equation

$$-\Delta_p v_\infty = 0, \quad \text{ in } B_{1/2}.$$

From a) we immediately conclude v_∞ attains its minimum value at 0, and hence, by the strong maximum principle of Vázquez ([21]), we conclude that $v_\infty \equiv v_\infty(0)$. This gives a contradiction with c) if we take $j \gg 1$. The lemma is proven. ∎

Proof of Theorem 6.6. With no loss of generality, we can assume ξ_0 is the origin and $u(0) = 0$. Let $\varphi(x)$ be the $p\prime$-cap touching u at 0 from below. Define $v(x) = u(\lambda x)$. One simply checks that v verifies

$$-\Delta_p v(x) = \lambda^p f(\lambda x).$$

Owing to Lemma 6.7, we choose $\lambda > 0$ such that

$$2\lambda^p \|f\|_{L^\infty(B_1)} \le \delta_\star, \tag{6.3}$$

where $\delta_\star > 0$ is the closeness number given by Lemma 6.7 when one takes $\eta = 2^{-p\prime}$. From uniform Lipchitz continuity,

$$|v(x)| = |u(\lambda x) - u(0)| \le C(\|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})\lambda.$$

Hence, selecting a smaller $\lambda > 0$ if necessary, we can assume $|v| \le 1$, for all $x \in B_1$. Yet by uniform continuity,

$$\begin{aligned}\left(v(0) - \inf_{B_1} v\right) &= -\inf_{B_\lambda} u \\ &\le \frac{\delta_\star}{2},\end{aligned}$$

if $\lambda > 0$ is once more diminished, if necessary. Finally, v is touched from below by the $p\prime$-cap $\tilde{\varphi}(x) := \varphi(\lambda x)$. We can estimate

$$\sup_{B_r} \tilde{\varphi} \leq [\varphi]_{p\prime} \lambda^{p\prime} \cdot r^{p\prime}.$$

Thus, if we take $\lambda > 0$ even smaller, if necessary, we can further assume

$$[\tilde{\varphi}]_{p\prime} \leq \frac{\delta_\star}{2}.$$

Now, we aim to prove that for any $j \in \mathbb{N}$, there holds

$$\operatorname*{osc}_{B_{2^{-j}}} v(x) \leq 2^{-jp\prime}. \tag{6.4}$$

We argue by finite induction. By our previous selections, Lemma 6.7 gives the first step in the induction process. Suppose we have verified (6.4) for $j = 1, 2, \cdots, k$. Define

$$v_{k+1}(x) := 2^{kp\prime} \cdot v(2^{-k} x).$$

Initially, from the induction process we readily verify that $v_{k+1}(0) = 0$, $|v_{k+1}| \leq 1$ and

$$\begin{array}{rcl} |\Delta_p v_{k+1}| & \leq & \lambda^p \left| f(2^{-k}\lambda x) \right| \\ & \leq & \dfrac{\delta_\star}{2}, \end{array} \tag{6.5}$$

by the decision made in (6.3). Also, from the $p\prime$-cap control from below, we can estimate

$$\begin{array}{rcl} \left(v_{k+1}(0) - \inf\limits_{B_1} v_{k+1} \right) & = & -2^{kp\prime} \inf\limits_{B_{2^{-k}}} v \\ & \leq & -2^{kp\prime} \inf\limits_{B_{2^{-k}}} \varphi \\ & \leq & \dfrac{\delta_\star}{2}. \end{array}$$

We can now apply Lemma 6.7 to v_{k+1}, which yields

$$\operatorname*{osc}_{B_{1/2}} v_{k+1}(x) = 2^{kp\prime} \operatorname*{osc}_{B_{2^{-k}}} v(x) \leq 2^{-p\prime}, \tag{6.6}$$

and the induction chain is complete. Finally, given any $0 < r \ll 1$, let $k \in \mathbb{N}$ be such that

$$2^{-k-1} < r \leq 2^{-k}.$$

We estimate, defining $\rho = \lambda r$,

$$\begin{aligned} \operatorname*{osc}_{B_\rho} u &= \operatorname*{osc}_{B_r} v \\ &\leq \operatorname*{osc}_{B_{2^{-k}}} v \\ &\leq 2^{-kp'} \\ &\leq \left(\frac{r}{2}\right)^{p'} \\ &\leq \frac{1}{(2\lambda)^{p'}} \cdot \rho^{p'}, \end{aligned}$$

and the theorem is proven. ∎

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